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The determinants of q-distance matrices of trees and two quantities relating to permutations

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Abstract

In this paper we prove that two quantities relating to the length of permutations defined on trees are independent of the structures of trees. We also find that these results are closely related to the results obtained by Graham and Pollak [R.L. Graham, H.O. Pollak, On the addressing problem for loop switching, Bell System Tech. J. 50 (1971) 2495–2519] and by Bapat, Kirkland, and Neumann [R. Bapat, S.J. Kirkland, M. Neumann, On distance matrices and Laplacians, Linear Algebra Appl. 401 (2005) 193–209].

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1. Introduction

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and let $\mathcal{S}_n$ be the set of permutations of $[n]$. Partition $\mathcal{S}_n$ into $\mathcal{S}_n = \mathcal{E}_n \cup \mathcal{O}_n$, where $\mathcal{E}_n$ (respectively $\mathcal{O}_n$) is the set of even (respectively odd) permutations

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in $\mathcal{S}_n$. It is well known that $|\mathcal{E}_n| = |\mathcal{O}_n|$. Let σ and π be two elements of $\mathcal{S}_n$. Diaconis and Graham [4] defined a metric called Spearman's measure of disarray on the set $\mathcal{S}_n$ as follows:

$$D(\sigma, \pi) = \sum_{i=1}^{n} \big|\sigma(i) - \pi(i)\big|.$$

They derived the mean, variance, and limiting normality of $D(\sigma, \pi)$ when σ and π are chosen independently and uniformly from $\mathcal{S}_n$. In particular, the authors in [4] characterized those permutations $\sigma \in \mathcal{S}_n$ for which $D(\sigma) =: D(1, \sigma)$ takes on its maximum value. Some related work appears in [12,16]. The *length* $|\sigma|$ of a permutation σ is defined to be $D(1, \sigma)$, that is, $|\sigma| = \sum_{i=1}^{n} |i - \sigma(i)|$. For an arbitrary nonnegative integer k, let

$$\mathcal{A}_{n,k} = \big\{\sigma \in \mathcal{S}_n \mid |\sigma| = k\big\},$$
$$N_{n,k} = \sum_{\sigma \in \mathcal{A}_{n,k}} \operatorname{sgn}(\sigma) = |\mathcal{A}_{n,k} \cap \mathcal{E}_n| - |\mathcal{A}_{n,k} \cap \mathcal{O}_n|.$$

Furthermore, we define $\phi_{\sigma,k} = 0$ if σ has at least one fixed point, otherwise, let $\phi_{\sigma,k}$ be the number of nonnegative integer solutions of the equation $x_1 + x_2 + \cdots + x_n = k$ which satisfy $0 \leqslant x_i < |i - \sigma(i)|$ for $1 \leqslant i \leqslant n$. Let

$$M_{n,k} = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma)\phi_{\sigma,k}. \tag{1}$$

It is natural to pose the following problem:

Problem 1.1. Find closed expressions for $N_{n,k}$ and $M_{n,k}$.

We may generalize the concept of the length of a permutation defined in Problem 1.1 as follows. Let T be a weighted tree with the vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. For two vertices u and v in T, there exists a unique path $u = v_{i_1} - v_{i_2} - \cdots - v_{i_l} - v_{i_{l+1}} = v$ from u to v in T. Define the *distance* $d(u, v)$ between u and v as zero if $u = v$, otherwise, let $d(u, v)$ be the sum $x_1 + x_2 + \cdots + x_l$, where x_k is the weight of edge $v_{i_k} v_{i_{k+1}}$ for $k = 1, 2, \ldots, l$. Let T be a simple tree (i.e., the weight of each edge equals one) and let $\sigma \in \mathcal{S}_n$. The *length* $|\sigma_T|$ of σ on T is defined as the sum of all $d(v_i, v_{\sigma(v_i)})$, that is, $|\sigma_T| = \sum_{i=1}^{n} d(v_i, v_{\sigma(i)})$. Let

$$\mathcal{A}_{n,k}(T) = \big\{\sigma \in \mathcal{S}_n \mid |\sigma_T| = k\big\},$$
$$N_{n,k}(T) = \sum_{\sigma \in \mathcal{A}_{n,k}(T)} \operatorname{sgn}(\sigma) = \big|\mathcal{A}_{n,k}(T) \cap \mathcal{E}_n\big| - \big|\mathcal{A}_{n,k}(T) \cap \mathcal{O}_n\big|.$$

