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On a Set of Lattice Points not Containing the Vertices of a Square

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Let $S_n = \{(x, y): x, y$ integers, $0 \leq x, y \leq n-1\}$, so that S_n is an $n \times n$ square array of lattice points in the plane. Denote by $g(n)$ the size of a smallest subset X of S_n which does not contain the vertices of a square with its sides parallel to the sides of S_n but which is such that the addition of any new point to X forces the appearance of such a square. In [1] it was shown that

$$g(2^k) \leq 3^k \text{ for } k = 1, 2, 3, \ldots \tag{1}$$

and it was pointed out that the construction which leads to (1) implies

$$g(n) \leq 3n^\alpha \quad \text{where} \quad \alpha = \log 3/\log 2 = 1.58\ldots. \tag{2}$$

For further background information on this and related problems, see [1–3].

In this note we obtain a lower bound for $g(n)$ by proving the following theorem.

THEOREM. *For every $\varepsilon > 0$, $g(n) > n^{4/3-\varepsilon}$, provided $n \geq n_0(\varepsilon)$.*

PROOF. Let X be a minimal subset of S_n with the desired property. Color the points of X red and the points of $S_n - X$ blue. With each blue point b we may associate a square Q_b, one of whose vertices is b and whose remaining three vertices are red. There may be several choices for Q_b; it does not matter which one is selected. The set of squares Q_b, $b \in S_n - X$, may be split into four classes according to the location of b. One of these classes must contain at least $\frac{1}{4}(n^2 - |X|)$ members. Denote this class by C and observe that we may, without loss of generality, suppose that if $Q_b \in C$ then b is the upper left corner of Q_b. The points in the upper left and lower right corners of a member of C will be called a special pair. The red and blue points which are members of special pairs will also be called special.

Let $k \geq 2$ be a positive integer and let the numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ be defined by

$$\alpha_1 = \frac{2^k - 1}{3.2^{k-1} - 2}, \tag{3}$$

$$\alpha_2 = 3\alpha_1 - 2, \tag{4}$$

$$\alpha_i = 2^{i-2}\alpha_2 \quad \text{for} \quad i = 3, 4, \cdots, k. \tag{5}$$

It is easy to check that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1 \tag{6}$$

and we note also that

$$\lim_{k\to\infty} \alpha_1 = \tfrac{2}{3}. \tag{7}$$

Let

$$\beta_i = \alpha_1 + \alpha_2 + \cdots + \alpha_i, \; i = 1, 2, \ldots, k. \tag{8}$$

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Split the special red points into k classes $R_1, R_2, \ldots, R_k$ where

$$R_1 = \{r: r \text{ belongs to } d \text{ special pairs, } d < n^{\alpha_1}\}$$

and for $i = 2, \ldots, k$

$$R_i = \{r: r \text{ belongs to } d \text{ special pairs, } n^{\beta_{i-1}} \leq d < n^{\beta_i}\}.$$

Let B_i be the set of special blue points which are paired with points in R_i. Then for some i, $1 \leq i \leq k$,

$$|B_i| \geq \frac{n^2 - |X|}{4k}. \tag{9}$$

If (9) holds for $i = 1$ we have

$$|R_1| n^{\alpha_1} \geq |B_1| \geq \frac{n^2 - |X|}{4k}$$

so that, by (2), for all sufficiently large n

$$|R_1| \geq \frac{n^{2-\alpha_1}}{5k} \tag{10}$$

and the desired conclusion follows from (7) and (10).

We may therefore suppose that (9) holds for some $i \geq 2$. Since each $r \in R_i$ belongs to fewer than n^{β_i} special pairs, we have

$$|R_i| n^{\beta_i} \geq |B_i| \geq \frac{n^2 - |X|}{4k}$$

so that, for large enough n,

$$|R_i| \geq \frac{n^{2-\beta_i}}{5k}. \tag{11}$$

If at least $|R_i|^{\frac{1}{2}}$ of the points in R_i lie one some horizontal line L, then, since each vertical line containing a member of $R_i \cap L$ must contain at least $n^{\beta_{i-1}}$ red points, it follows that the number q of red points exceeds $|R_i|^{\frac{1}{2}} n^{\beta_{i-1}}$.

