Countable paracompactness versus normality in $\omega_1^2$

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Abstract

We will see that:
(1) In ZFC, for each subspace $X \subseteq \omega_1^2$, the following are equivalent;
   (a) $X$ is normal,
   (b) $X$ is countably paracompact and strongly collectionwise Hausdorff,
   (c) $X$ is expandable.
(2) Under a variety of different set-theoretic assumptions (including $V = L$ and PMEA) all countably paracompact subspaces of $\omega_1^2$ are normal.
(3) All subspaces of $\omega_1^2$ are collectionwise Hausdorff. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction and basic lemmas

All spaces considered in this paper are regular and $T_1$. It is well known that all subspaces of an ordinal with the order topology are (collectionwise) normal and countably paracompact, and that the product space $(\omega_1 + 1) \times \omega_1$ is countably paracompact but not normal. In [6], it is proved that, for $X = A \times B$, where $A$ and $B$ are subspaces of an ordinal:
(1) normality, collectionwise normality and shrinking property of $X = A \times B$ are equivalent,
(2) countable paracompactness and expandability of $X = A \times B$ are equivalent,
(3) normality of $X = A \times B$ implies its countable paracompactness,

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(4) in particular if \( A \) and \( B \) are subspaces of \( \omega_1 \), then normality and countable paracompactness of \( X = A \times B \) are equivalent.

Furthermore the following problems were asked:

**Problem A.** For all subspaces of products of two ordinals, are normality, collectionwise normality and shrinking property equivalent?

**Problem B.** For all subspaces of products of two ordinals, are countable paracompactness and expandability equivalent?

**Problem C.** Is \( A \times B \) countably metacompact for any subspace \( A \) and \( B \) of an ordinal?

Recently Kemoto et al. gave affirmative answers for Problems A and C in [5] and [7], respectively. Actually it is proved in [7] that every subspace of a product of two ordinals is countably metacompact. Moreover in [5], the following question was asked:

**Problem D.** For all subspaces of \( \omega_1^2 \), are normality and countable paracompactness equivalent?

In this paper we give partial answers to Problems B and D as described in the abstract.

In the rest of this section, we introduce some specific notation and recall some basic definitions and lemmas. Let \( X \) be a space. Subsets \( H \) and \( K \) of \( X \) are separated in \( X \) if there are disjoint open sets \( U \) and \( V \) containing \( H \) and \( K \), respectively. Let \( D \) be a closed discrete subspace of \( X \). \( D \) is said to be separated (respectively, strongly separated) in \( X \) if there is a pairwise disjoint (respectively discrete) collection \( \{ U(x) : x \in D \} \) of open sets with \( x \in U(x) \) for each \( x \in D \). A space \( X \) is said to be (strongly) collectionwise Hausdorff if all closed discrete subspaces are (strongly) separated. A space \( X \) is normal if each pair of disjoint closed sets is separated. A space \( X \) is countably paracompact if every countable open cover has a locally finite open refinement. A space \( X \) is said to be expandable if every locally finite collection \( F \) of closed sets has a locally finite open expansion

\[
\mathcal{U} = \{ U(F) : F \in \mathcal{F} \},
\]

that is, \( \mathcal{U} \) is a locally finite collection of open sets such that \( F \subset U(F) \) for each \( F \in \mathcal{F} \).

For \( A \subset \omega_1 \), put

\[
\text{Lim}(A) = \{ \alpha \in \omega_1 : \sup(A \cap \alpha) = \alpha \},
\]

where \( \sup \emptyset = -1 \), \( \text{Succ}(A) = A \setminus \text{Lim}(A) \), \( \text{Lim}(\omega_1) \) and \( \text{Succ} = \text{Succ}(\omega_1) \).

Observe that \( \text{Lim}(A) \) is closed and unbounded (cub) in \( \omega_1 \) whenever \( A \) is unbounded in \( \omega_1 \).

For a cub set \( C \subset \omega_1 \) and \( \alpha \in C \), put \( \text{pC}(\alpha) = \sup(C \cap \alpha) \). Observe that \( \text{pC}(\alpha) \in C \cup \{-1\} \), and \( \text{pC}(\alpha) = \alpha \) iff \( \alpha \in \text{Lim}(C) \), and \( \text{pC}(\alpha) \) is the immediate predecessor of \( \alpha \) in \( C \cup \{-1\} \) whenever \( \alpha \in \text{Succ}(C) \). It is easy to show

\[
\omega_1 \setminus C = \bigcup_{\alpha \in \text{Succ}(C)} (\text{pC}(\alpha), \alpha),
\]

where \((\alpha, \beta)\) denotes the usual open interval.
Assume that a cub set \( C_\alpha \) is defined for each \( \alpha \in A \), where \( A \subset \omega_1 \). Then
\[
\triangle_{\alpha \in A} C_\alpha = \{ \beta \in \omega_1 : \forall \alpha \in A \cap \beta (\beta \in C_\alpha) \}
\]
is a cub set in \( \omega_1 \) (see [8, II, Lemma 6.14]).

For sets \( X \) and \( Y \), \( X \mathrel{\text{\ CouncilScript}} Y \) denotes the set of all function from \( X \) to \( Y \), and \( \mathcal{P}(X) \) denotes the set of all subsets of \( X \). A partial function from \( X \) to \( Y \) is any function \( f \) such that \( \text{dom}(f) \subseteq X \) and \( \text{ran}(f) \subseteq Y \).

We use the following specific notation: Let \( X \subset \omega_1^2 \), \( \alpha \in \omega_1 \) and \( \beta \in \omega_1 \). Let
\[
V_\alpha(X) = \{ \beta \in \omega_1 : (\alpha, \beta) \in X \},
\]
\[
H_\beta(X) = \{ \alpha \in \omega_1 : (\alpha, \beta) \in X \} \quad \text{and}
\]
\[
\triangle(X) = \{ \alpha \in \omega_1 : (\alpha, \alpha) \in X \}.
\]
Finally, for subsets \( C \) and \( D \) of \( \omega_1 \), let \( X_C = X \cap C \times \omega_1 \), \( X_D = X \cap \omega_1 \times D \) and \( X^D_C = X \cap C \times D \).

**Lemma 1.1.** Let \( H \) and \( K \) be disjoint closed sets in a countably paracompact space \( X \).

1. If there is a sequence \( \{ G(n) : n \in \omega \} \) of open sets such that
\[
K \subseteq \bigcap_{n \in \omega} G(n) \subseteq \bigcap_{n \in \omega} \text{Cl} G(n) \subseteq X \mathrel{\text{\ CouncilScript}} H,
\]
then \( H \) and \( K \) are separated.

2. If \( H = \bigcup_{n \in \omega} H(n) \) where \( H(n) \) and \( K \) are separated for each \( n \in \omega \), then \( H \) and \( K \) are separated.

