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C^1-stably expansive flows

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Abstract

In this paper, the C^1 interior of the set of vector fields whose integrated flows are expansive is characterized as the set of vector fields without singularities satisfying both Axiom A and the quasi-transversality condition, and it is proved that the above vector fields possessing the shadowing property must be structurally stable. As a corollary, there exists a non-empty C^1 open set of vector fields whose integrated flows do not have the shadowing property.

MSC: 37C27; 37C50; 37D20

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1. Introduction

We are interested in characterizing the geometrical structure of dynamical systems possessing a topological property of Anosov systems such as topological stability under

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the C^1 open condition (see [9]). The C^1 open condition signifies that the topological property under consideration is preserved with respect to C^1 small perturbations of the system.

In this paper, we consider the set of expansive flows (vector fields), and investigate its geometric structure from the above point of view. More precisely, the C^1 interior of the set of vector fields whose integrated flows are expansive is characterized as the set of vector fields without singularities satisfying both Axiom A and the quasi-transversality condition. Furthermore, we prove that such vector fields possessing the shadowing property must be structurally stable. As a corollary, it follows from Robinson's example (see [14]) that there exists a non-empty C^1 open set of vector fields whose integrated flows do not have the shadowing property.

Let M be a C^∞ closed manifold, and denote by $\mathcal{X}^1(M)$ the set of C^1 vector fields on M endowed with the C^1 topology. Denote by $\mathcal{E}(M)$ the set of $X \in \mathcal{X}^1(M)$ whose integrated flow is expansive, and by $\operatorname{int}\mathcal{E}(M)$ the C^1 interior of $\mathcal{E}(M)$ in $\mathcal{X}^1(M)$.

The following result is obtained.

Theorem A. *For $X \in \mathcal{X}^1(M)$, the following conditions are mutually equivalent:*
(i) $X \in \operatorname{int}\mathcal{E}(M)$,
(ii) *X is quasi-Anosov,*
(iii) *X has no singularities, and satisfies both Axiom A and the quasi-transversality condition.*

A similar result is obtained by Mañé in [7,8] for diffeomorphisms on M. When $\dim M = 3$, it is easy to see that every quasi-Anosov vector field on M is Anosov. Thus, every $X \in \operatorname{int}\mathcal{E}(M)$ is Anosov when $\dim M = 3$. However, in higher dimensions that is not true by Robinson's example (see [14]).

In the present paper, we also prove the following.

Theorem B. *For $X \in \mathcal{X}^1(M)$, the following conditions are mutually equivalent:*
(i) *$X \in \operatorname{int}\mathcal{E}(M)$ and has the shadowing property,*
(ii) *$X \in \operatorname{int}\mathcal{E}(M)$ and is structurally stable,*
(iii) *X is Anosov.*

In [15] the second author showed an analogue of the above theorem for diffeomorphisms by making use of a result proved in [8].

Let $X \in \mathcal{X}^1(M^{11})$ be Robinson's example of a quasi-Anosov vector field that is not Anosov on an 11-dimensional manifold M^{11} (for diffeomorphisms, see [2]). Since the set of quasi-Anosov vector fields is C^1 open in $\mathcal{X}^1(M)$ (see Remark 1), it is easy to see that every C^1 nearby system $Y \in \mathcal{X}^1(M^{11})$ of X is also quasi-Anosov but not Anosov by construction. Thus, combining these facts with Theorem B we have the following.

Corollary. *There exists a non-empty C^1 open set $\mathcal{U} \subset \mathcal{X}^1(M^{11})$ whose any element does not have the shadowing property.*

Thus the set of vector fields having the shadowing property on M is not C^1 dense in $\mathcal{X}^1(M)$ in general.

2. Preliminaries

Let M and $\mathcal{X}^1(M)$ be as before, and let d be the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Every $X \in \mathcal{X}^1(M)$ generates a C^1 flow $X_t : M \times \mathbb{R} \to M$; that is a C^1 map such that $X_t : M \to M$ is a diffeomorphism satisfying $X_0(x) = x$ and $X_{t+s}(x) = X_t(X_s(x))$ for all $s, t \in \mathbb{R}$, and $x \in M$.

We say that a (continuous) flow X_t is *expansive* if for any $\varepsilon > 0$ there is $\delta > 0$ with the property that if $d(X_s(x), X_{\alpha(s)}(y)) \leqslant \delta$ for all $s \in \mathbb{R}$, for a pair of points $x, y \in M$, and for a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = X_s(x)$ where $|s| \leqslant \varepsilon$ (see [1,16,17]). In this case, the above δ is called an *expansive constant* corresponding to ε (with respect to X_t).

An orientation preserving (increasing) homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$ is called a *reparametrization* of $\mathbb{R}$. Denote by $\mathrm{Rep}(\mathbb{R})$ the set of reparametrizations of $\mathbb{R}$. Given $\delta > 0$ and $a > 0$, a pair of sequences

$$(\{x_i\}_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty})$$

is called a (δ, a)-*pseudo-orbit* of X_t if

$$t_i \geqslant a \quad \text{and} \quad d(X_{t_i}(x_i), x_{i+1}) < \delta$$

for all $i \in \mathbb{Z}$. Let $s_0 = 0$, $s_n = \sum_{i=0}^{n-1} t_i$, and $s_{-n} = \sum_{i=-n}^{-1} t_i$ for any sequence $\{t_i\}_{i=-\infty}^{\infty} \subset \mathbb{R}$. Given $\varepsilon > 0$, a (δ, a)-pseudo-orbit $(\{x_i\}_{i=-\infty}^{\infty}, \{t_i\}_{i=-\infty}^{\infty})$ is ε-*shadowed* by an orbit $X_{\mathbb{R}}(z) = \{X_t(z) : t \in \mathbb{R}\}$ $(z \in M)$ if there exists $\alpha \in \mathrm{Rep}(\mathbb{R})$ such that

$$d(X_{\alpha(t)}(z), X_{t-s_n}(x_n)) < \varepsilon$$

whenever $t \geqslant 0$ and $s_n \leqslant t < s_{n+1}$ for all $n \geqslant 0$, and

$$d(X_{\alpha(t)}(z), X_{t+s_{-n}}(x_{-n})) < \varepsilon$$

whenever $t \leqslant 0$ and $-s_{-n} \leqslant t \leqslant -s_{-n+1}$ for all $n \geqslant 1$.

