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A bound of generalized competition index of a primitive digraph

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ABSTRACT

For a positive integer m, where $1 \leq m \leq n$, the m-competition index (generalized competition index) of a primitive digraph D is the smallest positive integer k such that for every pair of vertices x and y, there exist m distinct vertices $v_1, v_2, \ldots, v_m$ such that there exist directed walks of length k from x to v_i and from y to v_i for $1 \leq i \leq m$. The m-competition index is a generalization of the scrambling index and the exponent of a primitive digraph. In this paper, we study the upper bound of the m-competition index of a primitive digraph using its order and girth.

1. Preliminaries and notations

In this paper, we follow the terminology and notation used in [1,3,4,6]. Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$, and order n. Loops are permitted but multiple arcs are not. A *walk* from x to y in a digraph D is a sequence of vertices $x, v_1, \ldots, v_t, y \in V(D)$ and a sequence of arcs $(x, v_1), (v_1, v_2), \ldots, (v_t, y) \in E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is a walk from x to y where $x = y$. A *cycle* is a closed walk from x to y with distinct vertices except for $x = y$.

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The *length of a walk W* is the number of arcs in W. The notation $x \xrightarrow{k} y$ is used to indicate that there exists a walk from x to y of length k. An l-cycle is a cycle of length l, denoted by C_l. If the digraph D has at least one cycle, the length of a shortest cycle in D is called the *girth* of D, and denote this by $s(D)$. The notation $x \rightarrow y$ indicates that there exists an arc (x, y). The *distance* from vertex x to vertex y in D is the length of the shortest walk from x to y, and it is denoted by $d_D(x, y)$.

A digraph D is called *strongly connected* if for each pair of vertices x and y in $V(D)$, there exists a walk from x to y. For a strongly connected digraph D, the *index of imprimitivity* of D is the greatest common divisor of the lengths of the cycles in D, and it is denoted by $l(D)$. If D is a trivial digraph of order 1, $l(D)$ is undefined. For a strongly connected digraph D, D is *primitive* if $l(D) = 1$.

If D is a primitive digraph of order n, there exists some positive integer k such that there exists a walk of length exactly k from each vertex x to each vertex y. The smallest such k is called the *exponent* of D, and it is denoted by exp(D). For a positive integer m where $1 \leq m \leq n$, we define the *m-competition index* of a primitive digraph D, denoted by $k_m(D)$, as the smallest positive integer k such that for every pair of vertices x and y, there exist m distinct vertices $v_1, v_2, \ldots, v_m$ such that $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ for $1 \leq i \leq m$ in D.

Kim [7] introduced the m-competition index as a generalization of the competition index presented in [5,6]. Akelbek and Kirkland [1,2] introduced the scrambling index of a primitive digraph D, denoted by $k(D)$. In the case of primitive digraphs, the definitions of the scrambling index and 1-competition index are identical. We have $k(D) = k_1(D)$.

For a positive integer k and a primitive digraph D, we define the *k-step outneighborhood* of a vertex x as

$$N^+(D^k : x) = \left\{ v \in V(D) | x \xrightarrow{k} v \right\}.$$

We define the *k-step common outneighborhood* of vertices x and y as

$$N^+(D^k : x, y) = N^+(D^k : x) \cap N^+(D^k : y).$$

We define the *local m-competition index* of vertices x and y as

$$k_m(D : x, y) = \min\{k : |N^+(D^t : x, y)| \geq m \text{ where } t \geq k\}.$$

We also define the *local m-competition index* of x as

$$k_m(D : x) = \max_{y \in V(D)} \{k_m(D : x, y)\}.$$

Then, we have

$$k_m(D) = \max_{x \in V(D)} k_m(D : x) = \max_{x, y \in V(D)} k_m(D : x, y).$$

From the definitions of $k_m(D)$, $k_m(D : x)$, and $k_m(D : x, y)$, we have $k_m(D : x, y) \leq k_m(D : x) \leq k_m(D)$. On the basis of the definitions of the m-competition index and the exponent of D of order n, we can write $k_m(D) \leq \exp(D)$, where m is a positive integer with $1 \leq m \leq n$. Furthermore, we have $k_n(D) = \exp(D)$ and

$$k(D) = k_1(D) \leq k_2(D) \leq \cdots \leq k_n(D) = \exp(D).$$

This is a generalization of the scrambling index and exponent. There exist many researches about exponents and their generalization; for example, [8,10].

Let $D_{n,s} = (V, E)$ be the digraph where $n \geq 3$ such as

$$V = \{v_0, v_1, , \ldots, v_{n-1}\},$$
$$E = \{(v_i, v_{i+1}) \mid 0 \leq i \leq n-2\} \cup \{(v_{n-1}, v_0), (v_{n-1}, v_{n-s})\}.$$

Proposition 1 [1,2]**.** *Let D be a primitive digraph with n vertices and girth s. Then,*

$$k_1(D) \le \begin{cases} n - s + \left(\frac{s-1}{2}\right) n, & \text{when } s \text{ is odd,} \\ n - s + \left(\frac{n-1}{2}\right) s, & \text{when } s \text{ is even.} \end{cases}$$

If the equality holds and $s \ge 2$, then $\gcd(n, s) = 1$ and D contains $D_{n,s}$ as a subgraph.

