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Large Time Behaviour of Solutions of the Porous Medium Equation with Convection

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1. INTRODUCTION

In this paper we present some results about the large time behaviour of solutions of the equation

$$u_t + (u^\lambda)_x = (u^m)_{xx}, \qquad u \geqslant 0, \tag{1.1}$$

where $m > 1$ and $\lambda > 0$ are constants and where the subscripts t and x denote differentiation with respect to the variable t (time) and x (space coordinate), respectively.

Equation (1.1) arises in a model describing the unsaturated flow of a fluid through a homogeneous porous column (x denotes the vertical distance) under influence of capillary pressure and gravity. The unknown u denotes the moisture content in the porous material. When $u > 0$ both liquid and gas (air) are present, and when $u = 0$ no liquid is present and the material is dry. The case $\lambda > 1$ describes a wetting process and the case $0 < \lambda < 1$ describes a drying process in the column; e.g., see Diaz and Kersner [2], and further references are given there.

Throughout this paper, T denotes a fixed positive number which eventually may tend to infinity. We consider two problems for Eq. (1.1).

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THE CAUCHY PROBLEM.

$$\mathrm{C}\begin{cases} u_t + (u^\lambda)_x = (u^m)_{xx} & \text{in } S_T \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{R}, \end{cases}$$

where $S_T = \{(x, t) : x \in \mathbb{R},\ 0 < t \leqslant T\}$ and where $u_0 : \mathbb{R} \to \bar{\mathbb{R}}^+$.

THE CAUCHY–DIRICHLET PROBLEM.

$$\mathrm{CD}\begin{cases} u_t + (u^\lambda)_x = (u^m)_{xx} & \text{in } H_T \\ u(0, \cdot) = u^0 & \text{on } (0, T] \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \mathbb{R}^+, \end{cases}$$

where $H_T = \{(x, t) : x \in \mathbb{R}^+,\ 0 < t \leqslant T\}$, $u^0 \in \bar{\mathbb{R}}^+$, and $u_0 : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ such that $u_0(0) = u^0$.

Since $m > 1$, Eq. (1.1) degenerates at points where $u = 0$. Therefore we introduce the concept of weak solutions.

DEFINITION 1.1. A function $u : \bar{S}_T \to \mathbb{R}$ is called a weak solution of Problem C if it satisfies

(i) $u \in C(\bar{S}_T)$, uniformly bounded and nonnegative in S_T;

(ii) u^m has a bounded generalized derivative with respect to x in S_T;

(iii) $\int_{S_T}\int \{u\zeta_t + (u^\lambda - (u^m)_x\, \zeta_x\}\, dx\, dt = \int_{\mathbb{R}} u_0(x)\, \zeta(x, 0)\, dx$, for all $\zeta \in C^1(\bar{S}_T)$ which vanish for large $|x|$ and $t = T$.

DEFINITION 1.2. A function $u : \bar{H}_T \to \mathbb{R}$ is called a weak solution of Problem CD if it satisfies

(i) $u \in C(\bar{H}_T)$, uniformly bounded and nonnegative in H_T;

(ii) u^m has a bounded generalized derivative with respect to x in H_T;

(iii) $u(0, \cdot) = u^0$ on $[0, T]$;

(iv) $\int_{H_T}\int \{u\zeta_t + (u^\lambda - u^m_x)\, \zeta_x\}\, dx\, dt = \int_{\mathbb{R}^+} u_0(x)\, \zeta(x, 0)\, dx$, for all $\zeta \in C^1(\bar{H}_T)$ which vanish for $x = 0$, for large $|x|$, and for $t = T$.

Gilding and Peletier [10] proved existence for Problem C provided u_0 is nonnegative and bounded on $\mathbb{R}$ and u_0^m is uniformly Lipschitz continuous on $\mathbb{R}$. Existence for Problem CD was proven by Gilding [6; 8, Theorem 9] under the previous assumptions on u_0 ($\mathbb{R}$ replaced by $\mathbb{R}^+$) and $u_0(0) = u^0$. The proofs are based on the early work of Oleinik, Kalashnikov, and Chzhou [14]. Uniqueness was proven in [10, 6] for $m \geqslant 1$ and $\lambda \geqslant \frac{1}{2}(m+1)$. Later this was improved by Diaz and Kersner [2], who showed uniqueness for $m \geqslant 1$ and $\lambda > 0$. In their proof they first introduced the notion of generalized solutions. Then they showed uniqueness of such solutions and the equivalence between generalized and weak solutions. Below we given an example of a weak solution of (1.1) in the form of a travelling wave which satisfies

$$\lim_{x \to -\infty} u(x, t) = u^- \qquad \text{and} \qquad \lim_{x \to \infty} u(x, t) = u^+, \tag{1.2}$$

where u^- and u^+ are given nonnegative constants.

DEFINITION 1.3. A function u which satisfies (1.2) is called a travelling wave solution of (1.1) if

$$u(x, t) = f(x - kt) \qquad \text{for} \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+,$$

where $k \in \mathbb{R}$ and $f: \mathbb{R} \to [0, \infty)$ satisfy

(i) $(f^m)'$, f, and f^λ are absolutely continuous;

(ii) $-kf' + (f^\lambda)' = (f^m)''$ a.e. on $\mathbb{R}$;

(iii) $f(-\infty) = u^-$ and $f(+\infty) = u^+$.

Following van Duijn and de Graaf [4] we obtain:

If $\lambda > 1$ and $u^- > u^+ \geqslant 0$, then there exists a unique (modulo translations) function f, satisfying (i)–(iii), which is strictly decreasing and C^∞ at points where $f > 0$. If $u^+ = 0$, then there exists a number s_0 such that $f(s) > 0$ for $s < s_0$ and $f(s) = 0$ for $s \geqslant s_0$. Furthermore,

$$\lim_{s \uparrow s_0} (f^{m-1})'(s) = -\frac{m-1}{m} k.$$

If $\lambda \in (0, 1)$ and $0 \leqslant u^- < u^+$, then there exists a unique function f, satisfying (i)–(iii), which is strictly increasing and C^∞ at points where $f > 0$. If $u^- = 0$, then there exists a number s_1 such that $f(s) > 0$ for $s > s_1$ and $f(s) = 0$ for $s \leqslant s_1$. Furthermore

$$\lim_{s \downarrow s_1} (f^{m-\lambda})'(s) = +\frac{m-\lambda}{m} k.$$

In both cases the speed of the travelling wave satisfies the Rankine–Hugoniot condition

$$k = \frac{(u^+)^\lambda - (u^-)^\lambda}{u^+ - u^-}.$$

It is easy to check that a travelling wave is indeed a weak solution. Thus if $u(x, t) = f(x - kt)$, then u is the unique weak solution corresponding to the initial function $u_0 = f$. When $u > 0$ it is C^∞ and it satisfies the equation in a classical sense. In the degenerate case $u^+ = 0$ or $u^- = 0$, however, the solution vanishes identically in part of the (x, t) plane. Across the interfaces $x = kt + s_i$ $(i = 0, 1)$, the travelling wave satisfies

$$u(\cdot, t) \in C^\nu(\mathbb{R}) \qquad \text{for all} \quad t \geqslant 0,$$

where $\nu = \min\{1, 1/\beta\}$ and $\beta = \max\{(m-1), (m-\lambda)\}$. This is precisely the x-regularity for weak solutions, as obtained in [2]. Properties of interfaces of weak solutions were studied by Gilding [7, 9].

