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Deformation quantization with generators and relations ☆

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A B S T R A C T

In this paper we prove a conjecture of B. Shoikhet which claims that two quantization procedures arising from Fourier dual constructions actually coincide.

1. Introduction

There are two ways to quantize a polynomial Poisson structure π on the dual V^* of a finite-dimensional complex vector space V, using Kontsevich's formality as a starting point.

The first (obvious) way is to consider the image $\mathcal{U}(\pi_\hbar)$ of $\pi_\hbar = \hbar\pi$ through Kontsevich's L_∞-quasi-isomorphism

$$\mathcal{U} : \mathrm{T}_{\mathrm{poly}}(V^*) \longrightarrow \mathrm{D}_{\mathrm{poly}}(V^*),$$

and to take $\mathrm{m}_\star := \mathrm{m} + \mathcal{U}(\pi_\hbar)$ as a $\star$-product quantizing π, m being the standard product on $\mathrm{S}(V) = \mathcal{O}_{V^*}$.

The main idea, due to B. Shoikhet [8], behind the second (less obvious) way is to deform the relations of $\mathrm{S}(V)$ instead of the product m itself.

Consider for example a constant Poisson structure π on V^*: the deformation quantization of $\mathrm{S}(V)$ w.r.t. $\hbar\pi$ is the Moyal–Weyl algebra $\mathrm{S}(V)[[\hbar]]$ with the Moyal product $\star$ given by

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$$f_1 \star f_2 = \mathrm{m}\left(\exp \frac{\hbar\pi}{2}(f_1 \otimes f_2)\right),$$

where π is understood here as a bidifferential endomorphism of $\mathrm{S}(V) \otimes \mathrm{S}(V)$. On the other hand, it is well known that the Moyal–Weyl algebra associated to π is isomorphic to the free associative algebra over $\mathbb{C}[[\hbar]]$ with generators x_i (for $\{x_i\}$ a basis of V) by the relations

$$x_i \star x_j - x_j \star x_i = \hbar\pi_{ij}.$$

The construction we are interested in generalizes to any polynomial Poisson structure π on V^* the two ways of characterizing the Moyal–Weyl algebra associated to π.

More conceptually, $\mathrm{S}(V)$ is a quadratic Koszul algebra of the form $T(V)/\langle R\rangle$, where R is the subspace of $V^{\otimes 2}$ spanned by vectors of the form $x_i \otimes x_j - x_j \otimes x_i$, $\{x_i\}$ as in the previous paragraph. The right-hand side of the identity $\mathrm{S}(V) = \mathrm{T}(V)/\langle R\rangle$ can be viewed as the 0-th cohomology of the free associative dg (short for differential graded from now on) algebra $\mathrm{T}(\wedge^-(V))$ over $\mathbb{C}$, where $\wedge^-(V)$ is the graded vector space $\wedge^-(V) = \bigoplus_{p=-d+1}^{0} \wedge^-(V)_p = \bigoplus_{p=-d+1}^{0} \wedge^{-p+1}(V)$ and differential δ on generators $\{x_{i_1}, x_{i_1,i_2}, \ldots\}$ of $\wedge^-(V)$ specified by

$$\delta(x_{i_1}) = 0, \qquad \delta(x_{i_1,i_2}) = x_{i_1} \otimes x_{i_2} - x_{i_2} \otimes x_{i_1}, \text{ etc.}$$

Observe that the differential δ dualizes the product of the graded commutative algebra $\wedge(V^*)$: in fact, $\wedge(V^*)$ is the Koszul dual of $\mathrm{S}(V)$, and the above complex comes from the identification $\mathrm{S}(V) = \mathrm{Ext}_{\wedge(V^*)}(\mathbb{C}, \mathbb{C})$ by explicitly computing the cohomology on the right-hand side w.r.t. the bar resolution of $\mathbb{C}$ as a (left) $\wedge(V^*)$-module (the above dg, short for differential graded, algebra is the cobar construction of $\mathrm{S}(V)$, and δ is the cobar differential). The above dg algebra is acyclic except in degree 0; the 0-th cohomology is readily computed from the above formulæ and equals precisely $\mathrm{T}(V)/\langle R\rangle$.

Therefore, the idea is to prove that the property of being Koszul and the Koszul duality between $\mathrm{S}(V)$ and $\wedge(V^*)$ is preserved (in a suitable sense, which will be clarified later on) by deformation quantization.

Namely, one makes use of the graded version [3] of Kontsevich's formality theorem, applied to the Fourier dual space $V[1]$. We then have an L_∞-quasi-isomorphism

$$\mathcal{V} : T_{\mathrm{poly}}(V^*) \cong T_{\mathrm{poly}}(V[1]) \longrightarrow D_{\mathrm{poly}}(V[1]),$$

and the image $\mathcal{V}(\widehat{\pi_\hbar})$ of $\widehat{\pi_\hbar}$, where $\widehat{\bullet}$ is the isomorphism $\mathrm{T}_{\mathrm{poly}}(V^*) \cong \mathrm{T}_{\mathrm{poly}}(V[1])$ of dg Lie algebras (graded Fourier transform), defines a deformation quantization of the graded commutative algebra $\wedge(V^*)$ as a (possibly curved) A_∞-algebra.

In the context of the formality theorem with two branes [2], the deformation quantization of $\wedge(V^*)$ is the Koszul dual (in a suitable sense) w.r.t. the first deformation quantization of $\mathrm{S}(V)$, and the (possibly curved) A_∞-structure on the deformation quantization of $\wedge(V^*)$ induces a deformation $\delta_\hbar$ of the cobar differential δ, which in turn produces a deformation $\mathcal{I}_\star$ of the two-sided ideal $\mathcal{I} = \langle R\rangle$ in $\mathrm{T}(V)$ of defining relations of $\mathrm{S}(V)$.

