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Convex central configurations for the n-body problem ☆

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Abstract

We give a simple proof of a classical result of MacMillan and Bartky (Trans. Amer. Math. Soc. 34 (1932) 838) which states that, for any four positive masses and any assigned order, there is a convex planar central configuration. Moreover, we show that the central configurations we find correspond to local minima of the potential function with fixed moment of inertia. This allows us to show that there are at least six local minimum central configurations for the planar four-body problem. We also show that for any assigned order of five masses, there is at least one convex spatial central configuration of local minimum type. Our method also applies to some other cases.

Keywords: n-Body problem; Central configurations; Relative equlibrium; Morse theory

1. Introduction

Central configurations play an important role in the study of the Newtonian n-body problem (cf. [8,9]). Let $m_1, \dots, m_n$ be n point masses moving in $\mathbb{R}^3$ and let $q_1, \dots, q_n$ in $\mathbb{R}^3$ be their positions. We say that the n bodies form a central configuration if there exists a constant λ such that

$$\lambda m_i q_i = \sum_{1 \leqslant j \leqslant n} \frac{m_i m_j}{|q_i - q_j|^3} (q_j - q_i)$$

for all $1 \leqslant i \leqslant n$. One can easily verify that a central configurations remains a central configuration after a rotation in $\mathbb{R}^3$ and a scalar multiplication. More precisely,

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let $A \in SO_3$ and $a > 0$, if $q = (q_1, \dots, q_n)$ is a central configuration, then so are $Aq = (Aq_1, \dots, Aq_n)$ and $aq = (aq_1, \dots, aq_n)$.

Two central configurations are said to be equivalent if one can be transformed to the other by a scalar multiplication and a rotation. In this paper, when we say a central configuration, we mean a class of central configurations as defined by the above equivalence relation.

The study of central configurations goes back to Euler and Lagrange. For $n = 3$, it is a classical result that there are three collinear, called Euler, central configurations and one equilateral triangular, called Lagrange, central configurations. For $n \geqslant 4$, Moulton [6] proved that there are exactly one collinear central configuration for each arrangement of the particles on the line. As for the planar case with given arbitrary n masses, little is known as far as the exact numbers and positions of the central configurations. Only very recently, we know the exact number and positions of the central configurations for four equal masses [1]. Moeckel [5] showed that for generic four masses, the number of central configurations is finite. On the other hand, for any given n, Xia [10] found the exact number of central configurations for some open sets of n positive masses.

One can reformulate the central configurations as critical points of certain functions. Let

$$I = \sum_{i=1}^{n} m_i |q_i|^2$$

be the moment of inertia of the n-body system and let

$$U = \sum_{1 \leqslant i < j \leqslant n} \frac{m_i m_j}{|q_i - q_j|}$$

be the potential function. Then the central configurations are the critical points of the function U on the ellipsoid

$$S = \{q = (q_1, \dots, q_n) \in \mathbb{R}^{3n} \mid I = 1\}.$$

The group SO_3 acts on the ellipsoid I and this action is free on all non-collinear configurations. The potential function U is invariant under this action. We say that a (non-collinear) central configuration is non-degenerate if it is a non-degenerate critical point of U on S/SO_3. Using Morse theory, one can obtain a lower bound on the number of central configurations when all central configurations are non-degenerate. See [3,7].

In this paper, we will use an equivalent definition of central configurations. One easily verifies that central configurations are critical points of the function IU^2 on R^{3n}.

We first consider the planar central configurations. A planar configuration for the n bodies is said to be (strictly) convex if the n masses form a (strictly) convex configuration in $\mathbb{R}^2$. Each convex configuration defines a cyclic order of the n bodies on S^1. Let σ be any such cyclic order and let R_σ be the set of all convex configuration with this cyclic order σ. Obviously, if $\sigma_1 \neq \sigma_2$, then $\text{int}(R_{\sigma_1}) \cap \text{int}(R_{\sigma_2}) = \emptyset$. For any fixed σ, the boundary of R_σ consists of configurations where three or more bodies are collinear. The number of distinct cyclic orders for n bodies is $(n-1)!$.

For the four body problem, MacMillan and Bartly [2] proved that for any cyclic order of the four bodies, there exist a convex central configuration with that order. Their proof is quite long and involved. In this note, we give a simple proof of this result and moreover, the convex central configurations we found are local minima of the function IU^2. We also expect our method to work for the problem with more than four bodies, at least for some open set of masses.