Results about the large time behaviour of solutions of (1.1) are only known when $\lambda > 1$. Il'in and Oleinik [11] proved convergence of classical solutions of Problem C (with $m = 1$) towards travelling waves when $u^- > u^+ > 0$ and they gave an exponential rate of convergence. Similar results were obtained by Khusnytdinova [12] for the Cauchy–Dirichlet problem when $m \geqslant 1$. More recently, Osher and Ralston [15] obtained L^1-stability of travelling waves in the degenerate case $u^+ = 0$. Their result applied to Problem C gives

THEOREM A. *Assume $\lambda > 1$ and $u^- > 0$. Let $u_0 : \mathbb{R} \to [0, u^-]$ and u_0^m uniformly Lipschitz continuous such that $u^- - u_0 \in L^1(\mathbb{R}^-)$ and $u_0 \in L^1(\mathbb{R}^+)$. If $a \in \mathbb{R}$ is chosen such that*

$$\int_{\mathbb{R}} \{u_0(x) - f(x-a)\}\, dx = 0,$$

then

$$\lim_{t \to \infty} \|u(\cdot, t) - f_a\|_{L^1(\mathbb{R})} = 0,$$

where u is the weak solution of Problem C *and f_a is the translated travelling wave $(f_a(x - kt) = f(x - a - kt))$.*

The main purpose of this paper is to study the large time behaviour of solutions of Problem C and Problem CD in the non-travelling wave cases

$$\lambda > 1 \qquad \text{and} \qquad 0 \leqslant u^- \ (\text{or } u^0) < u^+, \tag{1.3}$$

and

$$\lambda \in (0, 1) \qquad \text{and} \qquad u^- \ (\text{or } u^0) > u^+ \geqslant 0. \tag{1.4}$$

For these cases we prove the convergence of weak solutions of Problem C towards a rarefaction wave u^* of the first order problem

$$\mathrm{C}^\infty \begin{cases} u_t + (u^\lambda)_x = 0 & \text{in } \mathbb{R} \times \mathbb{R}^+, \\ u(-\infty, \cdot) = u^-, \ u(+\infty, \cdot) = u^+ & \text{on } \mathbb{R}^+. \end{cases} \tag{1.5a}$$

For solutions of Problem CD we prove convergence towards the rarefaction wave u^* of the problem

$$\mathrm{CD}^\infty \begin{cases} u_t + (u^\lambda)_x = 0 & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^+ \\ u(0, \cdot) = u^0, \ u(\infty, \cdot) = u^+ & \text{on } \mathbb{R}^+. \end{cases} \tag{1.5b}$$

This function u^*, for (1.5b) say, is given by

$$u^*(x, t) = w^*(\eta) \qquad \text{with} \quad \eta = \frac{x}{t+1}, \tag{1.6}$$

and

$$w^*(\eta) = \begin{cases} u^0, & 0 \leqslant \eta \leqslant \lambda(u^0)^{\lambda-1} \\ (\eta/\lambda)^{1/(\lambda-1)}, & \lambda(u^0)^{\lambda-1} < \eta < \lambda(u^+)^{\lambda-1} \\ u^+, & \lambda(u^+)^{\lambda-1} \leqslant \eta < \infty \end{cases} \tag{1.7}$$

or a suitably redefined version when $u^+ = 0$ and $\lambda \in (0, 1)$. In the definition of η in (1.6) we take $t+1$ instead of t in order to avoid discontinuities near $x = 0$ at $t = 0$.

In Section 2 we prove in detail the convergence for the Cauchy–Dirichlet problem in the degenerate case $u^0 = 0$ with $\lambda > 1$. The result is

THEOREM B. *Assume* $\lambda > 1$. *Let* $u_0: \bar{\mathbb{R}}^+ \to [0, u^+]$, $u_0(0) = 0$, u_0^m *uniformly Lipschitz continuous on* $\mathbb{R}^+$, *and* $u_0 - u^+ = 0(x^{-\alpha})$ *as* $x \to \infty$ *for some* $\alpha > 1$.

Let u be the weak solution of Problem CD. *Then*

$$\|u^\mu(\cdot, t) - u^{*\mu}(\cdot, t)\|_{L^\infty(\mathbb{R}^+)} \leqslant C(1 + \log(t+1))^{1/2}(t+1)^{-1/2}$$

for all $t \geqslant 0$. *Here* $\mu = \max\{m, \lambda - 1\}$, u^* *is given by* (1.6), (1.7), *and C is a positive constant depending only on the data of the problem.*

In Section 3, similar convergence results are given for the Cauchy–Dirichlet problem when $\lambda \in (0, 1)$ and for the Cauchy problem when $\lambda > 1$ and $\lambda \in (0, 1)$.

The method of proof used in Section 2 is based on ideas developed in [4], where the large time behaviour of solutions of the equation $(u + u^p)_t + u_x = u_{xx}$ $(p \in (0, 1))$ was considered.

Remark 1.4. Let g: $\mathbb{R} \to \mathbb{R}$ be given by

$$g(\eta) = \begin{cases} 0, & -\infty < \eta \leqslant 0 \\ (\eta/\lambda)^{1/(\lambda-1)}, & 0 < \eta < \lambda(u^+)^{\lambda-1} \\ u^+, & \lambda(u^+)^{\lambda-1} \leqslant \eta < \infty. \end{cases}$$

Then for any $a \geqslant 0$, the function

$$u_a^*(x, t) := g\left(\frac{x-a}{t+1}\right) \qquad \text{with} \quad (x, t) \in \bar{\mathbb{R}}^+ \times \bar{\mathbb{R}}^+$$

is a solution of Problem CD$^\infty$. These solutions satisfy for every $a_1, a_2 \geqslant 0$

$$\|u_{a_1}^{*\mu}(\cdot, t) - u_{a_2}^{*\mu}(\cdot, t)\|_{L^\infty(\mathbb{R}^+)} \leqslant C(t+1)^{-1} \qquad \text{for} \quad t \geqslant 0,$$

where C is a constant depending only on a_1, a_2, and μ.

Thus Theorem B does not tell us to which solution of Problem CD$^\infty$ the function u converges. For convenience we take the one with $a = 0$.

Remark 1.5. It was pointed out to us by D. Hilhorst that the result of Osher and Ralston (Theorem A) can be extended to the case $0 < \lambda < 1$ and $0 \leqslant u^- < u^+$ by a small modification of their proof.

2. The Cauchy–Dirichlet Problem: Case $\lambda > 1$

Without loss of generality we take $u^+ = \lim_{x \to \infty} u_0(x) = 1$. Throughout this section we assume with respect to the initial function u_0:

H1: $u_0 : [0, \infty) \to [0, 1]$, u_0^m is uniformly Lipschitz continuous on $\mathbb{R}^+$, and $u_0(0) = 0$.

H2: $1 - u_0 = O(x^{-\alpha})$ as $x \to \infty$ for some $\alpha > 1$.