We are then able to prove the following result, first conjectured by Shoikhet in [7, Conjecture 2.6]:

Theorem 1.1. *(See Theorem 2.7.) Given a polynomial Poisson structure π on V^* as above, the algebra $A_\hbar := (\mathrm{S}(V)[[\hbar]], \mathrm{m}_\star)$ is isomorphic to the quotient of $\mathrm{T}(V)[[\hbar]]$ by the two-sided ideal $\mathcal{I}_\star$; the isomorphism is an $\hbar$-deformation of the standard symmetrization map from $\mathrm{S}(V)$ to $\mathrm{T}(V)$.*

Remark 1.2. We mainly consider here a formal polynomial Poisson structure of the form $\hbar\pi$, but all the arguments presented here apply as well to any formal polynomial Poisson structure $\pi_\hbar = \hbar\pi_1 + \hbar^2\pi_2 + \cdots$, where π_i is a polynomial bivector field.

The paper is organized as follows. In Section 2 we start with a recollection on A_∞-algebras and bimodules. We then formulate the formality theorem with two branes of [2] in a form suitable for the application at hand. After this we describe the deformation of the cobar complex obtained from $\mathcal{V}(\widehat{\pi_\hbar})$ and prove Theorem 1.1. We conclude the paper with three examples, see Section 3: the cases of constant, linear, and quadratic Poisson structures.

2. A deformation of the cobar construction of the exterior coalgebra

2.1. A_∞-algebras and (bi)modules of finite type

We first recall the basic notions of the theory of A_∞-algebras and modules, see [2,5] to fix the conventions and settle some finiteness issues. Note that we allow non-flat A_∞-algebras in our definition. Let $\mathrm{T}(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus \cdots$ be the tensor coalgebra of a $\mathbb{Z}$-graded complex vector space V with coproduct $\Delta(v_1, \ldots, v_n) = \sum_{i=0}^n (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n)$ and counit $\eta(1) = 1$, $\eta(v_1, \ldots, v_n) = 0$ for $n \geqslant 1$. Here we write $(v_1, \ldots, v_n)$ as a more transparent notation for $v_1 \otimes \cdots \otimes v_n \in \mathrm{T}(V)$ and set $() = 1 \in \mathbb{C}$. Let $V[1]$ be the graded vector space with $V[1]^i = V^{i+1}$ and let the suspension $s : V \to V[1]$ be the map $a \mapsto a$ of degree -1. Then an A_∞-algebra over $\mathbb{C}$ is a $\mathbb{Z}$-graded vector space B together with a codifferential $\mathrm{d}_B : \mathrm{T}(B[1]) \to \mathrm{T}(B[1])$, namely a linear map of degree 1 which is a coderivation of the coalgebra and such that $\mathrm{d}_B \circ \mathrm{d}_B = 0$. A coderivation is uniquely given by its components $\mathrm{d}_B^k : B[1]^{\otimes k} \to B[1]$, $k \geqslant 0$ and any set of maps $: B[1]^{\otimes k} \to B[1]$ of degree 1 uniquely extends to a coderivation. This coderivation is a codifferential if and only if $\sum_{j+k+l=n} \mathrm{d}_B^n \circ (\mathrm{id}^{\otimes j} \otimes \mathrm{d}_B^k \otimes \mathrm{id}^{\otimes l}) = 0$ for all $n \geqslant 0$. The maps d_B^k are called the *Taylor components* of the codifferential d_B. If $\mathrm{d}_B^0 = 0$, the A_∞-algebra is called *flat*. Instead of d_B^k it is convenient to describe A_∞-algebras through the product maps $\mathrm{m}_B^k = s^{-1} \circ \mathrm{d}_B^k \circ s^{\otimes k}$ of degree $2-k$. If $\mathrm{m}_B^k = 0$ for all $k \neq 1, 2$ then B with differential m_B^1 and product m_B^2 is a differential graded algebra. A *unital* A_∞-algebra is an A_∞-algebra B with an element $1 \in B^0$ such that

$$\begin{aligned} &\mathrm{m}_B^2(1, b) = \mathrm{m}_B^2(b, 1) = b, && \forall b \in B, \\ &\mathrm{m}_B^j(b_1, \ldots, b_j) = 0, && \text{if } b_i = 1 \text{ for some } 1 \leqslant i \leqslant j \text{ and } j \neq 2. \end{aligned}$$

The first condition translates to $\mathrm{d}_B^2(s1, b) = b = (-1)^{|b|-1} \mathrm{d}_B^2(b, s1)$, if $b \in B[1]$ has degree $|b|$. A *right A_∞-module* M over an A_∞-algebra B is a graded vector space M together with a degree one codifferential d_M on the cofree right $\mathrm{T}(B[1])$-comodule $M[1] \otimes \mathrm{T}(B[1])$ cogenerated by M. The Taylor components are $\mathrm{d}_M^j : M[1] \otimes B[1]^{\otimes j} \to M[1]$ and in the unital case we require that $\mathrm{d}_M^1(m, s1) = (-1)^{|m|-1} m$ and $\mathrm{d}_M^j(m, b_1, \ldots, b_j) = 0$ if some b_j is $s1$. Left modules are defined similarly. An A_∞-A-B-bimodule M over A_∞-algebras A, B is the datum of a codifferential on the $\mathrm{T}(A[1])$-$\mathrm{T}(B[1])$-bicomodule $\mathrm{T}(A[1]) \otimes M[1] \otimes \mathrm{T}(B[1])$, given by its Taylor components $\mathrm{d}_M^{j,k} : A[1]^{\otimes j} \otimes M[1] \otimes B[1]^k \to M[1]$. The following is a simple but important observation.

Lemma 2.1. *If M is an A_∞-A-B-bimodule and A is a flat A_∞-algebra then M with the Taylor components $\mathrm{d}_M^{0,k}$ is a right A_∞-module over B.*

Morphisms of A_∞-algebras (A_∞-(bi)modules) are (degree 0) morphisms of graded counital coalgebras (respectively, (bi)comodules) commuting with the codifferentials. Morphisms of tensor coalgebras and of free comodules are again uniquely determined by their Taylor components. For instance a morphism of right A_∞-modules $M \to N$ over B is uniquely determined by the components $f_j : M[1] \otimes B[1]^{\otimes j} \to N[1]$.