We say that a flow X_t has the *shadowing property* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that every $(\delta, 1)$-pseudo-orbit is ε-shadowed by some orbit of X_t (see [10,16,17]).

Let $X \in \mathcal{X}^1(M)$ have no singularities, and let $\mathcal{N} \subset TM$ be the subbundle such that the fiber N_x at $x \in M$ is the orthogonal linear subspace of $\langle X(x) \rangle$ in T_xM; that is, $N_x = \langle X(x) \rangle^{\perp}$. Here $\langle X(x) \rangle$ is the linear subspace spanned by $X(x)$ for $x \in M$. Let $\pi : TM \to \mathcal{N}$ be the projection along X, and let

$$F_x^t(v) = \pi(D_x X_t(v))$$

for $v \in N_x$ and $x \in M$. It is well known that $F^t : \mathcal{N} \to \mathcal{N}$ is a one-parameter transformation group (cf. [6]).

We say that $X \in \mathcal{X}^1(M)$ is *quasi-Anosov* if X has no singularities and for $v \in \mathcal{N}$, if $\sup_{t\in\mathbb{R}} \|F^t(v)\| < \infty$, then $v = 0$ (see [14]).

Let X_t be the flow of $X \in \mathcal{X}^1(M)$, and let Λ be a X_t-invariant compact set. The Λ is called *hyperbolic* for X_t if there are constants $C > 0$, $\lambda > 0$ and a splitting $T_xM = E^s_x \oplus \langle X(x)\rangle \oplus E^u_x$ $(x \in \Lambda)$ such that the tangent flow $DX_t : TM \to TM$ leaves invariant the continuous splitting and

$$\|DX_{t_{|E^s_x}}\| \leqslant Ce^{-\lambda t} \quad \text{and} \quad \|DX_{-t_{|E^u_x}}\| \leqslant Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$ (see [3,12]). The set of non-wandering points of X is denoted by $\Omega(X_t)$. Clearly,

$$Sing(X) \cup PO(X_t) \subset \Omega(X_t).$$

Here $Sing(X)$ is the set of singularities of X and $PO(X_t)$ is the set of periodic orbits of X_t.

We say that $X \in \mathcal{X}^1(M)$ satisfies *Axiom A* if $PO(X_t)$ is dense in $\Omega(X_t) \setminus Sing(X)$ and $\Omega(X_t)$ is hyperbolic. We say that $X \in \mathcal{X}^1(M)$ is *Anosov* if M is hyperbolic for X_t.

Let $X \in \mathcal{X}^1(M)$ satisfy Axiom A. In the present paper, we say that X satisfies the *quasi-transversality condition* if

$$T_xW^s(x) \cap T_xW^u(x) = \{O_x\} \quad \text{for any } x \in M.$$

Here $W^s(x)$ is the stable manifold and $W^u(x)$ is the unstable manifold of x defined as usual (cf. [12]).

As before, denote by $\mathcal{E}(M)$ the set of $X \in \mathcal{X}^1(M)$ whose integrated flow X_t is expansive. Remark that each singular point of $X \in \mathcal{E}(M)$ is an isolated point in M by definition (see [1, Lemma 1]) so that $Sing(X) = \emptyset$ for $X \in \mathcal{E}(M)$.

Other authors do give the definition of expansive for flows slightly differently. For example, in [5] the author introduced the notion of $K^\star$-expansive for flows which is weaker than our definition, and proved therein that the geometric Lorenz flow possesses this property on its attractor, while in our definition it is not. In the present paper, a change in definition (from our definition to weaker one) would not invalidate the theorems, but would mean that they have to be slightly reworded. Indeed, to prove the same results in such weaker definition, we need to assume that the vector field under consideration has no singularities.

Let $\mathcal{X}^*(M)$ be the set of $X \in \mathcal{X}^1(M)$ with the property that there exists a C^1 neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X such that every singularity and every periodic orbit of $Y \in \mathcal{U}$ are hyperbolic. Write

$$\mathcal{L}(M) = \{X \in \mathcal{X}^*(M) : X \text{ has no singularities}\}.$$

A proof of Theorem A(i) $\Rightarrow$ (ii) is based on the following remarkable result obtained by Gan and Wen in [3]. The assertion will be proved by showing $\operatorname{int}\mathcal{E}(M) \subset \mathcal{L}(M)$.

Theorem. *Every $X \in \mathcal{L}(M)$ satisfies both Axiom A and the no-cycle condition.*

Denote by $\mathcal{QA}(M) \subset \mathcal{X}^1(M)$ the set of quasi-Anosov vector fields. It is easy to see that if $X \in \mathcal{X}^1(M)$ has no singularities and satisfies both Axiom A and the quasi-transversality condition, then $X \in \mathcal{QA}(M)$ by definition. Thus, the proof of Theorem A is divided into the following two propositions.

Proposition 1. *Every $X \in \operatorname{int}\mathcal{E}(M)$ satisfies both Axiom A and the quasi-transversality condition.*

Proposition 2. *Let $X \in \mathcal{QA}(M)$. Then there exists a C^1 neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X such that for every $Y \in \mathcal{U}$, the integrated flow Y_t is expansive. More strongly, for any $\varepsilon > 0$, there exists a common expansive constant $\delta = \delta(\mathcal{U}, \varepsilon) > 0$ corresponding to ε with respect to Y_t for every $Y \in \mathcal{U}$.*

We say that $Y \in \mathcal{X}^1(M)$ is *semiconjugate* to $X \in \mathcal{X}^1(M)$ if Y_t is semiconjugate to X_t; that is, there are a continuous surjection $h : M \to M$ and a continuous map $\tau : M \times \mathbb{R} \to \mathbb{R}$ such that

- for all $x \in M$, $\tau_x \in \operatorname{Rep}(\mathbb{R})$,
- for all $x \in M$ and $t \in \mathbb{R}$, $h(Y_t(x)) = X_{\tau_x(t)}(h(x))$,

where X_t and Y_t are the flows induced from X and Y, respectively. The pair (h, τ) is called a *semiconjugacy* from Y to X. If the map h can be taken as a homeomorphism, then we say that Y is *conjugate* to X.