Because the expected asymptotic profile is a similarity solution with x and t combined as in (1.6), it appears convenient to introduce the new independent variables

$$\eta = x/(t+1), \qquad \tau = \log(t+1).$$

Then u is a weak solution of Problem CD if and only if $w(\eta, \tau) := u(x, t)$ is a weak solution of the transformed problem (cf. [4])

$$\text{P}\begin{cases} w_\tau + (\lambda w^{\lambda-1} - \eta)\, w_\eta = e^{-\tau}(w^m)_{\eta\eta} & \text{in } H_{T'} \\ w(0, \tau) = 0 & \text{on } (0, T') \\ w(\eta, 0) = u_0(\eta) & \text{on } \mathbb{R}^+, \end{cases}$$

where $H_{T'} = \mathbb{R}^+ \times (0, T')$ with $T' = \log(T+1)$.

Below we show that $w(\cdot, \tau)$ converges as $\tau \to \infty$ to the function w^*, given by (1.7) with $u^0 = 0$ and $u^+ = 1$, which is the unique solution of the reduced problem (cf. [4])

$$\text{P}_\infty \begin{cases} (\lambda w^{\lambda-1} - \eta)\, w_\eta = 0, & \eta > 0 \\ w(0) = 0;\ w(\infty) = 1. \end{cases}$$

The main result of this section is

THEOREM 2.1. *Let w be the weak solution of Problem* P *and let w^* be given by* (1.7) *with $u^0 = 0$ and $u^+ = 1$. Then*

$$\|w^\mu(\cdot, \tau) - w^{*\mu}\|_{L^\infty} \leqslant (A_1 + A_2\tau)^{1/2} e^{-\tau/2}$$

for all $\tau \in [0, T']$. Here A_1 and A_2 are positive constants independent of T', and $\mu = \max\{m, \lambda - 1\}$.

Remark 2.2. When returning to the original variables x and t, Theorem B follows immediately.

Remark 2.3. Without loss of generality we may assume that u_0 is nondecreasing on $\mathbb{R}^+$. For a given u_0 one can find two nondecreasing functions f_1 and f_2, both satisfying $H_{1,2}$, such that $f_1 \leqslant u_0 \leqslant f_2$ on $\mathbb{R}^+$. Since a comparison principle holds for Problem CD, it is clear that if Theorem 2.1 holds for initial functions f_1 and f_2, then it also holds for the initial function u_0.

To prove the theorem we consider the cases $m - \lambda + 1 \geqslant 0$ and $m - \lambda + 1 < 0$ separately.

Proof of Theorem 2.1: *The Case $m - \lambda + 1 \geqslant 0$*

Let

$$v := w^m \qquad \text{on } \bar{H}_{T'}.$$

Then v is the pointwise limit of the sequence of functions $\{v_n\}_{n \in \mathbb{N}}$, which satisfy for each $n \in \mathbb{N}$ the problem

$$\tilde{\mathrm{P}}_n \begin{cases} A(v) := v_\tau + (\lambda v^{(\lambda - 1)/m} - \eta) v_\eta - e^{-\tau} m v^{(m-1)/m} v_{\eta\eta} = 0 & \text{in } H_{T'}, \\ v(0, \cdot) = 1/n & \text{on } (0, T'], \\ v(\cdot, 0) = v_{0n} & \text{on } \mathbb{R}^+. \end{cases} \tag{2.1}$$

Here the functions v_{0n} are chosen such that

(i) $v_{0n} \in C^\infty[0, \infty)$ and $0 \leqslant v'_{0n} \leqslant L$, where L is the Lipschitz constant from H1;

(ii) $v_{0n}(0) = 1/n$ and $v_{0n}(\infty) = 1$;

(iii) $v_{0n} \to u_0^m$ as $n \to \infty$, uniformly on $\bar{\mathbb{R}}^+$.

For each $n \in \mathbb{N}$, the unique classical solution v_n of Problem $\tilde{\mathrm{P}}_n$ satisfies $1/n \leqslant v_n \leqslant 1$ in $\bar{H}_{T'}$. This follows directly from the observation that $w_n := v_n^{1/m}$ is the transformed of a function u_n which is the solution of Problem CD with $u^- = (1/n)^{1/m}$ and $u_0 = v_{0n}^{1/m}$; cf. [4]. In fact we have

LEMMA 2.4. *For all $n \in \mathbb{N}$ and for all $(\eta, \tau) \in \bar{H}_{T'}$*

$$\text{(a)} \quad \frac{1}{n} \leqslant v_n(\eta, \tau) \leqslant \min\left\{\frac{1}{n} + L_0\eta, 1\right\}, \qquad \text{where} \quad L_0 = \max\left\{L, \frac{1}{\lambda}\right\};$$

$$\text{(b)} \quad 0 \leqslant v_{n\eta}(\eta, \tau) \leqslant \max\left\{L_0, \frac{m}{\lambda(\lambda-1)}\right\} =: L_1.$$

Proof. Part (a) follows from a standard barrier function argument applied to Eq. (2.1) in the domain $\{(\eta, \tau) \mid 0 \leqslant \eta \leqslant 1/L_0, 0 \leqslant \tau \leqslant T'\}$. Part (b) follows from a comparison argument (Theorem 1 of Cosner [1]) applied to the equation for $v_{n\eta}$ in the domain $H_{T'}$. Both results only hold if $m - \lambda + 1 \geqslant 0$. ∎

The next result is about the behaviour for large η of the classical solutions v_n.

LEMMA 2.5. *There exist constants $\gamma > 0$, $1 < k < \alpha$, and $\eta_1 > 1$ such that the function $s(\eta)$, given by*

$$s(\eta) = \begin{cases} 0, & 0 < \eta \leqslant \eta_1, \\ \max\{0, 1 - \gamma(\eta - \eta_1)^{-k}\}, & n > \eta_1, \end{cases}$$

satisfies

$$s(\cdot) \leqslant v_n(\cdot, \tau) \qquad \text{on } \bar{\mathbb{R}}^+$$

for all $n \in \mathbb{N}$ and for all $\tau \in [0, T']$.

Proof. A similar result was also proven in [4]. We therefore omit the details here. ∎

Now define

$$v^* = w^{*m} \qquad \text{on } [0, \infty)$$

and let

$$v_n^* = \max\{1/n, v^*\} \qquad \text{for} \quad n \in \mathbb{N}.$$

Next we show that $v_n(\cdot, \tau)$ converges to v_n^* as $\tau \to \infty$, uniformly in $n \in \mathbb{N}$, and we give an L^1-estimate for the rate of convergence.

First consider for each $n \in \mathbb{N}$ the auxiliary problem

$$\mathrm{A}_1 \begin{cases} A(v) = 0 & \text{in } H_{T'}^{\lambda}, \\ v(0, \cdot) = 1/n, \; v(\lambda, \cdot) = 1 & \text{on } (0, T'], \\ v(\cdot, 0) = v_{0n}^+ & \text{on } (0, \lambda), \end{cases}$$

where $H^{\lambda}_{T'} = (0, \lambda) \times (0, T']$, $v^{+}_{0n} = \max\{v_{0n}, v^{*}_{n}\}$ and where the operator A is defined in (2.1).