Definition 2.2. A morphism between cofree (left-, right-, bi-) comodules over the cofree tensor coalgebra is said to be of *finite type* if all but finitely many of its Taylor components vanish. Therefore, by abuse of terminology, we may speak of a morphism of finite type between (left-, right-, bi-) A_∞-modules over an A_∞-algebra.

The identity morphism is of finite type and the composition of morphisms of finite type is again of finite type.

The unital algebra of endomorphisms of finite type of a right A_∞-module M over an A_∞-algebra B is the 0-th cohomology of a differential graded algebra $\underline{\mathrm{End}}_{-B}(M) = \bigoplus_{j\in\mathbb{Z}} \underline{\mathrm{End}}^j_{-B}(M)$. The component of degree j is the space of endomorphisms of degree j of finite type of the comodule $M[1]\otimes \mathrm{T}(B[1])$. The differential is the graded commutator $\delta f = [\mathrm{d}_M, f] = \mathrm{d}_M \circ f - (-1)^j f \circ \mathrm{d}_M$ for $f \in \underline{\mathrm{End}}^j_{-B}(M)$. If M is an A_∞-A-B-bimodule and A is flat, then $\underline{\mathrm{End}}_{-B}(M)$ is defined and the left A-module structure induces a *left action* L_A, which is a morphism of A_∞-algebras $A \to \underline{\mathrm{End}}_{-B}(M)$: its Taylor components are $\mathrm{L}^j_A(a)^k(m\otimes b) = \mathrm{d}^{j,k}_M(a\otimes m\otimes b)$, $a \in A[1]^{\otimes j}$, $m \in M[1]$, $b \in B[1]^{\otimes k}$.

Lemma 2.3. *Let M be a right A_∞-module over a unital A_∞-algebra B. Then the subspace $\underline{\mathrm{End}}_{-B^+}(M)$ of endomorphisms f such that $f_j(m, b_1, \dots, b_j) = 0$ whenever $b_i = s1$ for some i, is a differential graded subalgebra.*

We call this differential graded subalgebra the subalgebra of *normalized* endomorphisms.

Proof. It is clear from the formula for the Taylor components of the composition that normalized endomorphisms form a graded subalgebra: $(f\circ g)^k = \sum_{i+j=k} f^j \circ (g^i \otimes \mathrm{id}^{\otimes j}_{B[1]})$. The formula for the Taylor components of the differential of an endomorphism f is

$$(\delta f)^k = \sum_{i+j=k} \bigl(\mathrm{d}^j_M \circ \bigl(f^i \otimes \mathrm{id}^{\otimes j}_{B[1]}\bigr) - (-1)^{|f|} f^i \circ \bigl(\mathrm{d}^j_M \otimes \mathrm{id}^{\otimes i}_{B[1]}\bigr)$$
$$- (-1)^{|f|} f^{k-j+1} \circ \bigl(\mathrm{id}_{M[1]} \otimes \mathrm{id}^{\otimes i}_{B[1]} \otimes \mathrm{d}^j_B \otimes \mathrm{id}^{\otimes(k-i-j)}_{B[1]}\bigr)\bigr).$$

If f is normalized and $b_i = s1$ for some i, then only two terms contribute non-trivially to $(\delta f)^k(m, b_1, \dots, b_k)$, namely $f^{k-1}(m, b_1, \dots, \mathrm{d}^2_B(s1, b_{i+1}), \dots)$ (or $\mathrm{d}^1_M(f^{k-1}(m, b_1, \dots, b_{k-1}), s1)$ if $i = k$) and $f^{k-1}(m, b_1, \dots, \mathrm{d}^2_B(b_{i-1}, s1), \dots)$ (or $f^{k-1}(\mathrm{d}^1_M(m, s1), b_2, \dots)$ if $i = 1$). Due to the unital condition these two terms are equal up to the sign, hence cancel together. $\square$

The same definitions apply to A_∞-algebras and A_∞-bimodules over $\mathbb{C}[[\hbar]]$ with completed tensor products and continuous homomorphisms for the $\hbar$-adic topology, so that for vector spaces V, W we have $V[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} W[[\hbar]] = (V \otimes_{\mathbb{C}} W)[[\hbar]]$ and $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(V[[\hbar]], W[[\hbar]]) = \mathrm{Hom}_{\mathbb{C}}(V, W)[[\hbar]]$. A flat deformation of an A_∞-algebra B is an A_∞-algebra $B_\hbar$ over $\mathbb{C}[[\hbar]]$ which, as a $\mathbb{C}[[\hbar]]$-module, is isomorphic to $B[[\hbar]]$ and such that $B_\hbar/\hbar B_\hbar \simeq B$. Similarly we have flat deformations of (bi)modules. A right A_∞-module $M_\hbar$ over $B_\hbar$ which is a flat deformation of M over B is given by the Taylor coefficients $\mathrm{d}^j_{M_\hbar} \in \mathrm{Hom}_{\mathbb{C}}(M[1] \otimes B[1]^{\otimes j}, M[1])[[\hbar]]$. The differential graded algebra $\underline{\mathrm{End}}_{-B_\hbar}(M_\hbar)$ of endomorphism of finite type is then defined as the direct sum of the homogeneous components of $\mathrm{End}^{\mathrm{finite}}_{\text{comod-}\mathrm{T}(B[1])}(M[1] \otimes \mathrm{T}(B[1]))[[\hbar]]$ with differential $\delta_\hbar = [\mathrm{d}_{M_\hbar}, \ ]$. Thus its degree j part is the $\mathbb{C}[[\hbar]]$-module

$$\underline{\mathrm{End}}^j_{B_\hbar}(M_\hbar) = \Biggl(\bigoplus_{k\geqslant 0} \mathrm{Hom}^j\bigl(M[1] \otimes B[1]^{\otimes k}, M[1]\bigr)\Biggr)[[\hbar]],$$

where Hom^j is the space of homomorphisms of degree j between graded vector spaces over $\mathbb{C}$.

