ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 248 (2007) 344–386

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Range descriptions for the spherical mean Radon transform ☆

Mark Agranovsky [a], Peter Kuchment [b,*], Eric Todd Quinto [c]

[a] *Department of Mathematics, Bar-Ilan University, Ramat-Gan, 52900, Israel*

[b] *Department of Mathematics, Texas A&M University, College Station, TX 77845, USA*

[c] *Department of Mathematics, Tufts University, Medford, MA 02155, USA*

Received 7 September 2006; accepted 22 March 2007

Available online 7 May 2007

Communicated by H. Brezis

Abstract

The transform considered in the paper averages a function supported in a ball in $\mathbb{R}^n$ over all spheres centered at the boundary of the ball. This Radon type transform arises in several contemporary applications, e.g. in thermoacoustic tomography and sonar and radar imaging. Range descriptions for such transforms are important in all these areas, for instance when dealing with incomplete data, error correction, and other issues. Four different types of complete range descriptions are provided, some of which also suggest inversion procedures. Necessity of three of these (appropriately formulated) conditions holds also in general domains, while the complete discussion of the case of general domains would require another publication.

Keywords: Spherical mean operator; Radon transform; Darboux equation; Tomography; Range; Inversion

Contents

☆ The work of the second author was supported in part by the NSF Grants DMS-9971674, DMS-0002195, and DMS-0604778. The work of the third author was supported in part by the NSF Grants DMS-0200788 and DMS-0456868.

* Corresponding author. Fax: +1 979 862 4190.

E-mail addresses: agranovs@macs.biu.ac.il (M. Agranovsky), kuchment@math.tamu.edu (P. Kuchment), todd.quinto@tufts.edu (E.T. Quinto).

URLs: http://www.math.biu.ac.il/people/ (M. Agranovsky), http://www.math.tamu.edu/~kuchment (P. Kuchment), http://www.tufts.edu/~equinto/index.html (E.T. Quinto).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2007.03.022

1. Introduction

The spherical mean Radon transform, which integrates a function over all spheres centered at points of a given set, has been studied for quite a while in relation to PDE problems (e.g., [17, 24,47]). However, there has been a recent surge in its studies, due to demands of manifold applications. These include, among others, the recently developed thermoacoustic and photoacoustic tomography (e.g., [14,31,51,54,59,82,87,93,100–104]), as well as radar and sonar imaging, approximation theory, mathematical physics, and other areas (e.g., [3–11,24,27,47,53,62,63,66,76, 79,80]). For instance, in thermoacoustic (and photoacoustic) tomography, the spherical mean data of an unknown function (the radiofrequency energy absorption coefficient) is measured by transducers, and the imaging problem is to invert that transform (e.g., [51,54,59,100–103]). These applications also brought about mathematical problems that had not been studied before. Many issues of uniqueness and stability of reconstruction, inversion formulas, incomplete data problems, etc., are still unresolved, in spite of a substantial body of research available (e.g., [3–11,14,16,26, 27,29,31,32,35,36,53,54,57–59,62,63,66,72,75,76,79,80,83–85,87,89–91,100–104]). In this text we address the problem which has been recently considered for the first time ([15,30,87], see also related discussions in [73,74]), namely the range conditions for the spherical mean transform. In fact, as we will mention below, in an implicit form, a part of range conditions was already present in [62,63] and later in [6].

For someone coming from PDEs and mathematical physics, the range description question might seem somewhat unusual. However, it is well known in the areas of integral geometry and tomography that range descriptions are of crucial theoretical and practical importance [27, 33–35,43,55,56,71,72,81,82,93,94]. The ranges of Radon type transforms usually have infinite co-dimension (e.g., in spaces of smooth functions, or in appropriate Sobolev scales), and thus infinitely many range conditions appear. One might wonder, what is the importance of knowing the range conditions. The answer is that, besides their analytic usefulness for understanding the transform, they have been used for a variety of purposes in tomography (as well as in radiation therapy planning [22,23,52]): completing incomplete data, correcting measurement errors and hardware imperfections, recovering unknown parameters of the medium, etc. [45,55,56,67,69–71,77,78,88,95,96]. Thus, as soon as the spherical mean transform started attracting a lot of attention, researchers started looking for range descriptions. Some range conditions (albeit they were not called this way) were already present in [62,63] and in [6] (see also [20]), where the sequence of polynomials was considered arising as moments of the spherical Radon data. In an explicit form, these conditions were formulated recently in [87]. They, however, as it was discovered in [15], do not describe the range completely. A complete set of conditions was found in the two-dimensional case in the recent paper [15] and for odd dimensions (albeit, for somewhat different transforms) in [30]. In all these papers, the centers of spheres of integration (i.e., the location of tomographic transducers) were assumed to belong to a sphere.

In this paper, we obtain range descriptions in arbitrary dimension for the case of centers on a sphere. Moreover, we obtain several different range descriptions that shed new light on the meaning of the range conditions (in particular, onto the appearance of two seemingly different subsets of conditions).

In Section 2, we introduce main notations and preliminary facts that will be needed further on. Section 3 contains the formulation of the main results. Theorem 10 provides three different types of range descriptions. Theorem 11 establishes that in odd dimensions moment conditions of Theorem 10 can be dropped (a situation analogous to the one in [30]). On the other hand, it is shown in Theorem 12 that a strengthened version of moment conditions alone describes the range. It is shown in Theorem 13 that the results also hold in Sobolev scale, rather than in C^∞ category. The next four sections are devoted to the proofs of these theorems. It is noticed in Section 8 that necessity of most of the range conditions is in fact proven for general domains, not just for a ball. This is described in Theorem 22. Section 9 contains proofs or alternative proofs of some technical lemmas. The alternative proofs are provided, since the authors believe that they might shed extra light onto the problem. The final two sections contain additional remarks and discussions, and acknowledgments.

2. Main notions and preliminary information

In this section we introduce main notions and notations that will be used throughout the paper. We also remind the reader of some known facts that we will need to use.

We will be dealing with domains in $\mathbb{R}^n$. The closure of a domain Ω is denoted by $\overline{\Omega}$ and its boundary is $\partial\Omega$. We denote the n-dimensional unit ball $B = \{x \in \mathbb{R}^n \mid |x| \leqslant 1\}$ and the unit sphere is $S = \partial B = \{x \in \mathbb{R}^n \mid |x| = 1\}$. The area of S is ω and the area measure on S will be dS (this notation will also be used for the surface measure on the boundaries of other domains). The notation $C_0^\infty(B)$ stands for the class of smooth functions with the compact support in the closed unit ball. For partial derivatives, the notations $\frac{\partial f}{\partial t}$, $\partial_t f$, and f_t will be used.

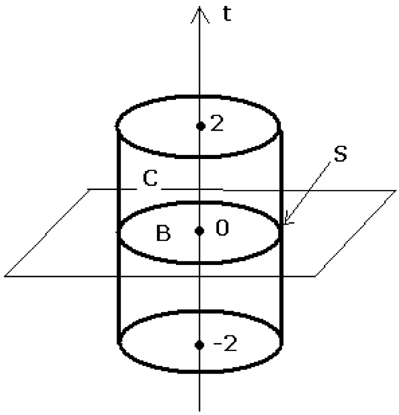

Fig. 1. The geometry of domains.

2.1. Spherical means

The main object of study in this paper is the *spherical mean transform* R (with centers on $S = \partial B$) that takes any function $f \in C_0^\infty(B)$ to

$$Rf(p,t) := \frac{1}{\omega} \int\limits_{y \in S} f(p+ty)\, dS(y), \quad p \in S. \tag{1}$$

In terms of thermoacoustic tomography [51,54,100–104] function $f(x)$ represents the unknown microwave energy absorption function (high values of which indicate tumors), and $Rf(p,t)$ has a simple relation to the pressure measured at the time t by the transducer located at the point p.

One might wonder why we require the support of f to belong to B. It will be explained in Section 10 that there is not much hope for explicit range descriptions, if one allows the support of the function to spill outside the surface S of the centers.

We will also consider the cylinder $C = B \times [0,2]$ and its lateral boundary $S \times [0,2]$ (see Fig. 1).

Notice that the sphere S enters here in two different ways: as the set of centers of the spheres of integration ($p \in S$), and as a parametrization of the spheres of integration ($\int_{y \in S} \dots\ dS(y)$). The reader should not confuse the two, since sometimes in the text we will change the set of centers to a more general surface S, while keeping the integration surfaces spherical.

The results can be easily re-scaled to the case when the set of centers is a sphere of an arbitrary radius ρ. We avoid doing this here, in order to simplify the expressions.

2.2. Darboux equation

Allowing in (1) the centers x of the sphere of integration be arbitrary, one arrives to a function

$$G(x,t) = \frac{1}{\omega} \int\limits_{y \in S} f(x+ty)\, dS(y), \quad x \in \mathbb{R}^n. \tag{2}$$

It is well known [17,24,47] that the function $G(x,t)$ defined by (2) satisfies the Darboux (Euler–Poisson–Darboux) equation

$$\frac{\partial^2 G}{\partial t^2} + \frac{n-1}{t}\frac{\partial G}{\partial t} = \Delta_x G, \tag{3}$$

as well as the initial conditions

$$G(x,0) = f(x), \qquad \frac{\partial G}{\partial t}(x,0) = 0, \quad x \in \mathbb{R}^n. \tag{4}$$

Moreover, any such solution of (3), (4) in $\mathbb{R}^n \times \mathbb{R}^+$ is representable as the spherical mean (2) of $f(x)$ (Asgeirsson's theorem, see [17,24,47]).

An important remark about the initial conditions (4) is that they mean that the solution G can be extended to all values of time as an even solution on $\mathbb{R}^n_x \times \mathbb{R}_t$ [24, Chapter VI.13].

One notices that the restriction g of G to $S \times \mathbb{R}^+$ coincides with Rf:

$$g = G|_{S\times\mathbb{R}^+} = Rf. \tag{5}$$

Another observation concerning the mean $G(x,t)$ of a function $f(x)$ supported in B is that it vanishes for $x \in B$, $t \geqslant 2$. Indeed, the value $G(x,t)$ is the average of f over the sphere centered at x and of radius t, while such a sphere for $x \in B$, $t \geqslant 2$ does not intersect the support of f, which is contained in B. So, G satisfies the terminal conditions

$$G(x,2) = 0, \qquad \frac{\partial G}{\partial t}(x,2) = 0, \quad x \in B. \tag{6}$$

We also need to mention some other known properties of the Darboux equation, which we will need to utilize further on in the text.

The Darboux equation has a useful connection with the wave equation (e.g., [24, Chapter VI.13]). This relation comes from existence of transformations that intertwine the second derivative operator $\frac{d^2}{dt^2}$ with the Bessel operator $\mathcal{B}_p = \frac{d^2}{dt^2} + \frac{2p+1}{t}\frac{d}{dt}$. A general approach to constructions of such transformation operators (not just for the Bessel case) can be found, for instance in [61] (see also [13,50]). Among those, the most commonly used ones are the Poisson transform (also called Delsarte or Riemann–Liouville transform [24,25,60,64,98]) and Sonine transform (all such transforms that we use are partial cases of transforms of Abel type). The formula we will need for the Poisson transform is

$$(\mathcal{P}_p U)(t) = c_p t^{-2p} \int_0^t U(y)\bigl(t^2 - y^2\bigr)^{p-1/2}\, dy. \tag{7}$$

Here c_p is a non-zero constant, whose specific value is of no relevance to us. We will use this transform for even functions U, in which case it can be rewritten as

$$(\mathcal{P}_p U)(t) = c_p \int_{-1}^{1} U(\mu t)\bigl(1 - \mu^2\bigr)^{p-1/2}\, d\mu. \tag{8}$$

The properties of this transform are well known (e.g., [50,60,61,98]). In particular, it is known to preserve evenness of functions. We will also need the inversion formula in the case when $2p$ is an odd integer:

$$\big(\mathcal{P}_p^{-1}G\big)(t) = const\, t\left(\frac{d}{d(t^2)}\right)^{p+1/2}\big(t^{2p}G(t)\big). \tag{9}$$

We will not use the Sonine transform here. We, however, will be interested in the intertwining operator sometimes called Weyl transform [98] (it is dual to a Poisson transform [50,98]):

$$U(t) = (\mathcal{W}_p G)(t) := \frac{2\Gamma(p+1)}{\sqrt{\pi}\,\Gamma(p+1/2)}\int_t^{\infty} G(s)\big(s^2-t^2\big)^{p-1/2} s\,ds, \tag{10}$$

which we will use for the specific value $p=(n-2)/2$. The inverse transform $\mathcal{W}^{-1}$ for this case is given by

$$G(t) = \big(\mathcal{W}^{-1}U\big)(t) = \begin{cases} const \int_t^\infty (s^2-t^2)^{-1/2}(\frac{d}{d(s^2)})^{n/2}U(s)s\,ds, & \text{for } n \text{ even},\\ const\ \frac{d}{d(t^2)}^{\frac{n-1}{2}}U(t), & \text{for } n \text{ odd}. \end{cases} \tag{11}$$

One has (e.g., [50,60,98]) the following intertwining identities:

$$\mathcal{P}_p\frac{d^2}{dt^2}U(t) = \left(\frac{d^2}{dt^2}+\frac{2p+1}{t}\frac{d}{dt}\right)\mathcal{P}_pU(t),$$
$$\frac{d^2}{dt^2}\mathcal{W}_pG(t) = \mathcal{W}_p\left(\frac{d^2}{dt^2}+\frac{2p+1}{t}\frac{d}{dt}\right)G(t). \tag{12}$$

These identities show that $G(x,t)$ (for $t \geqslant 0$) is a solution of Darboux equation if and only if its Weyl transform (with respect to t) $U(x,t) = \mathcal{W}_pG(x,t)$ solves the wave equation.

It will be important for us that the transform $\mathcal{W}$ (unlike the Poisson transform) involves integration from t to ∞. Hence, when applied to a function on $\mathbb{R}^+$ that vanishes for $t > a$, it preserves this property. It is clear from the inversion formulas that $\mathcal{W}^{-1}$ behaves the same way.

An important relation between Fourier, Fourier–Bessel, and Weyl transforms will be presented in the next subsection.

2.3. *Bessel functions and eigenfunctions of the Bessel operator and Laplacian*

It will be convenient for us to use a version of Bessel functions that is sometimes called *normalized*, and sometimes *spherical* Bessel functions (e.g., [60,68,98]):

$$j_p(\lambda) = \frac{2^p\Gamma(p+1)J_p(\lambda)}{\lambda^p}. \tag{13}$$

For $p=(n-2)/2$, according to Poisson's representations of Bessel functions [47,68], this is just the spherical average of a plane wave in $\mathbb{R}^n$.

We will also use standard expansions

$$J_p(\lambda) = \frac{\lambda^p}{2^p} \sum_{k=0}^{\infty} (-1)^k \frac{\lambda^{2k}}{2^{2k} k! \Gamma(p+k+1)},$$

$$j_p(\lambda) = \sum_{k=0}^{\infty} C_k \lambda^{2k}, \tag{14}$$

where C_k are non-zero constants.

The function $z(t) = j_p(\lambda t)$ satisfies the Bessel equation

$$\mathcal{B}_p z := \frac{d^2 z}{dt^2} + \frac{2p+1}{t} \frac{dz}{dt} = -\lambda^2 z \tag{15}$$

and initial conditions

$$z(0) = 1, \qquad z'_\nu(0) = 0.$$

The standard Fourier–Hankel (also called Hankel, or Fourier–Bessel) transform $g(t) \mapsto \mathcal{F}_p(g)(s\lambda)$ and its inverse can be nicely written in terms of j_p:

$$\mathcal{F}_p(g)(s\lambda) = \int_0^\infty g(t) j_p(\lambda t) t^{2p+1}\, dt$$

$$g(t) = \frac{1}{2^{2p} \Gamma^2(p+1)} \int_0^\infty \mathcal{F}_p(g)(t\lambda) j_p(\lambda t) \lambda^{2p+1}\, d\lambda. \tag{16}$$

We will use notation $\mathcal{F}$ for the standard one-dimensional Fourier transform.