Furthermore, we define $\phi_{\sigma,k}(T) = 0$ if σ has at least one fixed point, otherwise, let $\phi_{\sigma,k}(T)$ be the number of nonnegative integer solutions of the equation $x_1 + x_2 + \cdots + x_n = k$ which satisfy $0 \leqslant x_i < d(v_i, v_{\sigma(i)})$ for $1 \leqslant i \leqslant n$. Let

$$M_{n,k}(T) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma)\phi_{\sigma,k}(T). \tag{2}$$

A more general problem than Problem 1.1 is the following:

Problem 1.2. Let T be a simple tree with vertex set $\{v_1, v_2, \dots, v_n\}$. Find closed expressions for $N_{n,k}(T)$ and $M_{n,k}(T)$.

Remark 1.1. If we take $T = P_n$ (where P_n is a simple path with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{(v_i, v_{i+1}) \mid 1 \leqslant i \leqslant n-1\}$) in Problem 1.2, then Problem 1.1 is a special case of Problem 1.2. That is, $N_{n,k} = N_{n,k}(P_n)$ and $M_{n,k} = M_{n,k}(P_n)$.

The distance matrix $D(T)$ of the weighted tree T is an $n \times n$ matrix with its (i, j)-entry equal to the distance between vertices v_i and v_j. If T is a simple tree, Graham and Pollak [9] obtained the following result:

Theorem 1.1 *(Graham and Pollak [9]). Let T be a simple tree with n vertices. Then*

$$\det\bigl(D(T)\bigr) = -(n-1)(-2)^{n-2}, \tag{3}$$

which is independent of the structure of T.

Other proofs of Theorem 1.1 can be found in [1–3,6–8,19]. In particular, in [19] we gave a simple method to prove (3). If T is a weighted tree, Bapat, Kirkland, and Neumann [3] generalized the result in Theorem 1.1 as follows.

Theorem 1.2 *(Bapat, Kirkland, and Neumann [3]). Let T be a weighted tree with n vertices and with edge weights $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$. Then, for any real number x,*

$$\det\bigl(D(T) + xJ\bigr) = (-1)^{n-1}2^{n-2}\left(\prod_{i=1}^{n-1}\alpha_i\right)\left(2x + \sum_{i=1}^{n-1}\alpha_i\right), \tag{4}$$

where J is an $n \times n$ matrix with all entries equal to one.

A direct consequence of Theorem 1.2 is the following:

Corollary 1.1 *(Bapat, Kirkland, and Neumann [3]). Let $D(T)$ be as in Theorem 1.2. Then*

$$\det\bigl(D(T)\bigr) = (-1)^{n-1}2^{n-2}\left(\prod_{i=1}^{n-1}\alpha_i\right)\left(\sum_{i=1}^{n-1}\alpha_i\right). \tag{5}$$

Suppose T is a weighted tree with the vertex set $V(T) = \{v_1, v_2, \dots, v_n\}$, and suppose the distance $d(u, v)$ between two vertices u and v is α. Define two kinds of *q-distances* between u and v, denoted by $d_q(u, v)$ and $d_q^*(u, v)$, as $[\alpha]$ and q^α respectively, where

$$[\alpha] = \begin{cases} \frac{1-q^\alpha}{1-q} & \text{if } q \neq 1, \\ \alpha & \text{otherwise.} \end{cases}$$

By definition, $[0] = 0$ and $[\alpha] = 1 + q + q^2 + \dots + q^{\alpha-1}$ if α is a positive integer. We define two *q-distance matrices* on the weighted tree T, denoted by $D_q(T)$ and $D_q^*(T)$, as the $n \times n$ matrices with their (i,j)-entries equal to $d_q(v_i,v_j)$ and $d_q^*(v_i,v_j)$, respectively. If $q=1$ then $D_q(T)$ is the distance matrix $D(T)$ of T. Hence the distance matrix is a special case of the q-distance matrix $D_q(T)$.