We say that $X \in \mathcal{X}^1(M)$ is *structurally stable* if there is a C^1 neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1(M)$ such that every $Y \in \mathcal{U}$ is conjugate to X. It is proved by Robinson [12] that if X satisfies both Axiom A and the strong transversality condition, then X is structurally stable (remark that the converse is also true, see Hayashi [4] and Wen [18]).

A continuous flow X_t on M is said to be *topologically stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every perturbation flow Y_t on M with $d_{C^0}(X_t, Y_t) < \delta$, there exists a continuous map $h : M \to M$ such that $d(h(x), x) < \varepsilon$ $(x \in M)$ and

$$h(\text{orbit of } Y_t) \subset \text{orbit of } X_t.$$

Here

$$d_{C^0}(X_t, Y_t) = \sup_{t\in[0,1],\ x\in M} d(X_t(x), Y_t(x)).$$

Notice that the map h is surjection since M is connected and $d(h(x), x) < \varepsilon$ $(x \in M)$.

Some stability properties including topological stability of continuous flows on a compact metric space are systematically studied by Thomas (see [16,17]).

Now, suppose that the above h is a homeomorphism mapping orbits of X_t onto orbits of Y_t. If X_t has no fixed points, then for every $x \in M$, there is a unique $\sigma_x \in \mathrm{Rep}(\mathbb{R})$ such that $h \circ X_t(x) = Y_{\sigma_x(t)} \circ h(x)$ (see [17, p. 107]). Thus, if the integrated flow X_t of $X \in \mathcal{X}^1(M)$ is topologically stable, and for C^0 nearby system Y $(\in \mathcal{X}^1(M))$ of X if the map h of the topological stability is injective, then Y is conjugate to X.

It is proved in [16, Theorems 3 and 4] that every expansive flow X_t possessing the shadowing property is topologically stable, and if, in addition, a perturbation flow Y_t of X_t is also expansive, then the map h is injective. In the proof of latter result, to prove that h is one-to-one, we have to check the relationship between the expansive constant of X_t and that of Y_t. Unfortunately, in the original proof, the way of choice of the expansive constant for the perturbation flow Y_t is not so clear for the authors.

In this paper, following the proof of the original paper closely we give a proof for the above result for completeness. More precisely, we prove the following.

Proposition 3. *Let X_t be an expansive flow on M possessing the shadowing property. Then X_t is topologically stable, and for any continuous flow Y_t C^0 nearby X_t, if both X_t and Y_t have a common expansive constant, then the continuous map h between the orbits of X_t and the orbits of Y_t is injective.*

Let $X \in \operatorname{int} \mathcal{E}(M)$ have the shadowing property. Then, by Theorem A it will follow from Proposition 3 that X is structurally stable. Since every structurally stable vector field satisfies the strong transversality condition, the above X must be Anosov, so that Theorem B will be obtained.

3. Proof of Theorem A

Hereafter, for simplicity we assume that the exponential map

$$\exp_x : T_xM(1) \to M$$

is well defined for all $x \in M$, where $T_xM(r) = \{v \in T_xM : \|v\| < r\}$ for $r > 0$.

Let $X \in \mathcal{X}^1(M)$ have no singularities, and let X_t be the flow. For every $x \in M$, let

$$\Pi_{x,r} = \exp_x(N_{x,r}) \quad \text{and} \quad \Pi_x = \Pi_{x,1},$$

where $N_x = \langle X(x)\rangle^{\perp}$, and $N_{x,r} = N_x \cap T_xM(r)$ for $0 < r \leqslant 1$. Then, it is well known that for given $x' = X_{t_0}(x)$ $(t_0 > 0)$, there are $r_0 > 0$ and a C^1 map $\tau : \Pi_{x,r_0} \to \mathbb{R}$ such that $X_{\tau(y)}(y) \in \Pi_{x'}$ $(y \in \Pi_{x,r_0})$ with $\tau(x) = t_0$. The flow X_t uniquely defines the *Poincaré map* $f : \Pi_{x,r_0} \to \Pi_{x'}$ by $f(y) = X_{\tau(y)}(y)$ for all $y \in \Pi_{x,r_0}$. The map is C^1 embedding whose image is interior to $\Pi_{x'}$ if r_0 is small.

If $X_t(x) \neq x$ for $0 < t \leqslant t_0$ and r_0 is sufficiently small, then $(t, y) \mapsto X_t(y)$ C^1 embeds

$$\{(t, y) \in \mathbf{R} \times \Pi_{x,r} : 0 \leqslant t \leqslant \tau(y)\}$$

for $0 < r \leqslant r_0$. The image

$$\{X_t(y) : y \in \Pi_{x,r} \quad \text{and} \quad 0 \leqslant t \leqslant \tau(y)\}$$

is denoted by $F_x(X_t, r, t_0)$. For $\varepsilon > 0$, let $\mathcal{N}_\varepsilon(\Pi_{x,r})$ be the set of diffeomorphisms $\xi : \Pi_{x,r} \to \Pi_{x,r}$ such that $\mathrm{supp}(\xi) \subset \Pi_{x,r/2}$ and $d_{C^1}(\xi, id) < \varepsilon$. Here d_{C^1} is the usual C^1 metric, $id : \Pi_{x,r} \to \Pi_{x,r}$ is the identity map, and $\mathrm{supp}(\xi)$ is the closure of the set where it differs from *id*.

Lemma 1. *Let $X \in \mathcal{X}^1(M)$ have no singularities. Suppose $X_t(x) \neq x$ for $0 < t \leqslant t_0$, and let $f : \Pi_{x,r_0} \to \Pi_{x'}$ ($x' = X_{t_0}(x)$) be the Poincaré map ($r_0 > 0$ is sufficiently small). Then, for every C^1 neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X and $0 < r \leqslant r_0$, there is $\varepsilon > 0$ with the property that for every $\xi \in \mathcal{N}_\varepsilon(\Pi_{x,r})$, there exists $Y \in \mathcal{U}$ satisfying*

$$\begin{cases} Y(y) = X(y) & \text{if } x \notin F_x(X_t, r, t_0) \\ f_Y(y) = f \circ \xi(y) & \text{if } y \in \Pi_{x,r}. \end{cases}$$

Here $f_Y : \Pi_{x,r} \to \Pi_{x'}$ is the Poincaré map defined by Y_t.