The following relation between the Fourier, Fourier–Hankel, and Weyl transform (e.g., [98, p. 124]) helps to understand some parts of the further calculations:

$$\mathcal{F}_p = c_p \mathcal{F} \mathcal{W}_p, \tag{17}$$

where c_p is a non-zero constant, explicit value of which is of no relevance to this study. This relation shows that the Weyl transform is the ratio of the Fourier and Fourier–Bessel transforms.

We will need to use the following known Paley–Wiener theorem [12,38,50,98] for Fourier–Bessel transform. Albeit the statement of the Lemma 1 is formulated in the works cited only for the case $H = \mathbb{C}$, the proofs allow one to consider without any change H-valued functions as well.

Lemma 1. *Let $p = (n-2)/2$ and H be a Hilbert space. An H-valued function $\Phi(\lambda)$ on $\mathbb{R}$ can be represented as the transform* (16) *of an even function $g \in C_0^\infty(\mathbb{R}, H)$ supported on* $[-a, a]$, *if and only if the following conditions are satisfied*:

(1) $\Phi(\lambda)$ *is even.*

(2) $\Phi(\lambda)$ *extends to an entire function in* $\mathbb{C}$ *with Paley–Wiener estimates*

$$\|\Phi(\lambda)\| \leqslant C_N\big(1+|\lambda|\big)^{-N} e^{a|\operatorname{Im}\lambda|} \tag{18}$$

for any natural N.

Although this result is well known, for reader's convenience, we provide its proof in Section 9.

We will need some special solutions of Darboux equation in the cylinder $B \times \mathbb{R}$. Let $-\lambda^2$ be in the spectrum of the Dirichlet Laplacian in the ball B, and $\psi_\lambda(x)$ be the corresponding eigenfunction, i.e.

$$\begin{cases} \Delta\psi_\lambda(x) = -\lambda^2\psi_\lambda(x) & \text{in } B, \\ \psi_\lambda(x) = 0 & \text{on } S. \end{cases} \tag{19}$$

Equations (15) and (19) imply that the function

$$u_\lambda(x,t) = \psi_\lambda(x) j_{n/2-1}(\lambda t) \tag{20}$$

satisfies Darboux equation (3).

Finally, we need descriptions of (generalized) eigenfunctions

$$-\Delta\psi = \lambda^2\psi$$

of the Laplace operator in $\mathbb{R}^n$ and in B in terms of their spherical harmonics expansions.

Let $Y_l^m(\theta)$, $\theta \in S^{n-1}$, $m = 0, 1, \ldots, 1 \leqslant l \leqslant d(m)$, be an orthonormal basis of spherical harmonics, where m is the degree of the harmonic and

$$d(m) = (2m+n-2)\frac{\Gamma(m+n-2)}{\Gamma(n-1)\Gamma(m+1)}.$$

It is known (and can be easily shown by separation of variables) that any function

$$\begin{aligned} \phi_{m,l}(x) &= (\lambda r)^{1-n/2} J_{n/2-1+m}(\lambda r) Y_l^m(\theta) \\ &= (\lambda r)^m j_{n/2-1+m}(\lambda r) Y_l^m(\theta), \quad \text{where } x = r\theta \end{aligned} \tag{21}$$

is a generalized eigenfunction of the Laplace operator in $\mathbb{R}^n$ with the eigenvalue $-\lambda^2$. (As it is customary, we use the term "generalized eigenfunction" for any solution of the equation $\Delta u = -\lambda^2 u$ in $\mathbb{R}^n$.) In fact, one can show that any generalized eigenfunction of the Laplace operator in $\mathbb{R}^n$ has the following expansion into spherical harmonics:

$$u(r\theta) = \sum_{l,m} c_{l,m} (\lambda r)^m j_{n/2-1+m}(\lambda r) Y_l^m(\theta), \tag{22}$$

where $r = |x|$, $\theta = x/|x|$. One can describe precisely the conditions on the coefficients $c_{l,m}$, necessary and sufficient for (22) to provide all generalized eigenfunction, as well as generalized eigenfunctions with some prescribed growth condition at infinity [1,2].

If one chooses only the values of $\lambda \neq 0$ that are zeros of $J_{m+n/2-1}(\lambda)$, one arrives to the eigenfunctions of the Dirichlet Laplacian in the unit ball B. In particular, functions (21) for all m, l and $\lambda \neq 0$ such that $J_{m+n/2-1}(\lambda) = 0$, form a complete set of eigenfunctions.

With all these preparations in place, we can now set out to formulate and prove the results of this article.

2.4. Some preliminary results

We start considering the spherical mean transform R introduced in (1) with centers on the unit sphere S. The following moment conditions for this case were present in [62,63], as well as in [6] (see also [20]), and explicitly formulated as range conditions in [87].

Lemma 2. *Let $g(x,t) = Rf(x,t)$ for $f \in C_0^\infty(B)$. Then, for any non-negative integer k, the function M_k on S defined as*

$$M_k(x) := \int_0^2 t^{2k+n-1} g(x,t)\,dt, \quad x \in S, \tag{23}$$

has an extension to $\mathbb{R}^n$ as a polynomial $q_k(x)$ of degree at most $2k$.

Proof. One readily observes that, for $|x| = 1$,

$$M_k(x) = \int_{\mathbb{R}^n} |x-p|^{2k} f(p)\,dp. \tag{24}$$

Applying (24) to arbitrary $x \in \mathbb{R}^n$ (not necessarily on the unit sphere), one clearly gets a polynomial $q_k(x)$ of x of degree at most (not necessarily equal to) $2k$. $\square$

Remark 3. In fact, noticing that

$$M_k(x) := \int_{\mathbb{R}^n} |x-p|^{2k} f(p)\,dp = \int_{\mathbb{R}^n} \left(1 - 2p \cdot x + |p|^2\right)^k f(p)\,dp,$$

we conclude that $M_k(x)$ has an extension to $\mathbb{R}^n$ as a polynomial of degree at most k.

The possibility of reducing to degree k comes from the fact that we use centers of the spheres of integration that belong to the (unit) sphere. It is clear that when the centers run over a non-spherical surface, this reduction is not possible anymore, and one cannot guarantee degree less than $2k$. However, in this case an extra condition can be found that straightens up the situation. We thus provide here an alternative reformulation of the moment conditions, which will be handy in more general considerations.

Lemma 4. *Let $D \subset \mathbb{R}^n$ be a smooth bounded domain and R denote the spherical mean transform with centers on $S = \partial D$.*

Let $g(x,t) = Rf(x,t)$ for $f \in C_0^\infty(D)$. Then, for any non-negative integer k, the function M_k on S defined as in (23) *has an extension $Q_k(x)$ to $\mathbb{R}^n$ as a polynomial of degree at most $2k$, satisfying the following additional condition*:

$$\Delta Q_k = c_k Q_{k-1} \quad \textit{for any } k = 1, \ldots, \tag{25}$$

where $c_k = 2k(2k+n-2)$.

Proof. Let us notice that $M_k = |x|^{2k} * f(x)|_S$. We can now define the polynomials $Q_k(x) = |x|^{2k} * f(x)$. Then the relation follows from the easily verifiable identity $\Delta|x|^{2k} = c_k|x|^{2(k-1)}$. □

In fact, in the case when D is a ball B, the condition (25) is not needed.

Lemma 5. *In the case when $D = B$, the condition* (25) *of Lemma* 4 *on a function $g(x,t)$ on $S \times [0,2]$ can be dropped, and thus conditions of Lemmas* 2 *and* 4 *are equivalent.*

To prove the lemma, we need the following well known fact (e.g., [48]), which we prove here for the sake of completeness.

Proposition 6. *The solution of the boundary value problem*

$$\Delta u = v, \quad |x| < 1,$$
$$u = 0, \quad |x| = 1,$$

where v is a polynomial, is a polynomial of degree $\deg u = \deg v + 2$.

Proof. Let us prove first that there exists a polynomial solution $\tilde{u}$ of Poisson equation

$$\Delta \tilde{u} = v$$

in the unit ball, such that $\deg \tilde{u} = \deg v + 2$. Clearly, it suffices to do this for each homogeneous term of v, so we can assume the polynomial v to be homogeneous.

Let us represent v in the form:

$$v(x) = \sum_{\nu=0}^{[\deg v/2]} |x|^{2\nu} h_\nu(x),$$

where each h_ν is either zero, or a homogeneous harmonic polynomial of degree $\deg h_\nu = \deg v - 2\nu$, and brackets $[\ldots]$ denote the integer part. This representation is well known to be always possible (e.g., [44,97]). A solution $\tilde{u}$ can be found in the similar form (where we denote for brevity $k = [\frac{\deg v+2}{2}]$):

$$\tilde{u}(x) = \sum_{\nu=0}^{k} |x|^{2\nu} \tilde{h}_\nu(x), \quad \deg \tilde{h}_\nu = \deg v + 2 - 2\nu.$$

Here again, each $\tilde{h}_\nu$ is either zero, or a homogeneous harmonic polynomial of degree $\deg \tilde{h}_\nu = \deg v + 2 - 2\nu$. Then direct calculation shows

$$\Delta\tilde{u}(x) = \sum_{\nu=0}^{k} (c_\nu + 2\nu \deg \tilde{h}_\nu)|x|^{2\nu-2}\tilde{h}_\nu(x),$$

with the coefficients c_ν defined in Lemma 3. Thus, the needed polynomial solution $\tilde{u}$ that we are looking for can be obtained by choosing

$$\tilde{h}_\nu = \left[c_\nu + 2\nu(\deg v + 2 - 2\nu)\right]^{-1} h_{\nu-1}.$$

To finish the proof of proposition, introduce $U = \tilde{u} - u$. Then one obtains

$$\Delta U = 0, \quad |x| < 1,$$
$$U = \tilde{u}, \quad |x| = 1,$$

for the newly defined function U. The boundary value $\tilde{u}$ is the polynomial of degree $\deg \tilde{u} = \deg v + 2$. Its harmonic extension U from the unit sphere is obtained from the above decomposition of $\tilde{u}$ by replacing $|x|$ by 1:

$$U(x) = \sum_{\nu=0}^{k} \tilde{h}_\nu(x).$$

Since $\deg \tilde{h}_\nu \leqslant \deg v + 2$, we have $\deg U \leqslant \deg v + 2$.

Thus, the solution $u = \tilde{u} - U$ is a polynomial of degree at most $\deg v + 2$, which proves the proposition. □

Remark 7. Polynomial solvability of the Poisson problem with polynomial data is rather unique and essentially holds only for balls (e.g., [21,42,49]). Hence, in a general domain, the conditions of Lemma 4 are stronger than the ones of Lemma 2.

Proof of Lemma 5. We want to prove that among all polynomial extensions q_k, $\deg q_k \leqslant 2k$, of the functions M_k defined in (17), there is a sequence of extensions Q_k satisfying the additional recurrence relation (25). Let q_k be some extensions. Any other sequence Q_k of extensions can be represented as

$$Q_k = q_k + u_k,$$

where $u_k = 0$ on the unit sphere. The additional requirement (25) yields the relation

$$\Delta u_k = c_k Q_{k-1} - \Delta q_k.$$

The existence of polynomial solutions $u_k, \deg u_k \leqslant 2k$ follows now by inductive application of Proposition 6. Then the modified sequence of polynomials $Q_k = q_k + u_k$ satisfies all the requirements of the lemma. □

Remark 8. The sequence Q_k of polynomial extensions of the functions M_k satisfying the chain relation (19) is unique. Indeed, if Q'_k is another sequence of such extensions, then the polynomials $R_k = Q_k - Q'_k$ vanish on the unit sphere and still possess (25). If not all R_k are identically zero, then, due to (25), the first nonzero polynomial R_{k_0} is harmonic and vanishes on the unit sphere. Thus, it must be zero, due to the maximum principle. This contradiction shows that $R_k = 0$ and hence $Q_k = Q'_k$ for all k.

In the future, we will need the moment condition on the unit sphere in a different form [15].

Lemma 9. *The function $g(x,t)$, $x \in S$, $t \in [0,2]$, satisfies the moment condition of Lemma 2 if and only if, for any spherical harmonic $Y^m(\theta)$ of degree m, the function*

$$\widehat{g}_m(\lambda) = \int_0^2 \int_{\theta \in S} g(\theta,t)\, j_{n/2-1+m}(\lambda t) Y^m(\theta)\, dS(\theta)\, t^{n-1}\, dt$$

has at $\lambda = 0$ a zero of order at least m.

Proof. The moment conditions require that

$$\int_0^\infty t^{2k+n-1} g(\theta,t)\, dt$$

is extendable to a polynomial of degree at most k. Let us expand $g(\theta,t)$ into an orthonormal basis Y_l^m of spherical harmonics on S (where m is the degree of the harmonic):

$$g(\theta,t) = \sum_{l,m} g_{l,m}(t) Y_l^m(\theta).$$

Due to smoothness and compactness of support of g, it is legitimate to integrate term-wise in computing the momenta to obtain

$$\int_0^\infty t^{2k+n-1} g(\theta,t)\, dt = \sum_{l,m} Y_l^m(\theta) \int_0^\infty t^{2k+n-1} g_{l,m}(t)\, dt. \tag{26}$$

A spherical harmonic of degree m can be extended to a polynomial of degree d if and only if $d \geqslant m$ and $d-m$ is even. Thus, the moment conditions require that the coefficients $\int_0^\infty t^{2k+n-1} g_{l,m}(t)\, dt$ in (26) with $m > 2k$ must vanish.

Let us turn to the function $\widehat{g}_m(\lambda)$. Using the expansion (14) for Bessel functions, one arrives to the formula

$$\widehat{g}_m(\lambda) = \sum_{k=0}^\infty C_k \lambda^{2k} \int_0^\infty t^{2k+n-1} \int_{\theta \in S} Y^m(\theta) g(\theta,t)\, dS(\theta)\, dt. \tag{27}$$

Now one sees that the moment conditions are equivalent to the requirement that all terms in this series with $2k < m$ vanish. Therefore, the series begins with the power at least λ^m. □

3. Statements of the main results

As it follows from [15,30], the moment range conditions of the preceding lemmas are insufficient. Necessary and sufficient conditions of different kinds were provided in [15] in dimension two and in [30] in odd dimensions (albeit for somewhat different transforms). We formulate below our main result that resolves the problem of range description in any dimension, as well as provides several alternative ways to describe the range.

Theorem 10. *The following four statements are equivalent*:

(1) *The function $g \in C_0^\infty(S \times [0,2])$ is representable as Rf for some $f \in C_0^\infty(B)$.*
(2) (a) *The moment conditions of Lemma* 2 (*or Lemma* 4) *are satisfied.*
  (b) *The solution $G(x,t)$ of the interior problem* (3), (5), (6) *in C* (*which always exists for $t > 0$*) *satisfies the condition*

$$\lim_{t\to 0}\int_B \frac{\partial G}{\partial t}(x,t)\phi(x)\,dx = 0$$

  for any eigenfunction $\phi(x)$ of the Dirichlet Laplacian in B.
(3) (a) *The moment conditions of Lemma* 2 (*or Lemma* 4) *are satisfied.*
  (b) *Let $-\lambda^2$ be an eigenvalue of the Dirichlet Laplacian in B and u_λ be the corresponding eigenfunction solution* (20). *Then the following orthogonality condition is satisfied*:

$$\int_{S\times[0,2]} g(x,t)\partial_\nu u_\lambda(x,t)t^{n-1}\,dx\,dt = 0. \tag{28}$$

  Here ∂_ν is the exterior normal derivative at the boundary of C.
(4) (a) *The moment conditions of Lemma* 2 (*or Lemma* 4) *are satisfied.*
  (b) *Let $\widehat{g}(x,\lambda) = \int g(x,t)j_{n/2-1}(\lambda t)t^{n-1}\,dt$. Then, for any integer m, the mth order spherical harmonic term $\widehat{g}_m(x,\lambda)$ of $\widehat{g}(x,\lambda)$ vanishes at non-zero zeros of the Bessel function $J_{m+n/2-1}(\lambda)$.*

Here the "mth order spherical harmonic term $\widehat{g}_m(x,\lambda)$ of $\widehat{g}(x,\lambda)$" is the orthogonal projection of the function $\widehat{g}(x,\lambda)$ on the unit sphere onto the subspace of spherical harmonics of degree m.

In fact, if dimension n is odd, one does not need to require the moment conditions of Lemma 2 or Lemma 4, since in this case the other range conditions of Theorem 10 alone are sufficient. This was first noticed in a different setting in [30].