In quantum chemistry, if T is a simple tree with vertex set $V(T)=\{v_1,v_2,\ldots,v_n\}$,

$$W(T,q)=\sum_{i<j} d_q^*(v_i,v_j)=\sum_{\{u,v\}\subseteq V(T)} q^{d(u,v)}$$

is called the *Wiener polynomial* of T [11], $D_1(T)$ is called the *Wiener matrix* [10], and the q-derivative $W'(T,1)$ is defined as the *Wiener index* of T [17,18]. The study of the Wiener index, one of the molecular-graph-based structure descriptors (so-called "topological indices"), has been undergoing rapid expansion in the last few years (see for example [13–15,20,21]).

In the next section, we compute the determinants of $D_q^*(T)$ and $D_q(T)$, and show that they are independent of the structure of T, and hence we generalize the results obtained by Graham and Pollak [9] and by Bapat, Kirkland, and Neumann [3]. In Section 3, based on the results of Section 2, we prove that the generating functions $F_n(q)=\sum_{k\geqslant 0} N_{n,k}(T)q^k$ and $G_n(q)=\sum_{k\geqslant 0} M_{n,k}(T)q^k$ of $\{N_{n,k}(T)\}_{k\geqslant 0}$ and $\{M_{n,k}(T)\}_{k\geqslant 0}$, as defined in Problem 1.2, are exactly $\det(D_q^*(T))$ and $\det(D_q(T))$, respectively. Hence, both $F_n(q)$ and $G_n(q)$ are independent of the structure of T, and this leads to a resolution of Problem 1.2.

2. Determinants of $D_q^*(T)$ and $D_q(T)$

First we compute the determinant of $D_q^*(T)$.

Theorem 2.3. *Let T be a weighted tree with n vertices and with edge weights $\alpha_1,\alpha_2,\ldots,\alpha_{n-1}$. Then, for any $n\geqslant 2$,*

$$\det\bigl(D_q^*(T)\bigr)=\prod_{i=1}^{n-1}\bigl(1-q^{2\alpha_i}\bigr), \tag{6}$$

which is independent of the structure of T.

Proof. We prove the theorem by induction on n. It is trivial to show that the theorem holds for $n=2$ or $n=3$. Hence we assume that $n\geqslant 4$. Without loss of generality, we suppose that v_1 is a pendant vertex and $e=(v_1,v_s)$ is a pendant edge with weight α_1 in T. Let d_i denote the ith column of $D_q^*(T)$ for $1\leqslant i\leqslant n$. Note that each entry along the diagonal is one. Hence, by the definition of $D_q^*(T)$, we have

$$\bigl(d_1-q^{\alpha_1}d_s\bigr)^T=\bigl(1-q^{2\alpha_1},0,\ldots,0\bigr).$$

Thus

$$\det\bigl(D_q^*(T)\bigr)=\det\bigl(d_1-q^{\alpha_1}d_s,d_2,d_3,\ldots,d_n\bigr)=\bigl(1-q^{2\alpha_1}\bigr)\det\bigl(D_q^*(T)_1^1\bigr), \tag{7}$$

where $D_q^*(T)_1^1$ equals $D_q^*(T-v_1)$. By induction, the theorem is immediate from (7). □

Corollary 2.2. *Let T be a simple tree with n vertices. Then*

$$\det\bigl(D_q^*(T)\bigr) = \bigl(1-q^2\bigr)^{n-1},$$

which is independent of the structure of T.