Proof. See [11, p. 296, Remark 2]. □

Let $X \in \mathcal{X}^1(M)$, and suppose $p \in \gamma \in PO(X_t)$ ($X_T(p) = p$, where $T > 0$ is the minimum period). If $f : \Pi_{p,r_0} \to \Pi_p$ is the Poincaré map ($r_0 > 0$), then $f(p) = p$. In this case, γ is hyperbolic if and only if p is a hyperbolic fixed point of *f*.

The following lemma plays an essential role in the proof of the hyperbolicity of the periodic orbits of X_t ($X \in \mathrm{int}\,\mathcal{E}(M)$).

Lemma 2. *Let $X \in \mathcal{X}^1(M)$ have no singularities, $p \in \gamma \in PO(X_t)$ ($X_T(p) = p$), and let $f : \Pi_{p,r_0} \to \Pi_p$ be the Poincaré map for some $r_0 > 0$. Let $\mathcal{U} \subset \mathcal{X}^1(M)$ be a C^1 neighborhood of X, and let $0 < r \leqslant r_0$ be given. Then there are $\delta_0 > 0$ and $0 < \varepsilon_0 < r/2$ such that for a linear isomorphism $\mathcal{O} : N_p \to N_p$ with $\|\mathcal{O} - D_p f\| < \delta_0$, there is $Y \in \mathcal{U}$ satisfying*

(i) $Y(x) = X(x)$ *if* $x \notin F_p(X_t, r, T/2)$,

(ii) $p \in \gamma \in PO(Y_t)$,

(iii) $g(x) = \begin{cases} \exp_p \circ \mathcal{O} \circ \exp_p^{-1}(x) & \text{if } x \in B_{\varepsilon_0/4}(p) \cap \Pi_{p,r} \\ f(x) & \text{if } x \notin B_{\varepsilon_0}(p) \cap \Pi_{p,r}, \end{cases}$

where $B_\varepsilon(x)$ ($x \in M$) is a closed ball in M center at x with radius $\varepsilon > 0$, and $g : \Pi_{p,r} \to \Pi_p$ is the Poincaré map defined by Y_t.

Proof. cf. [9, p. 3395, Lemma 1.3]. □

Proof of Proposition 1. Let $X \in \operatorname{int}\mathcal{E}(M)$. We show that X satisfies both Axiom A and the quasi-transversality condition.

Let $\mathcal{U} \subset \mathcal{E}(M)$ be a C^1 neighborhood of X and pick $p \in \gamma \in PO(X_t)$ $(X_T(p) = p, T > 0)$. The flow X_t defines the Poincaré map $f : \Pi_{p,r_0} \to \Pi_p$ (for some $r_0 > 0$). Assuming that there is an eigenvalue λ of $D_p f$ with $|\lambda| = 1$, we shall derive a contradiction.

Let $\delta_0 > 0$ and $0 < \varepsilon_0 < r_0$ be given by Lemma 2 for $\mathcal{U}$ and r_0. Then, for the linear isomorphism $\mathcal{O} = D_p f : N_p \to N_p$, there exists $Y \in \mathcal{U}$ such that

- $Y(x) = X(x)$ if $x \notin F_p(X_t, r_0, T/2)$,
- $g(x) = \begin{cases} \exp_p \circ D_p f \circ \exp_p^{-1}(x) & \text{if } x \in B_{\varepsilon_0/4}(p) \cap \Pi_{p,r_0} \\ f(x) & \text{if } x \notin B_{\varepsilon_0}(p) \cap \Pi_{p,r_0}. \end{cases}$

Since $Y \in \mathcal{E}(M)$, for a sufficiently small $0 < \varepsilon < \min\{\varepsilon_0/16, T/2\}$, there is $0 < \delta < \min\{\delta_0, \varepsilon\}$ with the property that if $d(Y_s(x), Y_{\alpha(s)}(y)) \leqslant \delta$ for all $s \in \mathbb{R}$, for a pair of points $x, y \in M$, and for a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ $(\alpha(0) = 0)$, then $y = Y_s(x)$ where $|s| \leqslant \varepsilon$.

Let $0 < \delta' < \delta$ be a number such that $d(x, y) < \delta'$ $(x, y \in M)$ implies

$$d(Y_t(x), Y_t(y)) < \delta$$

for $0 \leqslant t \leqslant T$. For simplicity, we suppose $\lambda = 1$ (other case is similar). If we take an eigenvector $v \neq 0$ corresponding to λ with $\|v\| < \delta'$, then, by construction

$$d(x, p) = d(g(x), p) < \delta'.$$

Here $x = \exp_p(v) \in \Pi_{p,r_0} \setminus \{p\}$. Hence we see that there is a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ $(\alpha(0) = 0)$ such that $d(Y_t(p), Y_{\alpha(t)}(x)) < \delta$ for all $t \in \mathbb{R}$. Thus $x = Y_t(p)$ for some $|t| \leqslant \varepsilon$. This is a contradiction, because $x \in \Pi_{p,r_0} \setminus \{p\}$. Hence, by Theorem of Gan and Wen every $X \in \operatorname{int}\mathcal{E}(M)$ satisfies both Axiom A and the no-cycle condition since X is singular points free.

A proof of the quasi-transversality condition for X follows readily. Indeed, suppose that $T_x W^s(x) \cap T_x W^u(x) \neq \{O_x\}$ for some $x \in M$. Then making use of Lemma 1, we can perturb X to C^1 nearby Y such that for any $\delta > 0$, there exists $y \notin Y_{\mathbb{R}}(x)$ satisfying $d(Y_s(x), Y_{\alpha(s)}(y)) \leqslant \delta$ for all $s \in \mathbb{R}$ for some continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ $(\alpha(0) = 0)$. This is a contradiction. □

To prove Proposition 2, we prepare more two lemmas that we need.

