A piecewise variational iteration method for treating a nonlinear oscillator of a mass attached to a stretched elastic wire

Fazhan Geng
Department of Mathematics, Changshu Institute of Technology, Changshu, Jiangsu 215500, China

A R T I C L E  I N F O
Article history:
Received 26 January 2011
Accepted 2 May 2011

Keywords:
Piecewise variational iteration method
Variational iteration method
Nonlinear oscillator

A B S T R A C T
In this paper, we introduce a piecewise variational iteration method for treating a nonlinear oscillator of a mass attached to a stretched elastic wire. For the nonlinear oscillator, the present method can produce a good approximation to the exact solution in a very large region, while the standard variational iteration method (VIM) gives a good approximation only in a small region. The numerical results obtained show that the present method does not share the drawback of the standard VIM and can provide very accurate analytical approximate solutions for both small and large values of the oscillation amplitude and parameter.

1. Introduction

The governing non-dimensional equation of motion for a mass attached to a stretched elastic wire is [1]

\[
\begin{cases}
  u''(t) + u(t) - \frac{\lambda u}{\sqrt{1 + u^2}} = 0, & 0 \leq \lambda < 1, 0 \leq t \leq T, \\
  u(0) = A, & u'(0) = 0
\end{cases}
\]  

(1.1)

which is an example of a conservative nonlinear oscillatory system having an irrational elastic item. The system oscillates between symmetric bounds \([-A, A]\), and its angular frequency and corresponding periodic solution are dependent on the amplitude \(A\).

The problem has attracted a lot of attention. However, deriving its analytical solution in an explicit form seems to be unlikely to be achievable except in certain special situations. Therefore, one has to go for the numerical techniques or approximate approaches to get its solution. Recently, Xu [2] applied He’s parameter-expansion method to system (1.1) and obtained good approximate solutions. Ganji and Jamshidi [3] obtained the approximate solution to system (1.1) by using the energy balance method and the variational iteration method. Belédez [4,5] introduced the energy balance method and the homotopy perturbation method for treating oscillator (1.1). Wu [6] obtained the approximate analytical solutions for oscillator (1.1) on the basis of combining Newton’s method with the harmonic balance method.

The variational iteration method, which was proposed originally by He [7–11], has been proved by many authors to be a powerful mathematical tool for treating various kinds of linear and nonlinear problems [12–22]. The reliability of the method and the reduction in the size of the computational workload led to this method having wider application.

However, the standard VIM gives a good approximation to the solution for the nonlinear oscillator (1.1) only in a small region. In this paper, we shall present a piecewise variational iteration method (PVIM) for solving nonlinear oscillator (1.1), which can overcome the drawback of the standard VIM and gives highly accurate approximate solutions in a large region.

The rest of the paper is organized as follows. In Section 2, the VIM is introduced. The PVIM is presented for nonlinear oscillator (1.1) in Section 3. The numerical results are given in Section 4. Section 5 ends this paper with a brief conclusion.

E-mail address: gengfazhan@sina.com.

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.05.004
2. **Analysis of the variational iteration method**

Consider the differential equation

\[ Lu + Nu = g(x), \]  

(2.1)

where \( L \) and \( N \) are linear and nonlinear operators, respectively, and \( g(x) \) is the source inhomogeneous term. In [7–11], the VIM was introduced by He where a correct functional for (2.1) can be written as

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{ Lu_n(t) + Nu_n(t) - g(t) \} dt, \]  

(2.2)

where \( \lambda \) is a general Lagrangian multiplier [7], which can be identified optimally via variational theory, and \( \tilde{u}_n \) is a restricted variation which means that \( \delta \tilde{u}_n = 0 \). By this method, it is firstly required to determine the Lagrangian multiplier \( \lambda \) that will be identified optimally. The successive approximations \( u_{n+1}(x), \ n \geq 0, \) of the solution \( u(x) \) will be readily obtained upon using the Lagrangian multiplier determined and any selective function \( u_0(x) \). Consequently, the solution is given by

\[ u(x) = \lim_{n \to \infty} u_n(x). \]

In fact, the solution of problem (2.1) is considered as the fixed point of the following functional under a suitable choice of the initial term \( u_0(x) \):

\[ u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{ Lu_n(t) + Nu_n(t) - g(t) \} dt. \]  

(2.3)

As a well known powerful tool, we have:

**Theorem 2.1 (Banach’s Fixed Point Theorem).** Assume that \( X \) is a Banach space and

\[ A : X \to X \]

is a nonlinear mapping, and suppose that

\[ \| A[u] - A[v] \| \leq \alpha \| u - v \|, \quad u, v \in X \]

for some constants \( \alpha < 1 \). Then \( A \) has a unique fixed point. Furthermore, the sequence

\[ u_{n+1} = A[u_n], \]

with an arbitrary choice of \( u_0 \in X \), converges to the fixed point of \( A \).

According to Theorem 2.1, for the nonlinear mapping

\[ A[u(x)] = u(x) + \int_0^x \lambda \{ Lu(t) + Nu(t) - g(t) \} dt, \]

a sufficient condition for convergence of the variational iteration method is strict contraction of \( A \). Furthermore, the sequence (2.3) converges to the fixed point of \( A \) which is also the solution of problem (2.1).

