Global attractivity for a family of nonlinear difference equations

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Abstract

In this note, we consider the following nonlinear difference equation:

$$x_{n+1} = \frac{f(x_{n-r_1}, \ldots, x_{n-r_k})g(x_{n-m_1}, \ldots, x_{n-m_l}) + h(x_{n-p_1}, \ldots, x_{n-p_s}) + 1}{f(x_{n-r_1}, \ldots, x_{n-r_k}) + g(x_{n-m_1}, \ldots, x_{n-m_l}) + h(x_{n-p_1}, \ldots, x_{n-p_s})}, \quad n = 0, 1, \ldots,$$

where $f \in C((0, +\infty)^k, (0, +\infty))$, $g \in C((0, +\infty)^l, (0, +\infty))$ and $h \in C((0, +\infty)^s, [0, +\infty))$ with $k, l, s \in \{1, 2, \ldots\}$, $0 \leq r_1 < \cdots < r_k$, $0 \leq m_1 < \cdots < m_l$ and $0 \leq p_1 < \cdots < p_s$, and the initial values are positive. We give sufficient conditions under which the unique equilibrium $x = 1$ of this equation is globally asymptotically stable, which extends and includes corresponding results obtained in the cited references.

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1. Introduction

The study of properties of nonlinear difference equations has been an area of intense interest in recent years (for example, see \cite{1–3}). In \cite{4}, Ladas suggested investigating the global asymptotic stability of the following nonlinear difference equation:

$$x_{n+1} = \frac{x_n + x_{n-1}x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots,$$

where the initial values $x_{-2}, x_{-1}, x_0 \in \mathbb{R}_+ \equiv (0, +\infty)$.

In \cite{5}, Nesemann utilized the strong negative feedback property to study the following difference equation:

$$x_{n+1} = \frac{x_{n-1} + x_n x_{n-2}}{x_n x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots,$$

where the initial values $x_{-2}, x_{-1}, x_0 \in \mathbb{R}_+$. 

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In [6, 7], Li studied the global asymptotic stability of the following two nonlinear difference equations:

\[ x_{n+1} = \frac{x_n x_{n-1} x_{n-3} + x_{n-2} + x_{n-3} + a}{x_n x_{n-1} x_{n-1} + x_{n-1} x_{n-3} + x_{n-2} x_{n-3} + 1 + a}, \quad n = 0, 1, \ldots, \]

(E3)

and

\[ x_{n+1} = \frac{x_n x_{n-1} x_{n-3} + x_{n-1} + x_{n-3} + a}{x_n x_{n-1} + x_{n-1} x_{n-3} + x_{n-2} x_{n-3} + 1 + a}, \quad n = 0, 1, \ldots, \]

(E4)

where \( a \in [0, +\infty) \) and the initial values \( x_{-3}, x_{-2}, x_{-1}, x_0 \in \mathbb{R}_+ \).

In [8], Papaschinopoulos and Schinas investigated the global asymptotic stability of the following nonlinear difference equation:

\[ x_{n+1} = \frac{\sum_{i \in \mathbb{Z}_k} x_{n-i} + x_{n-j} x_{n-j+1} + 1}{\sum_{i \in \mathbb{Z}_k} x_{n-i}}, \quad n = 0, 1, \ldots, \]

(E5)

where \( k \in \{1, 2, 3, \ldots\} \), \( \{j, j - 1\} \subseteq \mathbb{Z}_k \equiv \{0, 1, \ldots, k\} \) and the initial values \( x_{-k}, x_{-k+1}, \ldots, x_0 \in \mathbb{R}_+ \).

Recently, Li et al. [9] studied the global asymptotic stability of the following rational difference equation:

\[ x_{n+1} = \frac{x_{n-k} x_{n-m} + x_{n-l} + a}{x_{n-k} + x_{n-m} x_{n-l} + a}, \quad n = 0, 1, \ldots, \]

(E6)

where \( a \geq 0, b \geq 0, k, l, m \in \{0, 1, \ldots\} \) with \( k \neq l, k \neq m \) and \( m \neq l \), and the initial values are positive.

The main theorem in this note is motivated by the above studies. In this work, we consider the following nonlinear difference equation:

\[ x_{n+1} = \frac{f(x_{n-r_1}, \ldots, x_{n-r_k}) g(x_{n-m_1}, \ldots, x_{n-m_l}) + h(x_{n-p_1}, \ldots, x_{n-p_s}) + 1}{f(x_{n-r_1}, \ldots, x_{n-r_k}) + g(x_{n-m_1}, \ldots, x_{n-m_l}) + h(x_{n-p_1}, \ldots, x_{n-p_s})}, \quad n = 0, 1, \ldots, \]

(1)

where \( f \in C(R_+^k, R_+), g \in C(R_+^l, R_+) \) and \( h \in C(R_+^s, [0, +\infty)) \) with \( k, l, s \in \{1, 2, \ldots\} \), \( 0 \leq r_1 < \cdots < r_k, 0 \leq m_1 < \cdots < m_l \) and \( 0 \leq p_1 < \cdots < p_s \), and the initial values are positive real numbers.