Proposition 2 [7]**.** *Let D be a primitive digraph of order n (≥ 3) and let s be the girth of D. For a positive integer m such that $1 \le m \le n$, we have*

$$k_m(D) \le \begin{cases} n - s + \left(\frac{n+m-2}{2}\right) s, & \text{when } n + m \text{ is even,} \\ n - s - 1 + \left(\frac{n+m-1}{2}\right) s, & \text{when } n + m \text{ is odd.} \end{cases}$$

When $m = 1$, the result of Proposition 2 does not coincide the result of Proposition 1. In this paper, we provide a sharp upper bound for $k_m(D)$.

2. Main results

Let $L(D)$ denote the set of lengths of the cycles of D. Let n, s, and m be positive integers such that $s < n$ and $1 \le m \le n$. For a nonnegative integer x such that $\left\lceil \frac{n-m}{2} \right\rceil \le x \le \left\lfloor \frac{n+m}{2} \right\rfloor$, the remainder of xs divided by n is denoted by $r(x)$ and the minimum of $r(x)$ is denoted by $\bar{r}$. Let $M(n, s)$ be the nearest positive integer to $\frac{n}{s}$ such that its parity differs from n and $M(n, s) \ne \frac{n}{s} - 1$.

Lemma 3. *Let D be a primitive digraph of order n (≥ 3) and girth s. If s be odd, then we have*

$$k_m(D) \le n - s + \left(\frac{s-1}{2}\right) n + (m-1)s,$$

for a positive integer m such that $1 \le m \le n$. If the equality holds and $s \ge 2$, then $\gcd(n, s) = 1$ and D contains $D_{n,s}$ as a subgraph.

Proof. Let C_s be a cycle of length s, and x and y be vertices in $V(D)$.

According to the proof of Proposition 1 in [1], we can have vertices x' and y' in $V(C_s)$ such that

$$x \xrightarrow{n-s} x' \xrightarrow{\left(\frac{s-1}{2}\right)n} w, \quad y \xrightarrow{n-s} y' \xrightarrow{\left(\frac{s-1}{2}\right)n} w.$$

for a vertex w. Because D and D^s are primitive, we have $|N^+(D^t : x', y')| \ge m$ where $t = \left(\frac{s-1}{2}\right) n + (m-1)s$. Then we have $k_m(D) \le n - s + \left(\frac{s-1}{2}\right) n + (m-1)s$.

Suppose $\gcd(n, s) \ne 1$ or D does not contain $D_{n,s}$ as a subgraph where $s \ge 2$. According to the proof of Proposition 1 in [1], for a vertex w there exist walks

$$W_1 : x \xrightarrow{t'} w, \quad W_2 : y \xrightarrow{t'} w,$$

where $t' < n - s + \left(\frac{s-1}{2}\right) n$, and W_1 and W_2 contain a vertex in $V(C_s)$. Then we have $|N^+(D^{t'+(m-1)s} : x, y)| \ge m$. Therefore

$$k_m(D) \le t' + (m-1)s < n - s + \left(\frac{s-1}{2}\right) n + (m-1)s.$$

This establishes the result. □

Lemma 4. *Let n, s, and m be positive integers such that $s < n$ and $1 \le m \le n$. If s is odd and $m \le M(n, s)$, then we have $\bar{r} = r(x)$, where $x = \left\lceil \frac{n-m}{2} \right\rceil$.*

Proof. *Case* 1. $n + m$ is odd.

Let x_1 and x_2 be nonnegative integers such that $0 \le x_1 < x_2 \le m - 1$. We have $\frac{n-(m-1)s}{2} \ge 0$ and $\frac{n+(m-1)s}{2} \le n$ because $(m-1)s = ms - s \le n$. Then, we have

$$\left(\frac{n-m+1}{2} + x\right) s = n\left(\frac{s-1}{2}\right) + \frac{n-(m-1)s}{2} + xs.$$

If $\frac{n+(m-1)s}{2} = n$, then $r\left(\left\lceil \frac{n-m}{2} \right\rceil\right) = 0$. Suppose $\frac{n+(m-1)s}{2} < n$. Then, we have $r\left(\left\lceil \frac{n-m}{2} \right\rceil + x_1\right) = \frac{n-(m-1)s}{2} + x_1 s$ and $r\left(\left\lceil \frac{n-m}{2} \right\rceil + x_2\right) = \frac{n-(m-1)s}{2} + x_2 s$. Therefore, we have $r\left(\left\lceil \frac{n-m}{2} \right\rceil + x_1\right) < r\left(\left\lceil \frac{n-m}{2} \right\rceil + x_2\right)$.

Case 2. $n + m$ is even.

Let x_1 and x_2 be nonnegative integers such that $0 \le x_1 < x_2 \le m$. We have $\frac{n-ms}{2} \ge 0$ and $\frac{n+ms}{2} \le n$ because $ms \le n$ by $m \le M(n, s) - 1$ because of the parity. Then, we have

$$\left(\frac{n-m}{2} + x\right) s = n\left(\frac{s-1}{2}\right) + \frac{n-ms}{2} + xs.$$

If $\frac{n+ms}{2} = n$, then $r\left(\left\lceil \frac{n-m}{2} \right\rceil\right) = 0$. Suppose $\frac{n+ms}{2} < n$. Then, we have $r\left(\left\lceil \frac{n-m}{2} \right\rceil + x_1\right) = \frac{n-ms}{2} + x_1 s$ and $r\left(\left\lceil \frac{n-m}{2} \right\rceil + x_2\right) = \frac{n-ms}{2} + x_2 s$. Therefore, we have $r\left(\left\lceil \frac{n-m}{2} \right\rceil + x_1\right) < r\left(\left\lceil \frac{n-m}{2} \right\rceil + x_2\right)$.

