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Destroying automorphisms by fixing nodes

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Abstract

The fixing number of a graph G is the minimum cardinality of a set $S \subset V(G)$ such that every nonidentity automorphism of G moves at least one member of S, i.e., the automorphism group of the graph obtained from G by fixing every node in S is trivial. We provide a formula for the fixing number of a disconnected graph in terms of the fixing numbers of its components and make some observations about graphs with small fixing numbers. We determine the fixing number of a tree and find a necessary and sufficient condition for a tree to have fixing number 1.

Keywords: Fixing number; Symmetry breaking; Automorphism group

1. Introduction

We use [7] for notation and terminology. The idea of distinguishing the nodes in a graph from one another goes back to work by Sumner [18] and Entringer and Gassman [6]. Denote by $N(v)$ the open neighborhood of node v. In studying point determination in graphs, they consider the following question: which graphs G have the property that for every pair u, v of nodes of G, $N(u) = N(v)$ implies that $u = v$? In such a graph, every node v can be distinguished from every other node by the map $v \mapsto N(v)$.

Most of the work on the problem of distinguishing nodes has used ideas different from those of Sumner. Specifically, the bulk of the existing literature has focused on two approaches to the problem. In the first—*dimension/location*—introduced separately by Harary and Melter [10] and Slater [17] each node of a connected graph G is distinguished from every other node of G by labeling a subset S of $V(G)$ and using distances between the nodes of G and nodes of S to construct a one-to-one function on $V(G)$.

The second approach, within which the ideas in this article find a natural home, has its roots both in a method employed by Tutte [19] to enumerate plane maps, and a later unrelated article by Zuckerman [20]. The notion of *symmetry breaking* was formalized by Albertson and Collins [2] and, independently, by Harary [8,9]. In this approach, a subset of the node set is colored in such a way that the automorphism group of the graph is ‘destroyed’, i.e., the automorphism group of the resulting structure is trivial. One distinction between distance/location and symmetry breaking that is worth noting is that, in the former, a 1–1 function on $V(G)$ is usually explicitly present at the end of the process, while in the latter we are usually left only with an assurance that, somehow, every two nodes can be distinguished.

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Both of these areas have applications. Dimension/location has been applied both to drug design [5] and to the problem of a robot navigating through euclidean space using distinctive landmarks [12]. Symmetry breaking has been applied to the problem of programming a robot to manipulate objects [13].

We shall denote the automorphism group of G by $\Gamma(G)$ and by S_1 the trivial group. For a node v of G, the set $\{\phi(v) : \phi \in \Gamma(G)\}$ is the *orbit* of v under $\Gamma(G)$, and two nodes in the same orbit are *similar*. An automorphism ϕ *fixes* a node v if $\phi(v) = v$. The set of automorphisms that fix v is a subgroup of $\Gamma(G)$ called the *stabilizer* of v and will be denoted $\Gamma_v(G)$. Let $\mathcal{O}(v)$ be the orbit containing v. Then

$$|\mathcal{O}(v)| = |\Gamma(G)|/|\Gamma_v(G)|. \tag{1}$$

For an elementary reference on group actions we refer to [11]. Lastly, we note a well-established fact (see [4]) that will play a part in what follows: every automorphism is also an *isometry*, that is, for $u, v \in V(G)$ and $\phi \in \Gamma(G)$, $d(u, v) = d(\phi(u), \phi(v))$.

Let $S \subset V(G)$ and $\phi \in \Gamma(G)$. The automorphism ϕ is said to *fix* the set S if for every $v \in S$, we have $\phi(v) = v$. The set of automorphisms that fix S is a subgroup $\Gamma_S(G)$ of $\Gamma(G)$ and $\Gamma_S(G) = \bigcap_{v \in S} \Gamma_v(G)$. If S is a set of nodes for which $\Gamma_S(G) = S_1$, then S *fixes* the graph G and we say that S is a *fixing set* of G. The minimum cardinality of a set of nodes that fixes G is the *fixing number* $\mathit{fix}(G)$, introduced in [9], and a fixing set containing $\mathit{fix}(G)$ nodes is a *minimum fixing set* of G (or, for short, a $\mathit{fix}(G)$-set). As an example, we note that, for every positive integer n,

$$\begin{aligned}
&\mathit{fix}(K_n) = n - 1,\\
&\mathit{fix}(P_n) = 1, \quad n \geqslant 2,\\
&\mathit{fix}(C_n) = 2, \quad n \geqslant 3.
\end{aligned}$$

Let c be a (not necessarily proper) coloring of $V(G)$ and $\phi \in \Gamma(G)$. The automorphism ϕ is said to *fix* the coloring c if for every $v \in V(G)$, we have $c(\phi(v)) = c(v)$. The set of automorphisms that fix c is a subgroup $\Gamma(G, c)$ of $\Gamma(G)$. If c is a coloring for which $\Gamma(G, c) = S_1$, then c *fixes* G, and we say that c is a *fixing coloring* of G. The minimum number of colors in a coloring that fixes G is the *chromatic fixing number* $\chi_{\mathit{fix}}(G)$. The chromatic fixing number was defined independently by Albertson and Collins [2] and Harary [9], and is further studied in [3,16].