Finally, the following notation will be used: if $\phi : V_1[1] \otimes \cdots \otimes V_n[1] \to W[1]$ is a linear map and V_i, W are graded vector spaces or free $\mathbb{C}[[\hbar]]$-modules, we set

$$\phi\bigl(v_1|\cdots|v_n\bigr) = s^{-1}\phi(sv_1 \otimes \cdots \otimes sv_n), \quad v_i \in V_i.$$

2.2. Formality theorem for two branes and deformation of bimodules

Let $A = \mathrm{S}(V)$ be the symmetric algebra of a finite-dimensional vector space V, viewed as a graded algebra concentrated in degree 0. Let $B = \wedge(V^*) = \mathrm{S}(V^*[-1])$ be the exterior algebra of the dual space with $\wedge^i(V^*)$ of degree i.[1] For any graded vector space W, the augmentation module over $\mathrm{S}(W)$ is the unique one-dimensional module on which W acts by 0. Let $A_\hbar = (A[[\hbar]], \star)$ be the Kontsevich deformation quantization of A associated with a polynomial Poisson bivector field $\hbar\pi$. It is an associative algebra over $\mathbb{C}[[\hbar]]$ with unit 1. The graded version of the formality theorem, applied to the same Poisson bracket (more precisely, to the image of $\hbar\pi$ w.r.t. the isomorphism of dg Lie algebras $T_{\mathrm{poly}}(A) \cong T_{\mathrm{poly}}(B)$), also defines a deformation quantization $B_\hbar$ of the graded commutative algebra B. However $B_\hbar$ is in general a unital A_∞-algebra with non-trivial Taylor components $\mathrm{d}^k_{B_\hbar}$ for all k including $k = 0$. Still, the differential graded algebra $\underline{\mathrm{End}}_{-B_\hbar}(M_\hbar)$ is defined since $A_\hbar$ is an associative algebra and thus a flat A_∞-algebra. The following result is a consequence of the formality theorem for two branes (= submanifolds) in an affine space, in the special case where one brane is the whole space and the other a point, and is proved in [2]. It is a version of the Koszul duality between $A_\hbar$ and $B_\hbar$.

Proposition 2.4. *Let $A = \mathrm{S}(V)$, $B = \wedge(V^*)$ for some finite-dimensional vector space V and let $A_\hbar$, $B_\hbar$ be their deformation quantizations corresponding to a polynomial Poisson bracket bivector.*

(i) *There exists a one-dimensional A_∞-A-B-bimodule K, which, as a left A-module and as a right B-module, is the augmentation module, and such that $\mathrm{L}_A : A \to \underline{\mathrm{End}}_{-B}(K)$ is an A_∞-quasi-isomorphism.*
(ii) *The bimodule K admits a flat deformation $K_\hbar$ as an A_∞-$A_\hbar$-$B_\hbar$-bimodule such that $\mathrm{L}_{A_\hbar} : A_\hbar \to \underline{\mathrm{End}}_{-B_\hbar}(K_\hbar)$ is an A_∞-quasi-isomorphism.*
(iii) *The A_∞-$A_\hbar$-$B_\hbar$-bimodule $K_\hbar$ is in particular a right A_∞-module over the unital A_∞-algebra $B_\hbar$. The first Taylor component $\mathrm{L}^1_{A_\hbar}$ sends $A_\hbar$ to the differential graded subalgebra $\underline{\mathrm{End}}_{-B_\hbar^+}(K_\hbar)$ of normalized endomorphisms.*

The proof of (i) and (ii) is contained in [2]. The claim (iii) follows from the explicit form of the Taylor components $\mathrm{d}^{1,j}_{K_\hbar}$, given in [2], appearing in the definition of L^1_A:

$$\mathrm{L}^1_{A_\hbar}(a)^j\big(1|b_1|\cdots|b_j\big) = \mathrm{d}^{1,j}_{K_\hbar}\big(a|1|b_1|\cdots|b_j\big).$$

Namely $\mathrm{d}^{1,j}_{K_\hbar}$ is a power series in $\hbar$ whose term of degree m is a sum over certain directed graphs with m vertices in the complex upper half-plane (vertices of the first type) and $j+2$ ordered vertices on the real axis (vertices of the second type). To each vertex of the first type is associated a copy of $\hbar\pi$; to the first vertex of the second type is associated a, to the second 1, and to the remaining j vertices the elements b_i. An example of such a graph is depicted in Fig. 4, Section 3.2.

Each graph contributes a multidifferential operator acting on $a, b_1, \ldots, b_j$ times a weight, which is an integral of a differential form on a compactified configuration space of m points in the complex upper half-plane and $j+2$ ordered points on the real axis modulo dilations and real translations. The convention is that to each directed edge of such a graph is associated a derivative acting on the element associated to the final point of the said edge and a 1-form on the corresponding compactified configuration space.

Therefore, since each b_i may be regarded as a constant polyvector field on V^*, there is no edge with final point at a vertex of the second type where a b_i sits (and obviously also where the constant

[1] In the case at hand, V is a graded vector space concentrated in degree 0 and the identification $\wedge(V^*) = \mathrm{S}(V^*[-1])$ as *graded algebras* is canonical. For a more general graded vector space V, $\mathrm{S}(V^*[-1])$ and $\wedge(V^*)$ are different objects; still, $\mathrm{S}^n(V^*[-1])$ is canonically isomorphic to $\wedge^n(V^*)[-n]$ as a graded vector space for every n by the *décalage* isomorphism, which is simply the identity when V is concentrated in degree 0.

function 1 sits). If $j \geqslant 1$ and b_i belongs to $\mathbb{C}$ for some $1 \leqslant i \leqslant j$, the vertex of the second type where b_i sits is neither the starting nor the final point of any directed edge: since $j \geqslant 1$, the dimension of the corresponding compactified configuration space is strictly positive. We may use dilations and real translations to fix vertices (of the first and/or second type) distinct from the one where b_i sits: thus, there would be a one-dimensional submanifold (corresponding to the interval, where the vertex corresponding to b_i sits), over which there is nothing to integrate, hence the corresponding weight vanishes.