Theorem 11. *Let $n > 1$ be an odd integer. Then the following four statements are equivalent*:

(1) *The function $g \in C_0^\infty(S \times [0,2])$ is representable as Rf for some $f \in C_0^\infty(B)$.*

(2) *The solution $G(x,t)$ of the interior problem* (3), (5), (6) *in C* (*which always exists for $t>0$*) *satisfies the condition*

$$\lim_{t\to 0}\int_B \frac{\partial G}{\partial t}(x,t)\phi(x)\,dx = 0$$

for any eigenfunction $\phi(x)$ of the Dirichlet Laplacian in B.

(3) *Let $-\lambda^2$ be an eigenvalue of the Dirichlet Laplacian in B and u_λ be the corresponding eigenfunction solution* (20). *Then the following orthogonality condition is satisfied*:

$$\int_{S\times[0,2]} g(x,t)\partial_\nu u_\lambda(x,t)t^{n-1}\,dx\,dt = 0. \tag{29}$$

Here ∂_ν is the exterior normal derivative at the boundary of C.

(4) *Let $\widehat{g}(x,\lambda) = \int g(x,t)j_{n/2-1}(\lambda t)t^{n-1}\,dt$. Then, for any integer m, the mth order spherical harmonic term $\widehat{g}_m(x,\lambda)$ of $\widehat{g}(x,\lambda)$ vanishes at non-zero zeros of the Bessel function $J_{m+n/2-1}(\lambda)$.*

As we proved in Lemma 5, in the case of a ball the polynomial extendibility of the t-moments M_k of the data $g(x,t)$ (the moment condition) is equivalent to the stronger moment condition (Lemma 4) that requires existence of polynomial extensions Q_k linked by the additional relations (25).

In fact, if g is in the range $g = Rf$, $f \in C_0(B)$, then the polynomials Q_k obey not only algebraic relations (25), but also the following growth (in k) estimates in B:

$$\bigl|Q_k(x)\bigr| = \left|\int_B |x-y|^{2k} f(y)\,dy\right| \leqslant 2^{2k}\max_{y\in B}\bigl|f(y)\bigr|, \quad x\in B. \tag{30}$$

It turns out that the moment conditions of Lemma 4 with estimates of the above type are not only necessary, but also sufficient for g being in the range of the transform R. The following theorem can be regarded as an alternative form of Theorem 10.

Theorem 12. *Let $g \in C_0^\infty(S\times[0,2])$. The following condition is necessary and sufficient for the function g being representable as $g = Rf$ for some $f \in C_0^\infty(B)$*:

- *The moments*

$$M_k(x) = \int_0^\infty g(x,t)t^{2k+n-1}\,dt$$

extend from $x\in S$ to $x\in\mathbb{R}^n$ as polynomials $Q_k(x)$ satisfying the recurrent condition (25) *and the growth estimates*

$$\bigl|Q_k(x)\bigr| \leqslant M^k, \quad x\in B, \tag{31}$$

for some $M>0$.

The condition of infinite smoothness of functions under consideration is not truly necessary. One can prove similar range descriptions in appropriate Sobolev spaces, if the functions are supported strictly inside the ball B.

Theorem 13. *Let $s \geqslant 0$. The following statements are equivalent*:

(1) *The function $g \in H^{\mathrm{comp}}_{s+(n-1)/2}(S \times (0,2))$ is representable as Rf for some $f \in H^{\mathrm{comp}}_s(B)$.*
(2) *Any of the equivalent conditions* (2)–(4) *of Theorem* 10 *is satisfied.*

When n is odd, the moment conditions part (a) *in conditions* (2)–(4) *of Theorem* 10 *can be dropped.*

Here we used the notation $H^{\mathrm{comp}}_s(B)$ for the space of H_s-functions in the ball B with compact support in the open ball. Analogously, $H^{\mathrm{comp}}_s(S \times (0,2))$ consists of H_s-functions on $S \times (0,2)$ with support in $S \times (\varepsilon, 2-\varepsilon)$ for some positive ε.

4. Proof of Theorem 10

4.1. *Implication* (1) $\Rightarrow$ (2)

Assume that (1) is satisfied, i.e. $g(p,r) = Rf(p,r)$ for a smooth function f supported in B.

The implication (1) $\Rightarrow$ (2)(a) is the statement of Lemma 2.

The implication (1) $\Rightarrow$ (2)(b) is one of the statements of Asgeirsson's theorem [17,24,43,47], which has already been quoted before.

4.2. *Equivalence* (2) $\Leftrightarrow$ (3)

Since conditions (2)(a) and (3)(a) are the same, we only need to establish the equivalence (2)(b) $\Leftrightarrow$ (3)(b). This is done in the lemma below. Notice that this lemma applies to any bounded domain, not just to a ball.

Lemma 14. *Let D be a bounded domain in $\mathbb{R}^n$ with smooth boundary, Δ_D—the Laplacian in D with the Dirichlet boundary conditions, and $T > 0$ be such that all spheres of radius T centered in $\overline{D}$ do not intersect $\overline{D}$. Let also $g \in C_0^\infty(\partial D \times [0,T])$.*

(1) *For any eigenfunction $\phi(x) = \phi_k(x)$ of Δ_D with the eigenvalue $-\lambda^2 = -\lambda_k^2 \in \sigma(\Delta_D)$, the following two statements are equivalent*:

(a)
$$\int_0^\infty \int_{\partial D} g(x,t)\partial_\nu u_\lambda(x,t) t^{n-1}\, ds(x)\, dt = 0,$$
where
$$u_\lambda(x,t) = \phi(x) j_{n/2-1}(\lambda t).$$

(b) *The solution to the backward initial value boundary value problem*

$$\begin{cases} \frac{\partial^2 G}{\partial t^2} + \frac{n-1}{t}\frac{\partial G}{\partial t} = \Delta_x G, \quad (x,t) \in D \times (0,T]; \\ G(x,T) = 0, \quad \partial_t G(x,T) = 0; \\ G(x,t) = g(x,t) \quad (x \in \partial D) \end{cases} \tag{32}$$

(*which always exists*) *satisfies the condition* $\int_D \partial_t G(x,t)\phi(x)\,dx \to 0$ *as* $t \to 0+$.

(2) *If the equivalent conditions* (a) *and* (b) *hold for all Dirichlet eigenfunctions* $\phi = \phi_k$, *then there exists a smooth function* $f(x)$ *in* D, *such that* $\lim_{t\to 0+} G(x,t) = f(x)$, *and* $\lim_{t\to 0+} G_t(x,t) = 0$, *where the limits are understood as convergence in* $C^\infty(D)$ (*or, equivalently, in the Sobolev space* $H^s(D)$ *for arbitrary* s).
The converse statement also holds, i.e. this behavior of G *at* $t \to 0+$ *implies* (a) *and* (b) *for any eigenfunction* ϕ.

Proof of the lemma.

(1) *Proof of equivalence of conditions (1)(a) and (1)(b).* First of all, we need to be sure that a solution $G(x,t)$ of the problem (32) exists and is unique and regular for $t > 0$. This is immediate, due to the hyperbolic nature of this problem (at least, until one approaches the singularity at $t = 0$). One can also show this as follows. Applying the Weyl transform with respect to time to the functions $G(x,t)$ and $g(x,t)$ in (32), one arrives (as we have discussed already) to a similar problem for the wave equation, where the corresponding theorems are available in PDE textbooks (e.g., [28, Section 7.2, Theorem 6]). Then, applying the inverse Weyl transform, one obtains the needed solution of (32) (see, e.g., [50] for usage of such transformation techniques for various PDE problems). In fact, a more elaborate consideration of this kind can be found further on in this proof.

We will now prove the implication (a) $\to$ (b). We choose a small $\epsilon > 0$ and start with a straightforward equality

$$\int_\epsilon^T \int_{\partial D} g\partial_\nu u_\lambda t^{n-1}\,dS\,dt = \int_\epsilon^T j_{n/2-1}(\lambda t)t^{n-1}\,dt \int_{\partial D} g\partial_\nu \phi\,ds.$$

Using Stokes' formula, one rewrites the inner integral as

$$\int_{\partial D} g\partial_\nu \phi\,dS = \int_{\partial D} \partial_\nu g\phi\,dS + \int_D (G\Delta\phi - \Delta G\phi)\,dx. \tag{33}$$

The first integral on the right is zero, since ϕ vanishes on the boundary. Then in the second integral, we use the eigenfunction property for ϕ to get

$$\int_{\partial D} g\partial_\nu \phi\,dS = -\int_D \bigl(\Delta G + \lambda^2 G\bigr)\phi\,dx.$$

Since G satisfies Darboux equation, we can replace ΔG by the Bessel operator $\mathcal{B} := \mathcal{B}_{(n-2)/2} = \frac{\partial^2}{\partial t^2} + \frac{n-1}{t}\frac{\partial}{\partial t}$. This leads to the following form of the last expression:

$$-\int\limits_D \bigl(G_{tt} + (n-1)t^{-1}G_t + \lambda^2 G\bigr)\phi\, dx.$$

Substituting this into the right-hand side of (33) and changing the order of integration, one arrives to

$$\int\limits_\epsilon^T \int\limits_{\partial D} g\partial_\nu u_\lambda\, dS\, dt = -\int\limits_D \phi(x)\, dx \int\limits_\epsilon^T \left(G_{tt} + \frac{n-1}{t}G_t + \lambda^2 G\right) j_{n/2-1}(\lambda t)t^{n-1}\, dt. \tag{34}$$

Let us introduce a temporary notation

$$h(t) := \int\limits_D G(x,t)\phi(x)\, dx \quad \text{for } t > 0.$$

Integrating by parts with respect to t in the inner integral in (34), we can rewrite the resulting expression for $\int_\epsilon^T \int_{\partial D} g\partial_\nu u_\lambda\, ds\, dt$ as follows:

$$\int\limits_\epsilon^T \int\limits_{\partial D} g\partial_\nu u_\lambda\, ds\, dt = \epsilon^{n-1}\bigl(h_t(\epsilon) j_{n/2-1}(\epsilon t) - h(\epsilon)\bigl(j_{n/2-1}(\lambda t)\bigr)_t\big|_{t=\epsilon}\bigr). \tag{35}$$

We now need to investigate possible behavior of $h(t)$ and its derivative when $t \to 0$. In order to do so, let us derive from the Darboux equation for G a differential equation for $h(t)$. Applying the Bessel operator $\partial^2/\partial t^2 + (n-1)t^{-1}\partial/\partial t$ to the identity defining the function $h(t)$, one obtains

$$\frac{\partial^2 h}{\partial t^2} + \frac{n-1}{t}\frac{\partial h}{\partial t} = \int\limits_D \left(\frac{\partial^2 G}{\partial t^2} + \frac{n-1}{t}\frac{\partial G}{\partial t}\right)\phi\, dx = -\lambda^2 h - \int\limits_{\partial D} g\partial_\nu \phi\, dS. \tag{36}$$

We used here the Darboux equation for G, integration by parts, the fact that ϕ is an eigenfunction, and finally the vanishing of ϕ at ∂D.

Let us introduce a shorthand notation for the last integral in (36):

$$w(t) = -\int\limits_{\partial D} g(x,t)\partial_\nu \phi(x)\, dS(x).$$

Thus, we get the final non-homogeneous Bessel ODE for $h(t)$:

$$\frac{\partial^2 h}{\partial t^2} + \frac{n-1}{t}\frac{\partial h}{\partial t} + \lambda^2 h = w(t). \tag{37}$$

Due to the condition that g belongs to $C_0^\infty(\partial D \times [0,T])$, we conclude that $w(t)$ is smooth and vanishes to the infinite order at $t = 0$. It is a matter of simple consideration to show existence of a particular solution of (37) that vanishes to the infinite order at the origin. Thus, the type of the behavior at the origin is dictated by the solutions of the homogeneous equation. This behavior is well known (e.g., [60]). It depends on whether Bessel functions of the first or the second kind are involved.

Bessel functions of the first kind are smooth and have zero derivative at the origin. The ones of the second type, have singularity at zero. If there are no Bessel functions of the second kind involved, then the solution of the homogeneous equation is continuous at the origin and has zero derivative there. On the other hand, if there is a Bessel function of the second kind as a part of $h(t)$, then when $t \to 0+$, h behaves as follows (e.g., [60]): when $n = 2$, then $h(t) = \log t(C + o(t))$ and $h'(t) = t^{-1}(C + o(t))$ with non-zero constants C. In the case when $n > 2$, the corresponding behavior is $h(t) = t^{2-n}(C + o(t))$ and $h'(t) = t^{1-n}(C + o(t))$. We will now show that this type of behavior is impossible, due to (a). Indeed, (a) says that $\int_\epsilon^T \int_{\partial D} g \partial_\nu u_\lambda \, ds \, dt \to 0$ when $\epsilon \to 0$. Then, due to (35),

$$\epsilon^{n-1}\big(h_t(\epsilon) j_{n/2-1}(\epsilon t) - h(\epsilon)\big(j_{n/2-1}(\lambda t)\big)_t\big|_{t=\epsilon}\big) \to 0, \quad t \to 0.$$

On the other hand, if Bessel functions of the second kind were involved, then this expression would be $C + o(1)$ with a non-zero constant C, which is a contradiction. Thus, we conclude that $h(t)$ is continuous at $t = 0$, and also that $h'(0) = 0$. The latter statement is exactly the claim of (b).

Remark 15. In fact, we have proven more than we claimed in (b). Indeed, we showed not only that $h_t(t) = \int_D \partial_t G(x,t)\phi(x)\,dx \to 0$ as $t \to 0+$, but also that $h(t) = \int_D G(x,t)\phi(x)\,dx$ is continuous at $t = 0$.

The converse implication (b) $\to$ (a) is even simpler. Condition (b) means that $h_t(\epsilon) \to 0$, $\epsilon \to 0+$. Therefore, $h(\epsilon)$ has no singularity at $\epsilon = 0$ and is continuous there. Then the right-hand side in (35) tends to zero as $\epsilon \to 0$ and therefore the left-hand side does as well. This means that (a) holds.

(2) *Proof of statement (2): regularity of G at $t = 0$.* In this part of the proof, we will use the transformation technique already briefly mentioned above, which allows one to toggle between the solutions of the wave equation and Darboux equation. Ideologically, what we are about to do, is using the Weyl transform. This can be done, and has been done by the authors. However, it seemed to the authors, that using only Fourier and Fourier–Bessel transforms makes the proof less technical and more transparent. An alternative version of the proof, which uses Weyl transform explicitly is provided in Section 9.

First of all, it is well known (e.g., [24, Chapter 6.13]) that existence of the limit when $t \to 0$ of $G(x,t)$ and the equality $\lim_{t\to 0} G_t = 0$ (even in weaker topologies than C^∞) mean that $G(x,t)$ can be extended to an even with respect to t solution of the Darboux equation. Due to the zero conditions at $t = T$, this even solution will be supported in $D \times [-T, T]$.

Thus, our task, instead of studying the limits of G and G_t when $t \to 0$ (which we would need to do if using Weyl transform), will be to investigate existence and regularity (as a function of t with values in $H^s(D)$) of an even with respect to time solution G.

Suppose we do have such a solution $G(x,t)$ supported in $D \times [-T, T]$. Let us then take its Fourier–Bessel transform $\mathcal{F}_p$ (with $p = (n-2)/2$) with respect to time (16), to get a function $\widehat{G}(x,\lambda)$. According to the Lemma 1, this function, as an $H^s(D)$-valued function of λ, would be even with respect to λ and would satisfy the Paley–Wiener estimate (18) with $a = T$. Besides, the Darboux equation and the boundary conditions would also imply that the following equation and boundary conditions are satisfied:

$$\begin{cases} (\lambda^2 + \Delta_x)\widehat{G}(x,\lambda) = 0 & \text{in } D, \\ \widehat{G}(x,\lambda) = \widehat{g}(x,\lambda) & \text{for } x \in S. \end{cases} \tag{38}$$

Notice, that $\widehat{g}(x,\lambda)$ is even with respect to λ and of the appropriate Paley–Wiener class as a $H^s(S)$-valued function for any s, due to our conditions on smoothness and support of $g(x,t)$. So, our problem is now equivalently reformulated as showing existence of an even and entire with respect to λ solution of (38) of the appropriate Paley–Wiener class.