2. The fixing number

We now present some elementary results on the fixing number. We begin by establishing that we may restrict our attention to connected graphs.

2.1. Disconnected graphs

For a positive integer t, by tG we mean the disjoint union of t copies of G. A maximal set of pairwise isomorphic components of G is a *component class* of G, and if H is a component of G and ϕ an automorphism of G, then ϕ *acts nontrivially* on H if there is some $v \in V(H)$ having $\phi(v) \neq v$. A component isomorphic to H is a *H-component*.

Let T be the tree obtained from P_6 by adding a new node v and joining v to a central node of P_6; thus, T is the smallest nontrivial identity tree. Consider the graph $G = 3K_2 \cup 3T$, shown in Fig. 1.

Note that $\Gamma(K_2) = S_2$ and $\Gamma(T) = S_1$. Let $\phi \in \Gamma(G)$ and let H be a component of G on which ϕ acts nontrivially. Either (i) $\phi(V(H)) = V(H)$, i.e., ϕ induces a nontrivial automorphism of the nodes of H, or (ii) $\phi(V(H)) = V(H')$, where H' is a H-component of G distinct from H. If H is K_2, then either (i) or (ii) could be true, while if H is T, then (i) cannot be true since T admits no nontrivial automorphisms. Thus, if $S \subset V(G)$ fixes G, then every K_2-component contains at least $\mathit{fix}(K_2) = 1$ nodes of S, while two of the three T-components must each contain at least one node of S.

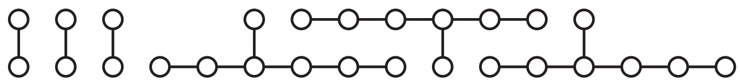

Fig. 1. The graph $G = 3K_2 \cup 3T$.

Consequently, $\mathit{fix}(G) \geqslant 5$. Choose nodes x_1, x_2, x_3 from different K_2-components of G and nodes y_1, y_2 from different T-components of G. The set $S' = \{x_1, x_2, x_3, y_1, y_2\}$ fixes G. It follows that $\mathit{fix}(G) = 5$. The ideas illustrated by the preceding example can be used to express the fixing number of a graph in terms of the fixing numbers of its components.

Observation 1. *Let A be the set of components X of G satisfying $|\Gamma(X)| = 1$ and let B be the set of components Y of G satisfying $|\Gamma(Y)| > 1$. Let k be the number of component classes in A. Then*

$$\mathit{fix}(G) = \sum_{Y \in B} \mathit{fix}(Y) + |A| - k.$$

2.2. Bounds on fix(G)

Two upper bounds on $\mathit{fix}(G)$ are easily established. The number of nodes in a graph is its *order*. Let S be the set constructed by choosing from each orbit of G every node except one. Then S fixes G. Hence, for every graph G having order n and α orbits under the action of $\Gamma(G)$, $\mathit{fix}(G) \leqslant n - \alpha$.

To see the second upper bound, for each nontrivial automorphism $\phi \in \Gamma(G)$, choose a node v_ϕ that is moved by ϕ. Then the union of all v_ϕ fixes G; consequently, $\mathit{fix}(G) \leqslant |\Gamma(G)| - 1$.

Still another upper bound on $\mathit{fix}(G)$ is given by a previously studied invariant. Let $S = \{s_1, s_2, \ldots, s_k\}$ be a k-subset of $V(G)$ and, for each node $v \in V(G)$, define $r(v|S) = (d(v, s_1), d(v, s_2), \ldots, d(v, s_k))$. A k-set S is a *resolving* or *locating set* for G if for every pair u, v of distinct nodes of G, $r(u|S) \neq r(v|S)$. The (*metric*) *dimension* or *location number* $\dim(G)$, considered in [5,10,12,14,17], is the smallest cardinality of a resolving subset $S \subset V(G)$. A resolving set of minimum cardinality is a *metric basis* for G.

Lemma 2. *If S is a metric basis for G, then $\Gamma_S(G)$ is trivial.*

Proof. Let $S = \{s_1, s_2, \ldots, s_k\}$ and suppose, to the contrary, that there is some node u and some $\phi \in \Gamma_S(G)$ for which $u \neq \phi(u)$. Since S is a resolving set for G, there is some integer i with $1 \leqslant i \leqslant k$ such that $d(u, s_i) \neq d(\phi(u), s_i)$. However, since ϕ fixes s_i, this contradicts the fact that ϕ is an isometry; thus, no such automorphism ϕ exists and $\Gamma_S(G)$ is trivial. □

Theorem 3. *For every connected graph G,*

$$\chi_{\mathit{fix}}(G) - 1 \leqslant \mathit{fix}(G) \leqslant \dim(G).$$

Proof. The upper bound on $\mathit{fix}(G)$ is Lemma 2. The lower bound follows directly from the definitions of $\mathit{fix}(G)$ and $\chi_{\mathit{fix}}(G)$. □

2.3. Graphs with small fixing number

A graph G has fixing number zero if and only if G is an identity graph, i.e., $\Gamma(G)$ is trivial. The problem of algebraically characterizing those graphs with fixing number 1 is almost as simple. The following result is essentially proved in [2] (where it provides a sufficient but not necessary condition for a graph G to have $\chi_{\mathit{fix}}(G) = 2$) and follows immediately from (1).