3. If \( H = \bigcup_{n \in \omega} H(n) \) and \( K = \bigcup_{m \in \omega} K(m) \) where \( H(n) \) and \( K(m) \) are separated for each \( n, m \in \omega \), then \( H \) and \( K \) are separated.

**Proof.** (1) See [3].

(2) Since \( H(n) \) and \( K \) are separated, take an open set \( G(n) \) with \( K \subset G(n) \subset \text{Cl} G(n) \subset X \mathrel{\text{\ CouncilScript}} H(n) \), and then apply (1).

(3) Apply (2) twice. \( \square \)

**Lemma 1.2.** Let \( H \) and \( K \) be disjoint closed sets in a subspace \( X \) of \( \omega_1 \). Then \( H \) and \( K \) are separated in \( \omega_1 \).

**Proof.** Put \( Y = \omega_1 \setminus (\text{Cl}_{\omega_1} H \cap \text{Cl}_{\omega_1} K) \). Then, since \( \omega_1 \) is hereditarily normal, \( \text{Cl}_{\omega_1} H \cap Y \) and \( \text{Cl}_{\omega_1} K \cap Y \) are separated in \( Y \). Since \( Y \) is open in \( \omega_1 \), \( H \) and \( K \) are separated in \( \omega_1 \). \( \square \)

**Lemma 1.3.** Let \( X \) be a countably paracompact subspace of \( \omega_1^2 \).

1. \( X_{[0, \alpha)} \) and \( X^{[0, \alpha)} \) are normal for each \( \alpha \in \omega_1 \).

2. If \( C \) is a cub set in \( \omega_1 \), then \( X_{\omega_1 \setminus C} \) and \( X^{\omega_1 \setminus C} \) are normal.

3. If \( \triangle(X) \) is stationary in \( \omega_1 \), then \( X \) is normal.

4. If there is a cub set \( C \) of \( \omega_1 \) such that \( X_C \) and \( X^C \) are separated in \( X \), then \( X \) is normal.
Proof. (1) Let \( \alpha \in \omega_1 \). First we shall show \( X_{[0,\alpha)} \) is countably paracompact. We may assume \( \alpha \in \text{Lim} \), otherwise it is a clopen subspace of \( X \). Fix a strictly increasing cofinal sequence \( \{ \alpha(n): n \in \omega \} \) in \( \alpha \). Since \( X_{[0,\alpha)} \) is represented as the topological sum \( \bigoplus_{n \in \omega} X_{(\alpha(n-1),\alpha(n))] \} \) of clopen subspaces of \( X \), where \( \alpha(-1) = -1 \), it is countably paracompact.

To show \( X_{[0,\alpha)} \) is normal, let \( H \) and \( K \) be disjoint closed sets of \( X_{[0,\alpha)} \). By Lemma 1.1(3), it suffices to separate \( H_{[\gamma]} \) and \( K_{[\delta]} \) for each \( \gamma, \delta \in \alpha \). Let \( \gamma, \delta \in \alpha \). First assume \( \gamma \neq \delta \), say \( \gamma < \delta \). Then \( X_{(\gamma,\gamma]} \) and \( X_{(\gamma,\delta]} \) separate \( H_{[\gamma]} \) and \( K_{[\delta]} \). Next assume \( \gamma = \delta \). It follows from Lemma 1.2 that there are disjoint open sets \( U \) and \( V \) in \( \omega_1 \) which separate \( V_\gamma(H) \) and \( V_\gamma(K) \), respectively. Then \( X^U_{(0,\alpha)} \) and \( X^V_{(0,\alpha)} \) separate \( H_{[\gamma]} \) and \( K_{[\delta]} \). The proof for \( X^{[0,\alpha)} \) is identical.

(2) Let \( C \) be a cub set in \( \omega_1 \). Since
\[
X_{\omega_1 \setminus \omega} = \bigoplus_{\alpha \in \text{Succ}(C)} X_{(\mu_{(\alpha \setminus \omega)}(\alpha), \omega)}
\]
it is normal by (1). The proof for \( X^{\omega_1 \setminus \omega} \) is identical.

(3) Assume \( \Delta(X) \) is stationary in \( \omega_1 \). Let \( U = \{ U(i): i \in 2 \} \) be an open cover of \( X \), where \( 2 = \{ 0, 1 \} \). For each \( \alpha \in \Delta(X) \), fix \( f(\alpha) < \alpha \) and \( i(\alpha) \in 2 \) such that \( X_{(f(\alpha), i(\alpha)])} \subset U(i(\alpha)). \) Applying the PDL, we find \( \gamma \in \omega_1 \) and \( i_0 \in 2 \) such that \( X_{(\gamma, \gamma]} \subset U(i_0) \). Since, by (1), \( X_{[0,\gamma]} \cup X^{[0,\gamma]} \) is a normal clopen subspace of \( X \), it is easy to find a closed shrinking of \( U \).

(4) Let \( C \) be a cub set such that \( X_C \) and \( X^C \) are separated and \( U = \{ U(i): i \in 2 \} \) an open cover of \( X \). Let
\[
\mathcal{H} = \{ H(X \setminus X_C), H(X \setminus X^C) \}
\]
be a closed shrinking of the open cover \( \{ X \setminus X_C, X \setminus X^C \} \). Since \( H(X \setminus X_C) \) is a closed subspace of \( X_{\omega_1 \setminus \omega} \), it is normal. Similarly \( H(X \setminus X^C) \) is normal. So \( \mathcal{U} \cup H(X \setminus X^C) = \{ U(i) \cap H(X \setminus X_C): i \in 2 \} \) has a closed shrinking \( \{ F_C(i): i \in 2 \} \). Similarly \( \mathcal{U} \cup H(X \setminus X^C) \) has a closed shrinking \( \{ F^C(i): i \in 2 \} \). Then \( \{ F_C(i) \cup F^C(i): i \in 2 \} \) is a closed shrinking of \( U \).

(5) Assume \( \Delta(X) \) is not stationary, but \( A \) is stationary. For each \( \alpha \in A \), fix \( h(\alpha) \in V_{\alpha}(X) \cap \Delta_{\gamma \in A} \text{Lim}(V_{\gamma}(X)) \) with \( \alpha < h(\alpha) \). Define \( h(\alpha) = 0 \) if \( \alpha \in \omega_1 \setminus A \). Take a cub set \( C^e \) disjoint from \( \Delta(X) \) and put
\[
C = \{ \alpha \in \omega_1: \forall \alpha' < \alpha \ (h(\alpha') < h(\alpha)) \} \cap C^e.
\]
Then \( C \) is a cub set. For each \( \alpha \in A', A \cap C \), put \( x_{\alpha} = (\alpha, h(\alpha)). \) Note that \( x_{\alpha} \in X \) for each \( \alpha \in A' \). We shall show \( D = \{ x_{\alpha}: \alpha \in A' \} \) is closed discrete. To show this, let
\((\gamma, \delta) \in X\). If \(\gamma \notin C\), then \(X_{\alpha_1 \setminus C}\) is a neighborhood of \((\gamma, \delta)\) which misses \(D\). So assume \(\gamma \in C\). Since \(\gamma \in C \subseteq C'\), we have \(\gamma \neq \delta\). If \(\gamma > \delta\), then \(X_{(0, \gamma]}^{(0, \delta]}\) is a neighborhood of \((\gamma, \delta)\) which does not meet \(D\). If \(\gamma < \delta\), then \(X_{(\gamma, \delta]}^{(0, \delta]}\) is a neighborhood of \((\gamma, \delta)\) which meets \(D\) at most one point.