In all cases, we have $\bar{r} = r(x)$, where $x = \lceil \frac{n-m}{2} \rceil$. This establishes the result. □

Theorem 5. *Let D be a primitive digraph of order n (≥ 3) and girth s. Let m be a positive integer such that $m \le M(n, s)$. If s is odd, then we have*

$$k_m(D) \le n - \bar{r} + \left\lceil \frac{n+m-4}{2} \right\rceil s.$$

If the equality holds and $s \ge 2$, then $\gcd(n, s) = 1$ and D contains $D_{n,s}$ as a subgraph.

Proof. By Lemma 4, we have

$$\bar{r} = \begin{cases} \frac{n-(m-1)s}{2}, & \text{when } n+m \text{ is odd,} \\ \frac{n-ms}{2}, & \text{when } n+m \text{ is even.} \end{cases}$$

Therefore, we have

$$n - \bar{r} + \left\lceil \frac{n+m-4}{2} \right\rceil s = n - s + \left(\frac{s-1}{2}\right) n + (m-1)s.$$

By Lemma 3, we have $k_m(D) \le n - \bar{r} + \left\lceil \frac{n+m-4}{2} \right\rceil s$, and the equality holds only if $\gcd(n, s) = 1$ and D contains $D_{n,s}$ as a subgraph. This establishes the result. □

Lemma 6. *Let n, s, and m be positive integers such that $s < n$ and $1 \le m \le n$. If s is even or $m > M(n, s)$, then we have*

$$\bar{r} \le \frac{s}{2}.$$

Proof. We show that there exists a nonnegative integer x such that $r(x) \le \frac{s}{2}$.

Case 1. s is even.

Let

$$x = \begin{cases} \frac{n}{2}, & \text{when } n \text{ is even,} \\ \frac{n+1}{2}, & \text{when } n \text{ is odd.} \end{cases}$$

Then, we have $\left\lceil \frac{n-m}{2} \right\rceil \leq x \leq \left\lfloor \frac{n+m}{2} \right\rfloor$ because $m \geq 1$, and we have

$$r(x) = \begin{cases} 0, & \text{when } n \text{ is even,} \\ \frac{s}{2}, & \text{when } n \text{ is odd.} \end{cases}$$

Therefore, we have $\bar{r} \leq r(x) \leq \frac{s}{2}$.

Case 2. s is odd.

Case 2.1. n is even.

In this case, we have $M(n, s) = 2\left\lfloor \frac{n/2}{s} \right\rfloor + 1$. Let $x_1 = \frac{n}{2} - \left\lfloor \frac{n/2}{s} \right\rfloor$ and $x_2 = \frac{n}{2} + \left\lfloor \frac{n/2}{s} \right\rfloor + 1$. Then, we have $\left\lceil \frac{n-m}{2} \right\rceil \leq x_1 < x_2 \leq \left\lfloor \frac{n+m}{2} \right\rfloor$, and

$$x_1 s \equiv \frac{n}{2} - \left\lfloor \frac{n/2}{s} \right\rfloor s \pmod{n},$$

$$x_2 s \equiv -\frac{n}{2} + \left\lfloor \frac{n/2}{s} \right\rfloor s + s \pmod{n}.$$

Therefore, we have $r(x_1) + r(x_2) = s$; this implies that

$$\bar{r} \leq \min(r(x_1), r(x_2)) \leq \frac{s}{2}.$$

Case 2.2. n is odd.

In this case, we have $M(n, s) = 2\left\lfloor \frac{(n-s)/2}{s} \right\rfloor + 2$. Let $x_1 = \frac{n-1}{2} - \left\lfloor \frac{(n-s)/2}{s} \right\rfloor$ and $x_2 = \frac{n+1}{2} + \left\lfloor \frac{(n-s)/2}{s} \right\rfloor + 1$. Then, we have $\left\lceil \frac{n-m}{2} \right\rceil \leq x_1 < x_2 \leq \left\lfloor \frac{n+m}{2} \right\rfloor$, and

$$x_1 s \equiv \frac{(n-s)}{2} - \left\lfloor \frac{(n-s)/2}{s} \right\rfloor s \pmod{n},$$

$$x_2 s \equiv -\frac{(n-s)}{2} + \left\lfloor \frac{(n-s)/2}{s} \right\rfloor s + s \pmod{n}.$$

Therefore, we have $r(x_1) + r(x_2) = s$; this implies that

$$\bar{r} \leq \min(r(x_1), r(x_2)) \leq \frac{s}{2}.$$

This establishes the result. □

Denote

$$K(n, s, m) = \begin{cases} n - \bar{r} + \left(\frac{n+m-3}{2}\right) s, & \text{when } n + m \text{ is odd,} \\ n - \bar{r} + \left(\frac{n+m-4}{2}\right) s, & \text{when } n + m \text{ is even, } s \text{ is odd, and } m < \frac{n}{s}, \\ n - s + \left(\frac{n+m-2}{2}\right) s, & \text{otherwise.} \end{cases}$$

Lemma 7. *Let D be a primitive digraph of order n and girth s such that* $n \in L(D)$ *and* $\gcd(n, s) = 1$*. For a positive integer m such that* $1 \le m \le n$*, we have*

$$k_m(D) \le K(n, s, m).$$

If the equality holds and $s \ge 2$*, then D contains* $D_{n,s}$ *as a subgraph.*

Proof. If s is odd and $m \le M(n, s)$, then we have the result from Theorem 5. Suppose s is even or $m > M(n, s)$. Let C_s be an s-cycle. There exists a positive integer k such that $1 \le k \le n-2$, where $D = (V, E)$ is

$$V = \{v_0, v_1, , \dots, v_{n-1}\},$$
$$E \supset \{(v_i, v_{i+1}) \mid 0 \le i \le n-2\} \cup \{(v_{n-1}, v_0), (v_{n-1}, v_k)\},$$

and $(v_{n-1}, v_k) \in E(C_s)$. There exists an n-cycle in D^s because $\gcd(n, s) = 1$. In this proof, we assume that all subscripts are taken by modulo n. Consider two vertices v_i and v_j, where $i < j$.