Observation 4. *Let G be a nonidentity graph. Then $\mathit{fix}(G) = 1$ if and only if G has an orbit of cardinality $|\Gamma(G)|$.*

As an immediate application of Observation 4 we determine the fixing number of the grid $P_s \times P_t$.

Theorem 5. *For every pair s, t of integers with $s, t \geqslant 2$,*

$$\mathit{fix}(P_s \times P_t) = \begin{cases} 2 & \text{if } s = t = 2 \text{ or } 3, \\ 1 & \text{otherwise}. \end{cases}$$

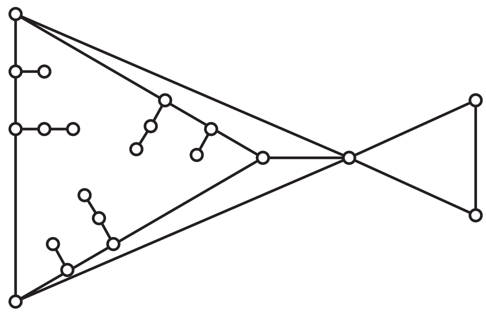

Fig. 2. A graph with fixing number 2 and automorphism group $\mathbb{Z}_6$.

Proof. If $s \neq t$, then $\Gamma(P_s \times P_t) = S_2 \oplus S_2$ and the four nodes of degree 2 are similar, so by Observation 4 the fixing number is 1. Similarly, if $s = t \geqslant 4$, then $\Gamma(P_s \times P_t) = D_4$ and the eight nodes that are adjacent to the four nodes of degree two are similar. If $s = t = 2$ or 3, then $\Gamma(P_s \times P_t) = D_4$ but neither $P_2 \times P_2$ nor $P_3 \times P_3$ contains an orbit of size eight. $\square$

The automorphism group of a graph with fixing number 1 can be arbitrarily large.

Observation 6. *For every positive integer t, there is a graph G_t with fixing number* 1 *and* $|\Gamma(G_t)| = t$.

Proof. Let $G_1 = K_1$ and $G_2 = K_2$. For $t \geqslant 3$, let G_t be the graph obtained from $C_{3t} : u_0, u_1, \ldots, u_{3t-1}$ by joining, for each integer $i \equiv 1, 2 \bmod 3$, the node u_i to a new node w_i (thus introducing $2t$ new nodes) and then, for each integer $i \equiv 1 \bmod 3$, subdividing the edge $u_i w_i$. The graph G_t has order $6t$, fixing number 1, and automorphism group $\mathbb{Z}_t$. $\square$

While every graph in the class constructed in Observation 6 has fixing number 1 and cyclic automorphism group, not every graph with cyclic automorphism group has fixing number 1.

Observation 7. *For every positive integer k, there are infinitely many graphs with fixing number k and cyclic automorphism group.*

Proof. When $k = 1$ the result is provided by Observation 6. Let $k \geqslant 2$ and $p_1, p_2, \ldots, p_k$ distinct prime numbers. For each i with $1 \leqslant i \leqslant k$, construct the graph G_{p_i} according to the instructions in Observation 6. Form a graph H from the graphs $G_{p_1}, G_{p_2}, \ldots, G_{p_k}$ as follows: introduce a new node z and, for every pair i, j of positive integers with $i \equiv 0 \bmod 3$ and $1 \leqslant j \leqslant k$, join z to the p_j nodes u_i of G_{p_j}. The graph H has fixing number k and automorphism group $\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_k} \cong \mathbb{Z}_{p_1 p_2 \cdots p_k}$. $\square$

The construction in Observation 7 is shown for the case $p_1 = 2$, $p_2 = 3$ in Fig. 2.

2.4. The fixing number of a tree

We now establish a formula for the fixing number of a tree. In order to do this, we first must prove some elementary results.

2.4.1. Preliminary results

If v is an endnode of tree T, then every node similar to v is an endnode of T. An orbit that consists of endnodes will be called an *endorbit*.

Lemma 8. *Let T be a tree and $S \subset V(T)$. Then S fixes T if and only if S fixes the endnodes of T.*

Proof. One direction of the proof is trivial; for the other, let u and v be distinct nodes of T and ϕ an automorphism mapping u to v. Let P be a maximal path that includes both u and v. The path P has an endnode u' satisfying $d(u, u') < d(v, u')$, and by assumption $\phi(u') = u'$. But then $d(u, u') < d(\phi(u), u')$, contradicting the fact that ϕ is an isometry. □

The *eccentricity* of a node u is $e(u) = \max\{d(u, v) : v \in V(G)\}$. The *radius* of G is $\operatorname{rad} G = \min\{e(u) : u \in V(G)\}$ and the *center* of G is the subgraph $\operatorname{Cen}(G)$ induced by those nodes with eccentricity equal to the radius. A node v *lies between* two nodes u, w if v lies on a $u - w$ *geodesic*, i.e., v lies on a $u - w$ path of length $d(u, w)$. A set S of nodes *lies between* u and v if some $u - v$ geodesic has a nonempty intersection with S.