To evaluate the determinant of $D_q(T)$ we must introduce some terminology and notation. Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix, and let $I = \{i_1, i_2, \ldots, i_l\}$ and $J = \{j_1, j_2, \ldots, j_l\}$ be two subsets of $\{1, 2, \ldots, n\}$. We use $A^{i_1 i_2 \ldots i_l}_{j_1 j_2 \ldots j_l}$ to denote the submatrix of A by deleting rows in I and columns in J.

Zeilberger [22] gave an elegant combinatorial proof of Dodgson's determinant-evaluation rule [5] as follows:

$$\det(A)\det\bigl(A^{1n}_{1n}\bigr) = \det\bigl(A^1_1\bigr)\det\bigl(A^n_n\bigr) - \det\bigl(A^n_1\bigr)\det\bigl(A^1_n\bigr), \tag{8}$$

where A is a matrix of order $n > 2$. Let

$$F(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = \frac{[\alpha_1][\alpha_2][\alpha_1+\alpha_2]}{[2\alpha_1][2\alpha_2]} + \frac{[\alpha_{n-2}][\alpha_{n-1}][\alpha_{n-2}+\alpha_{n-1}]}{[2\alpha_{n-2}][2\alpha_{n-1}]} + \sum_{i=1}^{n-3} \frac{[\alpha_i][\alpha_{i+2}][\alpha_i+\alpha_{i+2}]}{[2\alpha_i][2\alpha_{i+2}]}.$$

It is not difficult to prove the following lemma.

Lemma 2.1. (a) *If $n \geqslant 3$, $F(\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ is a symmetric function on $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$.*

(b) *If T is a weighted tree with two vertices and with edge weight α_1,* $\det(D_q(T)) = -[\alpha_1]^2$.

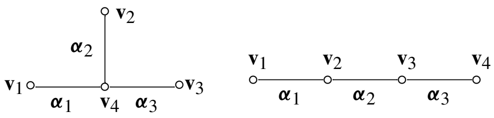
(c) *If T is a weighted tree with three vertices and with edge weights α_1, α_2,* $\det(D_q(T)) = 2[\alpha_1][\alpha_2][\alpha_1+\alpha_2]$.

Theorem 2.4. *Let T be a weighted tree with n vertices and with edge weights $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. Then, for any $n \geqslant 4$,*

$$\det\bigl(D_q(T)\bigr) = (-1)^{n-1}\left(\prod_{i=1}^{n-1}[2\alpha_i]\right) \times \left(\frac{[\alpha_1][\alpha_2][\alpha_1+\alpha_2]}{[2\alpha_1][2\alpha_2]} + \frac{[\alpha_{n-2}][\alpha_{n-1}][\alpha_{n-2}+\alpha_{n-1}]}{[2\alpha_{n-2}][2\alpha_{n-1}]} + \sum_{i=1}^{n-3} \frac{[\alpha_i][\alpha_{i+2}][\alpha_i+\alpha_{i+2}]}{[2\alpha_i][2\alpha_{i+2}]}\right), \tag{9}$$

which is independent of the structure of T.

Proof. We prove the theorem by induction on n. Note that there exist two trees with four vertices: the star $K_{1,3}$ and the path P_4. Let the edge weights of two weighted trees $K_{1,3}$ and P_4 with four



(a) (b)

Fig. 1. (a) The weighted tree $T_{1,3}$. (b) The weighted tree P_4.

vertices be as shown in Fig. 1(a) and (b), respectively. The q-distance matrices $D_q(T_{1,3})$ and $D_q(P_4)$ of $K_{1,3}$ and P_4 are as follows:

$$D_q(T_{1,3}) = \begin{pmatrix} 0 & [\alpha_1+\alpha_2] & [\alpha_1+\alpha_3] & [\alpha_1] \\ [\alpha_1+\alpha_2] & 0 & [\alpha_2+\alpha_3] & [\alpha_2] \\ [\alpha_1+\alpha_3] & [\alpha_2+\alpha_3] & 0 & [\alpha_3] \\ [\alpha_1] & [\alpha_2] & [\alpha_3] & 0 \end{pmatrix}$$

and

$$D_q(P_4) = \begin{pmatrix} 0 & \alpha_1 & [\alpha_1+\alpha_2] & [\alpha_1+\alpha_2+\alpha_3] \\ [\alpha_1] & 0 & [\alpha_2] & [\alpha_2+\alpha_3] \\ [\alpha_1+\alpha_2] & [\alpha_2] & 0 & [\alpha_3] \\ [\alpha_1+\alpha_2+\alpha_3] & [\alpha_2+\alpha_3] & [\alpha_3] & 0 \end{pmatrix}.$$