It is clear that (38) might not have any solution at all when $-\lambda^2$ belongs to the spectrum of the Dirichlet Laplacian in D. However, the necessary and sufficient condition for solvability of (38) for such values of λ are well known and easy to derive (they represent the Fredholm alternative):

$$\int_S \widehat{g}(x,\lambda)\partial_\nu \phi(x)\, dS = 0 \tag{39}$$

for any eigenfunction ϕ of Δ_D corresponding to the eigenvalue $-\lambda^2$. These conditions clearly are equivalent to 3(b) and thus satisfied in our case. Hence, one hopes to solve (38) for all λ and to eventually get the needed solution. This is exactly what we will endeavor now.

First of all, it will be convenient for us to apply the standard trick of moving the inhomogeneity in (38) from the boundary condition into the equation. Let us denote by E any "nice" extension operator of functions from S to D, for instance any one that would map Sobolev spaces $H^s(S)$ to $H^{s+1/2}(D)$. Existence of such operators is well known (see, e.g., [65]). The Poisson operator of harmonic extension is one of them. Let us denote $U(x,\lambda) = \widehat{G}(x,\lambda) - E\widehat{g}(x,\lambda)$. Then this function solves the problem

$$\begin{cases} \left(\lambda^2 + \Delta_x\right)U(x,\lambda) = \widehat{f}(x,\lambda) & \text{in } B, \\ U(x,\lambda) = 0 & \text{for } x \in S. \end{cases} \tag{40}$$

Here $\widehat{f}(x,\lambda) = -(\lambda^2 + \Delta_x)E\widehat{g}(x,\lambda)$ is of the same Paley–Wiener class with respect to λ, as $\widehat{g}(x,\lambda)$.

Let us apply to (40) the inverse Fourier (rather than Fourier–Bessel) transform with respect to λ (this amounts to applying the Weyl transform to the original Darboux equation). Then we arrive to the following evolution problem:

$$\begin{cases} U_{tt}(x,t) = \Delta U(x,t) + f(x,t), & x \in B,\ t \in \mathbb{R}, \\ U(x,t)|_{x\in S} = 0. \end{cases} \tag{41}$$

Here U and f are inverse Fourier transforms from λ to t of $\widehat{U}$ and $\widehat{f}$. Function f is even with respect to t, infinitely smooth as $H^s(D)$-valued function of t for any s, and is supported (due to the Paley–Wiener theorem) in $D \times [-T,T]$. Our goal now boils down to proving existence of an even with respect to time solution $U(x,t)$ that is smooth as $H^s(D)$-valued function of t and is supported in $D \times [-T,T]$. If this is done, then taking Fourier transform with respect to time first and the inverse Fourier–Bessel transform next, we will arrive to the solution $G(x,t)$ we need, which will finish the proof of the lemma.

Let us consider the following problem:

$$\begin{cases} U_{tt}(x,t) = \Delta U(x,t) + f(x,t), & x \in B,\ t > -T, \\ U(x,t)|_{x \in S} = 0, \\ U(x,-T) = U_t(x,-T) = 0. \end{cases} \tag{42}$$

According to the standard existence and uniqueness theorems for the wave equation (e.g., [28, Section 7.2, Theorem 6]), there exists unique (and smooth as $H^s(D)$-valued function of t) solution of this problem. Due to the type of the initial and boundary conditions we imposed, one can extend the solution to all times by assuming that it is zero for $t < -T$.

It only remains to prove that $U(x,t)$ vanishes for $t > T$ and is even with respect to time. To do so, let us consider a complete orthonormal set $\{\phi_k(x)\}$ of eigenfunctions of the Dirichlet Laplacian in D and denote by $-\lambda_k^2$ the corresponding eigenvalues. Let us also expand the functions U and f into this basis:

$$\begin{cases} U(x,t) = \sum u_k(t)\phi_k(x), \\ f(x,t) = \sum f_k(t)\phi_k(x). \end{cases} \tag{43}$$

It will be sufficient for our purpose to show that all functions $u_k(t)$ vanish for $t > T$.

Let us notice that the following initial value problem is satisfied by $u_k(t)$:

$$\begin{cases} u_k'' + \lambda_k^2 u_k(t) = f_k(t), \\ u_k(t) = u_k'(t) = 0 \quad \text{for } t \leqslant -2. \end{cases} \tag{44}$$

Taking Fourier transform (in distribution sense) in (44), we get

$$\left(\lambda_k^2 - \lambda^2\right)\widehat{u_k}(\lambda) = \widehat{f_k}(\lambda). \tag{45}$$

Here $\widehat{f_k}(\lambda)$ is even and from the Paley–Wiener class corresponding to smooth functions with support in $[-T,T]$. Consider the function

$$\widehat{v_k}(\lambda) := \frac{\widehat{f_k}(\lambda)}{(\lambda_k^2 - \lambda^2)}. \tag{46}$$

As it was mentioned above in this proof, conditions (3)(b) guarantee that $\widehat{f_k}(\lambda)$ vanishes at the points $\pm\lambda_k$. Thus, $\widehat{v_k}$ is entire, even, and by a simple estimate, belongs to the same Paley–Wiener class as $\widehat{f_k}$. Thus, it is Fourier transform of a smooth even function $v_k(t)$ supported in $[-T,T]$. Consider the difference $w_k(t) = u_k(t) - v_k(t)$. It satisfies then the homogeneous equation $w_k'' + \lambda_k^2 w_k = 0$ and zero initial conditions at $t = -T$. Thus, it is identically zero. Hence, $u_k = v_k$ is even and supported in $[-T,T]$ for any k, and thus $u(x,t)$ is also even and supported in $D \times [-T,T]$. This finishes the proof of the existence of a solution $G_+(x,t)$ of the Darboux equation inside the cylinder C that agrees with the spherical mean data on $S \times \mathbb{R}$ and which is even with respect to time, smooth as an $H^s(D)$-valued function of t, and supported in $t \in [-T,T]$.

What now remains to prove in the lemma, is the converse statement in its part (2). This is, however, trivial. Indeed, the strong convergence of G at $t \to 0$ we have derived clearly implies the statement (1)(b) for any eigenfunction.

This finishes the proof of Lemma 14. $\square$

4.3. Equivalence (3) $\Leftrightarrow$ (4)

Since conditions 3(a) and 4(a) are the same, it is sufficient to prove equivalence of (3)(b) and (4)(b). Implication (3)(b) $\Rightarrow$ (4)(b) is straightforward. Indeed, one can choose in (20) instead of ψ_λ one of the eigenfunctions $\phi_{m,l}$ introduced in (21), provided $\lambda \neq 0$ is a zero of the Bessel function $J_{m+n/2-1}$. In this case, the integral in (28) evaluates to be proportional to $\widehat{g}_{m,l}(\lambda) = \int_S \widehat{g}(\lambda,\theta) Y_l^m(\theta)\, d\theta$. Thus, vanishing of these expressions for all m, l and λ as described, is equivalent to the condition (4)(b).

The converse implication (4)(b) $\Rightarrow$ (3)(b) follows analogously, if one takes into account the completeness of the system of eigenfunctions $\phi_{m,l}$.

4.4. Implication (2) + (3) + (4) $\Rightarrow$ (1)

Our goal here is, assuming any of the equivalent assumptions (2), (3), (4) (or a combination of those), to show existence of a function $f(x)$ supported in B such that the restriction of its spherical mean Radon transform $G(x,t)$ onto the lateral boundary $S \times [0,2]$ of the cylinder C coincides with the function g.

Using the Darboux equation reformulation that we have mentioned before, this is equivalent to showing existence in $\mathbb{R}^n \times [0,\infty)$ of a solution $G(x,t)$ of the Darboux equation (3) such that $G(x,0) = f(x)$, $G_t(x,0) = 0$, and $G|_{S\times[0,2]} = g$, for a function f supported in B. Then this function f would be a pre-image under the spherical mean transform R of the data g.

Our strategy consists of solving the following sequence of problems:

- Showing that the solution $G_+(x,t)$ of the interior problem (3) is even and smooth with respect to t on the whole t-axis. This would, in particular, provide us with a candidate $f(x) = G(x,0)$, $x \in B$ for the pre-image.
- Using rotational invariance, reducing the problem to single spherical harmonic terms of g, G, and f.
- Showing that each such term G^m of G extends to the whole space $\mathbb{R}^n$ as a global solution of Darboux equation.
- Proving that the value $G^m(x,0)$ is supported inside the ball B and coincides with the corresponding harmonic term f_m of f. This will show that $Rf^m = g^m$.
- Now an immediate continuity argument will show that $Rf = g$, which will finish the proof of this implication, and thus of the whole theorem.

Let us start realizing this program.

4.4.1. Interior solution G_+

Conditions (2)(b) and (3)(b) mean that the equivalent requirements (1)(a) and (1)(b) of Lemma 14 are satisfied. Then the second claim of this lemma guarantees that the interior solution $G_+(x,t)$ can be continued to an even, smooth as $H^s(B)$-valued function of t solution of Darboux equation in the infinite cylinder $B \times \mathbb{R}$. This resolves the first step of our program.

4.4.2. Projection to the $O(n)$-irreducible representations

Each irreducible sub-representation X^m of the representation of the orthogonal group $O(n)$ on functions on $\mathbb{R}^n$ by rotations consists of homogeneous harmonic polynomials of a fixed degree m (e.g., [97,99]). Restrictions of the elements of X^m to the unit sphere S are spherical harmonics of degree m. We denote, as before, by $Y_l^m(\theta)$, $l = 1, \dots, d(m)$, $m = 1, \dots$ an orthonormal basis in X^m. The orthogonal projection $L^2(S) \mapsto X^m$ will be denoted by $\mathcal{P}^m$:

$$\big(\mathcal{P}^m h\big)(x) = \int\limits_S h(y) Z_x^m(y)\, dS(y),$$

where Z_x^m is the zonal spherical harmonic of degree m with the pole x (e.g., [97, Chapter 4, Section 2]). Since Bessel operator $\mathcal{B}$ and Laplace operator $\Delta = \Delta_x$ both commute with the action of $O(n)$, the projection onto X^m

$$G^m(x,t) = \big(\mathcal{P}^m G\big)(x,t)$$

reduces the Darboux equation. I.e., G^m solves the same Darboux equation with the zero data for $t = 2$ and with the boundary data $g^m = \mathcal{P}^m g$. Clearly, we also have $G^m(x,0) = f^m(x)$. So, let us assume for now that $G = G^m$, $g = g^m$, and $f = f^m$.

Since functions Y_l^m, $l = 1, \dots, d(m)$, form an orthonormal basis of X^m, we have

$$\begin{aligned} G^m(x,t) &= \sum_{l}^{d(m)} g_l(r,t) Y_l^m(\theta), \\ g^m(\theta,t) &= \sum_{l}^{d(m)} g_l(t) Y_l^m(\theta). \end{aligned} \tag{47}$$

As we have already seen, the Fourier–Bessel transform takes the solution G^m of Darboux equation to a function $\widehat{G}^m(x,\lambda)$ that satisfies the equation

$$\Delta_x \widehat{G}^m(x,\lambda) = -\lambda^2 \widehat{G}^m(x,\lambda)$$

in B. Due to this and the special form (47) of G^m, its Fourier–Bessel transform can be written as

$$\widehat{G}^m(r\theta,\lambda) = j_{n/2-1+m}(\lambda r)(\lambda r)^m \sum_{l=1}^{d(m)} b_l(\lambda) Y_l^m(\theta), \tag{48}$$

and correspondingly

$$\widehat{g}^m(\theta,\lambda) = j_{n/2-1+m}(\lambda)(\lambda)^m \sum_{l=1}^{d(m)} b_l(\lambda) Y_l^m(\theta). \tag{49}$$

Now observe that the right-hand side of (48) is defined for all r, not only for $r \leqslant 1$ and therefore defines a smooth extension of $\widehat{G}^m(r\theta,\lambda)$ for $r > 1$. Thus, we can think of $\widehat{G}^m(x,\lambda)$ as smooth in $x = r\theta$ function defined for *all* $(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}$. For $x \in B$, due to Lemma 1, this function is of the Paley–Wiener class in λ, being Fourier–Bessel transform in t of the compactly supported smooth function $G^m(x,t)$. However, at this stage we do not know much about its behavior with respect to λ for $x \notin B$. To gain this knowledge, we need some control over the smoothness and growth of the coefficients $b_l(\lambda)$, which are defined for all $\lambda \in \mathbb{C}$.

Computing the Fourier coefficients with respect to the orthonormal basis Y_l^m of spherical harmonics, we obtain

$$b_l(\lambda)(\lambda r)^m j_{n/2-1+m}(\lambda r) = \int\limits_{\theta\in S} \widehat{G}^m(r\theta,\lambda) Y_l^m(\theta)\, dS(\theta). \tag{50}$$

The functions $b_l(\lambda)$ are clearly analytic at any point $\lambda_0 \neq 0$. Indeed, for any such λ_0 one can choose $r < 1$ so that $j_{n/2-1+m}(\lambda r) \neq 0$ for λ near λ_0. Thus, b_l is analytic near λ_0 as the ratio of two analytic functions with non-vanishing denominator.

This argument does not work at $\lambda = 0$. Moreover, smoothness of b_l at $\lambda = 0$ is not guaranteed immediately by (50), and requires the moment condition. Indeed, let us restrict (50) to the boundary S to get

$$b_l(\lambda)(\lambda)^m j_{n/2-1+m}(\lambda) = \int\limits_{\theta\in S} \widehat{g}^m(\theta,\lambda) Y_l^m(\theta)\, dS(\theta). \tag{51}$$

As we already established in Lemma 9, the moment condition is equivalent to the integral in the right-hand side in (51) vanishing at $\lambda = 0$ to the order at least m. On the other hand, the function $\lambda^m j_{n/2-1+m}(\lambda)$ in (51) has zero of order m at $\lambda = 0$. Then, dividing by $\lambda^m j_{n/2-1+m}(\lambda)$ in (51), we conclude that $b_l(\lambda)$ is smooth at $\lambda = 0$.

Thus, b_l is an entire function. Now we need to estimate its growth at infinity (looking for Paley–Wiener estimates). Due to the Paley–Wiener class estimates that we have for the expression in the right-hand side of (51) and known behavior of Bessel functions, it is a standard exercise to show that their ratio $b_l(\lambda)$ is of the same Paley–Wiener class as the numerator. Indeed, this was treated in [15]. The estimate from below for Bessel functions provided in [15, Lemma 6] and its consequent usage there show that this in fact is true (an alternative proof can be found in Section 9).

4.4.3. Extending G^m

Now we can apply the inverse Fourier–Bessel transform in λ to the extended function $\widehat{G}^m(x,\lambda), x \in \mathbb{R}^n$ in (48). We use the same notation $G^m(x,t)$ for the obtained function. This is justified, since by the construction this function satisfies the Darboux equation and it coincides with the original interior "m-irreducible" solution $G^m(x,t)$ in the cylinder $B \times \mathbb{R}$. One can also observe that, due to the Paley–Wiener Lemma 1, it is smooth with respect to t and supported in $B \times [-2,2]$.