Lemma 9. *Let u, v, w be three nodes of a tree T. If $d(u, v) = d(u, w)$, then $d(v, w)$ is even (and hence $vw \notin E(T)$).*

Proof. Let $P : u = v_0, v_1, \ldots, v_k = v$ be the $u - v$ path, Q the $u - w$ path, and t the largest integer $(0 \leqslant t \leqslant k - 1)$ for which $v_t \in V(P) \cap V(Q)$. Necessarily, $d(v, v_t) = d(w, v_t)$ and the result follows. □

Jordan observed that the center of a tree T is either K_1 or K_2 [7]. In the first instance, we call T *central* and in the second *bicentral*. For brevity we shall write C for $\operatorname{Cen}(T)$.

Lemma 10. *Let u, v be adjacent nodes of a tree T. If $v \notin V(C)$, then $d(u, V(C)) \neq d(v, V(C))$.*

Proof. Let T_u, T_v be the components of the tree $T - uv$ containing the nodes u and v, respectively. For all nodes $x \in T_u$ and $y \in T_v$, we have $d(u, x) = d(v, x) - 1$ and $d(v, y) = d(u, y) - 1$. Since C is connected, the result follows immediately. □

Lemma 11. *Let u be a node in a tree T and v satisfy $d(u, v) = e(u)$. Then $V(C)$ lies between u and v.*

Proof. Let $P : u = v_0, v_1, \ldots, v_k = v$ be the $u - v$ path in T and assume, to the contrary, that $V(P) \cap V(C) = \emptyset$. Then there is a unique node v_j on P that both (i) lies between u and $V(C)$ and (ii) lies between v and $V(C)$. Let $t = d(v_j, V(C))$. Since every node $c \in V(C)$ has $e(c) = \operatorname{rad} T = r$, we must have $t + k - j \leqslant r$. Furthermore, since $e(u) = d(u, v)$, we have $t + r - 1 \leqslant k - j$. However, upon combining these two inequalities we find that $t \leqslant \frac{1}{2}$, which implies that v_j is central. □

Corollary 12. *Let u, v be adjacent nodes in a tree T. If $d(u, V(C)) < d(v, V(C))$, then $e(u) < e(v)$.*

Corollary 13. *Let v be a node of a tree T. Then $e(v) = \operatorname{rad} T + d(v, V(C))$.*

Proof. If v is central then the result follows immediately. Hence, let $v \notin V(C)$. We consider two cases according to whether T is central or bicentral.

Case 1: T is central. Let c be the unique central node of T. Let e be that edge incident with c that lies between v and c, and T_c the component of $T - e$ containing c. From Lemma 11, it suffices to show that there is a node $c' \in V(T_c)$ which satisfies $d(c, c') = \operatorname{rad} T = r$. Suppose, to the contrary, that every node of T_c is distance at most $r - 1$ from c. Then necessarily every node in the component of $T - e$ containing v is distance at most r from c. However, if u is the node distinct from c that is incident with e, then $e(u) \leqslant r$, which contradicts our assumption that T is central.

Case 2: T is bicentral. Now let c_1, c_2 be the two central nodes of T. Since c_1 and c_2 are adjacent, we may from Lemma 9 let $d(v, c_1) < d(v, c_2)$. Let T_2 be the component of $T - uv$ containing the node c_2. Then there must be a node $c' \in V(T_2)$ that is distance at least $r - 1$ from c_2, and hence distance at least r from c_1. The result then follows from Lemma 11. □

Lemma 14. *Let u and v be two nodes of a tree T having $e(u) = e(v)$.*

(i) *If $d(u, v)$ is odd, then T is bicentral and both central nodes of T lie between u and v.*
(ii) *If $d(u, v)$ is even, then T may be central or bicentral, and the middle node of the $u - v$ path P lies between u and C.*

Proof. We consider two cases.

Case 1: $d(u, v)$ is odd. If the center of T does not lie between u and v or if the tree is central, then there is a unique node z on the $u - v$ path that both lies between u and $V(C)$ and between v and $V(C)$. Thus, from Corollary 13, $d(u, z) = d(v, z)$. This contradicts our assumption that $d(u, v)$ is odd. Consequently, the center lies between u and v and T is bicentral.

Case 2: $d(u, v)$ is even. Since $e(u) = e(v)$, it follows from Corollary 13 that the two central nodes of T cannot both lie on P. If only a single node of C lies on P, the result is immediate. Suppose then that the center of T does not lie between u and v. Then there is a unique node z of P that lies between u and C and between v and C. Moreover, from Corollary 13, $d(u, z) = d(v, z)$, so that z is the central node of P. Consequently, the center of the $u - v$ path lies between u and v and the center of T. $\square$

If u, v are similar nodes of a graph G, then since every automorphism is an isometry, we must have $e(u) = e(v)$.