Theorem 1. *Fix four positive masses m_1, m_2, m_3 and m_4. For any fixed cyclic order σ of four bodies, the minimum of IU^2 over R_σ is always attained in the interior of R_σ. Thus there are at least $3! = 6$ local minimum central configurations for the planar four-body problem.*

Using the topology of the space I/SO_2, Palmore [7] obtained a lower bound on the number of planar, non-collinear central configurations when all central configurations are non-degenerate. McCord [3] further improved this lower bound. For the four body problem, this lower bound is 14. Note that we count the reflection of a planar, non-collinear central configuration as a distinct central configuration in $\mathbb{R}^2$. By Morse inequality, each extra local minimum central configuration we found here increases the lower bound estimate by two. The following theorem is a corollary of Theorem 1.

Theorem 2. *For any set of four positive masses, if all the central configurations are non-degenerate, then there are at least 22 planar, non-collinear central configurations.*

Similar results can be proved for the spatial central configurations. For the four-body problem, there are exactly two spatial (non-planar) central configurations where four bodies form a regular tetrahedron, with two different orientations. The first non-trivial case for the spatial central configurations is for the five body problem. Each convex arrangement defines an ordering for the five bodies. There are total of eight such distinct orderings of the spatial convex central configurations.

Theorem 3. *For any given five positive masses $m_1, \ldots, m_5$ and for any ordering of strictly convex configurations for the five bodies, there is a central configuration of local minimum type with that ordering.*

2. Proof of the theorems and other results

Our proof of the theorems is quite simple. Fix four positive masses $m_1, \ldots, m_4$ and a cyclic order σ. We may assume that the cyclic order is (1234). Suppose that the minimum of IU^2 is not attained in the interior of R_σ. Since IU^2 is invariant under dilation and R_σ with fixed I is contained in a compact set, the infimum of IU^2 must be attained at some point $q^* = (q_1^*, \ldots, q_4^*)$ in the boundary of R_σ. At least three of points in $q_1^*, \ldots, q_4^*$ must be collinear. Since collinear central configurations are not local minimizers (cf. [4]), this implies exactly three points, say, q_1^*, q_2^* and q_3^*, are collinear. It is easy to see that such minimizer can not take place at a collision. By rotating the configuration, we may assume that the line formed by q_1^*, q_2^* and q_3^* are parallel to and on the right side of the y-axis. The fourth body will be on the left of y-axis. More precisely, we have $x_1^* = x_2^* = x_3^* > x_4^*$ and $y_1^* < y_2^* < y_3^*$.

We claim that $\partial(IU^2)/\partial x_2 < 0$ at $q = q^*$. Suppose this is not true and $\partial(IU^2)/\partial x_2 \geqslant 0$ at $q = q^*$. We must have

$$2m_2 x_2^* U^2 + 2IU \frac{m_2 m_4 (x_4^* - x_2^*)}{|q_2^* - q_4^*|^3} \geqslant 0.$$

Now, we have either $y_4^* \geqslant y_2^*$ or $y_4^* \leqslant y_2^*$. We may assume that $y_4^* \geqslant y_2^*$. The other case can be dealt in the similar way. Thus $|q_1^* - q_4^*| > |q_2^* - q_4^*|$ and

$$2m_1 x_1^* U^2 + 2IU \frac{m_1 m_4 (x_4^* - x_1^*)}{|q_1^* - q_4^*|^3} > 0,$$

because $x_4^* - x_1^* < 0$. This implies that $\partial(IU^2)/\partial x_1 > 0$ at $q = q^*$. This implies that by reducing x_1, one can actually reduce IU^2. But reducing x_1 pushes q^* to the interior of R_σ. This contradicts to our choice of q^*. This contradiction proves our claim.

We therefore must have $\partial(IU^2)/\partial x_2 < 0$ at $q = q^*$. This implies that by increasing x_2 from q^*, one can actual reduce IU^2. But increasing x_2 pushes q^* into the interior of R_σ, this again contradicts to our choice of q^*. This contradiction implies that no such q^* exists.

This proves Theorem 1.