We turn to the description of the differential graded algebra $\underline{\mathrm{End}}^j_{-B^+_\hbar}(K_\hbar)$. Let $B^+ = \bigoplus_{j\geqslant 1} \wedge^j(V^*) = \wedge(V^*)/\mathbb{C}$. We have

$$\underline{\mathrm{End}}^j_{-B^+_\hbar}(K_\hbar) = \left(\bigoplus_{k\geqslant 0} \mathrm{Hom}^j\big(K[1]\otimes B^+[1]^{\otimes k}, K[1]\big)\right)[[\hbar]],$$

with product

$$(\phi\cdot\psi)\big(1|b_1|\cdots|b_n\big) = \sum_k \psi\big(1|b_1|\cdots|b_k\big)\phi\big(1|b_{k+1}|\cdots|b_n\big).$$

It follows that the algebra $\underline{\mathrm{End}}^j_{-B^+_\hbar}(K_\hbar)$ is isomorphic to the tensor algebra $\mathrm{T}(B^+[1]^*)[[\hbar]]$ generated by $\mathrm{Hom}(K[1]\otimes B^+[1], K[1]) \simeq B^+[1]^*$. In particular it is concentrated in non-positive degrees.

Lemma 2.5. *The restriction $\delta_\hbar : B^+[1]^* \to \mathrm{T}(B^+[1]^*)[[\hbar]]$ of the differential of $\underline{\mathrm{End}}_{-B^+_\hbar}(K_\hbar) \simeq \mathrm{T}(B^+[1]^*)[[\hbar]]$ to the generators is dual to the A_∞-structure $\mathrm{d}_{B_\hbar}$ in the sense that*

$$(\delta_\hbar f)^k(z\otimes b) = (-1)^{|f|} f\big(z\otimes \mathrm{d}^k_{B_\hbar}(b)\big), \quad z\in K[1],\ b\in B[1]^{\otimes k},$$

for any $f \in \mathrm{Hom}(K[1]\otimes B^+[1], K[1]) \simeq B^+[1]^*$.

Proof. The A_∞-structure of $B_\hbar$ is given by the Taylor components $\mathrm{d}^k_{B_\hbar} : B[1]^{\otimes k} \to B[1]$. By definition the differential on $\underline{\mathrm{End}}^j_{-B^+_\hbar}(K_\hbar)$ is the graded commutator $\delta_\hbar f = [\mathrm{d}_{K_\hbar}, f]$. In terms of the Taylor components,

$$\begin{aligned}(\delta_\hbar f)^k(z\otimes b_1\otimes\cdots\otimes b_k) &= \mathrm{d}^{k-1}_{K_\hbar}\big(f(z\otimes b_1)\otimes b_2\otimes\cdots\otimes b_k\big)\\ &\quad - (-1)^{|f|} f\big(\mathrm{d}^{k-1}_{K_\hbar}(z\otimes b_1\otimes\cdots\otimes b_{k-1})\otimes b_k\big)\\ &\quad + (-1)^{|f|} f\big(z\otimes \mathrm{d}^k_{B_\hbar}(b_1\otimes\cdots\otimes b_k)\big).\end{aligned}$$

The first two terms vanish if $b_i \in B^+[1]$ for degree reasons. □

Thus $\mathrm{L}_{A_\hbar}$ induces an isomorphism from $A_\hbar$ to the cohomology in degree 0 of $\underline{\mathrm{End}}_{-B^+_\hbar}(K_\hbar) \simeq \mathrm{T}(B^+[1]^*)[[\hbar]]$.

Remark 2.6. For $\hbar = 0$ this complex is Adam's cobar construction of the graded coalgebra B^*, which is a free resolution of $\mathrm{S}(V)$.

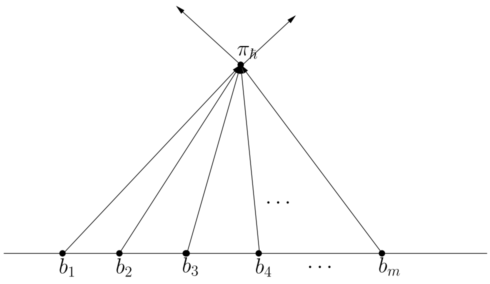

Fig. 1. The only admissible graph contributing to $\mathrm{d}^m_{B_\hbar}$ at order 1 in $\hbar$.

Theorem 2.7. *The composition*

$$\mathrm{L}^1_{A_\hbar} : A_\hbar \to \underline{\mathrm{End}}_{-B^+_\hbar}(K_\hbar) \xrightarrow{\simeq} \mathrm{T}\big(B^+[1]^*\big)[[\hbar]],$$

induces on cohomology an algebra isomorphism

$$\mathrm{L}^1_{A_\hbar} : A_\hbar \to \mathrm{T}(V)/\big(\mathrm{T}(V) \otimes \delta_\hbar\big(\big(\wedge^2 V^*\big)^*\big) \otimes \mathrm{T}(V)\big),$$

where $\delta_\hbar : (\wedge^2 V^*)^* \to \mathrm{T}(V)[[\hbar]]$ *is dual to* $\bigoplus_{k\geqslant 0} \mathrm{d}^k_{B_\hbar} : (B^+[1]^0)^{\otimes k} = V^{\otimes k} \to B^+[1]^1 = \wedge^2 V^*$.

Proof. The fact that the map is an isomorphism follows from the fact that it is so for $\hbar = 0$, by the classical Koszul duality. As the cohomology is concentrated in degree 0 it remains so for the deformed differential $\delta_\hbar$ over $\mathbb{C}[[\hbar]]$.