Next as \(A'\) is stationary, decompose \(A'\) into countably many disjoint stationary sets \(T(n), n \in \omega\). Put \(D(n) = \{x_\alpha : \alpha \in T(n)\}\). Since \(\{D(n) : n \in \omega\}\) is a countable discrete collection of closed sets, by countable paracompactness, we find a locally finite collection \(U = \{U(n) : n \in \omega\}\) of open sets with \(D(n) \subseteq U(n)\) for each \(n \in \omega\). Fix \(n \in \omega\). For each \(\alpha \in T(n)\), since \(x_\alpha = (\alpha, h(\alpha)) \in D(n) \subseteq U(n)\), take \(f(\alpha) < \alpha\) and \(g(\alpha) < h(\alpha)\) such that \(X_{[f(\alpha), g(\alpha)]}^{(0, \delta]} \subseteq U(n)\). Applying the PDL to \(T(n)\), we find a stationary set \(T(n)' \subseteq T(n)\) and \(\gamma(n) < \alpha_1\) such that \(f(\alpha) = \gamma(n)\) for each \(\alpha \in T(n)\). Put \(\gamma = \sup\{\gamma(n) : n \in \omega\}\). Then \(X_{[\gamma, \alpha_1]}^{(0, \alpha_1)} \subseteq U(n)\) for each \(\alpha \in T(n)'\) with \(n \in \omega\).

Pick \(a_0 \in A\) with \(\gamma < a_0\) and \(\beta_0 \in \bigcap_{n \in \omega} \text{Lim}(T(n))' \cap V_{a_0}(X)\) with \(a_0 < \beta_0\). Let \(V\) be a neighborhood of \((a_0, \beta_0)\) and again fix \(n \in \omega\). There is \(\beta < \beta_0\) such that \(a_0 \leq \beta\) and \(X_{[a_0 \beta]}^{(0, \alpha_1]} \subseteq V\). As \(\beta_0 \in \text{Lim}(T(n))\), take \(\delta, \delta' \in T(n)\) with \(\beta < \delta < \delta' < \beta_0\). It follows from \(T(n)' \subseteq C\) and \(\delta < \delta'\) that \(h(\delta) < \delta'\). On the other hand, it follows from \(h(\delta) \in \Delta_{\gamma} \subseteq \text{Lim}(V_{\gamma}(X))\) and \(a_0 \in A \cap h(\delta)\) that \(h(\delta) \in \text{Lim}(V_{a_0}(X))\). Moreover since \(\beta < h(\delta)\) and \(g(\delta) < h(\delta)\), take \(v \in V_{a_0}(X)\) such that \(\max\{\beta, g(\delta)\} < v < h(\delta)\). Then

\[(a_0, v) \in X_{(a_0, \beta_0]}^{(0, \alpha_1]} \cap X_{(\gamma, \delta]}^{(0, \delta]} \subseteq V \cap U(n).\]

This shows \((a_0, \beta_0) \in \text{Cl} U(n)\), and this holds for each \(n \in \omega\). Therefore it contradicts the local finiteness of \(U\). So \(A\) is not stationary. Similarly \(B\) is not stationary. \(\square\)

2. Equivalents of normality

In this section, we will prove that a subspace of \(\omega^2\) is normal if and only if it is expandable if and only if it is countable paracompact and strongly collectionwise Hausdorff. As a corollary, in any model where first countable, countably paracompact spaces are strongly collectionwise Hausdorff, we have that countably paracompactness, normality and expandability are all equivalent for subspaces of \(\omega^2\). We will consider such models in Section 3.

**Theorem 2.1.** Let \(X \subseteq \omega^2\). Then the following are equivalent:

1. \(X\) is countably paracompact and strongly collectionwise Hausdorff.
2. \(X\) is expandable.
3. \(X\) is normal.

**Proof.** To show (3) \(\Rightarrow\) (2), assume \(X\) is normal. By [5] and [7], \(X\) is collectionwise normal and countably paracompact, respectively. Therefore, by [4], \(X\) is expandable (and also (1) holds). Clearly (2) implies (1) so it suffices to prove that (1) implies (3). So assume \(X\) is countably paracompact and strongly collectionwise Hausdorff. Assume by
way of contradiction that \( X \) is not normal. By Lemma 1.3(3), we may assume \( \triangle(X) \) is not stationary. Take a cub set \( D \) which is disjoint from \( \triangle(X) \).

**Claim 1.** \( \mathcal{X} = \{ X^{(p_D(a),a)} | a \in \text{Succ}(D) \} \) is a discrete collection of clopen sets of \( X \).

**Proof.** Let \( \langle \gamma, \delta \rangle \in X \). If \( \gamma \notin D \), then there is \( a \in \text{Succ}(D) \) such that \( \gamma \in (p_D(a),a) \). Then \( X^{(p_D(a),a)} \) is a neighborhood of \( \langle \gamma, \delta \rangle \) which meets at most one member of \( \mathcal{X} \). If \( \gamma \in D \), then by \( D \cap \triangle(X) = \emptyset \), we have \( \gamma \neq \delta \). We may assume \( \gamma < \delta \). Then \( X^{(p_D(a),a)} \) is a neighborhood of \( \langle \gamma, \delta \rangle \) which meets no member of \( \mathcal{X} \). \( \square \)

Now by Lemma 1.3(1) and Claim 1, \( \bigcup \mathcal{X} \) is a normal clopen subspace of \( X \). Let

\[
Y = \{ (\alpha, \beta) \in \omega_1^2 : \alpha < \beta \} \cap (X \setminus \bigcup \mathcal{X}),
\]

\[
Z = \{ (\alpha, \beta) \in \omega_1^2 : \alpha > \beta \} \cap (X \setminus \bigcup \mathcal{X}).
\]

Then \( Y \) and \( Z \) are clopen subspace of \( X \) and \( X = Y \uplus Z \oplus (\bigcup \mathcal{X}) \). So it suffices to show that both \( Y \) and \( Z \) are normal. Since the proofs are the same, we may assume without loss of generality that \( X^{(\alpha,\beta)} \) is stationary in \( \omega_1 \). Moreover for each \( \alpha \in \omega_1 \setminus A \), fix a cub set \( C_\alpha \) which is disjoint from \( V_\alpha(X) \). Then \( C = C^{(\alpha,\beta)} \setminus \bigcup_{\alpha \in \omega_1 \setminus A} C_\alpha \) is a cub set.