We calculate

$$\det\bigl(D_q(K_{1,3})\bigr) = \det\bigl(D_q(P_4)\bigr) = -[2\alpha_1][2\alpha_2][2\alpha_3] \times \left(\frac{[\alpha_1][\alpha_2][\alpha_1+\alpha_2]}{[2\alpha_1][2\alpha_2]} + \frac{[\alpha_2][\alpha_3][\alpha_2+\alpha_3]}{[2\alpha_2][2\alpha_3]} + \frac{[\alpha_1][\alpha_3][\alpha_1+\alpha_3]}{[2\alpha_1][2\alpha_3]}\right).$$

Hence the theorem holds for $n=4$.

Now we assume that T is a weighted tree with n vertices and $n \geqslant 5$. We denote the q-distance matrix $D_q(T)$ of T by D. Note that T has least two pendant vertices. Without loss of generality, we assume both v_1 and v_n are pendant vertices of T. The unique neighbor of v_1 (respectively v_n) is denoted by v_s (respectively v_t). For convenience, we may suppose that the weights of two edges v_1v_s and v_nv_t are β_1 and β_{n-1}, and the weights of the edges in $T - v_1 - v_n$ are $\beta_2, \beta_3, \ldots, \beta_{n-2}$. Obviously, $\{\beta_1, \beta_2, \ldots, \beta_{n-1}\} = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ ($\{\beta_1, \beta_2, \ldots, \beta_{n-1}\}$ may be a multiset). Let d_i denote the ith column of $D_q(T)$. By the definition of v_1, v_s, v_t, and v_n, we have

$$\bigl(d_1 - q^{\beta_1} d_s\bigr)^T = \bigl(-q^{\beta_1}[\beta_1], [\beta_1], [\beta_1], \ldots, [\beta_1]\bigr)$$

and

$$\bigl(d_n - q^{\beta_{n-1}} d_t\bigr)^T = \bigl([\beta_{n-1}], [\beta_{n-1}], \ldots, [\beta_{n-1}], -q^{\beta_{n-1}}[\beta_{n-1}]\bigr),$$

which imply the following:

$$\overline{d_1}^T = \bigl(d_1 - q^{\beta_1} d_s\bigr)^T + \frac{-[\beta_1]}{[\beta_{n-1}]}\bigl(d_n - q^{\beta_{n-1}} d_t\bigr)^T = \bigl(-[2\beta_1], 0, 0, \ldots, 0, \bigl(1+q^{\beta_{n-1}}\bigr)[\beta_1]\bigr),$$

where d_1^T denotes the transpose of d_1. Hence

$$\det(D) = \det(d_1, d_2, \ldots, d_n) = \det(\overline{d_1}, d_2, d_3, \ldots, d_{n-1}, d_n).$$

So we have

$$\det(D) = -[2\beta_1]\det(D_1^1) + (-1)^{n+1}(1+q^{\beta_{n-1}})[\beta_1]\det(D_1^n). \tag{10}$$

Similarly, we have

$$\det(D) = -[2\beta_{n-1}]\det(D_n^n) + (-1)^{n+1}(1+q^{\beta_1})[\beta_{n-1}]\det(D_n^1). \tag{10$'$}$$

On the other hand, by Dodgson's determinant-evaluation rule (6), we have

$$\det(D)\det(D_{1n}^{1n}) = \det(D_1^1)\det(D_n^n) - \det(D_1^n)\det(D_n^1). \tag{11}$$