As a graded vector space, $B^+[1]^* = V \oplus (\wedge^2 V^*)^* \oplus \cdots$, with $(\wedge^i V^*)^*$ in degree $1-i$. Therefore the complex $\mathrm{T}(B^+[1]^*)[[\hbar]]$ is concentrated in non-positive degrees and begins with

$$\cdots \to \big(\mathrm{T}(V) \otimes \big(\wedge^2 V^*\big)^* \otimes \mathrm{T}(V)\big)[[\hbar]] \to \mathrm{T}(V)[[\hbar]] \to 0.$$

Thus to compute the degree 0 cohomology we only need the restriction of the Taylor components $\mathrm{d}^k_{B_\hbar}$ on $\mathrm{T}(V^*) = \mathrm{T}(B^+[1])^0$, whose image is in $B[1]^1 = \wedge^2 V^*$. $\square$

This theorem gives a presentation of the algebra $A_\hbar$ by generators and relations. Let $x_1, \ldots, x_d \in V$ be a system of linear coordinates on V^* dual to a basis $e_1, \ldots, e_d$. Let for $I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, d\}$, $x_I \in (\wedge^k V^*)^*$ be dual to the basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$. Then $A_\hbar$ is isomorphic to the algebra generated by $x_1, \ldots, x_d$ subject to the relations $\delta_\hbar(x_{ij}) = 0$. Up to order 1 in $\hbar$ the relations are obtained from the cobar differential and the graph of Fig. 1.

$$\delta_\hbar(x_{ij}) = x_i \otimes x_j - x_j \otimes x_i - \hbar\, \mathrm{Sym}(\pi_{ij}) + O\big(\hbar^2\big).$$

Here Sym is the symmetrization map $\mathrm{S}(V) \to \mathrm{T}(V)$.

The lowest order of the isomorphism induced by L^1_A on generators $x_i \in V$ of $A_\hbar = \mathrm{S}(V)[[\hbar]]$ was computed in [2]:

$$\mathrm{L}^1_A(x_i) = x_i + O(\hbar).$$

The higher order terms $O(\hbar)$ are in general non-trivial (for example in the case of the dual of a Lie algebra, see below).

By comparing our construction with the arguments in [7], we see that $\delta_\hbar$ corresponds to the image of $\mathcal{V}(\widehat{\pi_\hbar})$, where the notations are as in the introduction, by the quasi-isomorphism Φ_1 in [7, Subsection 1.4]. Hence, Theorem 2.7 provides a proof of [7, Conjecture 2.6] with the amendment that the isomorphism $A_\hbar \to \mathrm{T}(V)/\mathcal{I}_\star$ is not just given by the symmetrization map but has non-trivial corrections.

3. Examples

We now want to examine more closely certain special cases of interest. We assume here that the reader has some familiarity with the graphical techniques of [2,3,6]. To obtain the relations $\delta_\hbar(x_{ij})$ we need $\mathrm{d}^m_{B_\hbar}(b_1|\cdots|b_m) \in \wedge^2 V^*[[\hbar]]$, for $b_i \in V^* \subset B^+$. The contribution at order n in $\hbar$ to this is given by a sum over the set $\mathcal{G}_{n,m}$ of admissible graphs with n vertices of the first type and m of the second type.

3.1. The Moyal–Weyl product on V

Let $\pi_\hbar = \hbar\pi$ be a constant Poisson bivector on V^*, which is uniquely characterized by a complex, skew-symmetric matrix $d \times d$-matrix π_{ij}.

In this case, Kontsevich's deformed algebra $A_\hbar$ has an explicit description: the associative product on $A_\hbar$ is the Moyal–Weyl product

$$(f_1 \star f_2) = \mathrm{m} \circ \exp \frac{1}{2}\pi_\hbar,$$

where $\pi_\hbar$ is viewed here as a bidifferential operator, the exponential has to be understood as a power series of bidifferential operators, and m denotes the ($\mathbb{C}[[\hbar]]$-linear) product on polynomial functions on V^*. On the other hand, it is possible to compute explicitly the complete A_∞-structure on $B_\hbar$.

Lemma 3.1. *For a constant Poisson bivector $\pi_\hbar$ on V^*, the A_∞-structure on $B_\hbar$ has only two non-trivial Taylor components, namely*

$$\mathrm{d}^0_{B_\hbar}(1) = \hbar\pi, \qquad \mathrm{d}^2_{B_\hbar}(b_1|b_2) = (-1)^{|b_1|} b_1 \wedge b_2, \quad b_i \in B_\hbar,\ i = 1, 2. \tag{1}$$

Proof. We consider $\mathrm{d}^m_{B_\hbar}$ first in the case $m = 0$. Admissible graphs contributing to $\mathrm{d}^0_{B_\hbar}$ belong to $\mathcal{G}_{n,0}$, for $n \geqslant 1$. For $n \geqslant 2$, all graphs give contributions involving a derivative of π_{ij} and thus vanish. There remains the only graph in $\mathcal{G}_{1,0}$, whence the first identity in (1).

By the same reasons, $\mathrm{d}^m_{B_\hbar}$ is trivial, if $m \geqslant 1$ and $m \neq 2$: in the case $m = 1$, we have to consider contributions coming from admissible graphs in $\mathcal{G}_{n,1}$, with $n \geqslant 1$, which vanish for the same reasons as in the case $m = 0$.

For $m \geqslant 3$, contributions coming from admissible graphs in $\mathcal{G}_{n,m}$, $n \geqslant 1$, are trivial by a dimensional argument.

Finally, once again, the only possibly non-trivial contribution comes from the unique admissible graph in $\mathcal{G}_{0,2}$ which gives the product. □

As a consequence, the differential $\delta_\hbar$ can be explicitly computed, namely

$$\delta_\hbar(x_{ij}) = x_i \otimes x_j - x_j \otimes x_i - \hbar\pi_{ij}.$$

This provides the description of the Moyal–Weyl algebra as the algebra generated by x_i with relations $[x_i, x_j] = \hbar\pi_{ij}$.

We finally observe that the quasi-isomorphism $\mathrm{L}^1_{A_\hbar}$ coincides, by a direct computation, with the usual symmetrization morphism.