Case 1. $n + m$ is odd.

If $d_{D^s}(v_i, v_j) < \frac{n-m+1}{2}$ or $d_{D^s}(v_i, v_j) > \frac{n+m-1}{2}$, the number of vertices that can be reached from v_{i+n-s} and v_{j+n-s} within $\left(\frac{n+m-3}{2}\right)$-steps is greater than or equal to m in D^s. Because each of $v_i \overset{n-s}{\longrightarrow} v_{i+n-s}$ and $v_j \overset{n-s}{\longrightarrow} v_{j+n-s}$ contains a vertex in $V(C_s)$, we have $|N^+(D^{t_1} : v_i, v_j)| \ge m$, where $t_1 = n - s + \left(\frac{n+m-3}{2}\right) s$. Therefore, we have

$$k_m(D : v_i, v_j) \le t_1 < K(n, s, m), \tag{1}$$

because $\bar{r} < s$ by Lemma 6.

Suppose $\frac{n-m+1}{2} \le d_{D^s}(v_i, v_j) \le \frac{n+m-1}{2}$. Then, we have the following walks of length $(n - \bar{r})$:

$$W_1 : v_i \to v_{i+1} \to \cdots \to v_{n-\bar{r}+i},$$
$$W_2 : v_j \to v_{j+1} \to \cdots \to v_{n-\bar{r}+j},$$
$$W_3 : v_j \to v_{j+1} \to \cdots \to v_{n-1} \to v_k \to \cdots \to v_{k+j-\bar{r}},$$

where $v_{n-\bar{r}+j} \ne v_{k+j-\bar{r}}$. W_1 contains a vertex in $V(C_s)$ because $\bar{r} < s$. W_2 and W_3 also contain $v_{n-1} \in V(C_s)$. Then, we have

$$|N^+(D_{n,s}^{t_2} : v_i)| \ge \frac{n+m-1}{2}, \quad |N^+(D_{n,s}^{t_2} : v_j)| \ge \frac{n+m+1}{2},$$

where $t_2 = n - \bar{r} + \left(\frac{n+m-3}{2}\right) s$. Then, $|N^+(D_{n,s}^{t_2} : v_i, v_j)| \ge m$. Therefore, we have

$$k_m(D_{n,s}) \le t_2 = K(n, s, m). \tag{2}$$

If D does not contain $D_{n,s}$ as a subgraph, then there exists another arc (v_p, v_q) in the s-cycle, where $0 \le p \le n-2$ and $0 \le q \le n-1$. We have the following two walks of length $(n - \bar{r} - 1)$:

$$W_1' : v_i \to v_{i+1} \to \cdots \to v_{n-\bar{r}-1+i},$$
$$W_2' : v_j \to v_{j+1} \to \cdots \to v_{n-\bar{r}-1+j}.$$

In addition, we have $j - i < n - \bar{r}$ or $n - j + i < n - \bar{r}$. Then, there exists a walk among these walks of length $(n - \bar{r} - 1)$:

$$W_3' : v_j \to v_{j+1} \to \cdots \to v_p \to v_q \to \cdots \to v_{n-\bar{r}-1+j+q-p-1},$$
$$W_4' : v_j \to v_{j+1} \to \cdots \to v_{n-1} \to v_k \to \cdots \to v_{k+j-1-\bar{r}},$$
$$W_5' : v_i \to v_{i+1} \to \cdots \to v_p \to v_q \to \cdots \to v_{n-\bar{r}-1+i+q-p-1}.$$

W_1' and W_2' contain a vertex in the s-cycle because $\bar{r} < s$. One among W_3', W_4', and W_5' also contains a vertex in the s-cycle. If there exists a walk W_3' or W_4', we have

$$|N^+(D_{n,s}^{t_2-1} : v_i)| \geq \frac{n+m-1}{2}, \quad |N^+(D_{n,s}^{t_2-1} : v_j)| \geq \frac{n+m+1}{2}.$$

If there exists a walk W_5', we have

$$|N^+(D_{n,s}^{t_2-1} : v_i)| \geq \frac{n+m+1}{2}, \quad |N^+(D_{n,s}^{t_2-1} : v_j)| \geq \frac{n+m-1}{2}.$$

In all cases, we have $|N^+(D_{n,s}^{t_2-1} : v_i, v_j)| \geq m$. Therefore, we have

$$k_m(D_{n,s}) \leq t_2 - 1 < K(n, s, m). \tag{3}$$

By (1), (2), and (3), we have the result when $n + m$ is odd.

Case 2. Otherwise.

We have $k_m(D) \leq K(n, s, m)$ by Proposition 2. Suppose $k_m(D) = K(n, s, m)$. If $k \neq n - s$, then $v_i \xrightarrow{n-s-1} v_{i+n-s-1}$ contains a vertex in an s-cycle and $v_j \xrightarrow{n-s-1} v_{j+n-s-1}$ contains a vertex in an s-cycle. In D^s, the number of vertices that can be reached from $v_{i+n-s-1}$ and $v_{j+n-s+1}$ within $(\frac{n+m-2}{2})$-steps is greater than or equal to m. We have $|N^+(D^{t_3} : v_i, v_j)| \geq m$, where $t_3 = n - s - 1 + \left(\frac{n+m-2}{2}\right) s$. This is contradictory. Therefore, we have $k = n - s$. Therefore, D contains $D_{n,s}$ as a subgraph.