4.4.4. The size of the support of $G^m(x,0)$

Using the relation $G^m(x,0) = f^m(x), x \in B$ and applying the inverse Fourier–Bessel transform to (48), one finds

$$f^m(x) = f^m(r\theta) = const \int\limits_0^\infty (\lambda r)^m j_{n/2-1+m}(\lambda r) \left(\sum_{l=1}^{d(m)} b_l(\lambda) Y_l^m(\theta) \right) \lambda^{n-1}\, d\lambda. \tag{52}$$

We notice that now we can define an extension $F^m(x)$ of $f^m(x)$ to the whole space $\mathbb{R}^n$ by applying the formula (52) for $r > 1$:

$$F^m(x) = F^m(r\theta) := const \int_0^\infty (\lambda r)^m j_{n/2-1+m}(\lambda r) \left(\sum_{l=1}^{d(m)} b_l(\lambda) Y_l^m(\theta) \right) \lambda^{n-1}\, d\lambda, \tag{53}$$

which, according to known results (e.g., [97, Chapter IV, Theorem 3.10]), is just the inverse n-dimensional Fourier transform of the following function:

$$H(x) = H(\lambda\theta) = \lambda^m \sum_{l=1}^{d(m)} b_l(\lambda) Y_l^m(\theta) \tag{54}$$

in $\mathbb{R}^n$, written in polar coordinates $x = \lambda\theta$. Consider for a moment only real values of λ. One sees immediately that, due to the Paley–Wiener estimates on b_l, function $H(x)$ is smooth outside the origin and decays with all its derivatives faster than any power of $|x| = |\lambda|$. If we show smoothness at the origin, then H will be proven to belong to the Schwartz class. According to standard considerations of Radon transform theory [33–35,43], this function is smooth at the origin if and only if for any k the expression

$$\left.\frac{\partial^k H(\lambda\theta)}{\partial\lambda^k}\right|_{\lambda=0} = \sum_{l=1}^{d(m)} \left.\frac{\partial^k(\lambda^m b_l(\lambda))}{\partial\lambda^k}\right|_{\lambda=0} Y_l^m(\theta)$$

is the restriction to the unit sphere of a homogeneous polynomial of degree k with respect to θ. Due to the form of the last expression, this means that $\frac{\partial^k(\lambda^m b_l(\lambda))}{\partial\lambda^k}(0) = 0$ for any k such that either $k < m$ or $k - m$ is odd. The case $k < m$ is obvious, due to smoothness of b_l and the presence of the factor λ^m. Due to the structure of the functions $\widehat{g}(\lambda)$ and Bessel functions, discussed already, the condition for $k - m$ odd is automatic.

Hence, H belongs to the Schwartz space. Then its inverse Fourier transform, which we previously denoted by $F^m(x)$, is in Schwartz space itself. We now need to establish that F^m is supported inside B.

By its construction, F^m is the value at $t = 0$ of a global solution G^m of the Darboux equation. Since $G^m|_{S\times\mathbb{R}^+} = g^m$, Asgeirsson theorem [17,47] implies that $RF^m = g^m$. In particular, the integrals of F^m over all spheres centered inside B and of radii $t \geqslant 2$ are equal to zero. Indeed, such an integral over a sphere centered at $x \in B$ of radius t is equal to $G^m(x,t)$, which is known to be zero by construction of G^m.

Lemma 2.7 in [43, Chapter 1] claims that if a function decays faster than any power and its integrals over all spheres surrounding a convex body B are equal to zero, then the function is zero outside of B. In our case we do not have all such spheres, but only the ones of radii at least 2 and centered in B, where B is the unit ball. However, a simple exercise is to check that the proof of the cited lemma still holds and thus F^m is supported in B. This means that in fact $F^m(x)$ is the zero extension of $f^m(x)$ outside the ball B.

4.4.5. Final step: proving $Rf = g$

We already have constructed a function $f \in C_0^\infty(B)$ such that for each its component f^m corresponding to an irreducible representation X^m, the equality $Rf^m = g^m$ holds. Since we have expansions $f = \sum_m f^m$ and $g = \sum_m g^m$ converging in any Sobolev space, the equality $Rf = g$ immediately follows by continuity.

This finishes the proof of implication (2) + (3) + (4) $\Rightarrow$ (1) and thus completes the proof of Theorem 10.

5. Proof of Theorem 11

We assume now that the dimension n is odd. The claim of Theorem 11 is that the statement of Theorem 10 can be proven without using moment conditions of Lemma 2 or Lemma 4. In other words, we claim that in odd dimensions, the moments conditions (2)(a)–(4)(a) of Theorem 10 follow from the equivalent orthogonality conditions (2)(b)–(4)(b). Of course, this effects only the sufficiency part of this theorem, since we have proven that in any dimension the moment conditions are necessary for g to be in the range. Thus, in odd dimensions the moment conditions are redundant.

In order to prove Theorem 11, we will follow the part of the proof of Theorem 10 where the moment conditions were not used, and then will finish the proof avoiding the moment conditions.

As we have proven in Section 4, the equivalent conditions (2)–(4) of Theorem 11 (which coincide with conditions (2)(b)–(4)(b) of Theorem 10), imply the existence of a smooth solution $G_+(x,t)$ of Darboux equation in the solid cylinder $C = B \times \mathbb{R}$, even with respect to t and such that $G_+(x,t) = 0$ for $|t| \geqslant 2$ and $G_+|_{S\times\mathbb{R}} = g$. Here $g = g(x,t)$ is the function introduced in Theorem 11. We emphasize again that the moment conditions were not used in this derivation.

The key point in the proof of Theorem 11 is the following auxiliary statement, which in odd dimensions can be proven without using moment conditions.

Proposition 16. *The function G_+ and all its partial derivatives in x vanish on the sphere $S_0 = \{|x| = 1, t = 0\} \subset \mathbb{R}^n \times \{0\}$.*

We will postpone the proof of this proposition, and will show now that it implies the statement of Theorem 11.

5.1. Derivation of Theorem 11 from Proposition 16

Let $f(x) := G_+(x,0)$. This functions is smooth in B and, according to Proposition 16, vanishes to the infinite order at $S = \partial B$. Hence, the function obtained by zero extension of $f(x)$ to the whole $\mathbb{R}^n$, is smooth. We will use the same notation $f(x)$ for this function:

$$f(x) := \begin{cases} f(x) & \text{for } |x| \leqslant 1, \\ 0 & \text{for } |x| \geqslant 1. \end{cases}$$

Let us now define

$$G(x,t) = \frac{1}{\omega} \int\limits_{y\in S} f(x+ty)\, dS(y), \qquad x \in \mathbb{R}^n,$$

as the spherical mean transform of this function f. Theorem 11 will be proven, if we show that $G = G_+$ inside C. Indeed, then $Rf|_{S\times[0,\infty)} = g$, and thus g is in the range.

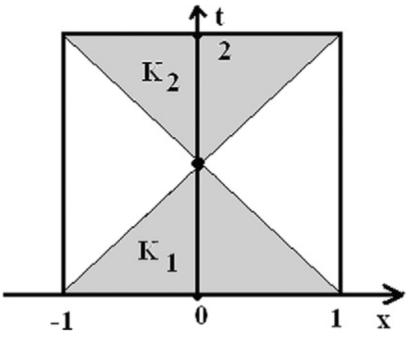

Fig. 2. The cones K_1 and K_2.

The function G solves Darboux equation in the entire space, and therefore the difference

$$Z(x,t) = G_+(x,t) - G(x,t)$$

solves this equation in the cylinder $B \times [0,\infty)$ (in fact, it extends to an even solution in $B \times \mathbb{R}$).

Our goal is to prove that the solutions G and G_+ agree on the cylinder C. At the initial moment $t = 0$, we have for $x \in B$:

$$Z(x,0) = f(x) - f(x) = 0,$$
$$Z_t(x,0) = G_{+,t}(x,0) - G_t(x,0) = 0.$$

Then Z vanishes inside the corresponding characteristic cone:

$$Z(x,t) = 0, \quad (x,t) \in K_1 := \{0 \leqslant t \leqslant 1 - |x|\}.$$

On the other hand, the interior solution $G_+(x,t)$, according to its construction, vanishes for $t \geqslant 2$. The spherical means $G(x,t)$ also vanish for $|x| \leqslant 1$, $t \geqslant 2$, since then the sphere $\{y\colon |y-x| = t\}$ does not intersect B, and hence also the support of f. Thus, the difference $Z = G_+ - G$ vanishes in the cylinder $\{|x| \leqslant 1,\ t \geqslant 2\}$. By the same dependence domain argument, Z vanishes on the backward characteristic cone K_2:

$$Z(x,t) = 0, \quad (x,t) \in K_2 := \{|x| + 1 \leqslant t \leqslant 2\}.$$

Thus, the difference Z of the two solutions vanishes on the union of two characteristic cones:

$$Z(x,t) = 0, \quad (x,t) \in K_1 \cup K_2,$$

that have the common vertex $(0,1)$ (see Fig. 2).

Notice that the union $K_1 \cup K_2$ of the two cones contains the segment $x = 0$, $0 \leqslant t \leqslant 2$. Each point $(0,t)$ of this segment, except the vertex $(0,1)$, belongs to the interior of $K = K_1 \cup K_2$. Since Z is smooth and $Z|_K = 0$, all partial derivatives $\partial_x^\alpha Z(0,t)$ vanish for all $t \neq 1$. By smoothness, this is also true for the vertex $(0,1)$. Thus, function Z vanishes, along with all its derivatives, on the entire line $x = 0$. We claim that this implies that $Z = 0$ for all $(x,t) \in C$. The following lemma does the job.

Lemma 17. *Let $Z(x,t)$ be an even and compactly supported in t smooth solution of Darboux equation in $B \times \mathbb{R}$ and x_0 an interior point in B. If $(\partial_x^\alpha Z)(x_0,t) = 0$ for any multi-index α and $t \in \mathbb{R}$, then $Z(x,t) = 0$ for all $(x,t) \in B \times \mathbb{R}$.*

Proof. Let us apply Fourier–Bessel transform with respect to t to the function $Z(x,t)$. Then Darboux equation transforms to Helmholtz equation and the resulting function $\widehat{Z}(x,\lambda)$ is an eigenfunction of Laplace operator:

$$\Delta \widehat{Z}(x,\lambda) = -\lambda^2 \widehat{Z}(x,\lambda).$$

By the condition of the lemma, $\widehat{Z}(x,\lambda)$ has at x_0 a zero of infinite order. Since eigenfunctions of the Laplace operator Δ are known to be real-analytic, we conclude that $\widehat{Z}(\cdot,\lambda) \equiv 0$. Taking inverse Fourier–Bessel transform, we get $Z \equiv 0$. □

This finishes the proof of Theorem 11, modulo the proof of Proposition 16, which we provide in the next subsection.

5.2. Proof of Proposition 16

The goal of this subsection is to prove that, as the Proposition 16 states, at the boundary points $(x,0)$, the infinite order zero of the boundary data $g(x,t)$ at $t = 0$ implies infinite order zero of $G(x,t)$ with respect to x at $t = 0$, $|x| = 1$.

This proof will be close to the proof of Proposition 7 in [30]. Following [30], we will reduce the initial-boundary value problem for Darboux equation to a problem for the one-dimensional wave equation. Again, as in [30], we will use separation of variables in polar coordinates and dimension reduction. However, we will do this in a somewhat different manner. Besides, since we are dealing with Darboux equation rather than with the wave equation, an additional integral transform will be required with respect to t.

We will break the proof into several steps.

5.2.1. Separation of variables

First of all, we decompose the solution G into spherical harmonic parts G^m that belong to the irreducible representations X^m of the rotation group $O(n)$:

$$G(x,t) = \sum_m G^m(x,t) = \sum_m \sum_{l=0}^{d(m)} \Psi_l(r,t) r^m Y_l^m(\theta), \quad x = r\theta.$$

It is clear that it is sufficient to prove the claim of the proposition for each of these components

$$G^m(x,t) = \sum_{l=0}^{d(m)} \Psi_l(r,t) r^m Y_l^m(\theta), \quad x = r\theta.$$

Indeed, we know that G belongs to any Sobolev space $H^s(B)$ and hence the convergence of the spherical harmonics expansion and Sobolev embedding theorems will deduce the claim for G from those for each G^m.

So, we will assume from now on that

$$G = G^m = \sum_{l=0}^{d(m)} \Psi_l(r,t) r^m Y_l^m(\theta).$$

Correspondingly, we also expand the data g and assume that

$$g(\theta,t) = \sum_{l=0}^{d(m)} \Psi_l(t) Y_l^m(\theta).$$

The Darboux equation for $G(x,t)$ implies that the coefficients $\Psi(r,t) = \Psi_l(r,t)$ satisfy the following PDE:

$$\partial_t^2 \Psi + \frac{n-1}{t}\partial_t \Psi = \partial_r^2 \Psi + \frac{n-1+2m}{r}\partial_r \Psi. \tag{55}$$

In order to prove the proposition, it suffices to prove that

$$\frac{\partial^p \Psi}{\partial r^p}(1,0) = 0, \quad p = 0, 1, \ldots.$$

So, we will concentrate now on studying the solution Ψ of (55). By its construction, $\Psi(r,t)$ satisfies the following conditions:

(a) $\Psi(r,t)$ is smooth and even with respect to $t \in \mathbb{R}$ and $r \in [-1,1]$,
(b) $\Psi(r,t) = 0$ for $|t| \geqslant 2$,
(c) $\Psi(1,t)$ vanishes at $t = 0$ to the infinite order.

5.2.2. Reduction to the one-dimensional wave equation

Let us set $p = \frac{n-2+2m}{2}$ and apply the inverse Poisson transform $\mathcal{P}_p^{-1}$ (see (9)) with respect to the variable r:

$$\left(\mathcal{P}^{-1}\Psi\right)(r,t) = const\, r\left(\frac{\partial}{\partial(r^2)}\right)^{(n-1+2m)/2} r^{n-2+2m}\Psi(r,t).$$

This will reduce the right-hand side of Eq. (55) to the second r-derivative. We also apply Weyl transform $\mathcal{W}_{(n-2)/2}$ (10) in the variable t. As the result, we obtain the function

$$U(r,t) = \left(\mathcal{P}^{-1}_{(n-2+2m)/2,r}\mathcal{W}_{(n-2)/2,t}\Psi\right)(r,t),$$

which, due to the intertwining properties (12), solves the one-dimensional wave equation

$$U_{tt} - U_{rr} = 0. \tag{56}$$

We can observe now that this new function U has the following properties:

(1) $U(\pm r, \pm t) = U(r,t)$.

(2) $U(r,t)=0$ for $|t|\geqslant 2$.
(3) $U(1,t)=q(t):=(\mathcal{P}^{-1}\mathcal{W}\Psi)(1,t)$ and $q(t)=0$, $|t|\geqslant 2$.

The evenness property (1) follows from the evenness of Ψ and the fact that both transforms we applied preserve it. Property (2) follows from the analogous property (b) of Ψ and the properties of the Weyl transform $\mathcal{W}$. Now, property (3) is just the definition of the boundary values of U for $r=\pm 1$ combined with (2).

The unique solution of (56) in the domain $-1\leqslant r\leqslant 1$, $t\geqslant 0$ with these properties, is the sum of the following two progressing waves:

$$U(r,t)=q(t-r+1)+q(t+r+1). \tag{57}$$

Indeed, this sum clearly satisfies (56). If $r=\pm 1$, $t\geqslant 0$, then $U(\pm 1,t)=q(t)+q(t+2)=q(t)$, due to (3). So, property (2) holds. Also, $q(t-r+1)+q(t+r+1)$ is obviously even in r. Taking into account that $q(t)=0$ for $|t|>2$, implies that $q(t-r+1)+q(t+r+1)=0$ for $t>2$. Thus, by standard uniqueness theorem, the solutions $U(r,t)$ and $q(t-r+1)+q(t+r+1)$ coincide.

The representation (57) has important consequences. The first is given in the following lemma.

Lemma 18. *For all $p\in\mathbb{Z}_+$, one has*

$$\partial_r^p U(\pm 1,0)=0.$$

Proof. Let us show first that the evenness of U with respect to t implies infinite order zero at 0 of the boundary value q. Indeed, observe that for any odd number $p=2s-1$, one has

$$q^{(2s-1)}(-r+1)+q^{(2s-1)}(r+1)=\big(\partial_t^{2s-1}U\big)(r,0)=0. \tag{58}$$

Substituting $r=1$ along with using (3) leads to

$$q^{(2s-1)}(0)=q^{(2s-1)}(0)+q^{(2s-1)}(2)=0.$$

On the other hand, differentiation of (58) with respect to r implies

$$-q^{(2s)}(-r+1)+q^{(2s)}(r+1)=0.$$

Again, substituting $r=1$ yields

$$-q^{(2s)}(0)=-q^{(2s)}(0)+q^{(2s)}(2)=0.$$

Thus, all derivatives $q^{(p)}(0)$ vanish. This, in turn, implies that $U(r,t)$ has a zero of infinite order (with respect to r) at $(1,0)$:

$$\partial_r^p U(1,0)=(-1)^p q^{(p)}(0)+q^{(p)}(2)=0. \tag{59}$$

Since U is even in r, also $\partial_r^p U(-1,0)=0$. $\square$

5.2.3. Proving that Ψ vanishes to infinite order at $|x| = 1$, $t = 0$

We can now finish the proof of the Proposition 16 by showing that $\partial_r^p \Psi(1,0) = 0$ for any $p = 0, 1, \ldots$. Notice, that it suffices to check this identity only for even p. Indeed, $\Psi(r,t)$ vanishes at $(1,0)$ with all derivatives with respect to t. Then Eq. (55) implies that all iterates of the Darboux operator acting in the variable r vanish:

$$\left(\partial_r^2 + \left((n-1+2m)/2\right)\partial_r\right)^N \Psi(1,0) = 0.$$

Now vanishing of all even order derivatives $\partial_r^{2k}\Psi(1,0)$ clearly implies vanishing of the derivatives of odd order as well.