2. Main result

In the sequel, write \( u^* = \max\{u, 1/u\} \) for any \( u \in \mathbb{R}_+ \).

**Lemma 1.** (i) Let \( u \geq w \geq 1, v \geq 1 \) and \( b \in [0, +\infty) \); then \( (uv + 1 + b)/(u + v + b) \leq (uv + 1)/(u + v) \) and \( (uv + 1)/(u + v) \leq (uv + 1)/(u + v) \).

(ii) Let \( u, v \in \mathbb{R}_+ \); then \( (uv)^* \leq \max\{u^*/v^*, u^*v^*, v^*/u^*\} \).

**Proof.** (i) is obvious.

(ii) If \( u/v \geq 1 \), then it follows that

\[
\left( \frac{u}{v} \right)^* = \frac{u}{v} = \begin{cases} 
  u^*, & \text{if } u \geq v \geq 1, \\
  \frac{u^*}{v^*}, & \text{if } u \geq 1 \geq v, \\
  \frac{v^*}{u^*}, & \text{if } 1 > u \geq v.
\end{cases}
\]

If \( u/v < 1 \), then it follows that

\[
\left( \frac{u}{v} \right)^* = \frac{v}{u} = \begin{cases} 
  v^*, & \text{if } u < v \leq 1, \\
  \frac{v^*}{u^*}, & \text{if } u \leq 1 \leq v, \\
  \frac{u^*}{v^*}, & \text{if } 1 \leq u < v.
\end{cases}
\]

Thus \( (uv)^* \leq \max\{u^*/v^*, u^*v^*, v^*/u^*\} \). This completes the proof. \( \square \)
Lemma 2. Let \( u, v \in R_+ \) and \( b \in [0, +\infty) \); then

(i) \([uv + 1 + b)/(u + v + b)]^* \leq (u^*v^* + 1)/(u^* + v^*) \leq \min\{u^*, \, v^*\}.

(ii) \([uv + 1)/(u + v)]^* = (u^*v^* + 1)/(u^* + v^*).

Proof. We only prove (i) (the proof for (ii) is similar). Let \( w = (uv + 1 + b)/(u + v + b) \); we have

\[
w - 1 = \frac{(u - 1)(v - 1)}{u + v + b}.
\]

If \((u - 1)(v - 1) \geq 0\), then by (2) we have \(w \geq 1\), which implies

\[
w^* = w = \frac{uv + 1 + b}{u + v + b} = \begin{cases} \frac{u^*v^* + 1 + b}{u^* + v^* + b}, & \text{if } u \geq 1 \text{ and } v \geq 1, \\ \frac{u^*v^* + 1 + u^*v^*b}{u^* + v^* + u^*v^*b}, & \text{if } u \leq 1 \text{ and } v \leq 1. \end{cases}
\]

It follows from Lemma 1 that \(w^* \leq (u^*v^* + 1)/(u^* + v^*)\).

On the other hand, since \(u^* \geq 1\) and \(v^* \geq 1\), it follows that \((u^*v^* + 1)/(u^* + v^*) \leq u^*\) and \((u^*v^* + 1)/(u^* + v^*) \leq v^*\). This completes the proof. \(\square\)

Now we formulate and prove the main result of this note.

Theorem 1. Let \( f, g \) satisfy the following two conditions:

(H1) There exists \( F \in C(R^k_+, \, R_+) \) such that \([f(u_1, \ldots, u_k)]^* \leq F(u_1^*, \ldots, u_k^*)\).

(H2) \([g(u_1, \ldots, u_l)]^* = g(u_1^*, \ldots, u_l^*)\) and \(g(u_1^*, \ldots, u_l^*) \leq u_1^*\).

Then \(\bar{\alpha} = 1\) is the unique positive asymptotically stable equilibrium of Eq. (1) which is globally asymptotically stable.