Lemma 3. *Let $X \in \mathcal{QA}(M)$, and let $F^t : \mathcal{N} \to \mathcal{N}$ be the transformation group induced by the flow X_t. Then there exists an integer $T > 0$ such that for any $v \in \mathcal{N}$,*

$$\|F^T(v)\| \geqslant 3\|v\| \quad \text{or} \quad \|F^{-T}(v)\| \geqslant 3\|v\|.$$

Proof. There exists an integer $T_0 > 0$ such that for any $v \in \mathcal{N}$, we have $\|F^t(v)\| \geqslant 3\|v\|$ for some $-T_0 < t < T_0$. Indeed, if this is not true, then, for any integer $n > 0$ there are $x_n \in M$ and $v_n \in N_{x_n}$ ($\|v_n\| = 1$) such that $\|F^t(v_n)\| < 3\|v_n\|$ for all $-n < t < n$. If we let $x_n \to x \in M$ and $v_n \to v_x \in N_x$ ($\|v_x\| = 1$) as $n \to \infty$, then $\|F^t(v_x)\| \leqslant 3$ for $t \in \mathbb{R}$. Thus, $v_x = 0$ since X is quasi-Anosov. This is a contradiction.

Fix any $v \in \mathcal{N}$ ($\|v\| \neq 0$), and take a real number $t_1 = t_1(v)$ such that

$$\sup_{|t| \leqslant T_0} \|F^t(v)\| = \|F^{t_1}(v)\|.$$

We suppose that $t_1 > 0$ (other case is similar). Thus $\|F^{t_1}(v)\| \geqslant 3\|v\|$. Since $-T_0 \leqslant t_1 - T_0 \leqslant t_1 \leqslant T_0$, there exists a real number $t_2 = t_2(v)$ with $0 < t_1 < t_2 \leqslant t_1 + T_0$ such that

$$\sup_{t_1 - T_0 \leqslant t \leqslant t_1 + T_0} \|F^t(v)\| = \|F^{t_2}(v)\|.$$

Thus we have $\|F^{t_2}(v)\| \geqslant 3\|F^{t_1}(v)\| \geqslant 3^2\|v\|$. Continuing this manner, we can choose an increasing sequence $t_k = t_k(v) \nearrow \infty$ of real numbers such that $t_{k+1} - t_k < T_0$ and $\|F^{t_k}(v)\| \geqslant 3^k\|v\|$.

Let $L > 0$ be a constant such that $\|F^t(v)\| \geqslant L\|v\|$ for all $v \in \mathcal{N}$ and $-T_0 \leqslant t \leqslant T_0$. Then for any $t \geqslant 0$ and $v \in \mathcal{N}$, we see $\|F^t(v)\| \geqslant L\|F^{t_k}(v)\|$ where $0 < t_k \leqslant t \leqslant t_{k+1}$, and so $\|F^t(v)\| \geqslant L3^k\|v\|$. Finally, pick an integer $T > 0$ such that $L3^{[T/T_0]} \geqslant 3$. $\square$

Let $X \in \mathcal{X}^1(M)$ have no singularities. Then, for any $Y \in \mathcal{X}^1(M)$, C^0-nearby X, we can define $F_Y^t : \mathcal{N}^Y \to \mathcal{N}^Y$ the transformation group induced from the flow Y_t as in the same way (X does not have singularities, neither does any C^0 nearby Y). Here $\mathcal{N}^Y = \cup_{x \in M} N_x^Y$, and N_x^Y is the orthogonal linear subspace of $\langle Y(x)\rangle$ in T_xM.

Remark 1. $\mathcal{QA}(M)$ is open in $\mathcal{X}^1(M)$.

Indeed, let $X \in \mathcal{QA}(M)$. Then, by Lemma 3 it is not hard to show that there exists $\nu > 0$ such that $d_{C^1}(X, Y) < \nu$ ($Y \in \mathcal{X}^1(M)$) implies

$$\|F_Y^T(v)\| \geqslant 2\|v\| \quad \text{or} \quad \|F_Y^{-T}(v)\| \geqslant 2\|v\|.$$

for any $v \in \mathcal{N}^Y$.

Suppose that $\|v\| \neq 0$ and $\sup_{t \in \mathbb{R}} \|F_Y^t(v)\| < \infty$. In case $\|F_Y^T(v)\| \geqslant 2\|v\|$ (other case is similar), we see $\|F_Y^{2T}(v)\| \geqslant 2\|F_Y^T(v)\|$. For, if $\|F_Y^{-T} \circ F_Y^T(v)\| \geqslant 2\|F_Y^T(v)\|$, then

$$\|v\| \geqslant 2\|F_Y^T(v)\| \geqslant 2^2\|v\|.$$

This is a contradiction, and so $\|F_Y^{2T}(v)\| \geqslant 2^2\|v\|$. Continuing this manner, we have $\|F_Y^{nT}(v)\| \geqslant 2^n\|v\|$ for all integer $n > 0$, which is a contradiction. Thus $\mathcal{QA}(M)$ is open.

Now, suppose that $X \in \mathcal{X}^1(M)$ has no singularities, and let X_t be the flow. Let $\mathcal{N}^X = \cup_{x\in M} N_x^X$, and let $F_X^t : \mathcal{N}^X \to \mathcal{N}^X$ be the transformation group induced from X_t. Here N_x^X is the orthogonal linear subspace of $\langle X(x)\rangle$ in T_xM. As before, we set $N_{x,r}^X = N_x^X \cap T_xM(r)$ $(r > 0)$ for $x \in M$, and put $\Pi_{x,r}^X = \exp_x(N_{x,r}^X)$.

For any Y $(\in \mathcal{X}^1(M))$ C^1 nearby X, we construct a Poincaré map with respect to $\Pi_{x,r}^X$ (for some r) in the whole space M modifying the technique used in [12, pp. 269–270]. The map will play an essential role in the proof of Proposition 2.

Lemma 4. *Under the above notation, there are a constant $\rho > 0$, a C^1 neighborhood $\mathcal{V} \subset \mathcal{X}^1(M)$ of X such that for any $Y \in \mathcal{V}$, there exists a C^1 Poincaré map $\varphi_{Y,x} : \Pi_{x,\rho}^X \to \Pi_{Y_1(x)}^X$ $(x \in M)$ satisfying*
(1) $\varphi_{Y,x}(x) = Y_1(x)$,
(2) $D_x\varphi_{X,x} = F_{X,x}^1$,
(3) $\varphi_{Y,x} \to \varphi_{X,x}$ $(x \in M)$ *as $Y \to X$ with respect to the C^1 topology.*
Here Y_t is the integrated flow of Y.