Lemma 15. *Let u, v, w be three similar nodes in a tree T and P the $u - v$ path in T. If P has odd order, z is the central node of P, and w is in the component of $T - z$ containing u, then $d(u, w) < d(v, w)$. Similarly, if P has even order, e is the central edge of P, and w is in the component of $T - e$ containing u, then $d(u, w) < d(v, w)$.*

Proof. Suppose first that P is odd. From Corollary 13 and Lemma 14, $d(u, z) = d(v, z) = d(w, z)$. However, $d(u, w) \leqslant 2d(v, z) - 2 < 2d(v, z) = d(v, w)$. A similar argument proves the other case. $\square$

2.4.2. Interchange equivalence classes

An automorphism ϕ *interchanges* two nodes u, v if $\phi(u) = v$ and $\phi(v) = u$. Let $P : u_0, u_1, \ldots, u_k$ be a path in G and ϕ an automorphism of G. If, for every integer i with $0 \leqslant i \leqslant k$, $\phi(u_i) = u_{k-i}$, then ϕ *flips* the path P.

Observation 16. *If an automorphism ϕ interchanges two nodes u, v in a graph G, then ϕ flips every $u - v$ path in G.*

The following result, due to Prins, will prove useful.

Theorem 17 (*Prins [15]*). *For every pair u, v of similar nodes in a tree T, there is an automorphism that interchanges u and v.*

We shall be interested in pairs of similar nodes that admit a special kind of interchange. Let u and v be similar nodes in an orbit $\mathcal{O}$. An automorphism ϕ that interchanges u and v while fixing every other node in $\mathcal{O}$ is a (u, v)-*interchange*. Define a relation $R_{\mathcal{O}}$ on $\mathcal{O}$ as follows: $uR_{\mathcal{O}}v$ if there is a (u, v)-interchange. Then $R_{\mathcal{O}}$ is an equivalence relation on $\mathcal{O}$ (the properties of reflexivity and symmetry follow immediately; for transitivity: if f is a (u, v)-interchange and g is a (v, w)-interchange, then fgf is a (u, w)-interchange). Two nodes that are related under $R_{\mathcal{O}}$ will be said to be in the same *interchange equivalence class*, which we abbreviate IEC. Every node v of T is in some orbit $\mathcal{O}(v)$ and hence in some IEC under $R_{\mathcal{O}(v)}$; we shall denote this IEC by $\overline{v}$. As usual, the set of IECs of $\mathcal{O}$ under $R_{\mathcal{O}}$ is written $\mathcal{O}/R_{\mathcal{O}}$.

Lemma 18. *For all $\phi \in \Gamma(G)$, $aR_{\mathcal{O}}b$ if and only if $\phi(a)R_{\mathcal{O}}\phi(b)$.*

Proof. If f is an (a, b)-interchange, then $\phi f \phi^{-1}$ is a $(\phi(a), \phi(b))$-interchange. A similar proof shows the converse. $\square$

Corollary 19. *Let $A, B \in \mathcal{O}/R(\mathcal{O})$. Then $|A| = |B|$ and, for every $\phi \in \Gamma(G)$, $\phi(A) \cap B \neq \emptyset$ if and only if $\phi(A) = B$.*

Corollary 20. *For every node $v \in V(G)$, $|\overline{v}|$ divides $|\mathcal{O}(v)|$ divides $|\Gamma(G)|$.*

Let u, v be nodes in a connected graph G. Then we say that u *fixes* v if, for all $\phi \in \Gamma(G)$, we have $\phi(u) = u$ implies that $\phi(v) = v$ (or, equivalently, we can say that u fixes v if $\Gamma_u(G) < \Gamma_v(G)$). The relation '*fixes*' is reflexive and transitive but not necessarily symmetric. The *fixing digraph* $F(G)$ is constructed as follows: $V(F(G)) = V(G)$ and $(u, v) \in E(F(G))$ if and only if u fixes v (in G). Note that for every node u, the loop (u, u) is an arc of $F(G)$.

For example, let G be the graph with $V(G)=\{a_1, a_2, b_1, b_2, c\}$ and $E(G)=\{a_1a_2, a_1c, a_2c, b_1b_2, b_1c, b_2c\}$. Then a_1 fixes a_2 (and vice versa), b_1 fixes b_2 (and vice versa), and every vertex fixes the vertex c. It follows that $F(G)$ has $V(F(G))=V(G)$ and $E(F(G))=\{(a_1,a_1), (a_1,a_2), (a_1,c), (a_2,a_2), (a_2,a_1), (a_2,c), (b_1,b_1), (b_1,b_2), (b_1,c), (b_2,b_2), (b_2,b_1), (b_2,c), (c,c)\}$.

Let G be a graph (or digraph) and $\mathscr{P}=\{X_1, X_2, \ldots, X_k\}$ be a collection of subsets of $V(G)$. Then the *$\mathscr{P}$-defective domination number* $\widetilde{\gamma}_{\mathscr{P}}(G)$ is the smallest cardinality of a set $S \subset V(G)$ such that for every integer i with $1 \leqslant i \leqslant k$, at most one node of X_i is not adjacent from a node of S.