It will be convenient to use the following simple relation.

Lemma 19. *For any smooth even function $v(t)$ the following relation holds*:

$$\left(\frac{d}{d(t^2)}\right)^j v(0) = \left(\frac{1}{2t}\frac{d}{dt}\right)^j v(0) = \frac{j!}{(2j)!} v^{(2j)}(0).$$

Proof. This equality follows from the Taylor formula. □

Let us write now the relation $\Psi = (\mathcal{P}\mathcal{W}^{-1})U$, using (8) and (10), explicitly:

$$\Psi(r,t) = const \int_{-1}^{1} \left(\frac{\partial}{\partial(t^2)}\right)^{(n-1)/2} U(\mu r, t)\left(1-\mu^2\right)^{(n-3)/2} d\mu. \tag{60}$$

According to the property (c) of the function Ψ, its all t-derivatives at $(1,0)$ are equal to zero. The odd order derivatives vanish due to the evenness of Ψ, so only the even order derivatives carry interesting information for us:

$$\partial_t^{2j}\Psi(1,0) = 0, \quad j = 0, 1, \ldots.$$

Let us translate, using (60), these equalities into the language of function U. Differentiating $2j$ times with respect to t at the point $r = 1, t = 0$ under the sign of the integral in (60) leads to the expression $\partial_t^{2j}\partial_{t^2}^{(n-1)/2}U(\mu,0)$. Since $U(\mu,t)$ is even with respect to t, Lemma 19 gives

$$\partial_t^{2j}\partial_{t^2}^{(n-1)/2}U(\mu,0) = const\, \partial_t^{2j+n-1}U(\mu,0).$$

Taking into account that $2j+n-1$ is an even number, the wave equation (56) (or the progressing wave expansion (57)) yields

$$\partial_t^{2j+n-1}U(r,0) = \partial_r^{2j+n-1}U(r,0). \tag{61}$$

Thus,

$$\partial_t^{2j}\Psi(1,0) = const \int_{-1}^{1} \partial_\mu^{2j+n-1}U(\mu,0)\left(1-\mu^2\right)^{(n-3)/2} d\mu = 0, \tag{62}$$

for all $j = 0, 1, \ldots$.

We can now reformulate the identities (62) as follows.

Lemma 20. *The function* $B(\mu) = (\partial_\mu^{(n+1)/2} U)(\mu, 0)$ *is orthogonal in* $L^2[-1,1]$ *to all polynomials* P *of degree* $\deg P \leqslant (n-3)/2$ *of the same parity as the natural number* $(n-3)/2$ (*i.e.* P *is even if* $(n-3)/2$ *is even, and odd if* $(n-3)/2$ *is odd*).

Proof. Integration by parts $2j + (n-3)/2$ times in (62) and vanishing of derivatives of U at $(1, 0)$ lead to

$$\int_{-1}^{1} B(\mu) L_{\frac{n-3}{2}}^{(2j)}(\mu)\, d\mu = 0, \tag{63}$$

where $L_m(\mu) = const\, \partial_\mu^m (\mu^2 - 1)^m$ are Legendre polynomials. Since the derivatives $L_m^{(2j)}$, $j = 0, 1, \ldots$ clearly span the space of polynomials P of degree $\deg P \leqslant m$ that have the same parity as m, this proves the lemma. □

Let us now prove that r-derivatives of Ψ vanish at $(1, 0)$. We differentiate $2k$ times the identity (60) in the variable r. The wave equation (56) and Lemma 19 imply

$$\left(\partial_\mu^{2k} \partial_{t^2}^{(n-1)/2} U\right)(\mu, 0) = const\left(\partial_t^{2k+n-1} U\right)(\mu, 0).$$

Then (60) leads to

$$\begin{aligned} \partial_r^{2k} \Psi(1, 0) &= const \int_{-1}^{1} \left(\partial_\mu^{2k} \partial_{t^2}^{(n-1)/2} U\right)(\mu, 0) \mu^{2k} \left(1 - \mu^2\right)^{(n-3)/2} d\mu \\ &= const \int_{-1}^{1} \left(\partial_\mu^{2k+n-1} U\right)(\mu, 0) \mu^{2k} \left(1 - \mu^2\right)^{(n-3)/2} d\mu. \end{aligned}$$

Integration by parts $2k + (n-3)/2$ times gives

$$\partial_r^{2k} \Psi(1, 0) = const \int_{-1}^{1} \left(\partial_\mu^{(n+1)/2} U\right)(\mu, 0) P(\mu)\, d\mu = const \int_{-1}^{1} B(\mu) P(\mu)\, d\mu,$$

where

$$P(\mu) = \partial_\mu^{2k+(n-3)/2} \left(\mu^{2k} \left(1 - \mu^2\right)^{(n-3)/2}\right)$$

is the polynomial of degree $(n-3)/2$ and of the same parity as $(n-3)/2$. Now Lemma 20 claims that all such integrals are equal to zero. Thus, $\partial_r^p \Psi(\pm 1, 0) = 0$, which finishes the proof of Proposition 16.

6. Proof of Theorem 12

The necessity was already established in (30). To prove sufficiency, let us check that the conditions (3)(a) and (3)(b) of Theorem 10 hold, which will imply that g is in the range of the transform R.

First of all, the moment condition (3)(a) is obviously weaker than our condition.

To check (3)(b), observe that the Fourier–Bessel transform

$$\widehat{g}(x,\lambda)=\int_0^\infty g(x,t)\,j_{n/2-1}(\lambda t)t^{n-1}\,dt,$$

which is an entire function of λ, expands, according to (14), into the power series with respect to the spectral parameter λ:

$$\widehat{g}(x,\lambda)=\int_0^\infty g(x,t)\,j_{n/2-1}(\lambda t)t^{n-1}\,dt=\sum_{k=0}^\infty C_k\lambda^{2k}M_k(x).$$

If we replace here the functions $M_k(x)$ by their extensions $Q_k(x)$ in the unit ball B, then, due to estimates (31) for Q_k, the extended series uniformly converges in B to a function

$$\psi_\lambda(x):=\sum_0^\infty C_k\lambda^{2k}Q_k(x).$$

This function ψ_λ is an eigenfunction of Laplace operator:

$$\Delta\psi_\lambda=\sum_{k=0}^\infty C_k\lambda^{2k}c_kQ_{k-1}=\sum_{s=0}^\infty C_{s+1}c_{s+1}\lambda^{2(s+1)}Q_s=-\lambda^2\psi_\lambda,$$

due to the relation

$$C_{s+1}c_{s+1}=-C_sc_s.$$

This relation between the coefficients can be easily verified using their explicit values

$$C_k=(-1)^k\frac{\Gamma(p+1)}{2^{2k}k!\,\Gamma(p+k+1)},\qquad c_k=2k(2k+n-2),\qquad p=\frac{n-2}{2}.$$

Therefore, the function $\widehat{g}_\lambda(x)=\widehat{g}(x,\lambda)$ extends to the ball B as an eigenfunction of Laplace operator with the eigenvalue $-\lambda^2$. Let us now apply Stokes formula to ψ_λ and a Dirichlet eigenfunction $\phi=\phi_\lambda$. Taking into account that $\widehat{g}_\lambda=\psi_\lambda$ on S, we obtain

$$\int_S\widehat{g}_\lambda\partial_\nu\phi_\lambda\,dS=\int_B(\psi_\lambda\Delta\phi_\lambda-\phi_\lambda\Delta\psi_\lambda)\,dV=0.$$

This provides the orthogonality condition (3)(b) (formula (28)). Thus, according to Theorem 10, function g belongs to the range of transform R.

7. Proof of Theorem 13

First of all, the necessity part of the proof of Theorem 10 clearly survives in the Sobolev case. Thus, we only need to establish that a function $g \in H^{\mathrm{comp}}_{s+(n-1)/2}(S \times (0,2))$ satisfying the range conditions, does belong to the range of R_S on $H^{\mathrm{comp}}_s(B)$. Here the following stability estimate is essential.

Proposition 21. *For any $\varepsilon > 0$, there exists a constant C_ε such that for any $f \in H^0_s(B_{1-\varepsilon})$ the following estimate holds*:

$$C_\varepsilon^{-1} \| R_S f \|_{H^0_{s+(n-1)/2}(S\times(\varepsilon,2-\varepsilon))} \leqslant \|f\|_{H^0_s(B_{1-\varepsilon})} \leqslant C_\varepsilon \| R_S f \|_{H^0_{s+(n-1)/2}(S\times(\varepsilon,2-\varepsilon))}. \tag{64}$$

Here $B_{1-\varepsilon}$ is the ball of radius $1-\varepsilon$ centered at the origin.

This proposition combines the known injectivity of the operator R_S on compactly supported functions (see, e.g., [6]) with the elliptic estimates obtained recently in [86]. Such estimates can also be derived from the known ellipticity theorem for the pseudo-differential normal operator $R_S^* R_S$ and FIO results of [46]. The ellipticity theorem was obtained in [40] (see also [37,39–41, 89]).

Let us now have a function $g(x,t) \in H^{\mathrm{comp}}_{s+(n-1)/2}(S \times (0,2))$ satisfying any of the range conditions formulated in the theorem. We can choose a positive ε such that $g \in H^0_{s+(n-1)/2}(S \times (2\varepsilon, 2-2\varepsilon))$. Let us extend g to a function $g_1(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n$, radial with respect to y, as follows:

$$g_1(x,y) = g\bigl(x, |y|\bigr).$$

The proof of Theorem 10 and lemmas preceding it show that the range conditions require the following: if one expands g_1 into a series of spherical harmonic terms with respect to x, and then takes the n-dimensional Fourier transform with respect to y of each term, the resulting functions have zeros at certain prescribed locations and of prescribed multiplicities. Let us now consider an even, smooth, compactly supported radial approximation ψ_k of the delta-function in $\mathbb{R}^n_y$, such that $\psi_k \to \delta$ in distributional sense when $k \to \infty$ and that the support of ψ_k shrinks to $\{0\}$ when $k \to \infty$. Then the convolution $g_k(x,y) = \psi_k(y) *_y g_1(x,y)$ for a large k is a C_0^∞, radial with respect to y function. If we now denote $g_k(x,t) = g_k(x,y)$ for $|y| = t$, we get for large k a smooth function with support in $S \times [\varepsilon, 2-\varepsilon]$ and such that $g_k \to g$ in $g \in H^0_{s+(n-1)/2}(S \times (\varepsilon, 2-\varepsilon))$. Since the Fourier transform of the convolution is the product of Fourier transforms, we see that the range conditions (being conditions on zeros of the Fourier transform) survive the convolution. Thus, functions $g_k(x,t)$ satisfy the range conditions.

According to Theorem 10, each of the functions g_k can be represented as $g_k = R_S f_k$ with $f_k \in C_0^\infty(B_{1-\varepsilon})$. Then the Proposition 21 shows existence of a limit $f = \lim_{k\to\infty} f_k$ in $H^0_s(B_{1-\varepsilon})$ such that $g = R_S f$. This finishes the proof.

8. The case of general domains

Formulation of the range conditions (2)(b) and (3)(b) of Theorem 10 do not use explicitly that the set S of centers is a sphere, and hence that the supports of functions under consideration are contained in the ball B. Only the range conditions (4)(b) (the ones involving Bessel functions) explicitly use such rotational invariance. The reader must have noticed that in fact we proved the necessity of conditions (2)(b) and (3)(b) of Theorem 10, as well as the range conditions of Theorem 12, for arbitrary domain D. The only caveat is that, as we have discussed already, the moment conditions (2)(a) and (3)(a) should be formulated in terms of Lemma 4 only, rather than in terms of Lemma 2. In other words, we have proven some necessary range conditions for the spherical mean transforms with centers on the boundary of an arbitrary smooth domain. In order to formulate them, let us introduce a number T such that every sphere centered in $\overline{D}$ and of radius at least T does not intersect the interior of D. We now consider the cylinder $C = D \times [0, T]$ and formulate the conditions

$$G(x, T) = G_t(x, T) = 0. \tag{65}$$

The following theorem was also proven while we were proving Theorem 10.

Theorem 22. *Let $D \subset \mathbb{R}^n$ be a bounded domain with the smooth boundary Γ. Consider the transform*

$$g(x, t) = R_\Gamma f(x, t) = \omega^{-1} \int\limits_{|y|=1} f(x + ty)\, dS(y), \quad x \in \Gamma,\ t \geqslant 0.$$

Then, if $f \in C_0^\infty(D)$ and $g(x, t) = R_\Gamma f$, the following three range conditions hold.

(1) (a) *The moment conditions of Lemma* 4 *are satisfied.*
   (b) *The solution $G(x, t)$ of the interior problem* (3), (5), (65) *in C* (*which always exists for $t > 0$*) *satisfies the condition*

$$\lim_{t \to 0} \int\limits_B \frac{\partial G}{\partial t}(x, t)\phi(x)\, dx = 0$$

   for any eigenfunction $\phi(x)$ of the Dirichlet Laplacian in D.

(2) (a) *The moment conditions of Lemma* 4 *are satisfied.*
   (b) *Let $-\lambda^2$ be an eigenvalue of the Dirichlet Laplacian in D and u_λ be the corresponding eigenfunction solution* (20). *Then the following orthogonality condition is satisfied*:

$$\int\limits_{\Gamma \times [0,T]} g(x, t)\partial_\nu u_\lambda(x, t) t^{n-1}\, dx\, dt = 0. \tag{66}$$

   Here ∂_ν is the exterior normal derivative at the lateral boundary of C.

(3) *The moments*

$$M_k(x) = \int\limits_0^\infty g(x, t) t^{2k+n-1}\, dt$$

extend from $x \in \Gamma$ to $x \in \mathbb{R}^n$ as polynomials $Q_k(x)$ satisfying the recurrence condition (25) *and the growth estimates*

$$\left|Q_k(x)\right| \leqslant M^k, \quad x \in D,$$

for some constant $M > 0$.

Moreover, range conditions (1) *and* (2) *on a function g are equivalent.*

One can ask whether these conditions are sufficient (together with appropriate smoothness and support conditions on g) for g being in the range of the transform R_Γ. The authors plan to address this topic in another publication.

One should also note here that in the case of a general domain D, the conditions (65) do not take into account geometry of the domain and can be specified further. Namely, let us define the following function in D:

$$\rho(x) = \sup_{y \in \partial D} |x - y|, \quad x \in \overline{D}. \tag{67}$$

Then clearly (65) can be replaced by

$$G(x,t) = 0 \quad \text{for } t \geqslant \rho(x). \tag{68}$$

9. Proofs of some lemmas

9.1. Proof of Lemma 1

As it has been mentioned in text, this is a known result, so we provide a quick sketch of the proof here for reader's convenience.

Let us prove the necessity of the conditions first. Evenness is immediate. Let us establish Paley–Wiener estimates. Consider the natural extension of the function $g(t)$ to a radial function $H(y) = g(|y|)$ on $\mathbb{R}^n$. Due to smoothness, evenness, and compactness of support of g, we see that H is a smooth function on $\mathbb{R}^n$ with the support in the ball of radius a. The standard Paley–Wiener theorem now claims that the n-dimensional Fourier transform $\widehat{H}(\xi)$ of H is an entire function on $\mathbb{C}^n$ with Paley–Wiener estimates analogous to (18). Since it is known that $\Phi(\lambda)$ is just the restriction of $\widehat{H}(\xi)$ to the set $\xi = \lambda\theta$, where θ is a unit vector in $\mathbb{R}^n$, we get the required estimate (18).