Proof. Suppose that $X \in \mathcal{X}^1(M)$ has no singularities, and let X_t be the integrated flow. Fix $\delta_1 > 0$, $r_1 > 0$, and a C^1 neighborhood $\mathcal{U}_0 \subset \mathcal{X}^1(M)$ of X small enough. For any $Y \in \mathcal{U}_0$, $-1 \leqslant s \leqslant 2$, and $-\delta_1 + s < t < \delta_1 + s$, we define

$$\Phi(x, v; s, t, Y) = \exp_{Y_s(x)}^{-1} \circ Y_t \circ \exp_x v \quad \text{for} \quad (x, v) \in TM(r_1).$$

Here Y_t is the integrated flow of Y.

Hereafter, we fix $s = 1$, and modify the flow $\Phi(x, v; t, Y) = \Phi(x, v; 1, t, Y)$ on *TM* to preserve $\mathcal{N}^X = \cup_{x\in M} N_x^X$.

Fix any $x \in M$ and let

$$\mu(x, v; t, Y) = \langle \Phi(x, v; t, Y), X(Y_1(x))\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the Riemannian inner product on *TM*. Then μ is C^1 such that

$$\mu(x, 0; 1, X) = 0$$

and

$$\frac{\partial}{\partial t}\mu(x, 0; t, X)_{|t=1} = \langle X(X_1(x)), X(X_1(x))\rangle \neq 0.$$

Therefore, by Implicit Function Theorem there exist a number $0 < r \leqslant r_1$ and a C^1 neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of X and a C^1 function

$$\tilde{\tau} : T_{U_r(x)}M(r) \times \mathcal{U} \to \mathbb{R}$$

such that $\mu(y, v; \tilde{\tau}(y, v; Y), Y) = 0$ for any $(y, v) \in T_{U_r(x)}M(r)$ and $Y \in \mathcal{U}$. Here $U_r(x)$ is an open ball in M center at x with radius r. By construction

$$\Phi(y, v; \tilde{\tau}(y, v; Y), Y) \in N^X_{Y_1(y)}$$

for $v \in N^X_{y,r}$ and $y \in U_r(x)$.

Since M is compact, we can pick a finite number of points $\{x_i\}_{i=1}^l \subset M$ and positive numbers $\{\rho_i\}_{i=1}^l$, and C^1 neighborhoods $\{\mathcal{U}_i\}_{i=1}^l$ such that $M = \bigcup_{i=1}^l U_{\rho_i}(x_i)$ and the above functions $\tilde{\tau}(x_i, v; Y)$ are defined for

$$(y, v; Y) \in T_{U_{\rho_i}(x_i)}M(\rho_i) \times \mathcal{U}_i.$$

If we set

$$\rho = \min_{1 \leqslant i \leqslant l} \rho_i \quad \text{and} \quad \mathcal{V} = \bigcap_{1 \leqslant i \leqslant l} \mathcal{U}_i,$$

then the function $\tilde{\tau}(x, v; Y)$ is well defined on the local tangent bundle

$$TM(\rho) = \bigcup_{x \in M} T_x M(\rho)$$

for any $Y \in \mathcal{V}$.

Define

$$\varphi_{Y,x}(y) = Y_{\tilde{\tau}(x, \exp_x^{-1} y; Y)}(y)$$

for $y \in \Pi^X_{x,\rho}$ $(x \in M)$. Then $\varphi_{Y,x}$ is C^1 and $\varphi_{Y,x}(\Pi^X_{x,\rho}) \subset \Pi^X_{Y_1(x)}$ for $x \in M$. Clearly, $\varphi_{Y,x} \to \varphi_{X,x}$ $(x \in M)$ as $Y \to X$ with respect to the C^1 topology.

Now we show that the derivative of $\varphi_{X,x}$ at the zero vector $0_x \in T_x M$ $(x \in M)$ coincides with $F^1_{X,x} : N^X_x \to N^X_{X_1(x)}$. That is

$$D_x \varphi_{X,x} = F^1_{X,x}$$

for all $x \in M$. Indeed, we see

$$D_x \varphi_{X,x}(w) = D_x X_1(w) + X(X_1(x)) \cdot \frac{d\tilde{\tau}}{dv}(w)$$

for $w \in N^X_x$ by definition. On the other hand, since $D_x \varphi_{X,x}(N^X_x) = N^X_{X_1(x)}$, we have

$$\pi(D_x X_1(w)) = D_x X_1(w) + X(X_1(x)) \cdot \frac{d\tilde{\tau}}{dv}(w) \in N^X_{X_1(x)},$$

where $\pi : TM \to \mathcal{N}^X$ is the projection along X. Thus $D_x\varphi_{X,x}(w) = F^1_{X,x}(w)$ for $w \in N^X_x$ and $x \in M$. $\square$

Remark 2. Suppose that $X \in \mathcal{X}^1(M)$ has no singularities, and let $\rho > 0$, $\mathcal{V}$, and $\varphi_{Y,x}$ $(x \in M,\ Y \in \mathcal{V})$ be given by Lemma 4.

(1) For simplicity, denote by $\varphi^n_{Y,x}$ the composition map

$$\varphi_{Y,Y_{n-1}(x)} \circ \varphi_{Y,Y_{n-2}(x)} \circ \cdots \circ \varphi_{Y,x}$$

for $n \geqslant 1$. Clearly, we can proceed in the same way for negative powers of $\varphi_{Y,x}$ for $x \in M$.

(2) Reducing $\mathcal{V}$ if necessary, we can see the following property; for any $c > 0$ and $\varepsilon > 0$ there exists $\delta = \delta(\mathcal{V}, c, \varepsilon) > 0$ such that if $Y \in \mathcal{V}$, a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ $(\alpha(0) = 0)$, and a pair of points $x, y \in M$ satisfy

$$d(Y_t(x), Y_{\alpha(t)}(y)) \leqslant \delta \quad \text{for all} \quad t \in \mathbb{R},$$

then there is $y' \in \Pi^X_{x,\rho}$ with

(i) $y' = Y_s(y)$ for some $|s| \leqslant \varepsilon$,

(ii) $d(\varphi^n_{Y,x}(x), \varphi^n_{Y,x}(y')) \leqslant c$ for all $n \in \mathbb{Z}$.