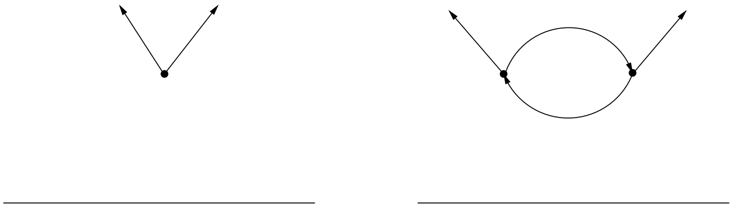

Fig. 2. The only admissible graphs in $\mathcal{G}_{1,0}$ and $\mathcal{G}_{2,0}$ respectively in the curvature of $B_\hbar$.

3.2. The universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$

We now consider a finite-dimensional complex Lie algebra $V = \mathfrak{g}$: its dual space $\mathfrak{g}^*$ with the Kirillov–Kostant–Souriau Poisson structure. With respect to a basis $\{x_i\}$ of $\mathfrak{g}$, we have

$$\pi = f_{ij}^k x_k \partial_i \wedge \partial_j,$$

where f_{ij}^k denote the structure constant of $\mathfrak{g}$ for the chosen basis.

It has been proved in [6, Subsubsection 8.3.1] that Kontsevich's deformed algebra $A_\hbar$ is isomorphic to the universal enveloping algebra $\mathrm{U}_\hbar(\mathfrak{g})$ of $\mathfrak{g}[[\hbar]]$ for the $\hbar$-shifted Lie bracket $\hbar[\,,\,]$.

On the other hand, we may, once again, compute explicitly the A_∞-structure on $B_\hbar$.

Lemma 3.2. *The A_∞-algebra $B_\hbar$ determined by $\pi_\hbar$, where π is the Kirillov–Kostant–Souriau Poisson structure on $\mathfrak{g}^*$, has only two non-trivial Taylor components, namely*

$$\mathrm{d}^1_{B_\hbar}(b_1) = \mathrm{d}_{\mathrm{CE}}(b_1), \qquad \mathrm{d}^2_{B_\hbar}(b_1|b_2) = (-1)^{|b_1|} b_1 \wedge b_2, \quad b_i \in B_\hbar,\ i = 1, 2, \tag{2}$$

where d_{CE} denotes the Chevalley–Eilenberg differential of $\mathfrak{g}$, endowed with the rescaled Poisson bracket $\hbar[\bullet, \bullet]$.

Proof. By dimensional arguments and because of the linearity of $\pi_\hbar$, there are only two admissible graphs in $\mathcal{G}_{1,0}$ and $\mathcal{G}_{2,0}$, which may contribute non-trivially to the curvature of $B_\hbar$, see Fig. 2 for a pictorial description of these two graphs.

The operator $\mathcal{O}^B_\Gamma$ for the graph in $\mathcal{G}_{1,0}$ vanishes, when setting $x = 0$. On the other hand, $\mathcal{O}^B_\Gamma$ vanishes in virtue of [6, Lemma 7.3.1.1].

We now consider the case $m \geqslant 1$. We consider an admissible graph Γ in $\mathcal{G}_{n,m}$ and the corresponding operator $\mathcal{O}^B_\Gamma$: the degree of the operator-valued form ω^B_Γ equals the number of derivations acting on the different entries associated to vertices either of the first or second type. Thus, the operator $\mathcal{O}^B_\Gamma$ has a polynomial part (since all the structures involved are polynomial on $\mathfrak{g}^*$): since the polynomial part of any of its arguments in $B_\hbar$ has degree 0, the polynomial degree of $\mathcal{O}^B_\Gamma$ must be also 0. A direct computation shows that this condition is satisfied if and only if $n + m = 2$, because $\pi_\hbar$ is linear.

Obviously, the previous identity is never satisfied if $m \geqslant 3$, which implies immediately that the only non-trivial Taylor components appear when $m = 1$ and $m = 2$. When $m = 1$, the previous equality forces $n = 1$: there is only one admissible graph Γ in $\mathcal{G}_{1,1}$, whose corresponding operator is non-trivial, in Fig. 3 is depicted the said graph.

The weight is readily computed, and the identification with the Chevalley–Eilenberg differential is then obvious.

Finally, when $m = 2$, the result is clear by the previous computations. $\square$

Thus $\delta_\hbar$ is given by

$$\delta_\hbar(x_{ij}) = x_i \otimes x_j - x_j \otimes x_i - \hbar \sum_k f_{ij}^k x_k.$$

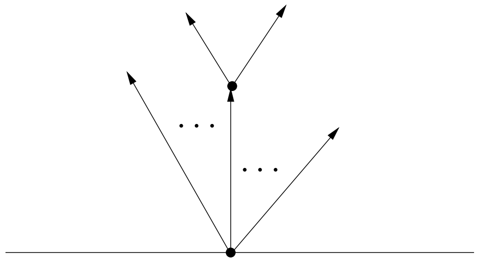

Fig. 3. The only admissible graph in $\mathcal{G}_{1,1}$ contributing to $\mathrm{d}^1_{B_\hbar}$.

Hence we reproduce the result that $A_\hbar$ is isomorphic to $\mathrm{U}_\hbar(\mathfrak{g})$. We now want to give an explicit expression for the isomorphism $\mathrm{L}^1_{A_\hbar}$.

We consider the expression $\mathrm{L}^1_{A_\hbar}(a)^m(1|b_1|\cdots|b_m) = \mathrm{d}^{1,m}_{K_\hbar}(a|1|b_1|\cdots|b_m)$. Degree reasons imply that the sum of the degrees of the elements b_i equals m; furthermore, if the degree of some b_i is strictly bigger than 1, the previous equality forces a different b_j to have degree 0, whence the corresponding expression vanishes by Proposition 2.4(iii). Hence, the degree of each b_i is precisely 1. We now consider a general graph Γ with n vertices of the first type and $m+2$ ordered vertices of the second type; to each vertex of the first type is associated a copy of $\hbar\pi$, while to the ordered vertices of the second type are associated a, 1 and the b_i's in lexicographical order. We denote by p the number of edges departing from the n vertices of the first type and hitting the first vertex of the second type (observe that in this situation edges departing from vertices of the first type can only hit vertices of the first type or the first vertex of the second type): in the present framework, edges have only one color (we refer to [2, Section 7] and [4, Subsection 3.2] for more details on the 4-colored propagators and corresponding superpropagators entering the formality theorem with two branes), thus there can be *at most* one edge hitting the first vertex of the second type, whence $p \leqslant n$. We now compute the polynomial degree of the multidifferential operator associated to the graph Γ: it equals $n - j - (2n - p) = p - j - n$, where $0 \leqslant j \leqslant m$ is the number of edges from the last m vertices of the second type hitting vertices of the first type. The first n come from the fact that π is a linear bivector field. As $p - j - n \geqslant 0$ and $p \leqslant n$, it follows immediately $p = n$ and $j = 0$, *i.e.* the edges departing from the last m vertices of the second type all hit the first vertex of the second type, and from each vertex of the first type departs exactly one edge hitting the first vertex of the second type; the remaining n edges must hit a vertex of the first type.