This establishes the result. □

Lemma 8. *Let* $\gcd(n, s) = 1$. *For a positive integer m such that* $1 \leq m \leq n$, *we have*

$$k_m(D_{n,s}) = K(n, s, m).$$

Proof. If $s = 1$, then we have $k_m(D_{n,s}) = n + m - 2 = K(n, s, m)$. Suppose $s \geq 2$. Let $S = \{v_{n-s}, v_{n-s+1}, \ldots, v_{n-1}\}$. There exists an n-cycle in $D_{n,s}^s$ because $\gcd(n, s) = 1$. By Lemma 7, we have $k_m(D_{n,s}) \leq K(n, s, m)$. We show $k_m(D_{n,s}) \geq K(n, s, m)$. In this proof, we assume that all subscripts are taken by modulo n.

Case 1. $n + m$ is odd.

Let $i = 0, j = \bar{r}$, and $t_1 = n - \bar{r} + \left(\frac{n+m-3}{2}\right) s$. Then, we have $N^+(D_{n,s}^{n-j-1} : v_i) = \{v_{n-j-1}\}$ and $N^+(D_{n,s}^{n-j-1} : v_j) = \{v_{n-1}\}$. We also have

$$N^+(D_{n,s}^{t_1-1} : v_i, v_j) = N^+\left(D_{n,s}^{\left(\frac{n+m-3}{2}\right)s} : v_{n-j-1}, v_{n-1}\right).$$

Because $\frac{n-m+1}{2} \leq d_{D_{n,s}^s}(v_i, v_j) \leq \frac{n+m-1}{2}$ by the definition of $j = \bar{r}$, we have

$$|N^+(D_{n,s}^{t_1-1} : v_i, v_j)| < m.$$

Therefore, we have

$$k_m(D_{n,s}) \geq t_1 = K(n, s, m).$$

Case 2. $n + m$ is even, s is odd, and $m < \frac{n}{s}$.

We have $\bar{r} = \frac{n-ms}{2}$. Let $i = 0, j = \bar{r} = \frac{n-ms}{2}$, and $t_2 = n - \bar{r} + \left(\frac{n+m-4}{2}\right) s$. Then, we have $N^+(D_{n,s}^{n-j-1} : v_i) = \{v_{n-j-1}\}$ and $N^+(D_{n,s}^{n-j-1} : v_j) = \{v_{n-1}\}$. We also have

$$N^+(D_{n,s}^{t_2-1} : v_i, v_j) = N^+\left(D_{n,s}^{\left(\frac{n+m-4}{2}\right)s} : v_{n-j-1}, v_{n-1}\right).$$

Because $\frac{n-m}{2} \le d_{D^s_{n,s}}(v_i, v_j) \le \frac{n+m}{2}$ by the definition of $j = \bar{r}$, we have

$$|N^+(D^{t_2-1}_{n,s} : v_i, v_j)| < m.$$

Therefore, we have

$$k_m(D_{n,s}) \ge t_2 = K(n, s, m).$$

Case 3. Otherwise.

Let $t_3 = n - s + \left(\frac{n+m-2}{2}\right) s$.

Case 3.1. $m < \frac{n}{s}$ and s is even.

Let $i = 0$ and $j = \frac{s}{2}$. Then, we have $N^+(D^{n-s-1}_{n,s} : v_i) = \{v_{n-s-1}\}$ and $N^+(D^{n-s-1}_{n,s} : v_j) = \{v_{j+n-s-1}\}$ because $j + n - s - 1 \le n - 1$. We also have $d_{D^s_{n,s}}(v_{n-s-1}, v_{j+n-s-1}) = \frac{n+1}{2}$ because $\frac{n+1}{2}s \equiv \frac{s}{2} \pmod{n}$. Therefore, we have

$$k_m(D_{n,s} : v_i, v_j) \ge t_3 = K(n, s, m)$$

because $|N^+(D^{t_3-1}_{n,s} : v_i)| \le \frac{n+m-2}{2}$ and $|N^+(D^{t_3-1}_{n,s} : v_j)| \le \frac{n+m}{2}$.

Case 3.2. $m > \frac{n}{s}$.

Let $i = 0$ and $j = \bar{r}$. Because $n + m$ is even, we have $m > M(n, s)$. We have $\bar{r} < s$ by Lemma 6. Then, we also have $N^+(D^{n-s-1}_{n,s} : v_i) = \{v_{n-s-1}\}$ and $N^+(D^{n-s-1}_{n,s} : v_j) = \{v_{\bar{r}+n-s-1}\}$ because $\bar{r} + n - s - 1 \le n - 1$. We also have $\frac{n-m}{2} \le d_{D^s_{n,s}}(v_{n-s-1}, v_{\bar{r}+n-s-1}) \le \frac{n+m}{2}$. Therefore, we have

$$k_m(D_{n,s} : v_i, v_j) \ge t_3 = K(n, s, m)$$

because $|N^+(D^{t_3-1}_{n,s} : v_i)| \le \frac{n+m-2}{2}$ and $|N^+(D^{t_3-1}_{n,s} : v_j)| \le \frac{n+m}{2}$.