By the definition of the q-distance matrix D $(= D_q(T))$ of T, $\det(D_1^n) = \det(D_n^1)$. In particular, D_1^1, D_n^n, and D_{1n}^{1n} denote the q-distance matrices $D_q(T - v_1)$, $D_q(T - v_n)$, and $D_q(T - v_1 - v_n)$ of trees $T - v_1$, $T - v_n$, and $T - v_1 - v_n$, respectively. Note that $T - v_1$ (respectively $T - v_n$) is a weighted tree with $n - 1$ vertices and with edge weights $\beta_2, \beta_3, \ldots, \beta_{n-1}$ (respectively $\beta_1, \beta_2, \ldots, \beta_{n-2}$). Hence, by induction, we have

$$\begin{aligned}\det(D_1^1) = (-1)^{n-2}\left(\prod_{i=2}^{n-1}[2\beta_i]\right) &\times \left(\frac{[\beta_2][\beta_3][\beta_2+\beta_3]}{[2\beta_2][2\beta_3]} + \frac{[\beta_{n-2}][\beta_{n-1}][\beta_{n-2}+\beta_{n-1}]}{[2\beta_{n-2}][2\beta_{n-1}]}\right. \\ &\left.+ \sum_{i=2}^{n-3}\frac{[\beta_i][\beta_{i+2}][\beta_i+\beta_{i+2}]}{[2\beta_i][2\beta_{i+2}]}\right)\end{aligned} \tag{12}$$

and

$$\begin{aligned}\det(D_n^n) = (-1)^{n-2}\left(\prod_{i=1}^{n-2}[2\beta_i]\right) &\times \left(\frac{[\beta_1][\beta_2][\beta_1+\beta_2]}{[2\beta_1][2\beta_2]} + \frac{[\beta_{n-3}][\beta_{n-2}][\beta_{n-3}+\beta_{n-2}]}{[2\beta_{n-3}][2\beta_{n-2}]}\right. \\ &\left.+ \sum_{i=1}^{n-4}\frac{[\beta_i][\beta_{i+2}][\beta_i+\beta_{i+2}]}{[2\beta_i][2\beta_{i+2}]}\right).\end{aligned} \tag{13}$$

Similarly,

$$\det\bigl(D_{1n}^{1n}\bigr) = (-1)^{n-3}\left(\prod_{i=2}^{n-2}[2\beta_i]\right) \times \left(\frac{[\beta_2][\beta_3][\beta_2+\beta_3]}{[2\beta_2][2\beta_3]} + \frac{[\beta_{n-3}][\beta_{n-2}][\beta_{n-3}+\beta_{n-2}]}{[2\beta_{n-3}][2\beta_{n-2}]} + \sum_{i=2}^{n-4}\frac{[\beta_i][\beta_{i+2}][\beta_i+\beta_{i+2}]}{[2\beta_i][2\beta_{i+2}]}\right). \tag{14}$$

From (10) and (10′),

$$\bigl[\det(D)\bigr]^2 + [2\beta_1]\det(D)\det\bigl(D_1^1\bigr) + [2\beta_{n-1}]\det(D)\det\bigl(D_n^n\bigr) + [2\beta_1][2\beta_{n-1}]\det\bigl(D_1^1\bigr)\det\bigl(D_n^n\bigr) = [2\beta_1][2\beta_{n-1}]\det\bigl(D_n^1\bigr)\det\bigl(D_1^n\bigr),$$

and hence by (11) we have

$$\bigl[\det(D)\bigr]^2 + [2\beta_1]\det(D)\det\bigl(D_1^1\bigr) + [2\beta_{n-1}]\det(D)\det\bigl(D_n^n\bigr) + [2\beta_1][2\beta_{n-1}]\det(D)\det\bigl(D_{1n}^{1n}\bigr) = 0. \tag{15}$$