Proof of Proposition 2. Let $X \in \mathcal{QA}(M)$, and let $\rho > 0$, $\mathcal{V} \subset \mathcal{X}^1(M)$ and $\varphi_{Y,x}$ $(x \in M,\ Y \in \mathcal{V})$ be given by Lemma 4. We show that there exist a C^1 neighborhood $\mathcal{V}_0$ $(\subset \mathcal{V})$ of X and a constant $c > 0$ such that for any $Y \in \mathcal{V}_0$, if $d(\varphi^n_{Y,x}(x), \varphi^n_{Y,x}(y)) \leqslant c$ $(x \in M,\ y \in \Pi^X_{x,\rho})$ for all $n \in \mathbb{Z}$, then $x = y$. If this is established, then for any $Y \in \mathcal{V}_0$, the flow Y_t must be expansive with a common expansive constant.

Indeed, for any $\varepsilon > 0$, let $\delta = \delta(\mathcal{V}, c, \varepsilon) > 0$ be the number given by Remark 2(2). Fix any $Y \in \mathcal{V}_0$ $(\subset \mathcal{V})$, and let Y_t be the flow. Then, for any continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ $(\alpha(0) = 0)$ and any pair of points $x, y \in M$, if $d(Y_t(x), Y_{\alpha(t)}(y)) \leqslant \delta$ for all $t \in \mathbb{R}$, then there is $y' = Y_s(y) \in \Pi^X_{x,\rho}$ with $|s| \leqslant \varepsilon$ such that

$$d(\varphi^n_{Y,x}(x), \varphi^n_{Y,x}(y')) \leqslant c \quad \text{for all} \quad n \in \mathbb{Z}$$

by Remark 2(2)(i) and (ii). Thus we have $x = y'$. Since $x = Y_s(y)$ with $|s| \leqslant \varepsilon$, the above δ is a common expansive constant corresponding to ε with respect to Y_t $(Y \in \mathcal{V}_0)$, and hence Proposition 2 is proved.

Now, let $F^t_X : \mathcal{N}^X \to \mathcal{N}^X$ be the transformation group induced from the flow X_t, and let $T > 0$ be the integer given by Lemma 3. Then, for any $v \in \mathcal{N}^X$,

$$\|F^T_X(v)\| \geqslant 3\|v\| \quad \text{or} \quad \|F^{-T}_X(v)\| \geqslant 3\|v\|.$$

Thus

$$\|D_x\varphi_X^T(v)\| \geqslant 3\|v\| \quad \text{or} \quad \|D_x\varphi_X^{-T}(v)\| \geqslant 3\|v\|$$

for any $v \in N_x^X$ since $\varphi_X = F_X^1$ (see Remark 2(1)). By Lemma 4(3), there exists a C^1 neighborhood $\mathcal{V}_0$ $(\subset \mathcal{V})$ of X such that for any $Y \in \mathcal{V}_0$,

$$\|D_x\varphi_Y^T(v)\| \geqslant 2\|v\| \quad \text{or} \quad \|D_x\varphi_Y^{-T}(v)\| \geqslant 2\|v\|$$

for any $v \in N_x^X$ and $x \in M$. Set

$$K = \sup_{x\in M, Y\in\mathcal{V}_0} \|D_x\varphi_Y\|.$$

Fix $\varepsilon' > 0$ with

$$\varepsilon'(1 + K + K^2 + K^3 + \cdots + K^{T-1}) < 1/2.$$

Then, reducing $\mathcal{V}_0$ if necessary, we can take $0 < c = c(\varepsilon', \mathcal{V}_0) < \rho$ such that

$$\| \exp^{-1}_{\varphi_Y^\sigma(x)} \circ \varphi_Y^\sigma \circ \exp_x v - D_x\varphi_Y^\sigma(v)\| < \|v\|\varepsilon' \quad (x \in M)$$

if $\|v\| \leqslant c$ $(v \in \mathcal{N}^X,\ \sigma = \pm 1)$ for any $Y \in \mathcal{V}_0$. We show that for any $Y \in \mathcal{V}_0$, if

$$d(\varphi_{Y,x}^n(x), \varphi_{Y,x}^n(y)) \leqslant c \quad \text{for all} \quad n \in \mathbb{Z}$$

$(x \in M,\ y \in \Pi_{x,\rho}^X)$, then $x = y$.

Hereafter, for simplicity, we denote φ_Y, $N_{x,\rho}^X$, and $\Pi_{x,\rho}^X$ by φ, $N_{x,\rho}$, and $\Pi_{x,\rho}$, respectively. If the above assertion is false, then there are distinct points x and $y \in \Pi_{x,\rho}$ such that $d(\varphi^n(x), \varphi^n(y)) \leqslant c$ for all $n \in \mathbb{Z}$. Let

$$c' = \sup_{n\in\mathbb{Z}} d(\varphi^n(x), \varphi^n(y)) \leqslant c$$

and take δ' with $0 < \delta' \leqslant c'/4$. Obviously, we see $c' - \delta' < d(\varphi^i(x), \varphi^i(y)) \leqslant c'$ for some $i \in \mathbb{Z}$.