This problem has a unique solution $v^{+}_{n} \in C^{\infty}(H^{\lambda}_{T'}) \cap C(\bar{H}^{\lambda}_{T'})$ for which $v^{+}_{n\eta}(\eta, \tau)$ exists up to $\eta = 0$ and $\eta = \lambda$ for all $\tau \in (0, T']$; see Ladyzhenskaja *et al.* [13]. Using the convexity of v^{*} (implied by $m - \lambda + 1 \geqslant 0$) and a comparison argument, it follows that

$$v^{+}_{n} \geqslant \max\{v^{*}_{n}, v_{n}\} \qquad \text{in } \bar{H}^{\lambda}_{T'}, \tag{2.2}$$

for all $n \in \mathbb{N}$.

Next define for each $n \in \mathbb{N}$

$$\bar{v}_{n}(\eta, \tau) = \begin{cases} v^{+}_{n}(\eta, \tau) & \text{for} \quad (\eta, \tau) \in [0, \lambda) \times [0, T'], \\ 1 & \text{for} \quad (\eta, \tau) \in [\lambda, \infty) \times [0, T']. \end{cases}$$

Then it follows directly from inequality (2.2) and Lemmas 2.4 and 2.5 that

$$0 \leqslant \bar{v}_{n}(\cdot, \tau) - v^{*}_{n} \in L^{1}(\mathbb{R}^{+}) \tag{2.3}$$

and

$$0 \leqslant \bar{v}_{n}(\cdot, \tau) - v_{n}(\cdot, \tau) \in L^{1}(\mathbb{R}^{+}) \tag{2.4}$$

for all $n \in \mathbb{N}$ and for all $\tau \in [0, T']$.

We estimate the L^{1}-norms of (2.3) and (2.4).

LEMMA 2.6. *There exist constants B_1 and B_2, independent of $T' \in \mathbb{R}^{+}$, such that*

$$\|\bar{v}_{n}(\cdot, \tau) - v_{n}(\cdot, \tau)\|_{L^1} \leqslant (B_1 + B_2 \tau)\, e^{-\tau}$$

and

$$\|\bar{v}_{n}(\cdot, \tau) - v^{*}_{n}\|_{L^1} \leqslant (B_1 + B_2 \tau)\, e^{-\tau}$$

for all $n \in \mathbb{N}$ and $\tau \in [0, T']$.

Proof. We prove here only the first inequality. The proof of the second one is almost identical and will be omitted.

Because the function $\bar{v}_{n}$ satisfies Eq. (2.1) for $(\eta, \tau) \in (\mathbb{R}^{+} \setminus \{\lambda\}) \times (0, T']$ we subtract this equation for $\bar{v}_{n}$ and v_{n} and integrate the result with respect to η from $\eta = 0$ to $\eta = \infty$. Using integration by parts and the asymptotic behaviour implied by Lemma 2.5, we find for each $n \in \mathbb{N}$ and $\tau \in (0, T']$

$$\frac{d}{d\tau}\int_0^\infty (\bar{v}_n - v_n) + \int_0^\infty (\bar{v}_n - v)$$
$$= e^{-\tau} m \bar{v}_n^{(m-1)/m}\, \bar{v}_{n\eta}\Big|_0^{\lambda^-}$$
$$- e^{-\tau}(m-1)\int_0^\infty \bar{v}_n^{-1/m}(\bar{v}_{n\eta})^2 - e^{-\tau} m v_n^{(m-1)/m}\, v_{n\eta}\Big|_0^\infty$$
$$+ e^{-\tau}(m-1)\int_0^\infty v_n^{-1/m}(v_{n\eta})^2. \tag{2.5}$$

Next use Lemma 2.4 and $\bar{v}_{n\eta}(0^+, \tau) = v_{n\eta}^+(0^+, \tau) \geq 0$ to obtain from (2.5) the inequality

$$\frac{d}{d\tau}\int_0^\infty (\bar{v}_n - v_n) + \int_0^\infty (\bar{v}_n - v) \leq e^{-\tau} m(\bar{v}_{n\eta}(\lambda^-, \tau) + L_0)$$
$$+ e^{-\tau}(m-1)\, L_1 \int_0^\infty v_n^{-1/m} v_{n\eta}.$$

Note that from (2.2) we have $\bar{v}_{n\eta}(\lambda^-, \tau) \leq v_n^{*\prime}(\lambda^-) = m/\lambda(\lambda - 1)$ and that $\int_0^\infty v_n^{-1/m} v_{n\eta}\, d\eta < \int_0^1 s^{-1/m}\, ds = 1 - 1/m$. Therefore it follows that there exists a constant $C > 0$, depending only on the data of Problem CD, such that for all $n \in \mathbb{N}$ and $\tau \in (0, T']$

$$\frac{d}{d\tau}\int_0^\infty (\bar{v}_n - v_n) + \int_0^\infty (\bar{v}_n - v_n) \leq C e^{-\tau}.$$

Clearly this implies the desired inequality with $B_2 = C$ and $B_1 \leq \int_0^\infty (1 - s(\eta))\, d\eta$ since

$$\int_0^\infty (\bar{v}_n - v_n)(\eta, 0)\, d\eta < \int_0^\infty (1 - s(\eta))\, d\eta \qquad \text{for all} \quad n \in \mathbb{N}. \quad \blacksquare$$

From Lemma 2.6 and the triangle inequality we obtain

$$\|v_n(\cdot, \tau) - v_n^*\|_{L^1} \leq 2(B_1 + B_2\tau)\, e^{-\tau}$$

for all $n \in \mathbb{N}$ and $\tau \in [0, T']$. As in [4] we can pass to the limit for $n \to \infty$ to find

PROPOSITION 2.7. *Let w and w^* be as in Theorem 2.1. Then*

$$\|w^m(\cdot, \tau) - w^{*m}\|_{L^1} \leq 2(B_1 + B_2\tau)\, e^{-\tau}$$

for all $\tau \in [0, T']$. Here B_1 and B_2 are the constants from Lemma 2.6.

Next we convert this L^1-estimate into the L^∞-estimate of Theorem 2.1. We use the following result.

LEMMA 2.8. *Let* $\phi: [0, \infty) \to [0, \infty)$ *satisfy* (i) $\phi(0)=0$, (ii) ϕ *is uniformly Hölder continuous on* $[0, \infty)$ *with exponent* α *and constant* A, *and* (iii) $\phi \in L^1(\mathbb{R}^+)$. *Then*

$$\|\phi\|_{L^\infty} \leqslant A^{1/(\alpha+1)} \left(\frac{\alpha+1}{\alpha} \|\phi\|_{L^1} \right)^{\alpha/(\alpha+1)}.$$

Proof. See the proof of Lemma 3 of Peletier [16]. ∎

The function

$$|w^m(\cdot, \tau) - w^{*m}|$$

satisfies for each $\tau \in [0, T']$ the conditions of Lemma 2.8. In particular $\alpha = 1$ and $A = L_1 + m/\lambda(\lambda - 1)$. Therefore the application of Lemma 2.8 to the inequality in Proposition 2.7 gives the desired L^∞-estimate. This concludes the first part of the proof.

Proof of Theorem 2.1: *The Case* $m - \lambda + 1 < 0$

As in the previous case we also consider here the family of functions $\{v_n\}_{n \in \mathbb{N}}$, which are solutions of Problem $\tilde{\mathrm{P}}_n$ for $n \in \mathbb{N}$. However, since $m - \lambda + 1 < 0$, the proof of the gradient bound (Lemma 2.4) is not valid. We therefore proceed as follows.