3. **The piecewise variational iteration method for treating nonlinear oscillator (1.1)**

In this section, the PVIM is introduced for obtaining an approximate solution to Eq. (1.1) in a piecewise fashion. For Eq. (1.1), according to the VIM, we can obtain the following iteration formula:

\[ u_{n+1}(t) = u_0(t) + \int_0^t (s-t)f_1(s)ds, \quad 0 \leq t \leq T. \]  

(3.1)

Unfortunately, iterative formula (3.1) gives a good approximation only in a small region. To overcome the drawback, we modify iterative formula (3.1) in the following way.

Divide \([0, T]\) into \( N \) equal subintervals \( \Delta t = t_{j+1} - t_j, \ j = 0, 1, \ldots, N - 1, \) with \( t_0 = 0 \) and \( t_N = T \).

On \([t_0, t_1]\), let

\[
\begin{cases}
    u_{1,n+1}(t) = u_{1,0}(t) + \int_{t_0}^{t} (s-t)f_1(s)ds, & 0 \leq t \leq t_1, n \geq 1, \\
    u_{1,0}(t) = u(t_0) + u'(t_0)(t - t_0) = A,
\end{cases}
\]

(3.2)

where

\[ f_1(s) = u_{1,n}(s) + \frac{\lambda u_{1,n}(s)}{\sqrt{1 + u_{1,n}^2(s)}} \]

Then we can obtain the \( n \)th-term approximation \( u_{1,n}(t) \) on \([t_0, t_1]\).
Fig. 1. Comparison of the approximate solution (dashed line) obtained by the PVIM with the numerical solution (solid line). (Left: \( A = 1, \lambda = 0.5 \); right: \( A = 5, \lambda = 0.5 \)).

Fig. 2. Absolute errors between the approximate solution obtained by the PVIM and the numerical solution. (Left: \( A = 1, \lambda = 0.5 \); right: \( A = 5, \lambda = 0.5 \)).

On \([t_1, t_2]\), let
\[
\begin{align*}
    u_{2,n+1}(t) &= u_{2,0}(t) + \int_0^t (s-t)f_2(s)\,ds, \quad t_1 \leq t \leq t_2, \quad n \geq 1, \\
    u_{2,0}(t) &= u_{1,n_1}(t_1) + u'_{1,n_1}(t_1)(t - t_1),
\end{align*}
\]
where \( f_2(s) = u_{2,n}(s) + \frac{\lambda u_{2,n}(s)}{\sqrt{1 + u_{2,n}^2(s)}} \). From (3.3), one can obtain the \( n \)-th-term approximation \( u_{2,n_2}(t) \) on \([t_1, t_2]\).

In a similar way, on \([t_{i-1}, t_i]\), \( i = 3, 4, \ldots, N \), let
\[
\begin{align*}
    u_{i,n+1}(t) &= u_{i,0}(t) + \int_0^t (s-t)f_i(s)\,ds, \quad t_{i-1} \leq t \leq t_i, \quad n \geq 1, \\
    u_{i,0}(t) &= u_{i-1,n_i-1}(t_{i-1}) + u'_{i-1,n_i-1}(t_{i-1})(t - t_i),
\end{align*}
\]
where \( f_i(s) = u_{i,n}(s) + \frac{\lambda u_{i,n}(s)}{\sqrt{1 + u_{i,n}^2(s)}} \) (3.4) yields the \( n \)-th-term approximation \( u_{i,n_i}(t) \) on \([t_{i-1}, t_i]\).

However, it may be difficult to perform the integrals that appear in (3.2)–(3.4) analytically. Here we employ the Maclaurin’s series expansions of \( f_i(s), i = 1, 2, \ldots, N \),
\[
f_i(s) = f_i(t_{i-1}) + \sum_{k=1}^{\max} \frac{f_{i(k)}(t_{i-1})}{k!} (s - t_{i-1})^k,
\]
and perform the resulting integrations analytically.

After obtaining solutions for all subintervals, these solutions are combined to obtain an approximate solution to Eq. (1.1) over the entire interval \([0, T]\).

It is easy to see that the present method provides a globally smooth approximate solution due to the continuity of the approximate solution obtained and its first-order derivative at the common end point of two adjacent intervals.

4. Results and analysis

In this section, we apply the PVIM presented in Section 3 to nonlinear oscillator (1.1). Numerical results show that the PVIM is very effective.

We choose different values of \( \lambda \) and \( A \) as examples. According to (3.2)–(3.5), taking \( N = 200, n_i = 4, i = 1, 2, \ldots, N \), we can obtain the approximate solutions to Eq. (1.1). The numerical results are shown in Figs. 1–4. Fig. 1 compares the approximate solutions obtained with numerical solutions when \( A = 1, 5, \lambda = 0.5 \). Fig. 2 shows the absolute errors between the obtained approximate solutions and numerical solutions when \( A = 1, 5, \lambda = 0.5 \). The comparison and absolute errors between the approximate solutions obtained and the numerical solutions are shown respectively in Figs. 3 and 4 for \( A = 10, 100, \lambda = 0.75 \).

Remark. Figs. 1–4 show that the PVIM can produce a good approximation to the solution of nonlinear oscillator (1.1) in a very large region. However, the standard VIM is invalid for nonlinear oscillator (1.1) in a large interval.
5. Conclusion

In this paper, the PVIM is presented, for treating a nonlinear oscillator of a mass attached to a stretched elastic wire. The major advantage of the method is that it can produce good globally smooth approximate solutions. Results of numerical examples demonstrate that the present method is quite effective.

Acknowledgments

The author would like to express thanks to the unknown referees for their careful reading and helpful comments. The work was supported by the NSFC (Tianyuan Fund for Mathematics, Grant No. 11026200).

References