In summary, a general graph Γ appearing in $\mathrm{L}^1_{A_\hbar}(a)(1|b_1|\cdots|b_m)$ is the disjoint union of wheel-like graphs $\mathcal{W}_n$, $n \geqslant 1$, and of the graph β_m, $m \geqslant 0$; such graphs are depicted in Fig. 4.

Observe that the 1-wheel $\mathcal{W}_1$ appears here explicitly because of the presence of short loops in the formality theorem with two branes [2]: the integral weight of the 1-wheel has been computed in [4] and equals $-1/4$, while the corresponding translation invariant differential operator is the trace of the adjoint representation of $\mathfrak{g}$. Any multiple of $c_1 = \mathrm{tr}_\mathfrak{g} \circ \mathrm{ad}$ defines a constant vector field on $\mathfrak{g}^*$: either as an easy consequence of the formality theorem of Kontsevich[2] or by an explicit computation using Stokes' Theorem, c_1 is a derivation of $(A_\hbar, \star)$, where $\star$ is the deformed product on $A_\hbar$ *via* Kontsevich's deformation quantization.

The integral weight of the graph β_m is $1/m!$ and the corresponding multidifferential operator is simply the symmetrization morphism; the integral weight of the wheel-like graph $\mathcal{W}_n$, $n \geqslant 2$, has been computed in [9,10] (observe that, except the case $n = 1$, the integral weights of $\mathcal{W}_n$ for n odd vanish) and equals the modified Bernoulli number of said index, and the corresponding translation invariant differential operators are $c_n = \mathrm{tr}_\mathfrak{g}(\mathrm{ad}^n(\bullet))$.

[2] Here, the formality morphism from [6] is applied to the MC element $\pi + \varepsilon c_1$ of the twisted DG algebra $T_{\mathrm{poly}}(\mathfrak{g}^*)[\varepsilon]$, where $\varepsilon^2 = 0$ and has degree 1; observe that c_1 is annihilated by $[\pi, \bullet]$ in view of its $\mathfrak{g}$-invariance, and the infinitesimal parameter ε makes $\pi + \varepsilon c_1$ of total degree 1, it also selects exactly one copy of c_1 is all relevant formulæ.

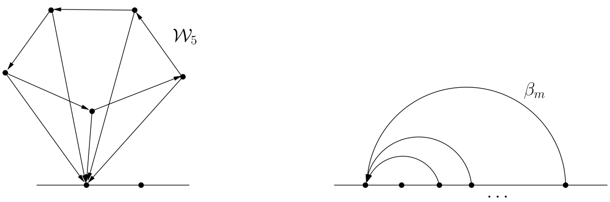

Fig. 4. The wheel-like graph $\mathcal{W}_5$ and the graph β_m.

Therefore, the isomorphism $\mathrm{L}^1_{A_\hbar}$ (for $\hbar = 1$) equals the composition of the PBW isomorphism from $\mathrm{S}(\mathfrak{g})$ to $\mathrm{U}(\mathfrak{g})$ with Duflo's strange automorphism; the derivation $-1/4\ c_1$ of the deformed algebra $(A, \star)$ is exponentiated to an automorphism of the same algebra. (The fact that π is linear permits to set $\hbar = 1$, see also [6, Subsubsection 8.3.1] for an explanation.)

3.3. *Quadratic algebras*

Here we briefly discuss the case where V^* is endowed with a quadratic Poisson bivector field π: this case has been already considered in detail in [2, Section 8], see also [8], where the property of the deformation associated $\pi_\hbar$ of preserving the property of being Koszul has been proved.

The main feature of the quadratic case is the degree 0 homogeneity of the Poisson bivector field, which reflects itself in the homogeneity of all structure maps. In particular the Kontsevich star-product on a basis of linear functions has the form

$$x_i \star x_j = x_i x_j + \sum_{k,l} S_{ij}^{kl}(\hbar) x_k x_l,$$

for some $S_{ij}^{kl} \in \hbar\mathbb{C}[[\hbar]]$. Our results imply that this algebra is isomorphic to the quotient of the tensor algebra in generators x_i by relations

$$x_i \otimes x_j - x_j \otimes x_i = \sum_{k,l} R_{ij}^{kl}(\hbar) x_k \otimes x_l,$$

for some $R_{ij}^{kl}(\hbar) \in \hbar\mathbb{C}[[\hbar]]$. The isomorphism sends x_i to

$$\mathrm{L}_{A_\hbar}(x_i) = x_i + \sum_j L_i^j(\hbar) x_j,$$

for some $L_i^j(\hbar) \in \hbar\mathbb{C}[[\hbar]]$.

3.4. *A final remark*

We point out that, in [1], the authors construct a flat $\hbar$-deformation between a so-called non-homogeneous quadratic algebra and the associated quadratic algebra: the characterization of the non-homogeneous quadratic algebra at hand is in terms of two linear maps α, β, from R onto V and $\mathbb{C}$ respectively, which satisfy certain cohomological conditions. In the case at hand, it is not difficult to prove that the conditions on α and β imply that their sum defines an affine Poisson bivector on V^*: hence, instead of considering α and β separately, as in [1], we treat them together. Both deformations are equivalent, in view of the uniqueness of flat deformations yielding the PBW property, see [1].

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