For each $n \in \mathbb{N}$ let

$$p_n := v_n^{(\lambda-1)/m} \qquad \text{on } \bar{H}_{T'}.$$

Then p_n satisfies the problem

$$\hat{\mathrm{P}}_n \begin{cases} B(p) := p_\tau + (\lambda p - \eta)\, p_\eta - e^{-\tau} m & \\ \qquad \left(\dfrac{m-\lambda+1}{\lambda-1}\, p^{(m-\lambda)/(\lambda-1)} (p_\eta)^2 + p^{(m-1)/(\lambda-1)}\, p_{\eta\eta} \right) = 0 & \text{on } H_{T'} \\ p(0, \cdot) = \left(\dfrac{1}{n} \right)^{(\lambda-1)/m} & \text{on } (0, T'] \\ p(\cdot, 0) = p_{0n} := v_{0n}^{(\lambda-1)/m} & \text{on } \mathbb{R}^+. \end{cases}$$

We immediately have

LEMMA 2.9. *For all* $n \in \mathbb{N}$ *and* $(\eta, \tau) \in \bar{H}_{T'}$

$$\text{(a)} \quad \left(\frac{1}{n} \right)^{(\lambda-1)/m} \leqslant p_n(\eta, \tau) \leqslant 1;$$

(b) $0 \leqslant p_{n\eta}(\eta, \tau) \leqslant \dfrac{\lambda - 1}{m} L e^{\tau}$;

(c) $\tilde{s}(\eta) \leqslant p_n(\eta, \tau)$,

where the function $\tilde{s}$ is obtained from the function s (see Lemma 2.5) when replacing γ by $((\lambda - 1)/m)\gamma$.

Proof. Parts (a) and (c) follow directly from corresponding properties of the functions v_n. Part (b) follows from the observation that $0 \leqslant (u_n^m)_x \leqslant L$ on $[0, \infty) \times [0, T]$. Transforming to the variables η and τ and using $(\lambda - 1)/m > 1$ gives the desired inequality. ∎

Being unable to obtain a uniform (in τ) gradient bound for the functions p_n, we cannot apply directly the method used for the case $m - \lambda + 1 \geqslant 0$. To overcome this difficulty we construct below sub- and supersolutions which have a uniform gradient bound. Then we use these functions to obtain the convergence result.

First introduce

$$p^* := w^{*^{\lambda - 1}} = \begin{cases} \eta/\lambda, & 0 \leqslant \eta < \lambda \\ 1, & \eta \geqslant \lambda \end{cases}$$

and

$$p_n^* = \max\{(1/n)^{(\lambda - 1)/m}, p^*\} \qquad \text{for} \quad n \in \mathbb{N}. \tag{2.6}$$

Then consider for each $n \in \mathbb{N}$ the auxiliary problem

$$\mathrm{A}_2 \begin{cases} C(y) := y_\tau + (\lambda y - \eta)\, y_\eta - e^{-\tau} m y^{(m-1)/(\lambda - 1)}\, y_{\eta\eta} = 0 & \text{in } H_{T'}^{\lambda} \\ y(0, \cdot) = (1/n)^{(\lambda - 1)/m},\ y(\lambda, \cdot) = 1 & \text{on } (0, T'] \\ y(\cdot, 0) = p_{0n}^{+}(\cdot) & \text{on} (0, \lambda), \end{cases}$$

where $p_{0n}^{+} = \max\{p_{0n}, p_n^*\}$. Clearly this problem has a unique solution $\tilde{y}_n \in C^{\infty}(H_{T'}^{\lambda}) \cap C(\bar{H}_{T'}^{\lambda})$ which is differentiable up to the boundaries $\eta = 0$ and $\eta = \lambda$ for every $\tau \in (0, T']$.

LEMMA 2.10. *For all $n \in \mathbb{N}$*

(a) $p_n^* \leqslant \tilde{y}_n \leqslant 1$ *in* $\bar{H}_{T'}^{\lambda}$;

(b) $0 \leqslant \tilde{y}_{n\eta} \leqslant L_0 = \max\{L, 1/\lambda\}$.

Proof. Part (a) follows from the observation that $C(p_n^*) = 0$, except at $\eta = \lambda(1/n)^{(m-1)/\lambda}$, and a comparison argument. To prove (b) we first use a barrier function argument at $\eta = 0$ and $\eta = \lambda$ and then a comparison argument for the equation for $\tilde{y}_{n\eta}$. ∎

Then introduce for each $n \in \mathbb{N}$

$$y_n(\eta, \tau) = \begin{cases} \tilde{y}_n(\eta, \tau) & \text{for} \quad (\eta, \tau) \in \bar{H}^{\lambda}_{T'} \\ 1 & \text{for} \quad (\eta, \tau) \in (\lambda, \infty) \times [0, T']. \end{cases}$$

The following lemma shows that y_n is a supersolution for Problem $\hat{\mathrm{P}}_n$ and also $y_n \geqslant p_n^*$.

LEMMA 2.11. *$y_n \geqslant \max\{p_n, p_n^*\}$ on $\bar{H}_{T'}$, for all $n \in \mathbb{N}$.*

Proof. The inequality $y_n \geqslant p_n$ is a direct consequence of the fact that

$$B(y_n) = -e^{-\tau} m \frac{m - \lambda + 1}{\lambda - 1} y_n^{(m-\lambda)/(\lambda-1)} y_{n\eta}^2 \geqslant 0 = B(p_n) \qquad \text{in } H^{\lambda}_{T'}$$

and $y_n \geqslant p_n$ on $\partial H^{\lambda}_{T'}$. The second inequality follows from Lemma 2.10. ∎

Next we construct a suitable subsolution. Let us first suppose that

$$\text{AH: } \liminf_{x \downarrow 0} \frac{u_0^m(x)}{x} > 0 \text{ and that } u_0 > 0 \qquad \text{on } \mathbb{R}^+.$$

This hypothesis implies that the *concave* function $\chi\colon [0, \infty) \to [0, 1)$, given by $\chi(\eta) = \beta\eta$ (β sufficiently small) for $0 \leqslant \eta \leqslant \eta^*$, where η^* is the second intersection point with the function $\tilde{s}$, and by $\chi(\eta) = \tilde{s}(\eta)$ for $\eta > \eta^*$, satisfies $\chi \leqslant v_0$ on $\mathbb{R}^+$; see Fig. 1.

Then consider the family of smooth, *concave* functions $\{\chi_n\}_{n \in \mathbb{N}}$, defined on $\bar{\mathbb{R}}^+$, where $\chi_1 = 1$ and where χ_n, $n \geqslant 2$, is obtained from χ by rotating the line $\beta\eta$ around the point $(1/\beta n, 1/n)$ such that $\chi_n(0) = 1/2n$, see Fig. 2, and by making a smooth, increasing connection of the rotated line with $\tilde{s}$ at the second intersection point.

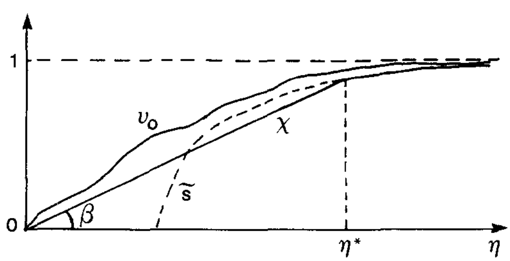

FIG. 1. Construction of concave function $\chi \leqslant v_0$.