**Claim 2.** \( X^C = X^C_\alpha \).

**Proof.** \( \Rightarrow \) is evident. Let \( (\alpha, \beta) \in X^C \). Then \( \alpha < \beta \in C \). If \( \alpha \notin A \), then by the definition of \( \Delta_{\alpha \in \omega_1 \setminus A} C_\alpha \), necessarily \( \beta \in C_\alpha \). So we have \( \beta \in V_\alpha(X) \cap C_\alpha \), a contradiction. Hence \( \alpha \in A \). This proves \( \Rightarrow \). \( \square \)

**Claim 3.** \( X_C \cap X^C = \emptyset \).

**Proof.** Assume \( (\alpha, \beta) \in X_C \cap X^C = X \cap C^2 \). By Claim 2, we have \( \alpha \in A \), this contradicts \( A \cap C = \emptyset \). \( \square \)

So it follows from Lemma 1.3(4) that it suffices to show that \( X_C \) and \( X^C \) are separated.

**Claim 4.** \( \mathcal{X} = \{ X^{(p_C(\gamma),\gamma)} | \gamma \in \text{Succ}(C) \} \) is a discrete collection of closed sets with \( \bigcup \mathcal{X} = X_C \).

**Proof.** \( X^{(p_C(\gamma),\gamma)} \) is clearly closed for each \( \gamma \in \text{Succ}(C) \). First we show the discreteness of \( \mathcal{X} \). Let \( (\alpha, \beta) \in X \). If \( \beta \notin C \), then there is \( \gamma \in \text{Succ}(C) \) such that \( \beta \in (p_C(\gamma), \gamma) \). Then \( X^{(p_C(\gamma),\gamma)} \) is a neighborhood of \( (\alpha, \beta) \) which meets at most one member of \( \mathcal{X} \). If \( \beta \in C \), then by Claim 3, we have \( \alpha \notin C \). Then \( X^{(\alpha,\beta)} \) is a neighborhood of \( (\alpha, \beta) \) which meets no member of \( \mathcal{X} \). Therefore \( \mathcal{X} \) is discrete.
Next assume \( \langle \alpha, \beta \rangle \in X_C \). Since \( \alpha \in C \), we have \( \beta \notin C \). Fix \( \gamma \in \text{Succ}(C) \) such that \( \beta \in (p_C(\gamma), \gamma) \). Then we have \( \langle \alpha, \beta \rangle \in X_C^{(p_C(\gamma), \gamma)} \subseteq \bigcup \mathcal{X} \). So \( X_C \subseteq \bigcup \mathcal{X} \). \( X_C \supset \bigcup \mathcal{X} \) is evident. \( \square \)

As \( X \subseteq \{ \langle \alpha, \beta \rangle \in \omega_1^2 : \alpha < \beta \} \), each member of \( \mathcal{X} \) is countable, say \( \mathcal{X} = \{ \langle \gamma, \beta \rangle : \beta \in \text{Succ}(\omega_1) \} \). Fix \( \langle \gamma, \beta \rangle \) such that \( \forall \gamma < \alpha < \omega_1 \). Then \( X_C = \bigcup_{n \in \omega} K(n) \) and \( |K(n) \cap X_C^{(p_C(\gamma), \gamma)}| \leq 1 \) for each \( \gamma \in \text{Succ}(C) \) and \( n \in \omega \). It follows from Claim 4 that each \( K(n) \) is closed discrete. By Lemma 1.1(2), it suffices to separate \( X_C \) and \( K(n) \) for each \( n \in \omega \).

Fix \( n \in \omega \) and put \( H = X_C^C \) and \( K = K(n) \). Now we use that \( X \) is strongly collectionwise Hausdorff. Since \( X \) is regular, there is a discrete collection \( \mathcal{U} = \{ U(x) : x \in K \} \) of open sets with \( x \in U(x) \subseteq X \setminus H \) for each \( x \in K \). Then \( X \setminus \text{Cl}(\bigcup \mathcal{U}) \) and \( \bigcup \mathcal{U} \) separate \( H \) and \( K \). \( \square \)

The proof of Theorem 2.1 shows the following lemma which will be used in Theorem 3.5.

**Lemma 2.2.** Countably paracompact subspaces of \( \omega_1^2 \) are normal if and only if the following statement holds:

Suppose that \( X \) is a countably paracompact subspace of \( \{ \langle \alpha, \beta \rangle \in \omega_1^2 : \alpha < \beta \} \), and \( C \) is a cub set of \( \omega_1 \) which misses \( A = \{ \alpha \in \omega_1 : V_\alpha(X) \) is stationary in \( \omega_1 \} \). If \( X_C \subseteq A \times \omega_1 \) and \( K \) is a discrete closed subset of \( X_C \), then \( X_C \) and \( K \) can be separated.

To conclude this section we prove the following related result.

**Proposition 2.3.** All subspaces of \( \omega_1^2 \) are collectionwise Hausdorff.

**Proof.** Let \( X \subset \omega_1^2 \) and \( D \) a closed discrete subspace of \( X \). For each \( \langle \alpha, \beta \rangle \in D \), fix \( f(\alpha, \beta) < \alpha \) and \( g(\alpha, \beta) < \beta \) such that \((f(\alpha, \beta), \alpha] \times (g(\alpha, \beta), \beta] \cap D = \{ \langle \alpha, \beta \rangle \} \). Let \( V(\alpha, \beta) = X_{C}^{(f(\alpha, \beta), \alpha]} \).

The following two claims are straightforward to prove:

**Claim 1.** If \( \langle \alpha, \beta \rangle \in D \), \( \langle \alpha, \beta' \rangle \in D \) and \( \beta \neq \beta' \), then \( V(\alpha, \beta) \cap V(\alpha, \beta') = \emptyset \).

**Claim 2.** If \( \langle \alpha, \beta \rangle \in D \), \( \langle \alpha', \beta \rangle \in D \) and \( \alpha \neq \alpha' \), then \( V(\alpha, \beta) \cap V(\alpha', \beta) = \emptyset \).

**Claim 3.** If \( \langle \alpha, \beta \rangle \in D \), \( \langle \alpha', \beta' \rangle \in D \), \( \langle \alpha, \beta \rangle \neq \langle \alpha', \beta' \rangle \) and \( V(\alpha, \beta) \cap V(\alpha', \beta') = \emptyset \), then \( \alpha' < \alpha \) and \( \beta < \beta' \) or \( \alpha < \alpha' \) and \( \beta' < \beta \).