Put $z = \varphi^i(x)$, $w = \varphi^i(y)$, and $v = \exp_z^{-1} w \in N_{z,\rho}$. Then

$$c' - \delta' < \|v\| = d(z, w)$$

and $\|D_z\varphi^n(v)\| \geqslant 2\|v\|$ for some $|n| = T$. We treat the case $\|D_z\varphi^T(v)\| \geqslant 2\|v\|$ (other case is similar). Since $\|v\| = d(z, w) \leqslant c'$ we have

$$\| \exp^{-1}_{\varphi(z)} \circ\varphi \circ \exp_z v - D_z\varphi(v)\| < \|v\|\varepsilon',$$

and so $\|D_z\varphi(v)\| < c'(1 + \varepsilon')$ since $\| \exp^{-1}_{\varphi(z)} \circ\varphi \circ \exp_z v\| = d(\varphi(z), \varphi(w)) \leqslant c'$. Moreover

$$\begin{aligned}
&\| \exp^{-1}_{\varphi^2(z)} \circ\varphi^2 \circ \exp_z v - D_z\varphi^2(v)\| \\
&\leqslant \| \exp^{-1}_{\varphi^2(z)} \circ\varphi^2 \circ \exp_z v - D_{\varphi(z)}\varphi(\exp^{-1}_{\varphi(z)} \circ\varphi \circ \exp_z v)\| \\
&\quad + \|D_{\varphi(z)}\varphi(\exp^{-1}_{\varphi(z)} \circ\varphi \circ \exp_z v) - D_z\varphi^2(v)\| \\
&\leqslant c'\varepsilon' + Kc'\varepsilon' \\
&= c'\varepsilon'(1 + K)
\end{aligned}$$

and hence,

$$\| \exp^{-1}_{\varphi^2(z)} \circ\varphi^2 \circ \exp_z v\| = d(\varphi^2(z), \varphi^2(w)) \leqslant c'$$

implies $\|D_z\varphi^2(v)\| \leqslant c'\{1 + \varepsilon'(1 + K)\}$. By induction we have

$$2\|v\| \leqslant \|D_z\varphi^T(v)\| \leqslant c'\{1 + \varepsilon'(1 + K + K^2 + K^3 + \cdots + K^{T-1})\}.$$

Thus $c' - \delta' < \|v\| \leqslant 3c'/4$ so that $c'/4 < \delta'$. This is a contradiction. □

4. Proof of Theorem B

Before starting a proof, we prove the following result which has been already stated in [16] for a continuous expansive flow X_t possessing the shadowing property.

As stated before, to show that the map h between the orbits of X_t and the orbits of a perturbation flow Y_t is injective, we have to clarify the relationship between the expansive constant of X_t and that of Y_t. However, in the original proof, the way of choice of the expansive constant for the perturbation flow Y_t seems not so clear for the authors.

In this paper, following [16] closely we show that the map h is injective in case both X_t and Y_t have a common expansive constant.

Proposition 3. *Let X_t be an expansive flow on M possessing the shadowing property. Then X_t is topologically stable, and for any continuous flow Y_t C^0 nearby X_t, if both*

X_t and Y_t have a common expansive constant, then the continuous map h between the orbits of X_t and the orbits of Y_t is injective.

Proof. Let X_t be a continuous expansive flow on M and assume that X_t has the shadowing property. Then, X_t has no fixed points so that there exists $T_0 > 0$ as in the assertion of [16, Lemma 3.4].

Fix $0 < \varepsilon < T_0/2$. Since X_t is expansive, for this ε, there is $e > 0$ such that if $d(X_{\alpha(t)}(x), X_t(y)) \leqslant 4e$ for all $t \in \mathbb{R}$, for a pair of points $x, y \in M$, and for a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = X_t(x)$ where $|t| \leqslant \varepsilon$. Thus $4e$ is an expansive constant corresponding to ε with respect to X_t.

Now, for ε, let $\zeta > 0$ be a number such that $d(X_\varepsilon(y), y) \geqslant 2\zeta$ for all $y \in M$ (see [16, Lemma 3.4]). For ε, let $0 < r < \min\{\varepsilon, e\}$ be the number given by [16, Lemma 3.3] such that if $x = X_t(y)$ $(x, y \in M)$ and $|t| < r$, then $d(x, y) \leqslant e$. Since $0 < r < T_0/2$, by [16, Lemma 3.4] there is $\gamma > 0$ such that $d(X_r(y), y) \geqslant \gamma$ for all $y \in M$.

Again, since X_t is expansive, for the above r, there is $0 < \varepsilon' < \min\{\zeta, r, \gamma\}$ such that if $d(X_{\alpha(t)}(x), X_t(y)) \leqslant \varepsilon'$ for all $t \in \mathbb{R}$, for a pair of points $x, y \in M$, and for a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = X_t(x)$ where $|t| \leqslant r$. Hence ε' is an expansive constant corresponding to r with respect to X_t.

Choose $\delta > 0$ with $0 < \delta < \min\{\zeta, \varepsilon/12\}$ such that

(a) every $(\delta, 1)$-pseudo-orbit is $\varepsilon/12$-shadowed by an orbit of X_t,

(b) for every $x, y \in M$, $d(x, y) < \delta$ implies $d(X_t(x), X_t(y)) < \varepsilon'/12$ for all $t \in [0, 1]$.

Let Y_t be a given perturbation flow on M with $d_{C^0}(X_t, Y_t) < \delta$. Then it is proved in [16, Proof of Theorem 3, pp. 491–496] that there exists a continuous map $h : M \to M$ so that $d(h(x), x) < \varepsilon$ $(x \in M)$ and

$$h(\text{orbit of } Y_t) \subset \text{orbit of } X_t.$$

Thus X_t is topologically stable. Furthermore, by the choice of δ we see that

$$d(Y_\varepsilon(x), x) \geqslant d(X_\varepsilon(x), x) - d(X_\varepsilon(x), Y_\varepsilon(x)) \geqslant \zeta$$

for all $x \in M$.

Notice that Y_t has a common expansive constant as X_t by assumption. Thus $4e$ is also an expansive constant corresponding to ε with respect to Y_t, so that if $d(Y_{\alpha(t)}(y_1), Y_t(y_2)) \leqslant 4e$ for all $t \in \mathbb{R}$, for a pair of points $y_1, y_2 \in M$, and a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y_2 = Y_t(y_1)$ for some t where $|t| \leqslant \varepsilon$. Therefore, by following the proof of Theorem 4 [16, pp. 497–499] we can see that the semiconjugacy h between X_t and Y_t is injective. $\square$

Proof of Theorem B. It is well known that every Anosov vector field has the shadowing property, and in [13] Robinson proved that if $X \in \mathcal{X}^1(M)$ satisfies both Axiom A and structural stability, then X satisfies the strong transversality condition. Thus, to prove Theorem B, it only remains to show that if $X \in \operatorname{int} \mathcal{E}(M)$ has the shadowing property, then X is structurally stable. However, this fact quickly follows combining Propositions 2 with 3 since $\mathcal{QA}(M) = \operatorname{int}\mathcal{E}(M)$ by Theorem A. $\square$

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