Proof. Let \(\{x_n\}_{n=-m}^\infty\) be a solution of Eq. (1) with initial conditions \(x_{-m}, \ldots, x_{-m+1}, \ldots, x_0 \in R_+\), where \(m = \max\{r_k, m_1, p_3\}\). From (1), (H2) and Lemma 2 it follows that for any \(n \geq 0\),

\[
1 \leq x_{n+1}^* \leq \frac{f(x_{n-1}, \ldots, x_{-m})g(x_{m-1}, \ldots, x_{-m}) + 1 + h(x_{n-1}, \ldots, x_{-m})}{f(x_{n-1}, \ldots, x_{-m}) + g(x_{m-1}, \ldots, x_{-m})} = \frac{f(x_{n-1}, \ldots, x_{-m}) + 1}{f(x_{n-1}, \ldots, x_{-m})}
\]

\[
\leq \frac{f(x_{n-1}, \ldots, x_{-m}) + 1}{f(x_{n-1}, \ldots, x_{-m})} = g(x_{n-1}, \ldots, x_{-m}) \leq x_{n-1}^*.
\]

from which we get that for any \(n \geq 0\) and \(0 \leq i \leq m_1\),

\[
1 \leq x_{i+n(m_1+1)}^* \leq x_{i+n(m_1+1)}^*.
\]

Let \(\lim_{n \to \infty} x_i^* = A_i\) for any \(0 \leq i \leq m_1\); then \(A_i \geq 1\). Write \(M = \max\{A_0, A_1, \ldots, A_{m_1}\}\) and \(A_{i+n(m_1+1)} = A_i\) for any integer \(n(0 \leq i \leq m_1)\). Then there exists some \(0 \leq j \leq m_1\) such that

\[
\lim_{n \to \infty} x_j^* = M.
\]
By (3) we have
\[ x_{j+(n+1)(m_1+1)}^* \leq g(x_{j+n(m_1+1)}^*, x_{j+(n+1)(m_1+1)-1-m_2}, \ldots, x_{j+(n+1)(m_1+1)-1-m_j}) \leq x_{j+n(m_1+1)}^*, \]
from which follows
\[ M = g(M, A_{j-1-m_2}, \ldots, A_{j-1-m_j}) = M. \]

Since
\[ 1 \leq x_{n+1}^* \leq \frac{f(x_{n-r_1}, \ldots, x_{n-r_k})^*}{f(x_{n-r_1}, \ldots, x_{n-r_k})^*} = \frac{g(x_{n-m_1}, \ldots, x_{n-m_j})^*}{g(x_{n-m_1}, \ldots, x_{n-m_j})^*}, \]
from which it follows that
\[ 1 \leq x_{n+1}^* \leq \frac{F(x_{n-r_1}, \ldots, x_{n-r_k}) g(x_{n-m_1}, \ldots, x_{n-m_j}) + 1}{F(x_{n-r_1}, \ldots, x_{n-r_k}) + g(x_{n-m_1}, \ldots, x_{n-m_j})}. \]

Therefore
\[ M \leq \frac{F(A_{j-1-r_1}, \ldots, A_{j-1-r_k}) g(M, A_{j-1-m_2}, \ldots, A_{j-1-m_j}) + 1}{F(A_{j-1-r_1}, \ldots, A_{j-1-r_k}) + g(M, A_{j-1-m_2}, \ldots, A_{j-1-m_j}) + M} M = \frac{F(A_{j-1-r_1}, \ldots, A_{j-1-r_k}) M + 1}{F(A_{j-1-r_1}, \ldots, A_{j-1-r_k}) + M}, \]
from which it follows that \( M = 1. \) This implies \( A_i = 1 \) for \( 0 \leq i \leq m_1 \) and \( \lim_{n \to \infty} x_n^* = 1. \) Since \( 1/x_n^* \leq x_n \leq x_n^*, \)
we obtain \( \lim_{n \to \infty} x_n = 1. \) By (1) it follows that
\[ 1 = \frac{f(1, 1, \ldots, 1) g(1, 1, \ldots, 1) + h(1, 1, \ldots, 1) + 1}{f(1, 1, \ldots, 1) + g(1, 1, \ldots, 1) + h(1, 1, \ldots, 1)}. \]

Thus \( \bar{x} = 1 \) is the unique positive equilibrium of Eq. (1) and all of its solutions converge to 1. In the following we show that \( \bar{x} = 1 \) is locally stable.

For any \( 1 > \varepsilon > 0, \) choose \( \delta = \varepsilon/(1 + \varepsilon) \) and let \( \{x_n\}_{n=-m}^{\infty} \) be a solution of Eq. (1) with initial conditions \( x_{-m}, x_{-m+1}, \ldots, x_0 \in (1-\delta, 1+\delta). \) Then for any \(-m \leq i \leq 0, \) we have that \( x_i < 1+\varepsilon \) and \( 1/x_i \leq 1/(1-\delta) = 1+\varepsilon. \) By (3) it follows that for any \( n \geq 0, \)
\[ 1 \leq x_{n+1}^* \leq x_{n-m_1} < 1 + \varepsilon. \]

Thus we get that for any \( n \geq 0, \)
\[ 1 - \varepsilon < \frac{1}{1 + \varepsilon} < \frac{1}{x_{n+1}^*} \leq x_{n+1} \leq x_{n+1}^* < 1 + \varepsilon. \]

This implies that \( \bar{x} = 1 \) is globally asymptotically stable. This completes the proof. \( \square \)

3. Examples

In this section, we shall give some applications of Theorem 1.