In all cases, we have $k_m(D_{n,s}) \ge K(n, s, m)$. This establishes the result. □

Remark 9. If $m = n - 1$, then we have $\bar{r} = 1$. By Lemma 8, we have

$$k_{n-1}(D_{n,s}) = n - 1 + (n-2)s = k_n(D_{n,s}) - 1.$$

Example 10 [7]. Let D be a primitive digraph whose adjacency matrix A is given as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The order of D is 5 and the girth of D is 3. Thus, we can check

$$k_1(D) = 7 = K(5, 3, 1),$$

$$k_2(D) = 10 = K(5, 3, 2),$$

$$k_3(D) = 11 = K(5, 3, 3),$$

$$k_4(D) = 13 = K(5, 3, 4),$$

$$k_5(D) = 14 = K(5, 3, 5).$$

Lemma 11. *Let D be a primitive digraph of order n and girth $s(\geq 2)$, and suppose $p \in L(D)$ such that $s < p < n$ and $\gcd(p, s) = 1$. For a positive integer m such that $1 \leq m \leq n$, we have*

$$k_m(D) < n - s + \left(\frac{n+m-2}{2}\right)s.$$

Proof. Let C_s and C_p be an s-cycle and a p-cycle, respectively. Consider two vertices x and y.

Case 1. $m \leq p$ and $s > 2$.

Case 1.1. $p + m$ is even.

There exist walks

$$x \xrightarrow{n-s} x_s \xrightarrow{n-p} x_p, \quad y \xrightarrow{n-s} y_s \xrightarrow{n-p} y_p,$$

where $x_s, y_s \in V(C_s)$ and $x_p, y_p \in V(C_p)$. Let $t_1 = n - s + n - p + \left(\frac{p+m-2}{2}\right)s$. Then, we have $|N^+(D^{t_1} : x, y) \cap V(C_p)| \geq m$ because $|N^+(D^{t_1} : x) \cap V(C_p)| \geq \frac{p+m}{2}$ and $|N^+(D^{t_1} : y) \cap V(C_p)| \geq \frac{p+m}{2}$. Therefore, we have

$$\begin{aligned} k_m(D : x, y) &\leq n - s + n - p + \left(\frac{p+m-2}{2}\right)s \\ &< n - s + \left(\frac{n+m-2}{2}\right)s. \end{aligned}$$

Case 1.2. $p + m$ is odd.

Case 1.2.1. $p \leq n - 2$.

There exists $x \xrightarrow{n-s-1} x_s \in V(C_s)$ or $y \xrightarrow{n-s-1} y_s \in V(C_s)$. Without loss of generality, we may assume that $x \xrightarrow{n-s-1} x_s \in V(C_s)$. Then, we can find a vertex y_s in $V(C_s)$ such that there exists $y \xrightarrow{n-1} y_s$. There exist walks such that $x_s \xrightarrow{n-p} x_p \in V(C_p)$ and $y_s \xrightarrow{n-p} y_p \in V(C_p)$. Let $t_2 = n-s+n-p-1+\left(\frac{p+m-1}{2}\right)s$. Then, we have $|N^+(D^{t_2} : x, y) \cap V(C_p)| \geq m$ because $|N^+(D^{t_2} : x) \cap V(C_p)| \geq \frac{p+m+1}{2}$ and $|N^+(D^{t_2} : y) \cap V(C_p)| \geq \frac{p+m-1}{2}$.

$$\begin{aligned} k_m(D : x, y) &\leq n - s + n - p - 1 + \left(\frac{p+m-1}{2}\right)s \\ &< n - s + \left(\frac{n+m-2}{2}\right)s. \end{aligned}$$

Case 1.2.2. $p = n - 1$.

We have $x \in V(C_p)$ or $y \in V(C_p)$. Without loss of generality, we assume that $x \in V(C_p)$. We also have $|V(C_s) \cap V(C_p)| \geq s - 1$. If $|V(C_s) \cap V(C_p)| = s$, we have $x \xrightarrow{n-s-1} x_s \in V(C_p)$ and $y \xrightarrow{n-1} y_s \in V(C_p)$, which contains a vertex in $V(C_s)$. If $|V(C_s) \cap V(C_p)| = s - 1$ and $y \notin V(C_p)$, we have $x \xrightarrow{n-1} x_s \in V(C_p)$ and $y \xrightarrow{n-s-1} y_s \in V(C_p)$, which contains a vertex in $V(C_s)$, because $n-s-1 \geq 1$. If $|V(C_s) \cap V(C_p)| = s - 1$ and $y \in V(C_p)$, we have $x \xrightarrow{n-s-1} x_s \in V(C_p)$ or $y \xrightarrow{n-s-1} y_s \in V(C_p)$, which contains a vertex in $V(C_s)$. In all cases, we may assume that

$$x \xrightarrow{n-s-1} x_s \in V(C_p), \quad y \xrightarrow{n-1} y_s \in V(C_p),$$

which contains a vertex in $V(C_s)$. Let $t_2 = n - s - 1 + (\frac{p+m-1}{2})s$. Then, we have $|N^+(D^{t_2} : x, y) \cap V(C_p)| \geq m$ because $|N^+(D^{t_2} : x) \cap V(C_p)| \geq \frac{p+m+1}{2}$ and $|N^+(D^{t_2} : y) \cap V(C_p)| \geq \frac{p+m-1}{2}$. Therefore, we have

$$k_m(D:x,y) \le n-s-1+\left(\frac{p+m-1}{2}\right)s$$
$$< n-s+\left(\frac{n+m-2}{2}\right)s.$$

Case 2. $m \le p$ and $s=2$.