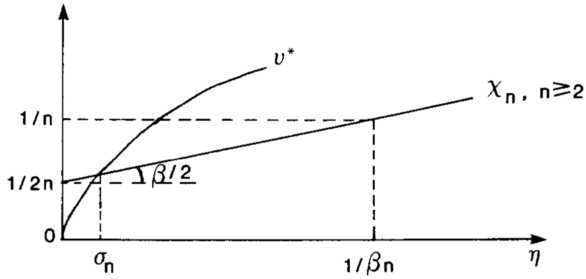

FIG. 2. Construction of functions χ_n, $n \geqslant 2$, near the origin.

By construction and from Fig. 2 we immediately see that

$$\chi_n \leqslant v_{0n}^- = \min\{v_n^*, v_{0n}\} \tag{2.7}$$

and

$$\chi_n \geqslant \tilde{s}(\eta - \eta_0) \qquad (\eta_0 \text{ sufficiently large}) \tag{2.8}$$

for all $n \in \mathbb{N}$.

If σ_n denotes the intersection point of χ_n with v^* then, from Fig. 2,

$$0 < \sigma_n < \lambda(1/n)^{(\lambda-1)/m} \qquad \text{for} \quad n \geqslant 2, \tag{2.9}$$

where the right hand side of the second inequality corresponds to the point where $v^* = 1/n$.

We now consider the third auxiliary problem $(n \geqslant 2)$

$$\mathrm{A}_3 \begin{cases} B(z) = 0 & \text{in} \quad (\sigma_n, \infty) \times (0, T'] \\ z(\sigma_n, \cdot) = \sigma_n/\lambda & \text{on } (0, T'] \\ z(\cdot, 0) = \chi_n^{(\lambda-1)/m} & \text{on } (\sigma_n, \infty), \end{cases}$$

where $\{\chi_n(\sigma_n)\}^{(\lambda-1)/m} = \{v^*(\sigma_n)\}^{(\lambda-1)/m} = \sigma_n/\lambda$ (see Fig. 2).

Let $\tilde{z}_n \in C^\infty((\sigma_n, \infty) \times (0, T']) \cap C([\sigma_n, \infty) \times [0, T'])$ denote the unique solution of Problem A_3. This function is differentiable up to the boundary $\eta = \sigma_n$ for all $\tau \in (0, T']$.

Introduce for $n \geqslant 2$ the functions

$$z_n(\eta, \tau) = \begin{cases} \sigma_n/\lambda & \text{for} \quad (\eta, \tau) \in [0, \sigma_n] \times [0, T'], \\ \tilde{z}_n(\eta, \tau) & \text{for} \quad (\eta, \tau) \in (\sigma_n, \infty) \times [0, T'], \end{cases}$$

and

$$z_n^* = \max\{\sigma_n/\lambda, p^*\} \qquad \text{on } \bar{H}_{T'} \tag{2.10}$$

and $z_1 = z_1^* = 1$. Further denote, for each $n \in \mathbb{N}$, by p_n^- the solution of Problem $\hat{\mathrm{P}}_n$ with initial function

$$p_{0n}^- = (v_{0n}^-)^{(\lambda-1)/m} = \min\{p_n^*, p_{0n}\}.$$

Then using inequalities (2.7) and (2.8), and applying a comparison argument, we immediately obtain

LEMMA 2.12. *There exists a constant $\eta_0 > 0$ such that*

$$\tilde{s}_{\eta_0} \leqslant z_n \leqslant \min\{p_n^-, z_n^*\} \leqslant p_n \qquad on\ \bar{H}_{T'},$$

for all $n \in \mathbb{N}$. Here $\tilde{s}_{\eta_0}$ is the lower bound from Lemma 2.9(c), *translated over η_0.*

The next lemma gives a gradient bound for the functions z_n.

LEMMA 2.13. *For each $n \geqslant 2$*

$$0 \leqslant z_{n\eta}(\eta, \tau) \leqslant \frac{\lambda-1}{m} \cdot \frac{z_n(\eta, \tau) - \sigma_n/\lambda}{\eta - \sigma_n} \leqslant \frac{\lambda-1}{m\lambda},$$

where $(\eta, \tau) \in (\sigma_n, \infty) \times [0, T']$.

Proof. The first inequality follows from a comparison argument applied to the equation for $z_{n\eta}$ (0 is a subsolution). To prove the second inequality, we consider for $n \geqslant 2$ the problem

$$\begin{aligned} A(v) &= 0 && \text{in} \quad (\sigma_n, \infty) \times (0, T'] \\ v(\sigma_n, \cdot) &= (\sigma_n/\lambda)^{m/(\lambda-1)} && \text{on } (0, T'] \\ v(\cdot, 0) &= \chi_n && \text{on } (\sigma_n, \infty). \end{aligned}$$

where the operator A is defined in (2.1).

Let h_n denote its solution. By construction $h_{n\eta\eta}(\cdot, 0) = \chi_n'' \leqslant 0$ on $[\sigma_n, \infty)$. Further, using the boundary condition and the equation at $\eta = \sigma_n$, we also have $h_{n\eta\eta}(\sigma_n, \cdot) = 0$ on $[0, T']$.

Next derive the equation for $h_{n\eta\eta}$. Using a comparison argument then gives $h_{n\eta\eta} \leqslant 0$ in $[\sigma_n, \infty) \times [0, T']$. A direct consequence of this concavity property is

$$h_{n\eta}(\eta, \tau) \leqslant \frac{h_n(\eta, \tau) - (\sigma_n/\lambda)^{m/(\lambda-1)}}{\eta - \sigma_n} \tag{2.11}$$

for $(\eta, \tau) \in (\sigma_n, \infty) \times (0, T']$. Now since $z_n = \tilde{z}_n = (h_n)^{(\lambda-1)/m}$ on $[\sigma_n, \infty) \times [0, T']$ (see Problem A_3) and since $z_{n\eta} \geqslant 0$, the result follows at once from (2.11). To obtain the third inequality, we replace z_n by z_n^*; see Lemma 2.12. ∎

We now turn to the large time behaviour. By Lemmas 2.11 and 2.12 we have for all $n \in \mathbb{N}$ on $\bar{H}_{T'}$,

$$|p_n - p_n^*| \leqslant y_n - z_n = (y_n - p_n^*) + (p_n^* - z_n^*) + (z_n^* - z_n), \qquad (2.12)$$

where the three terms on the right are nonnegative and belong to $L^1(\mathbb{R}^+)$. For the first and last term we derive an L^1-estimate, which is later converted into an L^∞-estimate. For the second term we have from (2.6) and (2.10)

$$0 \leqslant p_n^* - z_n^* \leqslant \left(\frac{1}{n}\right)^{(\lambda-1)/m} - \frac{\sigma_n}{\lambda} \qquad \text{on } [0, \infty) \qquad (2.13)$$

for all $n \geqslant 2$.