Thus

$$\begin{aligned}
q &> \frac{1}{\sqrt{5k}} n^{1+\beta_{i-1}-\frac{1}{2}\beta_i} \text{ by (11),} \\
&= \frac{1}{\sqrt{5k}} n^{1+\frac{1}{2}(\alpha_1+\cdots+\alpha_{i-1})-\frac{1}{2}\alpha_i} \text{ by (8),} \\
&= \frac{1}{\sqrt{5k}} n^{\frac{3}{2}-\frac{1}{2}(2\alpha_i+\alpha_{i+1}+\cdots+\alpha_k)} \text{ by (6),} \\
&= \frac{1}{\sqrt{5k}} n^{\frac{3}{2}-2^{k-2}\alpha_2} \text{ by (5),} \\
&= \frac{1}{\sqrt{5k}} n^{\frac{3}{2}-2^{k-2}(3\alpha_1-2)} \text{ by (4),} \\
&= \frac{1}{\sqrt{5k}} n^{2-\alpha_1} \text{ by (3).}
\end{aligned}$$

The desired conclusion now follows from (7).

Finally, if every horizontal line contains fewer than $|R_i|^{\frac{1}{2}}$ members of R_i then there must be at least $|R_i|^{\frac{1}{2}}$ horizontal lines each of which contains at least one member of R_i. Each of these lines must therefore contain at least $n^{\beta_{i-1}}$ red points and the same conclusion holds. This completes the proof of the theorem.

The question, posed in [1], as to whether there exists a number β such that $g(n) = n^{\beta+o(1)}$, as $n \to \infty$, remains unanswered. In this regard it may be useful to have some functional equation of inequality for g. However, we do not even know whether $g(n+1) \geq g(n)$ for all $n \geq 2$.

One may raise a similar question in higher dimensions. For example, let A_n denote an $n \times n \times n$ cubical array of lattice points in three dimensions. Denote by $f(n)$ the size of a smallest subset Z of A_n which does not contain the vertices of a cube with its faces parallel to the faces of A_n but which is such that the addition of any new point to Z forces the appearance of such a cube. It is easy to see that $f(n) > n^{\frac{7}{3}-\varepsilon}$ for every $\varepsilon > 0$, n sufficiently large. In fact, the proof of our theorem makes no use of the condition that X does not contain the vertices of a square. It uses only the condition that adding a point to X forces the appearance of a new square. Thus if we let $g^*(n)$ denote the size of a smallest such X, then $g^*(n) > n^{\frac{4}{3}-\varepsilon}$. Now if $f(n) < ng^*(n)$ then one of the horizontal planes of A_n would contain fewer than $g^*(n)$ points of Z. We could then add a new point y to this layer without forcing the appearance of a new square and thus also without forcing the appearance of a cube in $Z \cup \{y\}$. Thus $f(n) \geq ng^*(n)$ and hence $f(n) > n^{\frac{7}{3}-\varepsilon}$, if n is large enough. The method of proof of our theorem, unfortunately, gives a weaker result when directly applied in this situation.

We remark also that a variant of the argument used in [1] to prove (1) may be used to prove that for some constant c

$$f(n) < cn^{\alpha}, \ \alpha = \log 7/\log 2. \tag{12}$$

The construction may be described as follows:
Write the numbers $0, 1, \ldots, 2^k - 1$ in base 2, so that if $0 \leq a \leq 2^k - 1$ then $a = \Sigma_{i \in A} 2^i$ where $A \subseteq \{0, 1, \ldots, k-1\}$. Then put (a, b, c) in Z if $A \cup B \cup C = \{0, 1, \ldots, k-1\}$ where B and C are defined for b and c in the same manner that A is defined for a. An argument which is only slightly more complicated in detail than the one used in [1] may be used to show that Z has the desired property. Thus $f(2^k) \leq 7^k$ and (12) follows.

References

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