**Example 1.** Consider the equation
\[ x_{n+1} = \frac{(x_{n-r_0} x_{n-r_1} + 1) g(x_{n-r_2}, \ldots, x_{n-r_k}) + x_{a-r_0} + x_{a-r_1} + a}{x_{n-r_0} x_{n-r_1} + 1 + g(x_{n-r_2}, \ldots, x_{n-r_k}) (x_{a-r_0} + x_{n-r_1}) + a}, \quad n = 0, 1, \ldots, \tag{4} \]
where \( 0 \leq r_0 < r_1 < \cdots < r_k, \ a \in [0, +\infty), \) the initial conditions \( x_{-r_k}, \ldots, x_0 \in R_+ \) and \( g \in C(R_+^{k-1}, R_+) \) satisfies \([g(u_2, u_3, \ldots, u_k)]^* = g(u_2^*, u_3^*, \ldots, u_k^*) \leq u_2^* \). Then \( \bar{x} = 1 \) is the unique positive equilibrium of Eq. (4) which is globally asymptotically stable.
Lemma 1. Let $(x, y) = (x, y) = (x + 1)/(x + y)$ $(x > 0, y > 0)$ and $h(x, y) = a/(x + y)$ $(x > 0, y > 0)$. From Lemma 2, it follows that

$$\left[ f(x, y) \right]^* = \frac{x^* y^* + 1}{x^* + y^*} = F(x^*, y^*).$$

Thus conditions (H1) and (H2) hold. By Theorem 1 we know that $\bar{x} = 1$ is the unique positive equilibrium of Eq. (4) which is globally asymptotically stable.

Remark 1. Let $k = 2$ and $g(x) = x(x > 0)$; Eq. (4) reduces to Eq. (E3) and (E4).

Example 2. Consider the equation

$$x_{n+1} = \frac{x_n^b r_0 (x_n-r_2, \ldots, x_n-r_k) + x_n^b + a}{x_n^b + g(x_n-r_2, \ldots, x_n-r_k) x_n^b + a}, \quad n = 0, 1, \ldots,$$

where $r_0, r_1, \ldots, r_k \in \{0, 1, \ldots\}$ with $r_i \neq r_j$ for $i \neq j$, $a, b \in [0, +\infty)$, the initial values are positive and $g \in C(R_{+}^{k-1}, R_+)$ satisfies $g(u_2, u_3, \ldots, u_k))^* = g(u_2^*, u_3^*, \ldots, u_k^*) \leq u_2^*$. Then $\bar{x} = 1$ is the unique positive equilibrium of Eq. (5) which is globally asymptotically stable.

Proof. Let $f(x, y) = (x/y)^b (x > 0, y > 0)$ and $h(y) = a/y^b (y > 0)$. From Lemma 1, it follows that

$$\left[ f(x, y) \right]^* = \left[ \frac{x}{y} \right]^b \leq \max \left\{ (x^*)^b, (y^*)^b, \frac{(y^*)^b}{(x^*)^b} \right\} = F(x^*, y^*),$$

where $F(x, y) = \max\{a, (xy)^b, ((x/y)^b)/(y^b)\}$. Thus conditions (H1) and (H2) hold. By Theorem 1 we know that $\bar{x} = 1$ is the unique positive equilibrium of Eq. (5) which is globally asymptotically stable.

Remark 2. Let $k = 2$ and $g(x) = x(x > 0)$; Eq. (5) reduces to Eq. (E6).

Example 3. Consider the equation

$$x_{n+1} = \frac{x_n r_0 (x_n-r_2, \ldots, x_n-r_k) + 1 + h(x_n-r_2, \ldots, x_n-r_k)}{x_n r_0 + x_n r_1 + h(x_n-r_2, \ldots, x_n-r_k)}, \quad n = 0, 1, \ldots,$$

where $r_0, r_1, \ldots, r_k \in \{0, 1, \ldots\}$ with $r_i \neq r_j$ for $i \neq j$, the initial values are positive and $h \in C(R_{+}^{k-1}, [0, +\infty))$. Then $\bar{x} = 1$ is the unique positive equilibrium of Eq. (6) which is globally asymptotically stable.

Proof. Let $F(x) = f(x) = g(x) = x(x > 0)$; it is obvious that conditions (H1) and (H2) hold. By Theorem 1 we know that $\bar{x} = 1$ is the unique positive equilibrium of Eq. (6) which is globally asymptotically stable.

Remark 3. Let $h(u_2, \ldots, u_k) = u_2 + \cdots + u_k (u_i > 0$ for $2 \leq i \leq k)$; Eq. (6) reduces to Eq. (E5).

References