If $m=1$, then we have $k_1(D) < n-2+n-1$ by Proposition 1. Suppose $m \ge 2$. We have p is odd. Let $V(C_s)=\{v_1,v_2\}$. Let l_x and l_y be the smallest numbers such that there exist walks

$$x \xrightarrow{l_x} x_s, \quad y \xrightarrow{l_y} y_s, \tag{4}$$

where $x_s, y_x \in V(C_s)$. We may assume that $l_x \le n-3$.

If each walk of (4) contains a vertex in $V(C_p)$, then we have $V(C_s) \subset N^+(D^{n-2+p}:x,y)$. Therefore, we have $|N^+(D^{n-2+p+i}:x,y)| \ge 2+i$ for a nonnegative integer i such that $i \le n-2$. For $m \ge 2$, we have

$$k_m(D:x,y) \le n+p-2+m-2$$
$$< n-s+\left(\frac{n+m-2}{2}\right)s.$$

This holds even though $m > p$.

If a walk of (4), $x \xrightarrow{l_x} x_s$, does not contain a vertex in $V(C_p)$, then we have $l_x \le n-p-2$. There exist walks

$$x_s \xrightarrow{n-p} x_p, \quad y_s \xrightarrow{n-p} y_p,$$

where $x_p, y_p \in V(C_p)$. Let $t_3 = n-2+n-p+\left(\frac{p+m-3}{2}\right)s$. Then, $n-p-2+n-p+\left(\frac{p+m}{2}\right)s \le t_3$. We have $|N^+(D^{t_3}:x,y) \cap V(C_p)| \ge m$ because $|N^+(D^{t_3}:x) \cap V(C_p)| \ge \lfloor \frac{p+m+2}{2} \rfloor$ and $|N^+(D^{t_3}:y) \cap V(C_p)| \ge \lfloor \frac{p+m-1}{2} \rfloor$. Therefore, we have

$$k_m(D:x,y) \le n-2+n-p+\left(\frac{p+m-3}{2}\right)s$$
$$< n-s+\left(\frac{n+m-2}{2}\right)s.$$

Case 3. $m > p$.

If $V(C_p) \subset N^+(D^k:x,y)$ for a positive integer k, then we have

$$|N^+(D^{k+i}:x,y)| \ge p+i$$

for each nonnegative integer i such that $i \le n-p$. Therefore, we have

$$k_m(D:x,y) < n-s+\left(\frac{n+p-2}{2}\right)s+(m-p)$$
$$\le n-s+\left(\frac{n+m-2}{2}\right)s.$$

This establishes the result. □

Lemma 12. *Let D be a primitive digraph of order n and girth* $s(\ge 2)$, *and suppose* $L(D)=\{s,a_1,\ldots,a_h\}$ *such that* $\gcd(s,a_i) \ne 1$ *for each* $i=1,2,\ldots,h$, *where* $h \ge 2$. *For a positive integer m such that* $1 \le m \le n$, *we have*

$$k_m(D) < n-s+\left(\frac{n+m-2}{2}\right)s.$$

Proof. Because $\gcd(s, a_i) \neq 1$ for each $i = 1, 2, \ldots, h$, s is not prime and $s \geq 6$.

First, suppose $s \geq 8$. Then, there exists a cycle of length p such that $\gcd(s, p) \leq \frac{s}{4}$. Otherwise, $\gcd(s, a_i)$ is equal to one among s, $\frac{s}{2}$, and $\frac{s}{3}$. Then, we have $\gcd(s, a_1, \ldots, a_h) \geq \frac{s}{6}$. This contradicts the fact that D is primitive. Let $\gcd(s, p) = t \leq \frac{s}{4}$. We know that D^t is primitive because D is primitive. We also know that D^t contains t cycles of length $\frac{s}{t}$ and t cycles of length $\frac{p}{t}$.

Let $C(1), C(2), \ldots, C(t)$ be t disjoint cycles of length $\frac{p}{t}$ in D^t, that is, $V(C(i)) \cap V(C(j)) = \phi$ for $i \neq j$. Let $s' = \frac{s}{t}$ and $p' = \frac{p}{t}$; then, $\gcd(s', p') = 1$. Consider two vertices x and y in D. In D, there exist walks

$$x \xrightarrow{n-s} x', \quad y \xrightarrow{n-s} y',$$

where $x' \in V(C_s)$ and $y' \in V(C_s)$.

In D^t, for each $C(i)$, where $i = 1, 2, \ldots, t$, there exist vertices x_i and y_i in $C(i)$ such that there exist walks

$$x' \xrightarrow{n-p'} x_i, \quad y' \xrightarrow{n-p'} y_i.$$

Case 1. $m \leq p$.

Then, we have

$$\begin{aligned} k_m(D^t : x', y') &\leq n - p' + \left(\frac{p' + \left\lceil \frac{m}{t} \right\rceil - 1}{2} \right) s' \\ &\leq n - p' + \left(\frac{p' + \frac{m}{t}}{2} \right) s'. \end{aligned}$$

Because $k_m(D : x, y) \leq n - s + t \cdot k_m(D^t : x', y')$, we have

$$k_m(D : x, y) \leq n - s - p + nt + \left(\frac{p + m}{2t} \right) s. \tag{5}$$

Let $f(t) = n - s - p + nt + \left(\frac{p+m}{2t} \right) s$. Then, $f(t)$ is concave up on the interval $[2, \frac{s}{4}]$, and therefore, it attains its maximum at one of the end points.

$$\begin{aligned} f(2) &= 3n - s - p + \left(\frac{p + m}{4} \right) s \leq 2n - s + \left(\frac{n + m}{4} \right) s \\ &< n - 2s + \left(\frac{n + m}{2} \right) s. \end{aligned}$$

$$\begin{aligned} f\left(\frac{s}{4} \right) &= n - s + p + \frac{ns}{4} + 2m \leq 2n - s + \frac{ns}{4} + 2m \\ &< n - 2s + \left(\frac{n + m}{2} \right) s. \end{aligned}$$

Therefore, we have $k_m(D) < n - s + \left(\frac{n+m-2}{2} \right) s$.