For example, let G be the grid $P_2 \times P_4$ with $V(G)=\{a_{ij} : 1 \leqslant i \leqslant 2 \text{ and } 1 \leqslant j \leqslant 4\}$ and $E(G)=\{a_{ij}a_{i'j'} : |i-i'|=1$ or $|j-j'|=1\}$. Let $X_1=\{a_{11}\}$, $X_2=\{a_{12}, a_{22}\}$, $X_3=\{a_{21}, a_{22}, a_{23}\}$, $X_4=\{a_{14}, a_{24}\}$ and $\mathscr{P}=\{X_1, X_2, X_3, X_4\}$. Then $\{a_{23}\}$ is a minimum $\mathscr{P}$-defective dominating set, so $\widetilde{\gamma}_{\mathscr{P}}(G)=1$.

2.4.3. The main theorem

We are now ready to state and prove our main result.

Theorem 21. *Let T be a tree, let Θ be the set of endorbits of T, and let $\mathscr{P}=\bigcup_{\mathcal{O}\in\Theta} \mathcal{O}/R_{\mathcal{O}}$. Then $fix(T)=\widetilde{\gamma}_{\mathscr{P}}(F(T))$.*

Proof. We begin by showing that $fix(T) \leqslant \widetilde{\gamma}_{\mathscr{P}}(F(T))$. Let S be a $\mathscr{P}$-defective dominating set of $F(T)$. By Lemma 8, it suffices to prove that every endnode of T is fixed by S. Suppose to the contrary that there is an endorbit $\mathcal{O}$, a node $u \in \mathcal{O}$, and an automorphism $\phi \in \Gamma_S(T)$ such that $\phi(u) \neq u$. Since ϕ moves both u and $\phi(u)$, it follows from our choice of S that $\overline{u} \neq \overline{\phi(u)}$. We claim that $\overline{u}=\{u\}$; suppose, to the contrary, that u' is a node distinct from u having $u' \in \overline{u}$. From our choice of S, the automorphism ϕ fixes u'. But then, from Lemma 18, we have $\overline{u}=\overline{\phi(u')}=\overline{\phi(u)}$. This is a contradiction and our claim is proved.

Let v be a node in $\mathcal{O}$ distinct from u for which $d(u,v)$ is a minimum, and P the $u-v$ path in T. We now consider two cases.

Case 1a: P has odd order. Let z be the central node of P and T_u, T_v the components of $T-z$ containing u and v, respectively. Since T_u and T_v are necessarily isomorphic, there is an automorphism ϕ' that interchanges u and v, flips P (in the process moving every node in $T_u \cup T_v$), and fixes every node in $T-(T_u \cup T_v)$. Then, from Lemma 15 and our choice of v, we have $\mathcal{O} \cap V(T_u \cup T_v)=\{u, v\}$. However, the automorphism ϕ' then interchanges u and v without moving any other nodes in $\mathcal{O}$. Thus $v \in \overline{u}$, a contradiction.

Case 1b: P has even order. Then by Lemma 14, T is bicentral and both central nodes z_1, z_2 of T lie between u and v. Every automorphism that moves u to v moves every node of T. Let T_u and T_v be the two components of $T-z_1z_2$ containing u and v, respectively. From Lemma 15 and our choice of v, it must be the case that $\mathcal{O}=\{u,v\}$; thus, $v \in \overline{u}$, a contradiction, so $fix(T) \leqslant \widetilde{\gamma}_{\mathscr{P}}(F(T))$.

We now prove that $fix(T) \geqslant \widetilde{\gamma}_{\mathscr{P}}(F(T))$. Let S be a fixing set for T and suppose, to the contrary, that $|S| < \widetilde{\gamma}_{\mathscr{P}}(F(T))$. Then there is an IEC containing two endnodes not dominated by S. Amongst all such pairs of endnodes, let u, v be chosen such that $d(u,v)$ is a minimum. Let P be the $u-v$ path in T. We consider two cases.

Case 2a: P has odd order. Let z be the central node of P and T_u, T_v the components of $T-z$ containing the nodes u, v, respectively. Since S fixes T and the components T_u and T_v are isomorphic, there is (without loss of generality) some node $w \in S \cap V(T_u)$; on the other hand, since the node w does not fix u, there is some automorphism $\phi \in \Gamma_w(T)$ and some $u' \in V(T_u)-\{u\}$, such that $\phi(u)=u'$. However, every automorphism that interchanges u and v must necessarily move u', contradicting our assumption that u and v are in the same IEC.