LEMMA 2.14. *There exist constants B_1 and B_2, independent of $T' \in \mathbb{R}^+$, such that*

$$\|y_n(\cdot, \tau) - p_n^*\|_{L^1} \leqslant (B_1 + B_2\tau)\, e^{-\tau}$$

and

$$\|z_n^* - z_n(\cdot, \tau)\|_{L^1} \leqslant (B_1 + B_2\tau)\, e^{-\tau}$$

for all $n \in \mathbb{N}$ and $\tau \in [0, T']$.

Proof. With respect to the first inequality we observe that $y_n = p_n^* = 1$ on $[\lambda, \infty) \times (0, T']$. Therefore we only consider the difference in $H_{T'}^\lambda$. We have

$$y_{n\tau} + (\lambda y_n - \eta)\, y_{n\eta} = e^{-\tau} m y_n^{(m-1)/(\lambda-1)}\, y_{n\eta\eta} \qquad \text{in } H_{T'}^\lambda$$

and

$$(\lambda p_n^* - \eta)\, p_{n\eta}^* = 0 \qquad \text{on} \quad (0, \lambda)/\{\lambda(1/n)^{(\lambda-1)/m}\}.$$

Subtraction and integration (by parts) with respect to η from $\eta = 0$ to $\eta = \lambda$ gives

$$\frac{d}{d\tau}\int_0^\lambda (y_n(\cdot, \tau) - p_n^*) + \int_0^\lambda (y_n(\cdot, \tau) - p_n^*)$$

$$= e^{-\tau} m y_n^{(m-1)/(\lambda-1)}\, y_{n\eta}\Big|_0^{\lambda^-} - m e^{-\tau}\frac{m-1}{\lambda-1}\int_0^\lambda y_n^{(m-1)/(\lambda-1)}\, y_{n\eta}^2.$$

Then use Lemma 2.10(b) to obtain for each $\tau \in (0, T']$

$$\frac{d}{d\tau} \|y_n(\cdot, \tau) - p_n^*\|_{L^1} + \|y_n(\cdot, \tau) - p_n^*\|_{L^1} \leqslant mL_0 e^{-\tau},$$

from which the first L^1-estimate follows.

To prove the second inequality we first observe that $z_n = z_n^*$ on $[0, \sigma_n] \times [0, T']$. Subtracting the equations for z_n^* and z_n and integrating the result from $\eta = \sigma_n$ to $\eta = \infty$ gives

$$\frac{d}{d\tau} \int_{\sigma_n}^{\infty} (z_n^* - z_n(\cdot, \tau)) + \int_{\sigma_n}^{\infty} (z_n^* - z_n(\cdot, \tau))$$
$$= -e^{-\tau} m z_n^{(m-1)/(\lambda-1)} z_{n\eta} \Big|_{\sigma_n}^{\infty} + \frac{m(\lambda-2)}{\lambda-1} e^{-\tau} \int_{\sigma_n}^{\infty} z_n^{(m-\lambda)/(\lambda-1)} z_{n\eta}^2. \quad (2.14)$$

To estimate the right hand side of this equality we use Lemma 2.13. The first term can be bounded by $((\lambda-1)/\lambda)\, e^{-\tau}$. The second term we split:

$$\int_{\sigma_n}^{\lambda} z_n^{(m-\lambda)/(\lambda-1)} z_{n\eta}^2 \leqslant \frac{\lambda-1}{m\lambda} \int_{\sigma_n}^{\lambda} z_n^{(m-\lambda)/(\lambda-1)} z_{n\eta}$$
$$\leqslant \frac{\lambda-1}{m\lambda} \int_{\sigma_n/\lambda}^{1} s^{(m-\lambda)/(\lambda-1)}\, ds \leqslant \frac{(\lambda-1)^2}{\lambda m(m-1)}$$

and

$$\int_{\lambda}^{\infty} z_n^{(m-\lambda)/(\lambda-1)} z_{n\eta}^2 \leqslant \left(\frac{\lambda-1}{m}\right)^2 \int_{\lambda}^{\infty} z_n^{(m-\lambda)/(\lambda-1)} \cdot \frac{z_n^2}{(\eta-\sigma_n)^2}$$
$$\leqslant \left(\frac{\lambda-1}{m}\right)^2 \left\{\frac{-1}{\eta-\sigma_n}\right\}\Big|_{\lambda}^{\infty}$$
$$\leqslant \left(\frac{\lambda-1}{m}\right)^2 \frac{1}{\lambda-\sigma_n} < \left(\frac{\lambda-1}{m}\right)^2 \frac{1}{\lambda(1-2^{(1-\lambda)/m})},$$

for all $n \geqslant 2$, where we used (2.9). Thus from Eq. (2.14) and the inequalities underneath it we obtain

$$\frac{d}{d\tau} \|z_n^* - z_n(\cdot, \tau)\|_{L^1} + \|z_n^* - z_n(\cdot, \tau)\|_{L^1} \leqslant \mathscr{C} e^{-\tau}$$

for all $n \in \mathbb{N}$ $(z_1 = z_1^* = 1)$ and for all $\tau \in (0, T']$. Here $\mathscr{C}$ is a positive constant involving only λ and m. The desired estimate now follows after integration in τ. ▮

Because the integrants from the inequalities in Lemma 2.14 are Lipschitz continuous in η for each $\tau \in [0, T']$, we can use Lemma 2.8 to convert them into L^∞-estimates. Using also (2.13) we obtain from (2.12) that

$$|p_n(\eta, \tau) - p_n^*(\eta)| < (A_1 + A_2\tau)^{1/2} e^{-\tau/2} + (1/n)^{(\lambda-1)/m}$$

for all $n \in \mathbb{N}$ and $(\eta, \tau) \in \bar{H}_{T'}$. Here A_1 and A_2 are positive constants independent of n and T'. Finally, we let n tend to infinity, which concludes the proof of Theorem 2.1, under the additional hypothesis AH.

If this condition were not satisfied we would proceed as follows. Again we construct suitable sub- and supersolutions. For a supersolution we choose the function y_n as before. For a subsolution we choose $a > 0$ such that $u_0(a) > 0$ and we define $\eta' = (x-a)/(t+1)$ $(= \eta - ae^{-\tau})$. We now construct the functions z_n and z_n^* as before. As in Lemma 2.14 we can prove that z_n converges to z_n^*. In fact we have the pointwise estimate

$$|z_n(\eta', \tau) - p_n^*(\eta')| < (A_1 + A_2\tau)^{1/2} e^{-\tau/2} + 0(1) \qquad \text{as} \quad n \to \infty$$

for $(\eta', \tau) \in (-ae^{-\tau}, \infty) \times [0, T']$, where we have extended the functions z_n and p_n^* for $\eta' < 0$ by the constants σ_n/λ and $(1/n)^{(\lambda-1)/m}$, respectively. Now let n tend to infinity. Then the subsolution z_n converges to the translated $p^*((x-a)/(t+1)) = p^*(\eta - ae^{-\tau})$ for $(\eta, \tau) \in \bar{H}_{T'}$, where p^* has been extended by zero for negative values of the argument. The supersolution y_n converges to $p^*(x/(t+1)) = p^*(\eta)$. Then from Remark 1.4 the result follows. ∎

Remark 2.15. Note that our method of proof depends on the assumed regularity of the initial function u_0 (the estimates derived in the proof, and hence the constants A_1 and A_2, depend on L, the Lipschitz constant of u_0^m).