**Proof.** Assume not. Without loss of generality, we may assume \( \alpha' < \alpha \) and \( \beta' < \beta \). Since \( V(\alpha, \beta) \cap V(\alpha', \beta') = \emptyset \), we have \( \langle \alpha', \beta' \rangle \in V(\alpha, \beta) \), a contradiction. \( \square \)

**Claim 4.** \( V = \{ V(\alpha, \beta) : \langle \alpha, \beta \rangle \in D \} \) is star-countable.
Proof. Fix \( (\alpha, \beta) \in D \). Put
\[
D_0 = \{ (\gamma, \delta) \in D : \gamma < \alpha, \ V(\alpha, \beta) \cap V(\gamma, \delta) \neq \emptyset \}.
\]
We show the fact that if \( (\gamma, \delta) \in D_0 \) and \( (\gamma, \delta') \in D_0 \), then \( \delta = \delta' \). To show this assume \( \delta \neq \delta' \). Without loss of generality, we may assume \( \delta < \delta' \). By Claim 3, we have \( \delta < \delta' \). Then since \( V(\alpha, \beta) \cap V(\gamma, \delta') \neq \emptyset \), we have \( (\gamma, \delta) \in V(\gamma, \delta') \), a contradiction. Define \( F : D_0 \rightarrow \alpha \) by \( F(\gamma, \delta) = \gamma \). By the fact, \( F \) is well-defined and one-to-one. Therefore \( D_0 \) is countable. Next put
\[
D_1 = \{ (\gamma, \delta) \in D : \gamma > \alpha, \ V(\alpha, \beta) \cap V(\gamma, \delta) \neq \emptyset \}.
\]
Then similarly we can show \( D_1 \) is countable. So we see \( V \) is star-countable. \( \square \)

For each pair of \( (\alpha, \beta) \) and \( (\gamma, \delta) \) in \( D \), define \( (\alpha, \beta) \sim (\gamma, \delta) \) by there is a finite sequence \( (\alpha_i, \beta_i) \)'s \( (i \in \mathbb{N}) \) such that \( (\alpha, \beta) = (\alpha_0, \beta_0) \), \( (\gamma, \delta) = (\alpha_{n-1}, \beta_{n-1}) \) and \( V(\alpha_i, \beta_i) \cap V(\alpha_{i+1}, \beta_{i+1}) \neq \emptyset \) for each \( i \in n - 1 \). Then \( \sim \) is an equivalence relation on \( D \) and by Claim 4, each equivalence class is countable, so separated. Moreover since
\[
\{ \bigcup \{ V(\alpha, \beta) : (\alpha, \beta) \in E \} : E \in D / \sim \}
\]
is disjoint, \( D \) is separated. \( \square \)

3. Consistency results

In light of Theorem 2.1 we are interested in the question when a countably paracompact subspace of \( \omega_1^2 \) is strongly collectionwise Hausdorff. Indeed, under certain set-theoretic assumptions all first countable, countably paracompact subspaces are strongly collectionwise Hausdorff. For example, we have the following important theorem of Burke.

Theorem 3.1 [1]. If the PMEA is assumed, then first countable, countably paracompact spaces are strongly collectionwise Hausdorff.

In fact, for spaces of size \( \omega_1 \) we need only assume WMEA in Theorem 3.1. Moreover, in the models obtained by adding \( \omega_1^2 \) many random or Cohen reals to a model of CH, one obtains a model where the conclusion to Theorem 3.1 holds (for a discussion of PMEA, WMEA and the Cohen and random real results, see [10]). Therefore we have, for example, the following

Corollary 3.2. If WMEA is assumed, then countably paracompact subspaces of \( \omega_1^2 \) are normal.

It is open whether the conclusion of Theorem 3.1 can be established assuming Gödel’s Axiom of Constructibility, \( V = L \) (see Problem E’ in Section 4 below). The rest of this section is devoted to the partial answer that under \( \diamond_{SS} \) (a consequence of \( V = L \)) countably paracompact subspaces of \( \omega_1^2 \) are normal.
Definition 3.3. If $A$ is a set, a function $T$ from $^{\omega_1}A$ to $\mathcal{P}(\omega_1)$ is a \textit{stationary system} (for $A$) if

1. $T(f)$ is stationary for each $f \in ^{\omega_1}A$, and
2. $\forall \gamma \in \omega_1 \forall f, g \in ^{\omega_1}A(f|\gamma = g|\gamma \rightarrow T(f) \cap [0, \gamma] = T(g) \cap [0, \gamma])$.

Diamond for stationary systems at $\omega_1$, abbreviated $\diamond_{SS}$, is the assertion that, for each stationary system $T$ for $\omega_1$, there is a sequence $\{g_\gamma : \gamma \in \omega_1\}$ with $g_\gamma \in T(\gamma)$ such that $\{\gamma \in T(f) : f|\gamma = g_\gamma\}$ is stationary for each $f \in ^{\omega_1}\omega_1$.

It is well known that $\diamond_{SS}$ is a consequence of $V = L$ (see [2]). The following lemma can be proven by a standard coding argument (see [8]).

Lemma 3.4. $\diamond_{SS}$ is equivalent to (*) where

(*) For each stationary system $T : ^{\omega_1}(\omega \times \omega_1) \to \mathcal{P}(\omega_1)$ there is a sequence $\{g_\gamma : \gamma \in \omega_1\}$ with $g_\gamma \in T(\gamma)$ such that, for each $f \in ^{\omega_1}(\omega \times \omega_1)$, $\{\gamma \in T(f) : f|\gamma = g_\gamma\}$ is stationary.

Theorem 3.5. If $\diamond_{SS}$ is assumed, then countably paracompact subspaces of $\omega_1^2$ are normal.

Proof. Let $X$ be a countably paracompact subspace of $\{\alpha, \beta\} \in \omega_1^2 : \alpha < \beta$, and $C$ is a cub set of $\omega_1$ which misses $A = \{\alpha \in \omega_1 : V_\alpha(X) \text{ is stationary in } \omega_1\}$. Moreover assume $X^C \subset A \times \omega_1$ and $K$ is a discrete closed set of $X_C$. By Lemma 2.2, it suffices to show that $X^C$ and $K$ can be separated.