Now we prove the sufficiency. Consider function $F(x) = \Phi(|x|)$ on $\mathbb{R}^n$. Due to conditions on Φ, function $F(x)$ is smooth everywhere outside the origin and decays with all its derivatives faster than any power of $|x|$. So, if we can establish smoothness at the origin, this will mean that F belongs to the Schwartz class $\mathcal{S}$. Smoothness at the origin, however, immediately follows from the radial nature of F and evenness of Φ.

Thus, there exists a radial function H on $\mathbb{R}^n$ of the Schwartz class, such that its Fourier transform is equal to F. If now we define a function $g(t)$ such that $H(x) = g(|x|)$, then g is the function we need. We, however, need to establish that g has support in $[-a, a]$. This is equivalent to H having its support in the ball $|x| \leqslant a$. Consider the standard Radon transform of H:

$$u(s,\theta) = \mathcal{R}H(s,\theta) := \int\limits_{x\cdot\theta=s} H(x)\,dx, \quad |\theta| = 1.$$

Due to the projection-slice theorem [27,35,43,71,72], the one-dimensional Fourier transform $\widehat{u}(\lambda,\theta)$ of the Radon data $u(s,\theta)$ with respect to the linear variable s coincides (up to a non-zero constant factor) with the Fourier transform of H evaluated at the point $\lambda\theta$, i.e. with $F(\lambda\theta)$. Due to the Paley–Wiener estimates we have for $F(\lambda\theta)$ and standard 1D Paley–Wiener theorem, we conclude that $u(s,\theta)$ vanishes for any θ and any $|s| > a$. Now, since H is of the Schwartz class and its Radon transform vanishes for any $|s| > a$, the "hole theorem" [43,71,72] implies that $H(x) = 0$ for $|x| > a$. This concludes the proof of the lemma.

9.2. A Weyl transform proof of Lemma 14

We provide here a modification of a part of the proof of part (2) of Lemma 14. It is based on the same Weyl transformation from Darboux equation to the wave equation, with the inhomogeneity moved from the boundary conditions to the right-hand side. So, after the transform $G \mapsto U = \mathcal{W}G$, where $\mathcal{W}$ denotes the Weyl transform with respect to t, we, as before, arrive to the proving of evenness with respect to time of the solution of the following problem:

$$\begin{cases} U_{tt} - \Delta_x U = -\partial_t^2 \mathcal{W}(E) & \text{in } D \times [0,T], \\ U(x,t) = 0 & \text{for } x \in \partial D, \\ U(x,T) = U_t(x,T) = 0 & \text{for } x \in D. \end{cases} \tag{69}$$

Here $E(x,t)$ is a smooth function, even with respect to t, and having zero of infinite order at $t = 0$.

We need to establish that $U_t(x,0) = 0$. Now the argument starts to differ somewhat from what we had before.

Let us take any Dirichlet eigenfunction $\phi = \phi_k$ in D. It will be convenient to rewrite formula (10) using integration by parts and taking into account vanishing of the integrand in a neighborhood of ∞:

$$\big\langle U(\cdot,t),\phi\big\rangle = \big\langle (\mathcal{W}G)(\cdot,t),\phi\big\rangle = const \int\limits_t^\infty \big\langle \partial_s G(\cdot,s),\phi\big\rangle \big(s^2-t^2\big)^{(n-1)/2}\,ds,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(D)$.

Differentiating with respect to t yields

$$\big\langle U_t(\cdot,t),\phi\big\rangle = -const\,(n-1)t \int\limits_t^\infty \big\langle \partial_s G(\cdot,s),\phi\big\rangle \big(s^2-t^2\big)^{(n-3)/2}\,ds. \tag{70}$$

We have proven in part (1) that the conditions (a) and (b) imply that

$$\lim_{t\to 0+} \big\langle \partial_t G(\cdot,t),\phi\big\rangle = 0.$$

Therefore, the integral in the right-hand side tends to a finite limit as $t \to 0+$ and, taking into account the presence of the factor t in (70), we get

$$\langle \partial_t U(\cdot, 0), \phi \rangle = \lim_{t\to 0} \langle U_t(\cdot, t), \phi \rangle = 0.$$

Thus, the function $\partial_t U(x, 0)$ is orthogonal to arbitrary Dirichlet eigenfunction $\phi(x)$ and hence

$$\partial_t U(x, 0) = 0, \quad x \in D.$$

Then, due to the wave equation, all derivatives $\partial_t^p U(x, 0)$ of odd orders p vanish. Hence, iterates of the differential operator $\partial/\partial(t^2) = (2t)^{-1}\partial_t$ preserve smoothness of $U(x, t)$ at $t = 0$, as a function with values in $H^s(D)$. Formula (10) implies then that $G = \mathcal{W}^{-1}U$ is continuous and differentiable at $t = 0$. Moreover, $\partial_t G(x, 0) = 0$, since $\partial_t G(\cdot, 0)$ is orthogonal to all Dirichlet eigenfunctions ϕ. This finishes the proof of the sufficiency part (2) of Lemma 14.

9.3. An alternative proof of estimates on coefficients b_l

We provide an alternative growth estimate derivation for the coefficients b_l (see Eq. (51) and considerations after it). We would like to establish Paley–Wiener estimates for the functions $b_l(\lambda)$, $\lambda \in \mathbb{C}$. These estimates can be derived from the identity (50) by averaging over all $0 \leqslant r \leqslant 1$ the point-wise estimates that follow from (50). Namely, let us take the absolute value in both sides in (50) raised to a fixed power $\gamma > 1$ (to be specified later) and integrate with respect to r from 0 to 1. The triangle inequality leads to

$$\left|b_l(\lambda)\right|^\gamma \int_0^1 \left|j_{n/2-1+m}(\lambda r)\right|^\gamma |\lambda r|^{\gamma m}\, dr \leqslant \int_0^1 \int_{\theta\in S} \left|\widehat{G}^m(r, \lambda)\right|^\gamma \left|Y_l^m(\theta)\right|^\gamma dS(\theta)\, dr. \tag{71}$$

Since $G^m(x, t) = 0$ for $t > 2$, the Paley–Wiener estimate

$$\left|\widehat{G}^m(r\theta, \lambda)\right| \leqslant C_N\left(1 + |\lambda|\right)^{-N} e^{2|\mathrm{Im}\,\lambda|}$$

holds uniformly with respect to $r \in [0, 1]$, $\theta \in S$. Then (71) gives:

$$\left|b_l(\lambda)\right|^\gamma D(\lambda) \leqslant const_N\left(1 + |\lambda|\right)^{-\gamma N} e^{2\gamma|\mathrm{Im}\,\lambda|}, \tag{72}$$

where

$$D(\lambda) = \int_0^1 \left|j_{n/2-1+m}(\lambda r)\right|^\gamma |\lambda r|^{\gamma m}\, dr.$$

The integral $D(\lambda)$ satisfies the estimate

$$|\lambda| D(\lambda) \geqslant D_0 > 0. \tag{73}$$

Indeed, change of variable $r = |\lambda|^{-1} z$ in the integral $D(\lambda)$ yields

$$D(\lambda) = |\lambda|^{-1} \int_0^{|\lambda|} \left| z^m j_{n/2-1+m}\left(\frac{\lambda}{|\lambda|} z\right) \right|^\gamma dz.$$

The following estimate of Bessel functions at ∞ is well known:

$$\left| z^m j_{n/2-1+m}(z) \right|^\gamma \leqslant \frac{C}{|z|^{(n-1)/2}}.$$

Thus, if we take $\gamma > 2n/(n-1)$, the integral converges:

$$\int_0^\infty \left| z^m j_{n/2-1+m}(z) \right|^\gamma dz < \infty,$$

and therefore $|\lambda||D(\lambda)|$ tends, as $|\lambda| \to \infty$, to a finite positive constant. This gives us the required estimate (73) from below.

Now substituting (73) in (72) and raising both sides of the inequality to the reciprocal power $1/\gamma$ yields:

$$\left| b_l(\lambda) \right| \leqslant const_N \big(1 + |\lambda|\big)^{-N} e^{2|\operatorname{Im}\lambda|}.$$

Finally, combining this estimate for $b_l(\lambda)$ with Paley–Wiener estimate for $j_{n/2-1+m}(\lambda r)$, we obtain from (48) the needed Paley–Wiener estimate:

$$\left| \widehat{G}^m(r\theta, \lambda) \right| \leqslant const_N \big(1 + |\lambda|\big)^{-N} e^{(r+2)|\operatorname{Im}\lambda|}. \tag{74}$$

10. Final remarks

- The results of Theorems 10–13 easily rescale by change of variables to the ball B of arbitrary radius, which we leave as a simple exercise to the reader.
- It is necessary to note that the range condition (4) of Theorem 10 is an extension of the two-dimensional one in [15]. However, one notices a weaker formulation of the moment conditions than in [15].
  The condition (3) is a reformulation of (4), but, unlike (4), it is suitable for arbitrary domains. Condition (2) is not directly the one of [30], but they are definitely of the same spirit. The authors of [30] work with the wave equation, while we do with Darboux. In fact, except the dimension 3, we work with different (albeit related) transforms.
- The condition (1)(b) in Lemma 14 and thus (2)(b) in Theorems 10 and 13, as well as condition (2) in Theorem 11 can be replaced by much weaker ones on the behavior of $\partial_t G(x,t)$ or $G(x,t)$ at $t = 0$. One only needs to ensure that the function $h(t)$ constructed in the proof has singularities at $t = 0$ that are milder than those of Bessel functions of the second kind, and thus has no singularities at all. For instance, one can request that

$$\int_D \partial_t G(x,t)\phi(x)\, dx = o\big(t^{1-n}\big), \quad t \to 0+.$$

  For the same reasons, the above conditions for $\partial_t G(x,t)$ can be replaced by analogous conditions for $G(x,t)$, with t^{1-n} replaced by t^{2-n} for $n > 2$ and $\log t$ for $n = 2$.

- It is proven in Theorem 11 that, similarly to the results of [30], moment conditions are not needed in odd dimensions. We suspect that one cannot remove the moment conditions in even dimensions, albeit we do not have a convincing argument at this point. One notices, however, that Huygens' principle played significant role in the proof of Theorem 11.
  One can also notice that, as it is shown in [15], moment conditions alone do not suffice for the complete description of the range. However, Theorem 12 shows that a strengthened version of moment conditions does suffice.
- It is instructive to note that we proved in Section 5.1 the following statement:

  > Let $G_+(x,t)$ be the interior solution of the Darboux equation constructed using the orthogonality conditions inside the cylinder $C = B \times \mathbb{R}$ with zero data at $t = 2$. If $G_+(x,0)$ vanishes with all its derivatives at the boundary of B, then it extends to the global solution of the Darboux equation G, with the initial data supported in B.

  This holds in any (not necessarily odd, like in Section 5.1) dimension. Thus, the job of the moment conditions is to ensure smooth vanishing of the interior solution G_+ at the boundary, and as we proved, this is automatic in odd dimensions.
- An interesting interpretation of the range conditions (a) and (b) in Theorem 10 comes from a model presented in [4], where a link is established between the spherical mean Radon transform in $\mathbb{R}^d$ and the planar Radon transform of distributions supported on a paraboloid in $\mathbb{R}^{n+1}$. We plan to discuss this interpretation in detail elsewhere (see also a general approach to such relations in [35,36]).
- A natural question is: why do we need to restrict the support of the function f to the interior of the surface Γ of the centers? Cannot the range of R_Γ be reasonably described for compactly supported functions with supports reaching beyond the surface Γ? As it was explained in [15], this is not to be expected. Briefly, microlocal arguments of the type the ones in [66,92,104] show that the range would not be closed in reasonable spaces (e.g., in Sobolev scale), which is a natural precondition for range descriptions of the kind described in this paper.
- One notices that use of microlocal tools as in [15] was essentially avoided in this text (except of the proof of Theorem 13). What replaced it, is using properties of the solution of Darboux equation instead. Namely, existence of the solution $G(x,t)$ in C, and especially its regular behavior at $t = 0$ did the job. Thus, microlocal analysis was replaced by much simpler PDE tools (simple properties of the wave equation and Fourier transforms).
- Proposition 21 that concerns stability estimates [86], answers a question raised in the remark section of [15].
- The range condition (2) of Theorem 10 also provides a reconstruction procedure: one goes from $g(x,t)$ to the solution $G(x,t)$ of the reverse time initial-boundary value problem for Darboux equation in C, and then sets $f(x) := G(x,0)$. A similar effect was mentioned in [30] concerning their range conditions, where wave equation replaces Darboux.
  Such a procedure is not that abstract. It essentially corresponds to the time reversal reconstruction. Similar consideration lead the authors of [31] to their reconstruction formulas in odd dimensions (which has been extended recently to all dimensions [32,58]) and the authors of [101] to what they called "universal reconstruction formula" in 3D.
  One can notice that the proof of our range theorem also involves implicitly a reconstruction procedure based on eigenfunction expansions, assuming the knowledge of the full spectrum and eigenfunctions of the Dirichlet Laplacian in B. This procedure (which is somewhat sim-

ilar to the one of [79]) would involve at a certain stage division of analytic functions with the denominator having zeros. When the data is in the range of the spherical mean transform, the range theorem guarantees cancellation of zeros. However, such a procedure would be unstable to implement. A better version of an eigenfunction expansion inversion procedure, which does not involve unstable divisions, has been developed recently by L. Kunyansky [58]. The knowledge of the whole spectrum and eigenfunctions is available only in rare cases (besides when the domain is a ball), e.g. for crystallographic domains [18,19].

Acknowledgments

The work of the second author was supported in part by the NSF Grants DMS-9971674, DMS-0002195, and DMS-0604778. The work of the third author was supported in part by the NSF Grants DMS-0200788 and DMS-0456868. The authors thank the NSF for this support. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

The authors thank G. Ambartsoumian, W. Bray, D. Finch, F. Gonzalez, A. Greenleaf, S. Helgason, D. Khavinson, V. Kononenko, L. Kunyansky, V. Palamodov, S. Patch, B. Rubin, K. Trimeche, and B. Vainberg for useful information and discussions. The authors are also grateful to the reviewer for helpful remarks.

The first author would like to thank Texas A&M University and Tufts University for hospitality and support. The third author expresses his appreciation to the Tufts University FRAC and the Gelbart Institute of Bar Ilan University for their support.

References

[1] S. Agmon, A representation theorem for solutions of the Helmholtz equation and resolvent estimates for the Laplacian, in: P. Rabinowitz, E. Zehnder (Eds.), Analysis, et cetera, Academic Press, Boston, MA, 1990, pp. 39–76.

[2] S. Agmon, Representation theorems for solutions of the Helmholtz equation on $\mathbb{R}^n$, in: Differential Operators and Spectral Theory, M.Sh. Birman's 70th Anniversary Collection, in: Amer. Math. Soc. Transl. Ser. 2, vol. 189, Amer. Math. Soc., Providence, RI, 1999, pp. 27–43.

[3] M.L. Agranovsky, Radon transform on polynomial level sets and related problems, in: Israel Math. Conf. Proc., vol. 11, Bar-Ilan Univ., Ramat Gan, 1997, pp. 1–21.

[4] M. Agranovsky, On a problem of injectivity for the Radon transform on a paraboloid, in: Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis, Philadelphia, PA, 1998, in: Contemp. Math., vol. 251, Amer. Math. Soc., Providence, RI, 2000, pp. 1–14.

[5] M.L. Agranovsky, E.K. Narayanan, Injectivity of the spherical mean operator on the conical manifolds of spheres, Siberian Math. J. 45 (4) (2004) 597–605.

[6] M.L. Agranovsky, E.T. Quinto, Injectivity sets for the Radon transform over circles and complete systems of radial functions, J. Funct. Anal. 139 (1996) 383–413.

[7] M.L. Agranovsky, E.T. Quinto, Geometry of stationary sets for the wave equation in $\mathbb{R}^n$: The case of finitely supported initial data, Duke Math. J. 107 (1) (2001) 57–84.

[8] M.L. Agranovsky, E.T. Quinto, Stationary sets for the wave equation in crystallographic domains, Trans. Amer. Math. Soc. 355 (6) (2003) 2439–2451.

[9] M.L. Agranovsky, E.T. Quinto, Remarks on stationary sets for the wave equation, in: Integral Geometry and Tomography, in: Contemp. Math., vol. 405, Amer. Math. Soc., Providence, RI, 2006, pp. 1–11.