3. Other Convergence Results

THEOREM C. *Assume* $0 < \lambda < 1$. *Let* $u_0 : \bar{\mathbb{R}}^+ \to [0, u^0]$, $u_0(0) = u^0$, u_0^m *uniformly Lipschitz continuous on* $\mathbb{R}^+$, *and* $u_0 = O(|x|^{-\alpha})$ *as* $x \to \infty$ *for some* $\alpha > 1$.

Let u be the weak solution of Problem CD. *Then*

$$\|u^m(\cdot, t) - u^{*m}(\cdot, t)\|_{L^\infty(\mathbb{R}^+)} \leqslant C_2(1 + \log(t+1))^{1/2} (t+1)^{-1/2}$$

for all $t \geqslant 0$, *where* C_2 *is a positive constant and* u^* *is given by* (1.6), (1.7).

Proof. Without loss of generality we may assume that u_0 is nonincreasing on $\mathbb{R}^+$ (cf. Remark 2.3) and that $u^0 = 1$.

Note that, since $0<\lambda<1$, we are in the case $m-\lambda+1>0$. Because the proof of Theorem C is similar to the proof of Theorem B for that case, we give here only a brief sketch.

Again we introduce a sequence of approximating problems $\{\tilde{\mathrm{P}}_n\}_{n\in\mathbb{N}}$, in which the smooth approximation v_{0n} of $v_0=u_0^m$ satisfies: (i) $v_{0n}\in C^\infty([0,\infty))$ and $-L\leqslant v_{0n}'\leqslant 0$ where L is the Lipschitz constant of u_0^m, (ii) $v_{0n}(0)=1$, $v_{0n}(\infty)=1/n$, and (iii) $v_{0n}\to u_0^m$ as $n\to\infty$ uniformly on $\bar{\mathbb{R}}^+$.

For each $n\in\mathbb{N}$ the solution v_n of $\tilde{\mathrm{P}}_n$ has the following properties: (i) $1/n\leqslant v_n\leqslant 1$, (ii) $-L_1\leqslant v_{n\eta}\leqslant 0$ for some $L_1\geqslant L$ (cf. Lemma 2.4), and (iii) $v_n(\eta,\tau)\leqslant\min\{1,\gamma_1(\eta-\eta_2)^{-k_1}+1/n\}$ for all $\tau\in[0,T']$ and $\eta>\eta_2$, where η_2, γ_1, and k_1 are constants chosen as in Lemma 2.5.

In order to show that $v_n(\cdot,\tau)$ converges to $v_n^*=\max\{1/n,v^*\}$ uniformly in n as $\tau\to\infty$ (with $v^*=w^{*^m}$), we introduce the auxiliary problem

$$\begin{aligned} &A(v)=0 && \text{in}\quad (\lambda,\infty)\times(0,T'] \\ &v(\lambda,\tau)=1 && \text{on } (0,T'] \\ &v(\eta,0)=v_{0n}^+(\eta) && \text{on } (\lambda,\infty), \end{aligned}$$

where $v_{0n}^+=\max\{v_{0n},v_n^*\}$ and where the operator A is defined in (2.1). The solution v_n^+ of this problem satisfies $v_n^+\geqslant\max\{v_n^*,v_n\}$ for all $n\in\mathbb{N}$. Finally define

$$\bar{v}_n(\eta,\tau)=\begin{cases}1 & \text{in}\quad [0,\lambda)\times[0,T'] \\ v_n^+(\eta,\tau) & \text{in}\quad [\lambda,\infty)\times[0,T'].\end{cases}$$

Then (2.3), (2.4) hold for v_n, $\bar{v}_n$, and v_n^* as defined above. Now continue as in Section 2 $(m-\lambda+1>0)$. ∎

THEOREM D. *Assume $\lambda>1$. Let $u_0:\mathbb{R}\to[0,u^+]$, u_0^m uniformly Lipschitz continuous on $\mathbb{R}$, $u^+-u_0=O(|x|^{-\alpha})$ as $x\to\infty$, and $u_0=O(|x|^{-\alpha})$ as $x\to-\infty$ for some $\alpha>1$.*

Let u be the weak solution of Problem C. *Then*

$$\|u^\mu(\cdot,t)-u^{*\mu}(\cdot,t)\|_{L^\infty(\mathbb{R})}\leqslant C_3(1+\log(t+1))^{1/2}(t+1)^{-1/2}$$

for all $t\geqslant 0$, where C_3 is a positive constant, $\mu=\max\{m,\lambda-1\}$, and u^* is the solution (modulo translations) of Problem C^∞; *see* (1.5a).

Proof of Theorem D. Let $\tilde{u}$ be the solution of Problem CD with $\tilde{u}(0,t)=u^0=0$ for $t\in[0,T]$ and $\tilde{u}(x,0)=\min(u_0(x),u^*(x,0))$ for $x\in[0,\infty)$, where u^* is given by (1.6), (1.7). Then define

$$\hat{u}(x,t)=\begin{cases}0 & \text{in}\quad (-\infty,0)\times[0,T] \\ \tilde{u}(x,t) & \text{in}\quad [0,\infty)\times[0,T].\end{cases}$$

The norm $\|\hat{u}^m(\cdot, t) - u^{*^m}(\cdot, t)\|_{L^\infty(\mathbb{R})}$ can be estimated by applying Theorem B. To prove the result of the theorem it is sufficient to derive an estimate for $\|u^m(\cdot, t) - \hat{u}^m(\cdot, t)\|_{L^\infty(\mathbb{R})}$. This can easily be done with the methods developed in the previous section. We omit the details. ∎

THEOREM E. *Assume* $0 < \lambda < 1$. *Let* $u_0 : \mathbb{R} \to [0, u^-]$, u_0^m *uniformly Lipschitz continuous on* $\mathbb{R}$, $u^- - u_0 = O(|x|^{-\alpha})$ *as* $x \to -\infty$, *and* $u_0 = O(|x|^{-\alpha})$ *as* $x \to \infty$ *for some* $\alpha > 1$.

Let u be the weak solution of Problem C. *Then*

$$\|u^m(\cdot, t) - u^{*^m}(\cdot, t)\|_{L^\infty(\mathbb{R})} \leqslant C_4(1 + \log(t+1))^{1/2} (t+1)^{-1/2}$$

for all $t \geqslant 0$, *where* C_4 *is a positive constant and* u^* *is the solution* (*modulo translations*) *of Problem* C^∞; *see* (1.5a).

Proof. The proof is similar to the proof of Theorem D. We omit the details. ∎

Remark 3.1. In the nondegenerate cases $u^0, u^+, u^- > 0$, we have similar results as in the previous theorems. The constant μ can then be replaced by m.

Remark 3.2. When $\lambda = 1$ we use for the Cauchy problem (after a suitable transformation) the results derived in [5] (the nondegenerate case) and [3] (the degenerate case). Thus we find that the solution $u(x, t)$ of Problem C converges, as $t \to \infty$, to a function $f((x-t)/\sqrt{t})$, where f $(= f(\eta))$ is the weak solution of

$$(f^m)'' + \tfrac{1}{2}\eta f' = 0 \qquad \text{on } \mathbb{R}$$
$$f(-\infty) = u^-; \; f(+\infty) = u^+.$$

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