Note that, by Theorem 1.1, if $q = 1$ and $\beta_i = 1$ for $1 \leqslant i \leqslant n-1$, then $\det(D) = -(n-1) \times (-2)^{n-1}$, which implies that $\det(D) \neq 0$. Then by (12) we have

$$\det(D) + [2\beta_1]\det\bigl(D_1^1\bigr) + [2\beta_{n-1}]\det\bigl(D_n^n\bigr) + [2\beta_1][2\beta_{n-1}]\det\bigl(D_{1n}^{1n}\bigr) = 0. \tag{16}$$

From (12), (13), (14) and (16), it is immediate that

$$\det(D) = (-1)^{n-1}\left(\prod_{i=1}^{n-1}[2\beta_i]\right) \times \left(\frac{[\beta_1][\beta_2][\beta_1+\beta_2]}{[2\beta_1][2\beta_2]} + \frac{[\beta_{n-2}][\beta_{n-1}][\beta_{n-2}+\beta_{n-1}]}{[2\beta_{n-2}][2\beta_{n-1}]} + \sum_{i=1}^{n-3}\frac{[\beta_i][\beta_{i+2}][\beta_i+\beta_{i+2}]}{[2\beta_i][2\beta_{i+2}]}\right). \tag{17}$$

Note that $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} = \{\beta_1, \beta_2, \ldots, \beta_{n-1}\}$. The theorem follows immediately from (a) in Lemma 2.1 and (17). □

Let T be a weighted tree with the vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and with the edge weights $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, and let v_1 and v_n be two pendant vertices of T. The unique neighbor of v_1 (respectively v_n) is denoted by v_s (respectively v_t). The proof above also implies that

$$\det\bigl(D_q(T)_1^n\bigr) = [\alpha_1][\alpha_{n-1}]\prod_{i=2}^{n-2}[2\alpha_i],$$

where α_1 and α_{n-1} are the weights of edges v_1v_s and v_nv_t, respectively.

If we set $q = 1$ then the right-hand side of (9) in Theorem 2.4 equals

$$(-1)^{n-1}\prod_{i=1}^{n-1}(2\alpha_i)\left(\frac{\alpha_1\alpha_2(\alpha_1+\alpha_2)}{(2\alpha_1)(2\alpha_2)}+\frac{\alpha_{n-2}\alpha_{n-1}(\alpha_{n-2}+\alpha_{n-1})}{(2\alpha_{n-2})(2\alpha_{n-1})}+\sum_{i=1}^{n-3}\frac{\alpha_i\alpha_{i+2}(\alpha_i+\alpha_{i+2})}{(2\alpha_i)(2\alpha_{i+2})}\right)$$
$$=(-1)^{n-1}2^{n-2}\left(\prod_{i=1}^{n-1}\alpha_i\right)\left(\sum_{i=1}^{n-1}\alpha_i\right),$$

which implies Corollary 1.1 is a special case of Theorem 2.4. Hence we generalize the results obtained by Graham and Pollak [9], and by Bapat, Kirkland, and Neumann [3]. In particular, the following corollary is immediate from Theorem 2.4.

Corollary 2.3. *Let T be a simple tree with n vertices. Then*

$$\det\bigl(D_q(T)\bigr)=(-1)^{n-1}(n-1)(1+q)^{n-2},$$

which is independent of the structure of T.

3. The quantities $M_{n,k}(T)$ and $N_{n,k}(T)$

Let T be a simple tree and $\mathcal{A}_{n,k}(T)=\{\sigma\in\mathcal{S}_n \mid |\sigma_T|=k\}$. Partition $\mathcal{S}_n$ into $\mathcal{S}_n=\mathcal{A}_{n,0}(T)\cup\mathcal{A}_{n,1}(T)\cup\cdots\cup\mathcal{A}_{n,k}(T)\cup\cdots$.

Theorem 3.5. *Let T be a simple tree with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $N_{n,k}(T)$ be defined as in Problem 1.2. Then*

$$N_{n,k}(T)=\sum_{\sigma\in\mathcal{A}_{n,k}(T)}\operatorname{sgn}(\sigma)=\begin{cases}0 & \text{if } k \text{ is odd},\\ (-1)^{\frac{k}{2}}\binom{n-1}{\frac{k}{2}} & \text{if } k \text{ is even},\end{cases}$$

which is independent of the structure of T.