[10] M. Agranovsky, C.A. Berenstein, P. Kuchment, Approximation by spherical waves in L^p-spaces, J. Geom. Anal. 6 (3) (1996) 365–383.

[11] M.L. Agranovsky, V.V. Volchkov, L. Zalcman, Conical uniqueness sets for the spherical Radon transform, Bull. London Math. Soc. 31 (4) (1999) 363–372.

[12] N.I. Akhiezer, To the theory of coupled and integral equations, Zap. Kharkov. Mat. Obsch. 25 (1957) 5–31 (in Russian).

[13] N.I. Akhiezer, Lectures on Integral Transforms, Amer. Math. Soc., Providence, RI, 1988.
[14] G. Ambartsoumian, P. Kuchment, On the injectivity of the circular Radon transform arising in thermoacoustic tomography, Inverse Problems 21 (2005) 473–485.
[15] G. Ambartsoumian, P. Kuchment, A range description for the planar circular Radon transform, SIAM J. Math. Anal. 38 (2) (2006) 681–692.
[16] L.-E. Andersson, On the determination of a function from spherical averages, SIAM J. Math. Anal. 19 (1) (1988) 214–232.
[17] L. Asgeirsson, Über eine Mittelwerteigenschaft von Lösungen homogener linearer partieller Differentialgleichungen zweiter Ordnung mit konstanten Koeffizienten, Ann. of Math. 113 (1937) 321–346.
[18] P. Bérard, Spectres et groupes cristallographiques, C. R. Acad. Sci. Paris Sér. A–B 288 (23) (1979) A1059–A1060.
[19] P. Bérard, G. Besson, Spectres et groupes cristallographiques. II. Domaines sphériques, Ann. Inst. Fourier (Grenoble) 30 (3) (1980) 237–248.
[20] E. Bouzaglo-Burov, Inversion of spherical Radon transform, methods and numerical experiments, MS thesis, Bar-Ilan University, 2005, 1–30.
[21] M. Chamberland, D. Siegel, Polynomial solutions to Dirichlet problems, Proc. Amer. Math. Soc. 129 (1) (2001) 211–217.
[22] A. Cormack, E.T. Quinto, A problem in radiotherapy: questions of non-negativity, Internat. J. Imaging Syst. Technology 1 (1989) 120–124.
[23] A. Cormack, E.T. Quinto, The mathematics and physics of radiation dose planning, in: Contemp. Math., vol. 113, Amer. Math. Soc., Providence, RI, 1990, pp. 41–55.
[24] R. Courant, D. Hilbert, Methods of Mathematical Physics, vol. II, Partial Differential Equations, Interscience, New York, 1962.
[25] J. Delsarte, Une extension nouvelle de la théorie des fonctions presque-përiodiques de Bohr, Acta Math. 69 (1938) 259–317.
[26] A. Denisjuk, Integral geometry on the family of semi-spheres, Fract. Calc. Appl. Anal. 2 (1) (1999) 31–46.
[27] L. Ehrenpreis, The Universality of the Radon Transform, Oxford Univ. Press, Oxford, 2003.
[28] L.C. Evans, Partial Differential Equations, Amer. Math. Soc., Providence, RI, 1998.
[29] J.A. Fawcett, Inversion of n-dimensional spherical averages, SIAM J. Appl. Math. 45 (2) (1985) 336–341.
[30] D. Finch, Rakesh, The range of the spherical mean value operator for functions supported in a ball, Inverse Problems 22 (2006) 923–938.
[31] D. Finch, Rakesh, S. Patch, Determining a function from its mean values over a family of spheres, SIAM J. Math. Anal. 35 (5) (2004) 1213–1240.
[32] D. Finch, M. Haltmeier, Rakesh, Inversion of spherical means and the wave equation in even dimensions, preprint, 2006.
[33] I. Gelfand, M. Graev, N. Vilenkin, Generalized Functions, vol. 5: Integral Geometry and Representation Theory, Academic Press, New York, 1965.
[34] I. Gelfand, S. Gindikin, M. Graev, Integral geometry in affine and projective spaces, J. Soviet Math. 18 (1980) 39–167.
[35] I. Gelfand, S. Gindikin, M. Graev, Selected Topics in Integral Geometry, Transl. Math. Monogr., vol. 220, Amer. Math. Soc., Providence, RI, 2003.
[36] S. Gindikin, Integral geometry on real quadrics, in: Lie Groups and Lie Algebras: E.B. Dynkin's Seminar, in: Amer. Math. Soc. Transl. Ser. 2, vol. 169, Amer. Math. Soc., Providence, RI, 1995, pp. 23–31.
[37] A. Greenleaf, G. Uhlmann, Microlocal techniques in integral geometry, in: Integral Geometry and Tomography, Arcata, CA, 1989, in: Contemp. Math., vol. 113, Amer. Math. Soc., Providence, RI, 1990, pp. 121–135.
[38] J.L. Griffith, On Weber transforms, J. Proc. Roy. Soc. New South Wales 89 (1955) 109–115.
[39] V. Guillemin, On some results of Gelfand in integral geometry, Proc. Sympos. Pure Math. 43 (1985) 149–155.
[40] V. Guillemin, S. Sternberg, Geometric Asymptotics, Amer. Math. Soc., Providence, RI, 1977.
[41] V. Guillemin, S. Sterenberg, Some problems of integral geometry and some related problems in microlocal analysis, Amer. J. Math. 101 (4) (1979) 915–955.
[42] L. Hansen, H.S. Shapiro, Functional equations and harmonic extensions, Complex Variables 24 (1994) 121–129.
[43] S. Helgason, The Radon Transform, Birkhäuser, Basel, 1980.
[44] S. Helgason, Groups and Geometric Analysis, Amer. Math. Soc., Providence, RI, 2000.
[45] A. Hertle, The identification problem for the constantly attenuated Radon transform, Math. Z. 197 (1988) 13–19.
[46] L. Hörmander, Fourier integral operators I, Acta Math. 127 (1–2) (1971) 79–183.
[47] F. John, Plane Waves and Spherical Means, Applied to Partial Differential Equations, Dover, New York, 1971.
[48] D. Khavinson, Holomorphic Partial Differential Equations and Classical Potential Theory, Universidad de La Laguna Press, 1996.

[49] D. Khavinson, H.S. Shapiro, Dirichlet problem when the data is an entire function, Bull. London Math. Soc. 24 (1992) 456–468.

[50] I.A. Kipriyanov, Singular Elliptic Boundary Value Problems, Nauka, Moscow, 1997 (in Russian).

[51] R.A. Kruger, P. Liu, Y.R. Fang, C.R. Appledorn, Photoacoustic ultrasound (PAUS) reconstruction tomography, Med. Phys. 22 (1995) 1605–1609.

[52] P. Kuchment, On positivity problems for the Radon transform and some related transforms, in: Contemp. Math., vol. 140, Amer. Math. Soc., Providence, RI, 1993, pp. 87–95.

[53] P. Kuchment, Generalized transforms of radon type and their applications, in: G. Olafsson, E.T. Quinto (Eds.), The Radon Transform, Inverse Problems, and Tomography. American Mathematical Society Short Course, January 3–4, 2005, Atlanta, GA, in: Proc. Sympos. Appl. Math., vol. 63, Amer. Math. Soc., Providence, RI, 2006, pp. 67–91.

[54] P. Kuchment, L. Kunyansky, Mathematics of thermoacoustic and photoacoustic tomography, preprint arXiv: 0704.0286v1 [math.AP], submitted for publication.

[55] P. Kuchment, S. Lvin, Paley–Wiener theorem for exponential Radon transform, Acta Appl. Math. 18 (1990) 251–260.

[56] P. Kuchment, S. Lvin, The range of the exponential Radon transform, Soviet Math. Dokl. 42 (1) (1991) 183–184.

[57] P. Kuchment, E.T. Quinto, Some Problems of Integral Geometry Arising in Tomography, Oxford Univ. Press, Oxford, 2003, Chapter XI.

[58] L. Kunyansky, Explicit inversion formulas for the spherical mean Radon transform, Inverse Problems, in press.

[59] L. Kunyansky, A series solution and a fast algorithm for the inversion of the spherical mean Radon transform, arXiv: math.AP/0701236.

[60] B.M. Levitan, Expansion in Fourier series and integrals with Bessel functions, Uspekhi Mat. Nauk (N.S.) 6 (2(42)) (1951) 102–143 (in Russian).

[61] B.M. Levitan, Generalized Translation Operators and Some of Their Applications, Israel Progr. Sci. Transl., Davey, Jerusalem, 1964 (translation of the 1962 Russian edition).

[62] V.Ya. Lin, A. Pinkus, Fundamentality of ridge functions, J. Approx. Theory 75 (1993) 295–311.

[63] V.Ya. Lin, A. Pinkus, Approximation of multivariable functions, in: H.P. Dikshit, C.A. Micchelli (Eds.), Advances in Computational Mathematics, World Sci. Publ., Singapore, 1994, pp. 1–9.

[64] J.-L. Lions, Opérateurs de Delsarte et problèmes mixtes, Bull. Soc. Math. France 84 (1950) 9–95.

[65] J.-L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. 1, Springer, Berlin, 1972.

[66] A.K. Louis, E.T. Quinto, Local tomographic methods in Sonar, in: Surveys on Solution Methods for Inverse Problems, Springer, Vienna, 2000, pp. 147–154.

[67] S. Lvin, Data correction and restoration in emission tomography, in: E.T. Quinto, M. Cheney, P. Kuchment (Eds.), Tomography, Impedance Imaging, and Integral Geometry, in: Lectures in Appl. Math., vol. 30, Amer. Math. Soc., Providence, RI, 1994, pp. 149–155.

[68] W. Magnus, F. Oberhettinger, Raj Pal Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer, Berlin, 1966.

[69] C. Mennesier, F. Noo, R. Clackdoyle, G. Bal, L. Desbat, Attenuation correction in SPECT using consistency conditions for the exponential ray transform, Phys. Med. Biol. 44 (1999) 2483–2510.

[70] F. Natterer, Exploiting the range of Radon transform in tomography, in: P. Deuflhard, E. Hairer (Eds.), Numerical Treatment of Inverse Problems in Differential and Integral Equations, Birkhäuser, Basel, 1983.

[71] F. Natterer, The Mathematics of Computerized Tomography, Wiley, New York, 1986.

[72] F. Natterer, F. Wübbeling, Mathematical Methods in Image Reconstruction, Monogr. Math. Model. Comput., vol. 5, SIAM, Philadelphia, PA, 2001.

[73] M.M. Nessibi, L.T. Rachdi, K. Trimeche, Ranges and inversion formulas for spherical mean operator and its dual, J. Math. Anal. Appl. 196 (3) (1995) 861–884.

[74] M.M. Nessibi, L.T. Rachdi, K. Trimeche, Ranges and inversion formulas for spherical mean operator and its dual, C. R. Math. Rep. Acad. Sci. Canada 17 (1) (1995) 17–21.

[75] S. Nilsson, Application of fast backprojection techniques for some inverse problems of integral geometry, Linkoeping Studies in Science and Technology, Dissertation 499, Department of Mathematics, Linkoeping University, Linkoeping, Sweden, 1997.

[76] C.J. Nolan, M. Cheney, Synthetic aperture inversion, Inverse Problems 18 (2002) 221–235.

[77] F. Noo, J.-M. Wagner, Image reconstruction in 2D SPECT with 180° acquisition, Inverse Problems 17 (2001) 1357–1371.

[78] F. Noo, R. Clackdoyle, J.-M. Wagner, Inversion of the 3D exponential X-ray transform for a half equatorial band and other semi-circular geometries, Phys. Med. Biol. 47 (2002) 2727–2735.

[79] S.J. Norton, Reconstruction of a two-dimensional reflecting medium over a circular domain: Exact solution, J. Acoust. Soc. Am. 67 (1980) 1266–1273.
[80] S.J. Norton, M. Linzer, Ultrasonic reflectivity imaging in three dimensions: exact inverse scattering solutions for plane, cylindrical, and spherical apertures, IEEE Trans. Biomed. Engrg. 28 (1981) 200–202.
[81] R. Novikov, On the range characterization for the two-dimensional attenuated X-ray transform, Inverse Problems 18 (2002) 677–700.
[82] G. Olafsson, E.T. Quinto (Eds.), The Radon Transform, Inverse Problems, and Tomography. American Mathematical Society Short Course, January 3–4, 2005, Atlanta, GA, Proc. Sympos. Appl. Math., vol. 63, Amer. Math. Soc., Providence, RI, 2006.
[83] V.P. Palamodov, Reconstruction from limited data of arc means, J. Fourier Anal. Appl. 6 (1) (2000) 25–42.
[84] V.P. Palamodov, Reconstructive Integral Geometry, Birkhäuser, Basel, 2004.
[85] V.P. Palamodov, Characteristic problems for the spherical mean transform, in: Complex Analysis and Dynamical Systems II, in: Contemp. Math., vol. 382, Amer. Math. Soc., Providence, RI, 2005, pp. 321–330.
[86] V.P. Palamodov, Remarks on the general Funk transform, preprint, Tel Aviv University, August 2006.
[87] S.K. Patch, Thermoacoustic tomography—consistency conditions and the partial scan problem, Phys. Med. Biol. 49 (2004) 1–11.
[88] I. Ponomarev, Correction of emission tomography data. Effects of detector displacement and non-constant sensitivity, Inverse Problems 10 (1995) 1–8.
[89] E.T. Quinto, The dependence of the generalized Radon transform on defining measures, Trans. Amer. Math. Soc. 257 (1980) 331–346.
[90] E.T. Quinto, Null spaces and ranges for the classical and spherical Radon transforms, J. Math. Anal. Appl. 90 (2) (1982) 408–420.
[91] E.T. Quinto, The invertibility of rotation invariant Radon transforms, J. Math. Anal. Appl. 91 (1983) 510–522.
[92] E.T. Quinto, Singularities of the X-ray transform and limited data tomography in $\mathbf{R}^2$ and $\mathbf{R}^3$, SIAM J. Math. Anal. 24 (1993) 1215–1225.
[93] E.T. Quinto, An introduction to X-ray tomography and Radon transforms, in: G. Olafsson, E.T. Quinto (Eds.), The Radon Transform, Inverse Problems, and Tomography. American Mathematical Society Short Course January 3–4, 2005, Atlanta, GA, in: Proc. Sympos. Appl. Math., vol. 63, Amer. Math. Soc., Providence, RI, 2006, pp. 1–23.
[94] E.T. Quinto, M. Cheney, P. Kuchment (Eds.), Tomography, Impedance Imaging, and Integral Geometry, Lectures in Appl. Math., vol. 30, Amer. Math. Soc., Providence, RI, 1994.
[95] D. Solmon, Two inverse problems for the exponential Radon transform, in: P.S. Sabatier (Ed.), Inverse Problems in Action, Springer, Berlin, 1990, pp. 46–53.
[96] D. Solmon, The identification problem for the exponential Radon transform, Math. Methods Appl. Sci. 18 (1995) 687–695.
[97] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, NJ, 1971.
[98] K. Trimeche, Generalized Harmonic Analysis and Wavelet Packets, Gordon & Breach, Amsterdam, 2001.
[99] N.Ja. Vilenkin, Special Functions and the Theory of Group Representations, Amer. Math. Soc., Providence, RI, 1968.
[100] M. Xu, L.-H.V. Wang, Time-domain reconstruction for thermoacoustic tomography in a spherical geometry, IEEE Trans. Med. Imag. 21 (2002) 814–822.
[101] M. Xu, L.-H.V. Wang, Universal back-projection algorithm for photoacoustic computed tomography, Phys. Rev. E 71 (2005) 016706.
[102] Y. Xu, D. Feng, L.-H.V. Wang, Exact frequency-domain reconstruction for thermoacoustic tomography: I. Planar geometry, IEEE Trans. Med. Imag. 21 (2002) 823–828.
[103] Y. Xu, M. Xu, L.-H.V. Wang, Exact frequency-domain reconstruction for thermoacoustic tomography: II. Cylindrical geometry, IEEE Trans. Med. Imag. 21 (2002) 829–833.
[104] Y. Xu, L. Wang, G. Ambartsoumian, P. Kuchment, Reconstructions in limited view thermoacoustic tomography, Med. Phys. 31 (4) (2004) 724–733.