Case 2b: P has even order. Let z be the central edge of P; then by Lemma 14, z is also the central edge of T. Consequently, every (u,v)-interchange moves every node of T. Since u and v are in the same IEC, it must therefore be the case that $\mathcal{O}(u)=\{u,v\}$, from which it follows that any node of T fixes both u and v, contradicting our assumption that no node of S fixes either of them. $\square$

2.4.4. An alternative characterization of the IECs

In order to compute the fixing number of a tree using Theorem 21, we must be able to find the endorbits of that tree, compute the IECs, and then determine the $\mathscr{P}$-defective domination number of the associated fixing digraph. Computing the endorbits in a tree is computationally easy; see, for example, [1]. We now show that, once we have found the endorbits, it is possible to compute the IECs relatively easily, and certainly without finding the automorphism group. We then apply this result to obtain an interesting characterization of trees with fixing number 1. In the next section we note that the process of determining $fix(T)$ can be simplified as well.

Lemma 22. *Let T be a tree, $\mathcal{O}$ an orbit of T, and $u, v \in \mathcal{O}$ such that $d(u, v)$ is odd. Then every automorphism that sends u to v moves every node of T.*

Proof. The result follows from Lemma 14. □

Theorem 23. *Let T be a tree, $\mathcal{O}$ an orbit of T, and u, v distinct in $\mathcal{O}$.*

(i) *If $d(u, v)$ is odd, then $uR_{\mathcal{O}}v$ if and only if $\mathcal{O} = \{u, v\}$.*
(ii) *If $d(u, v)$ is even, then $uR_{\mathcal{O}}v$ if and only if $d(u, v) = d(u, \mathcal{O} - \{u\})$.*

Proof. Considering (i) first, we note that one direction is trivial and the other follows immediately from Lemma 22.

We now prove (ii). Assume first that $d(u, v) = d(u, \mathcal{O} - \{u\})$. Let P be the $u - v$ path in T and z the central node of P. Let T_u and T_v be the components of $T - z$ containing the nodes u and v, respectively. Then T_u and T_v are isomorphic and there is an automorphism ϕ of T that interchanges u and v while fixing every node in $V(T) - V(T_u \cup T_v)$. Moreover, from our assumption and Lemma 15, neither T_u nor T_v contains any nodes in $\mathcal{O} - \{u, v\}$. Thus, ϕ is a (u, v)-interchange and $uR_{\mathcal{O}}v$.

To see the converse, let $uR_{\mathcal{O}}v$ and assume that $d(u, v) > d(u, \mathcal{O} - \{u\})$. Let $u' \in \mathcal{O}$ be such that $d(u, u') = d(u, \mathcal{O} - \{u\})$. Since $uR_{\mathcal{O}}v$, there is a (u, v)-interchange ϕ that fixes u'. Since ϕ is an isometry, $d(u, u') = d(v, u')$. Thus, if P is the $u - v$ path in T and z the central node of P, then z lies between u and u' and between v and u' and, by assumption, $d(u, z) > d(u', z)$. However, from Lemma 14, z lies between u and the center of T. Thus, $d(u', V(C)) \leqslant d(u', z) + d(z, V(C)) < d(u, z) + d(z, V(C)) = d(u, V(C))$. This together with the fact that $e(u) = e(u')$ contradicts Corollary 13 and the result follows. □

An IEC $\overline{v}$ will be called *trivial* if $\overline{v} = \{v\}$ and *nontrivial* otherwise.

Corollary 24. *Let $\mathcal{O}$ be an orbit in a tree T. If $|\mathcal{O}| \geqslant 3$, then $\mathcal{O}$ contains a nontrivial IEC.*

Proof. Let $\mathcal{O}$ contain three nodes. If T is central, then by Lemma 14 the distance between any two of these nodes is even. If, on the other hand, T is bicentral, then let z be the central edge of T. Necessarily, some component of $T - z$ contains two nodes $u, v \in \mathcal{O}$, and $d(u, v)$ is even. Thus, in either case, there is a pair of nodes in $\mathcal{O}$ that are at even distance and we may apply Case (ii) of Theorem 23 to establish the existence of a nontrivial IEC. □

An immediate and interesting consequence of Corollary 24 is the following.

Theorem 25. *Let T be a tree. Then $fix(T) = 1$ if and only if $\Gamma(T) = S_2$.*

Proof. If $\Gamma(T) = S_2$, then clearly $fix(T) = 1$. Let $fix(T) = 1$ and $S = \{u\}$ a $fix(T)$-set. Suppose, to the contrary, that $|\Gamma(T)| > 2$. Since $\Gamma_u(T) = S_1$, from (1) the orbit $\mathcal{O}(u)$ containing u has cardinality $|\Gamma(T)| \geqslant 3$. Employing Corollaries 19 and 24, we see that there are nodes $v, w \in \mathcal{O}$ and an automorphism that interchanges v, w while fixing u. This contradicts our choice of u and the result follows. □

2.4.5. The minimum fixing set

If v is a noncentral node of a tree T, then there is a unique edge e incident with v that lies between v and the center of T. The component of $T - e$ that contains the node v will be denoted $T(v)$.