Proof. Let $F_n(q)=\sum_{k\geqslant 0}N_{n,k}(T)q^k$ be the generating function of $\{N_{n,k}(T)\}_{k\geqslant 0}$. Hence

$$F_n(q)=\sum_{k\geqslant 0}\left(\sum_{\sigma\in\mathcal{A}_{n,k}(T)}\operatorname{sgn}(\sigma)\right)q^k=\sum_{k\geqslant 0}\left(\sum_{\sigma\in\mathcal{A}_{n,k}(T)}\operatorname{sgn}(\sigma)\right)q^{|\sigma_T|}$$
$$=\sum_{k\geqslant 0}\left(\sum_{\sigma\in\mathcal{A}_{n,k}(T)}\operatorname{sgn}(\sigma)\right)q^{\sum_{i=1}^n d(v_i,v_{\sigma(i)})}=\sum_{\sigma\in\mathcal{S}_n}\left(\operatorname{sgn}(\sigma)q^{\sum_{i=1}^n d(v_i,v_{\sigma(i)})}\right)$$
$$=\sum_{\sigma\in\mathcal{S}_n}\left(\operatorname{sgn}(\sigma)\prod_{i=1}^n d_q^*(v_i,v_{\sigma(i)})\right).$$

By the definition of $D_q^*(T)$, we have

$$\det\big(D_q^*(T)\big) = \sum_{\sigma\in\mathcal{S}_n}\left(\operatorname{sgn}(\sigma)\prod_{i=1}^{n} d_q^*(v_i, v_{\sigma(i)})\right).$$

The theorem is immediate from Corollary 2.2. □

With notation as in the introduction, we state and prove our last result.

Theorem 3.6. *Let T be a simple tree with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $M_{n,k}(T)$ and $\phi_{\sigma,k}(T)$ be as in* (2). *Then*

$$M_{n,k}(T) = \sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\phi_{\sigma,k}(T) = (-1)^{n-1}(n-1)\binom{n-2}{k},$$

which is independent of the structure of T.

Proof. Let $G_n(q) = \sum_{k\geqslant 0} M_{n,k}(T)q^k$ be the generating function of $\{M_{n,k}(T)\}_{k\geqslant 0}$. Hence

$$\begin{aligned}
G_n(q) &= \sum_{k\geqslant 0}\left(\sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\phi_{\sigma,k}(T)\right)q^k = \sum_{\sigma\in\mathcal{S}_n}\left(\operatorname{sgn}(\sigma)\sum_{k\geqslant 0}\phi_{\sigma,k}(T)q^k\right)\\
&= \sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)\big(1+q+\cdots+q^{d(v_1,v_{\sigma(1)})-1}\big)\cdots\big(1+q+\cdots+q^{d(v_n,v_{\sigma(n)})-1}\big)\\
&= \sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)d_q(v_1, v_{\sigma(1)})d_q(v_2, v_{\sigma(2)})\cdots d_q(v_n, v_{\sigma(n)}).
\end{aligned}$$

By the definition of $D_q(T)$, we have

$$\det\big(D_q(T)\big) = \sum_{\sigma\in\mathcal{S}_n}\operatorname{sgn}(\sigma)d_q(v_1, v_{\sigma(1)})d_q(v_2, v_{\sigma(2)})\cdots d_q(v_n, v_{\sigma(n)}).$$

The theorem follows immediately from Corollary 2.3. □

By Remark 1.1, Theorems 3.5 and 3.6, $M_{n,k} = (-1)^{n-1}(n-1)\binom{n-2}{k}$, while $N_{n,k} = 0$ if k is odd and $N_{n,k} = (-1)^{\frac{k}{2}}\binom{n-1}{\frac{k}{2}}$ otherwise.

Our method to prove Theorems 3.5 and 3.6 is completely algebraic. Therefore it would be interesting to consider the following problem.

Problem 3.3. Give combinatorial proofs of Theorems 3.5 and 3.6.

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