For each $\alpha \in A \cap \text{Lim}$, as $A \cap \omega = \emptyset$ and $C$ is closed in $\omega_1$, fix a strictly increasing cofinal sequence $\{\alpha(k) : k \in \omega\}$ in $\omega$ with $\alpha(0), \alpha(1) \cap C = \emptyset$. Moreover for each $\alpha \in A \cap \text{Succ}$, put $\alpha(k) = \alpha - 1$ for each $k \in \omega$, where $\alpha - 1$ is the immediate predecessor of $\alpha$. For a partial function $f$ from $\omega_1$ to $\omega \times \omega_1$, let $f(\alpha) = (f_0(\alpha), f_1(\alpha))$ whenever $\alpha \in \text{dom}(f)$. Moreover for each $\alpha \in A \cap \text{dom}(f)$, put $U(f(\alpha)) = X^C_{(f_0(\alpha), \alpha)}$. For each $f \in ^{\omega_1}(\omega \times \omega_1)$ and $k \in \omega$, put

$$T(f, k) = \gamma \in \omega_1 : K(\gamma) \cap \text{Cl}(\bigcup \{U(f(\alpha)) : \alpha \in A \cap \gamma, f_0(\alpha) = k\}) \neq \emptyset$$

$$T(f) = \bigcup_{k \in \omega} T(f, k).$$

Then $T : ^{\omega_1}(\omega \times \omega_1) \to \mathcal{P}(\omega_1)$. There are two cases to consider.

Case 1. There is an $f \in ^{\omega_1}(\omega \times \omega_1)$ such that $T(f)$ is not stationary in $\omega_1$. In this case, take a cub set $E'$ which is disjoint from $T(f)$. And put

$$E = \{\gamma \in \omega_1 : \forall \gamma' < \gamma(f_1(\gamma') < \gamma)\} \cap E' \cap C,$$

$K' = K_E$ and $K'' = K \setminus K' = K_{\omega_1 \setminus E}$. Note that $E$ is a cub set in $\omega_1$. Since $X \cap C^2 = \emptyset$ and $E \subset C$, we have $H = X^C \subset X_{\omega_1 \setminus C} \subset X_{\omega_1 \setminus E}$. As $K'' = K_{\omega_1 \setminus E} \subset X_{\omega_1 \setminus E}$, $H$ and $K''$ are disjoint closed sets in the normal open subspace $X_{\omega_1 \setminus E}$ of $X$ by Lemma 1.3(2). So take an open set $G$ in $X$ such that $K'' \subset G \subset \text{Cl}_X G \subset X \setminus H$.
On the other hand, as $\emptyset = E \cap T(f) = E \cap \bigcup_{k \in \omega} T(f, k)$, we have
\[ K_{(\gamma)} \cap \text{Cl}\left( \bigcup \{ U(f(\alpha)) : \alpha \in A \cap \gamma, f_0(\alpha) = k \} \right) = \emptyset \]
for each $\gamma \in E$ and $k \in \omega$. So noting $K_{(\gamma)} \subset \{ \gamma \} \times (\omega, \omega_1)$, take an open set $G(\gamma, k)$ such that
\[ K_{(\gamma)} \subset G(\gamma, k) \subset X_{(\gamma, \omega_1)} \quad \text{and} \quad \quad \quad \quad \quad G(\gamma, k) \cap \left( \bigcup \{ U(f(\alpha)) : \alpha \in A \cap \gamma, f_0(\alpha) = k \} \right) = \emptyset, \]
for each $\gamma \in E$ and $k \in \omega$. Put $G(k) = (\bigcup_{\gamma \in E} G(\gamma, k)) \cup G$ for each $k \in \omega$. Then clearly $K \subset \bigcap_{k \in \omega} G(k)$ holds.

**Claim 1.** $\bigcap_{k \in \omega} \text{Cl}\ G(k) \cap H = \emptyset$.

**Proof.** Let $\langle \alpha, \beta \rangle \in H = X^C = X^C_A$. Put $k = f_0(\alpha)$. We shall show $\langle \alpha, \beta \rangle \not\in \text{Cl}\ G(k)$. Since $\alpha \in A$, note that the sequence $\{ \alpha(n) : n \in \omega \}$ is already defined with $\langle \alpha(0), \alpha \rangle \cap C = \emptyset$. Moreover note that $A \cap E \subset A \cap C = \emptyset$.

Let
\[ V = \begin{cases} \left. X_{(\alpha(0), \alpha]} \setminus \text{Cl}\ G \right. & \text{if } f_1(\alpha) \geqslant \beta, \\ \left. [X_{(\alpha(0), \alpha]} \cap U(f(\alpha))] \setminus \text{Cl}\ G \right. & \text{if } f_1(\alpha) < \beta. \end{cases} \]

We shall show $V \cap G(k) = \emptyset$. In either cases, as $V \cap G = \emptyset$, it suffices to show $V \cap G(\gamma, k) = \emptyset$ for each $\gamma \in E$. Let $\gamma \in E$. It follows from $\alpha \in A$, $\gamma \in E$ and $A \cap E = \emptyset$ that $\alpha \neq \gamma$. If $\gamma < \alpha$, then as $\gamma \in E \subset C$, $\langle \alpha(0), \alpha \rangle \cap C = \emptyset$ and $G(\gamma, k) \subset X_{(\gamma, \omega_1)}$, we have $X_{(\alpha(0), \alpha)} \cap G(\gamma, k) = \emptyset$. Therefore $V \cap G(\gamma, k) = \emptyset$. So we may assume $\gamma > \alpha$. First we consider the case $f_1(\alpha) \geqslant \beta$. In this case, it follows from $\alpha < \gamma \in E$ that $\beta < f_1(\alpha) < \gamma$. Since $G(\gamma, k) \subset X_{(\gamma, \omega_1)}$ and $V \subset X_{(0, \beta]} \subset X_{(0, \gamma]}$, we have $V \cap G(\gamma, k) = \emptyset$. Next we consider the case $f_1(\alpha) < \beta$. In this case, $U(f(\alpha))$ is a neighborhood of $\langle \alpha, \beta \rangle$. Moreover it follows from $\alpha \in A \cap \gamma$ and $f_0(\alpha) = k$ that, by the definition of $G(\gamma, k)$, $U(f(\alpha)) \cap G(\gamma, k) = \emptyset$. Since $V \subset U(f(\alpha))$, we have $V \cap G(\gamma, k) = \emptyset$. This completes the proof of Claim 1. \qed

By Lemma 1.1(1), $H$ and $K$ are separated. So this case is complete.