The proof Theorem 3 is similar. For any set of five positive masses $m_1, \ldots, m_5$, let R_σ be the set all convex configurations which correspond to an arrangement σ of the five masses. The boundary of R_σ consists of the configurations with four bodies in the same plan. We claim that the infimum of IU^2 over R_σ is attained in the interior of R_σ. Suppose that this is not true and the infimum is attained at $q^* \in \text{closure}(R_\sigma)$. Since co-planar configurations are not minimizers, without loss of generality, we may assume that $x_1^* = x_2^* = x_3^* = x_4^* > x_5^*$, and m_4 is in the closed triangle formed by m_1, m_2 and m_3.

Let's suppose that $\partial(IU^2)/\partial x_4 \geqslant 0$ at $q = q^*$. Since there exists at least one i, $i \in \{1, 2, 3\}$ such that $|q_i^* - q_5^*| > |q_4^* - q_5^*|$, we must have $\partial(IU^2)/\partial x_i > 0$ at $q = q^*$. This implies that at $q = q^*$ either $\partial(IU^2)/\partial x_4 < 0$ or $\partial(IU^2)/\partial x_i > 0$. Therefore we can decrease the value of IU^2 by either decreasing x_i or increasing x_4. In either case, one obtains a smaller value for IU^2 by moving to the interior of R_σ. This contradicts to the assumption that $IU^2(q^*)$ is the infimum of IU^2 over R_σ.

This proves Theorem 3.

We believe that our method also works for the planar five-body problem, or even planar and spatial n-body problem in general. However, this would requires careful estimates which we have not been able to carry out.

As an example, we consider four equal masses $m_i = 1$ for $1 \leqslant i \leqslant 4$ and a very small mass $m_5 = \mu$. For any cyclic order σ, say (15234), on five bodies, the minimizer of IU^2 over the closure of R_σ must have four big masses close to a square. How close it is to a square depends on the size of μ. As $\mu \to 0$, the shape of the first four bodies approaches to a square. Now, suppose that the infimum of IU^2 over R_σ is attained on the boundary, say at q^*. Then m_5 must be close to the middle of m_1 and m_2. For the limit case as $\mu \to 0$, we may choose the coordinates such that $q_1^* = (1, -1)$, $q_2^* = (1, 1)$, $q_3^* = (-1, 1)$, $q_4^* = (-1, -1)$ and $q_5^* = (1, 0)$. Now, we can easily verify that

$$\frac{(x_4^* - x_1^*)}{|q_1^* - q_4^*|^3} + \frac{(x_3^* - x_1^*)}{|q_1^* - q_3^*|^3} \equiv \frac{2}{2^3} + \frac{2}{(2\sqrt{3})^3} = 0.3384$$

is smaller than

$$\frac{(x_4^* - x_5^*)}{|q_5^* - q_4^*|^3} + \frac{(x_3^* - x_5^*)}{|q_5^* - q_3^*|^3} \equiv 2\frac{2}{(\sqrt{5})^3} = 0.3578.$$

This implies that either $\partial(IU^2)/\partial x_5 < 0$ at q^* or $\partial(IU^2)/\partial x_1 > 0$ at q^*. In either case, q^* cannot attain the infimum of IU^2 over R_σ.

This shows that for the five body problem, there exists $\varepsilon_1, \varepsilon_2 > 0$ such that if $|m_i - 1| \leqslant \varepsilon_1$, for $1 \leqslant i \leqslant 4$ and $m_5 \leqslant \varepsilon_2$, then for any cyclic order σ' of the five bodies, the minimum of IU^2 over R_σ is always attained in the interior of $R_{\sigma'}$. Therefore, for these masses, there are at least $4! = 24$ local minimum planar convex central configurations.

We give the above example to show that certain careful estimates are required for the five-body problem in general. If for example, instead of Newtonian inverse square force, we consider inverse k-power force with $k \geqslant 4$, then with four equal mass and one small mass, there is no convex central configurations.

We end the paper by stating the following conjecture.

Conjecture 4. *For any positive n masses*, $n \geqslant 5$, *and any cyclic order of n points on* S^1, *there is a convex planar central configuration of minimum type with the cyclic order. For* $n \geqslant 6$, *for any ordering of convex configurations of n points in the space*, *there is at least one spatial convex central configuration of local minimum type with that ordering*.

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