Case 2. $m > p$.

If $V(C_p) \subset N^+(D^k : x, y)$ for a positive integer k, then we have $|N^+(D^{k+i} : x, y)| \geq p + i$ for each nonnegative integer i such that $i \leq n - p$. Therefore, we have

$$k_m(D : x, y) < n - s + \left(\frac{n + p - 2}{2} \right) s + (m - p). \tag{6}$$

Therefore, we have $k_m(D : x, y) < n - s + \left(\frac{n+m-2}{2} \right) s$.

There is only remaining case, namely, $s = 6$. If $s = 6$, then there also exists a cycle of length p such that $\gcd(s, p) = 2$. Otherwise, $\gcd(s, a_i) = 3$ or 6 for all $i = 1, 2, \ldots, h$. This is contradictory. We also have $n \geq 9$. If $s = 6$ and $n = 9$, there exists a cycle of length $p = 8$. Then, we have $k_m(D) < n - s + \left(\frac{n+m-2}{2}\right) s$ by (5) and (6). If $s = 6$ and $n > 9$, then we also have $k_m(D) < n - s + \left(\frac{n+m-2}{2}\right) s$ by (5) and (6) because $p \leq n$.

This establishes the result. □

Theorem 13. *Let D be a primitive digraph of order $n(\geq 3)$ and girth s. For a positive integer m such that $1 \leq m \leq n$, we have*

$$k_m(D) \leq K(n, s, m).$$

If the equality holds and $s \geq 2$, then $\gcd(n, s) = 1$ and D contains $D_{n,s}$ as a subgraph. If $D = D_{n,s}$, then the equality holds.

Proof. Let $L(D) = \{s, a_1, \ldots, a_h\}$. If s is odd and $m \leq M(n, s)$, then we have the result by Theorem 5.

Suppose s is even or $m > M(n, s)$. Then, we have $K(n, s, m) \geq n - s + \left(\frac{n+m-2}{2}\right) s$ because $\bar{r} \leq \frac{s}{2}$ by Lemma 6. If $h \geq 2$ and $\gcd(s, a_i) \neq 1$ for each $i = 1, 2, \ldots, h$, then we have $k_m(D) < n - s + \left(\frac{n+m-2}{2}\right) s$ by Lemma 12. If there exists $p \in L(D)$ such that $s < p < n$ and $\gcd(p, s) = 1$, then we have $k_m(D) < n - s + \left(\frac{n+m-2}{2}\right) s$ by Lemma 11. If $n \in L(D)$ and $\gcd(n, s) = 1$, then we have the result by Lemma 7.

If $D = D_{n,s}$, then the equality holds by Lemma 8. This establishes the result. □

Corollary 14. *Let D be a primitive digraph of order n (≥ 3) and girth s. Let m be a positive integer such that $1 \leq m \leq n$. If $n + m$ is odd, then we have*

$$k_m(D) \leq n - s - 1 + \left(\frac{n+m-1}{2}\right) s.$$

Proof. If $s = 1$, then we have $k_m(D) \leq n + m - 2 \leq n - s - 1 + \left(\frac{n+m-1}{2}\right) s$ because $n \geq m + 1$.

Suppose $s \geq 2$, and let $L(D) = \{s, a_1, \ldots, a_h\}$. If $h \geq 2$ and $\gcd(s, a_i) \neq 1$ for each $i = 1, 2, \ldots, h$, then we have $k_m(D) < n - s - 1 + \left(\frac{n+m-1}{2}\right) s$ by Lemma 12. If there exists $p \in L(D)$ such that $s < p < n$ and $\gcd(p, s) = 1$, then we have $k_m(D) \leq n - s - 1 + \left(\frac{n+m-1}{2}\right) s$ by Lemma 11. If $n \in L(D)$ and $\gcd(n, s) = 1$, then we have $\bar{r} \geq 1$. Therefore, we have $k_m(D) \leq n - s - 1 + \left(\frac{n+m-1}{2}\right) s$ by Lemma 7. This establishes the result. □

Remark 15. In Theorem 13, the equality holds only if D contains $D_{n,s}$ as a subgraph. In addition, if $m = 1$, Theorem 13 and Proposition 1 give us the same bound because $m < \frac{n}{s}$. Corollary 14 is the same result as Proposition 2.

3. Closing remark

Akelbek and Kirkland [1] introduced the concept of the scrambling index of a primitive digraph. Kim [7] introduced a generalized competition index $k_m(D)$ as another generalization of the exponent $\exp(D)$ and scrambling index $k(D)$ for a primitive digraph D. Sim and Kim [9] studied the generalized competition index $k_m(T_n)$ of a primitive n-tournament T_n. In this paper, we study an upper bound of $k_m(D)$, where D is a primitive digraph. Akelbek and Kirkland [2] characterized a primitive digraph D where $k_1(D) = K(n, s, 1)$. It is also necessary to study the characterization of a primitive digraph D where $k_m(D) = K(n, s, m)$ for $1 \leq m \leq n$.

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