**Case 2.** $T(f)$ is stationary for every $f \in \omega_1(\omega \times \omega_1)$.

**Claim 2.** $T$ is a stationary system.

**Proof.** Let $\beta \in \omega_1$ and $f, g \in \omega_1(\omega \times \omega_1)$ with $f|\beta = g|\beta$. To show $T(f) \cap [0, \beta] \subset T(g) \cap [0, \beta]$, let $\gamma \in T(f) \cap [0, \beta]$, say $\gamma \in T(f, k)$. Then
\[ K_{(\gamma)} \cap \text{Cl}\left( \bigcup \{ U(f(\alpha)) : \alpha \in A \cap \gamma, f_0(\alpha) = k \} \right) \neq \emptyset. \]
It follows from $\gamma \leqslant \beta$ that $f|A \cap \gamma = g|A \cap \gamma$. Therefore $K_{(\gamma)} \cap \text{Cl}\left( \bigcup \{ U(g(\alpha)) : \alpha \in A \cap \gamma, g_0(\alpha) = k \} \right) \neq \emptyset$, so $\gamma \in T(g) \cap [0, \beta]$. The converse inclusion is similar. \qed
Applying the assertion (*) in Lemma 3.4, take a sequence \( \{g_\gamma : \gamma \in \omega_1\} \) with \( g_\gamma \in \mathcal{U}(\omega \times \omega_1) \) such that, for each \( f \in \omega_1(\omega \times \omega_1) \), \( \{\gamma \in T(f) : f|\gamma = g_\gamma\} \) is stationary. Say \( g_\gamma(\alpha) = (g_\gamma(0)(\alpha), g_\gamma(\alpha)) \) if \( \alpha \in \text{dom}(g_\gamma) \). Define \( F \in \omega_1 \omega \) by

\[
F(\gamma) = \min\{k \in \omega : K_{\gamma|\gamma} \cap \overline{\bigcup \{U(g_\gamma(\alpha)) : \alpha \in A \cap \gamma, g_\gamma(0)(\alpha) = k\}} \neq \emptyset\}
\]

if the minimum exists, and

\[
F(\gamma) = 0, \quad \text{otherwise.}
\]

Put \( L(k) = K_{F^{-1}(k)} \) for each \( k \in \omega \). Then \( \mathcal{L} = \{L(k) : k \in \omega\} \) is a countable partition of \( K \).

Since \( X \) is countably paracompact, there is a locally finite collection \( \mathcal{W} = \{W(k) : k \in \omega\} \) of open sets with \( L(k) \subset W(k) \) for each \( k \in \omega \). Let \( \alpha \in A \). As \( \mathcal{W} \) is locally finite and \( \{(\alpha(k), \alpha) : k \in \omega\} \) is a decreasing neighborhood base at \( \alpha \) in \( \omega_1(X) \), there are \( k(\alpha, \beta) \in \omega \) and \( \delta(\alpha, \beta) < \beta \) such that

\[
\{k \in \omega : X_{(\alpha(k(\alpha, \beta)), \alpha)}^{(\delta(\alpha, \beta), \alpha)} \cap W(k) \neq \emptyset\} \subset k(\alpha, \beta).
\]

Since \( V_\alpha(X) \) is stationary, applying the PDL, there are \( k(\alpha) \in \omega \) and \( \delta(\alpha) \in \omega_1 \) such that

\[
\{k \in \omega : X_{(\alpha(k(\alpha), \alpha))}^{(\delta(\alpha), \alpha)} \cap W(k) \neq \emptyset\} \subset k(\alpha).
\]

Let \( f \) be a function in \( \omega_1(\omega \times \omega_1) \) satisfying \( f(\alpha) = (k(\alpha), \delta(\alpha)) \) whenever \( \alpha \in A \). By (*) there is \( \gamma \in T(f) \) with \( f|\gamma = g_\gamma \). Say \( \gamma \in T(f, k) \). Then

\[
K_{\gamma|\gamma} \cap \overline{\bigcup \{U(f(\alpha)) : \alpha \in A \cap \gamma, f_0(\alpha) = k\}} \neq \emptyset,
\]

therefore

\[
K_{\gamma|\gamma} \cap \overline{\bigcup \{U(g_\gamma(\alpha)) : \alpha \in A \cap \gamma, g_\gamma(0)(\alpha) = k\}} \neq \emptyset.
\]

So by the definition of \( F \), we have

\[
\emptyset \neq K_{\gamma|\gamma} \cap \overline{\bigcup \{U(g_\gamma(\alpha)) : \alpha \in A \cap \gamma, g_\gamma(0)(\alpha) = F(\gamma)\}}
\]

\[
= K_{\gamma|\gamma} \cap \overline{\bigcup \{U(f(\alpha)) : \alpha \in A \cap \gamma, f_0(\alpha) = F(\gamma)\}}.
\]

It follows from \( K_{\gamma|\gamma} \subset L(F(\gamma)) \subset W(F(\gamma)) \) that there is \( \alpha \in A \cap \gamma \) with \( f_0(\alpha) = F(\gamma) \) and \( W(F(\gamma)) \cap U(f(\alpha)) \neq \emptyset \). Since \( f(\alpha) = (k(\alpha), \delta(\alpha)) \) and \( U(f(\alpha)) = X_{(\alpha(k(\alpha)), \alpha)}^{(\delta(\alpha), \alpha)} \), we have \( F(\gamma) \in k(\alpha) \). So \( F(\gamma) = k(\alpha) = f_0(\alpha) = F(\gamma) \), therefore we get a contradiction. These argument means that Case 2 cannot be happen.

\[\square\]

4. Examples and problems

Our results in Section 3 give a consistent affirmative answer to Problem D. It remains open whether an affirmative answer to Problem D is true in ZFC. In particular we would like to answer:

**Problem D’.** Assume MA + ¬CH. Are countably paracompact subspaces of \( \omega_1^2 \) normal?
Since subspaces of $\omega_1^2$ are collectionwise Hausdorff (Proposition 2.3), it is natural to ask:

**Problem E.** Are first countable countably paracompact collectionwise Hausdorff spaces strongly collectionwise Hausdorff?

No consistent counterexample to Problem E is known. An affirmative answer even under $V = L$ would be interesting. In particular it would answer the following question of Nyikos.

**Problem E’.** If $V = L$ (or $\diamondsuit_S$) is assumed, are first countable, countably paracompact collectionwise Hausdorff spaces strongly collectionwise Hausdorff?

In this connection, Watson [11] has constructed a countably paracompact not strongly collectionwise Hausdorff space in ZFC (it’s not first countable). And the third author [9] has constructed from $V = L$ a first countable paranormal not strongly collectionwise Hausdorff space.

The following proposition may be of interest when considering strongly collectionwise Hausdorff spaces. Since it is well known we omit its proof (it can be found, for example, in [1]). Recall a subset $D$ is a regular $G_\delta$ if there is a sequence $\{G(n): n \in \omega\}$ of open sets with $D = \bigcap_{n \in \omega} G(n) = \bigcap_{n \in \omega} \text{Cl} G(n)$.

**Proposition 4.1.** Let $X$ be a first countable countably paracompact collectionwise Hausdorff space and $D$ a closed discrete subspace of $X$. Then $D$ is a regular $G_\delta$ if and only if $D$ is strongly separated.