Lemma 26. *Let v be a noncentral node in a tree T and $\phi \in \Gamma(T)$. If ϕ moves v, then ϕ moves every node of $T(v)$.*

Proof. Let ϕ and v be as above and suppose, to the contrary, that there is a node $v' \in V(T(v))$ that is fixed by ϕ. Since ϕ is an isometry, v' is equidistant from v and $\phi(v)$. Thus v does not lie between v' and $\phi(v)$. By our choice of v', the $v - \phi(v)$ path does not include the edge e. However, this implies that v lies between $\phi(v)$ and the center of T, contradicting the fact that $e(v) = e(\phi(v))$. □

Corollary 27. *If v is a noncentral node in a tree T and $u \in V(T(v))$, then u fixes every node that is fixed by v.*

Lemma 28. *If S is a fix(T)-set and $v \in S$, then v fixes every node in $T(v)$.*

Proof. Certainly the result is true when v is an endnode; suppose, then, that v is not an endnode and to the contrary that there is an automorphism ϕ and a node $u \in V(T(v))$ having $\phi(u) \neq u$ and $\phi(v) = v$. Since S fixes T, there is a node $v' \in V(T(v)) \cap S$ that fixes u. Consider a maximal path P beginning at v and including the node v', and let w be the endnode of T that is the last node of P. Since $w \in V(T(v)) \cap V(T(v'))$, we know from Corollary 27 that w fixes every node fixed by v and every node fixed by v'. Thus $(S - \{v, v'\}) \cup \{w\}$ fixes T, which contradicts our choice of S. □

Theorem 29. *For every tree T, there is a fix(T)-set consisting only of endnodes of T.*

Proof. Amongst all $\widetilde{\gamma}_{\mathscr{P}}(F(T))$-sets, let S be one containing the maximum number of endnodes. We claim that every node in S is an endnode. Suppose, to the contrary, that this is not the case, and let $w \in S$ be a node of degree at least 2. We assume that $\Gamma(T)$ is nontrivial. If w is central, then the result is immediate, so we assume that w is not central. Following the notation of Lemma 26, choose an endnode x from T_w; then, from Lemma 26, x fixes w and thus fixes every node fixed by w. The set $(S \cup \{x\}) - \{w\}$ is a $\widetilde{\gamma}_{\mathscr{P}}(F(T))$-set containing more endnodes than S, which is a contradiction. □

3. Conclusion

In this article, it is shown that the fixing number of a tree is the $\mathscr{P}$-defective domination number of a special kind of digraph. However, so far as we are aware, little if anything is known about the $\mathscr{P}$-defective domination number. In particular, we would be interested in learning how difficult it is to compute $\widetilde{\gamma}_{\mathscr{P}}(G)$ for different classes of graphs G and collections $\mathscr{P}$ of subsets of $V(G)$.

Other questions suggested by our results include the following. We have seen that trees with fixing number 1 admit a wholly group-theoretic characterization. Is the same true for every tree? More generally, given a group Γ and a graph G on which Γ acts, what restrictions, if any, does the structure of Γ place on the value of *fix*(G)?

References

[1] A. Aho, J. Hopcroft, D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, MA, 1976.
[2] M. Albertson, K. Collins, Symmetry breaking in graphs, Electron. J. Combin. 3 (1996) R18.
[3] M. Albertson, K. Collins, A note on breaking the symmetries of tournaments, Congr. Numer. 136 (1999) 129–131.
[4] N. Biggs, Algebraic Graph Theory, second ed., Cambridge University Press, Cambridge, 1993.
[5] G. Chartrand, L. Eroh, M. Johnson, O. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Appl. Math. 105 (2000) 99–113.
[6] R.C. Entringer, L.D. Gassman, Line-critical point determinining and point distinguishing graphs, Discrete Math. 10 (1974) 43–55.
[7] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.
[8] F. Harary, Survey of methods of automorphism destruction in graphs, Invited address, Eighth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications, Kalamazoo, Michigan, 1996.
[9] F. Harary, Methods of destroying the symmetries of a graph, Bull. Malaysian Math. Sci. Soc. 24(2) (2001) 183–191.
[10] F. Harary, R.A. Melter, On the metric dimension of a graph, Ars Combin. 2 (1976) 191–195.
[11] T.W. Hungerford, Algebra, Springer, New York, 1996.
[12] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, Discrete Appl. Math. 70 (1996) 217–229.
[13] K. Lynch, Determining the orientation of a painted sphere from a single image: a graph coloring problem, URL: ⟨http://citeseer.nj.nec.com/469475.html⟩.
[14] C. Poisson, P. Zhang, The metric dimension of unicyclic graphs, J. Combin. Math. Combin. Comput. 40 (2002) 17–32.
[15] G. Prins, The automorphism group of a tree, Ph.D. Thesis, University of Michigan, 1957.
[16] A. Russell, R. Sundaram, A note on the asymptotics and computational complexity of graph distinguishability, Electron. J. Combin. 5 (1998) R23.
[17] P.J. Slater, Leaves of trees, Congr. Numer. 14 (1975) 549–559.
[18] D.P. Sumner, Point determination in graphs, Discrete Math. 5 (1973) 179–187.
[19] W.T. Tutte, A new branch of enumerative graph theory, Bull. Amer. Math. Soc. 68 (1962) 500–504.
[20] M.M. Zuckerman, Locating vertices of trees, Notre Dame J. Formal Logic 11 (3) (1970) 375–378.