In relation to Proposition 2.3 it is interesting to note that some subspaces of $(\omega_1 + 1)^2$ are not collectionwise Hausdorff:

**Example 4.2.** Let $X = \{ (\alpha, \beta): \alpha < \beta < \omega_1 \} \cup \{ (\alpha, \omega_1): \alpha \in \text{Succ} \}$. Then $D = \{ (\alpha, \alpha + 1): \alpha \in \text{Lim} \} \cup \{ (\alpha, \omega_1): \alpha \in \text{Succ} \}$ is clearly closed discrete in $X$. However, $D$ cannot be separated: Let $\mathcal{W} = \{ W_\alpha: \alpha \in \omega_1 \}$ be a collection of open sets such that $(\alpha, \alpha + 1) \in W_\alpha$ for each $\alpha \in \text{Lim}$, and $(\alpha, \omega_1) \in W_\alpha$ for each $\alpha \in \text{Succ}$. Applying the pressing down lemma to $\text{Lim}$, we can find a $\gamma < \omega_1$ and a stationary set $S \subseteq \text{Lim}$ such that $((\gamma, \alpha) \times (\alpha + 1) \cap X \subseteq W_\alpha$ for each $\alpha \in S$. Pick $\alpha_0 \in \text{Succ}$ with $\gamma < \alpha_0$, then $W_{\alpha_0}$ meets all $W_\alpha$’s $(\alpha \in S)$. Therefore $D$ cannot be separated.

Finally, we give the following example:

**Example 4.3.** A subspace $X$ of $\omega_1^2$ such that

1. $X_{[0, \alpha]}$ and $X^{[0, \alpha]}$ are normal for each $\alpha \in \omega_1$;
2. there is a cub set $C$ in $\omega_1$ such that $X_C$ and $X^C$ are disjoint but not separated;
3. there is a closed discrete subspace $D$ of $X$ such that $D$ is regular $G_\delta$ but not strongly separated.
First for each $\alpha \in \text{Lim}$, fix a strictly increasing cofinal sequence $\{\alpha(n): n \in \omega\}$ in $\alpha$ such that $\alpha(n) = \beta + (n + 1)$ for some $\beta \in \text{Lim} \cup \{0\}$. Let
\[
L = \bigcup_{\alpha \in \text{Lim}} \left( \{\alpha\} \cup \{\alpha(n): n \in \omega\} \right) \times \{\alpha + 1\}
\]
and
\[
N = \bigcup_{\alpha \in \text{Succ}} \{\alpha\} \times \{\beta \in \text{Lim}: \alpha < \beta\}.
\]
And let $X = L \cup N$. Then $X \subset \{(\alpha, \beta) \in \omega^2: \alpha < \beta\}$. We shall show this $X$ is as required.

(1) For each $\alpha \in \omega_1$, $X_{\{0,\alpha\}}$ is countable, so it is normal. We shall show $X_{\{0,\alpha\}}$ is normal for each $\alpha \in \omega_1$ by induction. Assume $X_{\{0,\beta\}}$ is normal for each $\beta < \alpha$. We may assume $\alpha$ is limit. As in the proof of Lemma 1.3(1), $X_{\{0,\alpha\}}$ is normal. Since $X_{\{\alpha+1\}}$ is a normal clopen subspace of $X$ and $X_{\{0,\alpha\}} \setminus X_{\{\alpha+1\}}$ is a clopen subspace of $X_{\{0,\alpha\}}$, $X_{\{0,\alpha\}} = (X_{\{0,\alpha\}} \setminus X_{\{\alpha+1\}}) \oplus X_{\{\alpha+1\}}$ is normal.

(2) and (3) Put $D = X_{\text{Lim}} = \{(\alpha, \alpha + 1): \alpha \in \text{Lim}\}$. It is straightforward to show $D$ is closed discrete in $X$, and $X_{\text{Lim}} \cap X^\text{Lim} = \emptyset$.

**Claim 1.** $D$ is a regular $G_\delta$.

**Proof.** Put
\[
V(n) = \bigcup_{\alpha \in \text{Lim}} \left( \{\alpha\} \cup \{\alpha(i): i \geq n\} \right) \times \{\alpha + 1\}
\]
for each $n \in \omega$. Then each $V(n)$ is open in $X$ and $D \subset \bigcap_{n \in \omega} V(n)$.

Let $\langle \gamma, \delta \rangle \notin D$. First assume $\delta \in \text{Succ}$. If $\delta = \alpha + 1$ for some $\alpha \in \text{Lim}$, then take $n \in \omega$ such that $\gamma < \alpha(n)$. Then $X_{\{0,\gamma\}}^{[\alpha]}$ is a neighborhood of $\langle \gamma, \delta \rangle$ which does not meet $V(n)$. If $\delta = \alpha + 1$ for some $\alpha \in \text{Succ}$, then $X_{\{\alpha\}}^{[\delta]}$ is a neighborhood of $\langle \gamma, \delta \rangle$ which does not meet any $V(n)$. Next assume $\delta \in \text{Lim}$. Note that $\gamma \in \text{Succ}$. Assume that for each $n \in \omega$, $\gamma \in \bigcup_{\alpha \in \text{Lim}} \{\alpha\} \cup \{\alpha(i): i \geq n\}$. Then for each $n \in \omega$, we can take $\alpha_n \in \text{Lim}$ and $i_n \geq n$ such that $\gamma = \alpha_n(i_n)$. Since for each $n \in \omega$, $\gamma = \beta + (i_n + 1)$ for some $\beta \in \text{Lim} \cup \{0\}$, we have a contradiction. These argument shows $\bigcap_{n \in \omega} \text{Cl} V(n) \subset D$. Therefore $D$ is a regular $G_\delta$. \(\square\)

**Claim 2.** $D = X_{\text{Lim}}$ and $X^\text{Lim}$ cannot be separated.

**Proof.** Let $W$ be an open set containing $X_{\text{Lim}}$. For each $\alpha \in \text{Lim}$, as $\langle \alpha, \alpha + 1 \rangle \in W$, pick $n(\alpha) \in \omega$ with $\langle \alpha(n(\alpha)), \alpha + 1 \rangle \in W$. Since $\alpha(n(\alpha)) < \alpha$, applying the PDL, we get $\gamma \in \omega_1$ and a stationary set $S \subset \text{Lim}$ such that $\alpha(n(\alpha)) = \gamma$ for each $\alpha \in S$. Pick $\delta \in \text{Lim}(\{\alpha + 1: \alpha \in S\})$. Then $\langle \gamma, \delta \rangle \in X^\text{Lim} \cap \text{Cl} W$. Therefore $X_{\text{Lim}}$ and $X^\text{Lim}$ cannot be separated. \(\square\)

It follows from Claim 2 that $D$ is not strongly separated (otherwise, $X_{\text{Lim}}$ and $X^\text{Lim}$ would be separated). Therefore, by Proposition 4.1, $X$ is not countably paracompact.
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