Quantities equivalent to the norm of a weighted Bergman space✩

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Abstract

Let $0 \leq \alpha < \infty$, $0 < p < \infty$, and $p - \alpha > -2$. If $f$ is holomorphic in the unit disc $D$ and if $\omega$ is a radial weight function of secure type, then the followings are equivalent:

\[
\int_D |f(z)|^p \omega(z) \, dA(z) < \infty,
\]
\[
\int_D |f(z)|^{p-\alpha} |\nabla f(z)|^\alpha \omega(z) \, dA(z) < \infty,
\]
\[
\frac{1}{\int_0^\infty \left( \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1-\alpha/p} \left( \int_0^{2\pi} |\nabla f(re^{i\theta})|^p \, d\theta \right)^{\alpha/p} \omega(r) \, r \, dr} < \infty.
\]

Here $\nabla f(z) = (1 - |z|^2) f'(z)$. Furthermore, if $f(0) = 0$ and $\omega$ is monotone, then three quantities on the left sides are mutually equivalent. This generalizes a classical result of Hardy–Littlewood.

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1. Introduction

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disc of the complex plane $\mathbb{C}$ and let $dA(z) = dx \, dy$ denote the Lebesgue area measure of $\mathbb{C}$. It follows from a theorem of Hardy and Littlewood that

\[
\int_D |f(z)|^p \left( 1 - |z|^2 \right)^\beta dA(z) \approx |f(0)|^p + \int_D |f'(z)|^p \left( 1 - |z|^2 \right)^{p+\beta} dA(z)
\] (1.1)

Generalizing (1.1), a number of authors have studied the question of finding radial weights $\omega$ for which we have

$$\int_D |f(z)|^p \omega(z) dA(z) \approx |f(0)|^p + \int_D |f'(z)|^p (1 - |z|^2)^\beta \omega(z) dA(z)$$

for $f$ holomorphic in $D$ and $0 < p < \infty$. In particular, we mention that results of Siskakis [13] and of Pavlovic and Peláez [12] show that this is true for the weights considered in Example 3.1 of [13].

In this paper, we improve these results by introducing a class of weights for which one can prove the stronger result

$$\int_D |f(z)|^p \omega(z) dA(z) \approx |f(0)|^p + \int_D |f(z)|^p (1 - |z|^2)^\alpha |f'(z)|^\alpha \omega(z) dA(z)$$

for $f$ holomorphic in $D$, $0 < p < \infty$ and $0 \leq \alpha < p + 2$. This class of weights include other previously considered in distinct settings.

We begin with introducing the following weight function $\omega(z)$ that generalize $(1 - |z|)^\beta$.

**Definition.** A weight on $D$ means a function $\omega : D \to [0, \infty)$ which is locally integrable. We call it “secure weight” if there is $r_s \in [0, 1)$ such that the following conditions are satisfied for $|z| = r \in [r_s, 1)$:

1. $\omega$ is radial, that is, $\omega(z) = \omega(|z|)$.
2. $\omega(r)$ is almost monotone.
3. $\omega(r) \approx \omega\left(\frac{1 + r}{2}\right)$.
4. $\int_r^1 \omega(\rho) d\rho \approx (1 - r)\omega(r)$.

Here and throughout, almost monotone means either almost increasing or almost decreasing in the sense of Bernstein [1]: a real valued function $\psi$ on an interval is called almost increasing if $\psi(r_1) \lesssim \psi(r_2)$ for all $r_1 \leq r_2$, and almost decreasing if $\psi(r_1) \gtrsim \psi(r_2)$ for all $r_1 \leq r_2$.

At first glance, (S3) and (S4) seems to be rather obscure. But if we set $\psi(r) = \omega(1 - r)$, (S3) says that $\psi$ has the doubling property: $\psi(r) \approx \psi\left(\frac{r}{2}\right)$ while (S4) says that the averaging property

$$\frac{1}{r} \int_0^r \psi(t) dt \approx \psi(r) \quad \text{for } 0 < r \leq 1 - r_s.$$  \hspace{1cm} (1.2)

The concept of secure weight has scope wider than several known concepts on weights, for example “admissible weight” in [7] and “majorant” in [3,4,11], as we shall see later. A typical example of secure weight $\omega(r)$ is

$$(1 - r)^a \left(1 + \log \frac{1}{1 - r}\right)^b$$

for some $a$: $-1 < a < \infty$ and $b$: $-\infty < b < \infty$.

Throughout this paper, $\omega$ always stands for a secure weight on $D$.

In terms of $\omega$, the weighted Bergman space $A^{p,\omega}$ ($0 < p < \infty$) is defined to consist of all holomorphic $f$ in $D$ satisfying

$$\|f\|_{p,\omega} := \int_D |f(z)|^p \omega(z) dA(z) < \infty.$$  

When $\omega(z) = (1 - |z|)^\beta$ ($\beta > -1$), we denote $\|f\|_{p,\omega}$ by $\|f\|_{p,\beta}$.
We let $\tilde{\nabla} f(z)$ stand for $(1 - |z|^2) f'(z)$ for the notational convenience, which is originated from the invariant complex gradient

$$\tilde{\nabla} f(z) = (f \circ \varphi_z)'(0), \quad \text{where } \varphi_z(w) = \frac{z - w}{1 - \overline{z} w}.$$ 

Then, in terms of the $p$-means

$$M_p(r, f) = \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad 0 < r < 1,$$

(1.1) can be stated as

$$\|f\|_{p, \omega}^p \approx \int_D |f'(z)|^p (1 - |z|^2)^{p+\beta} dA(z) = 2\pi \int_0^1 M_p(r, \tilde{\nabla} f)^p (1 - r^2)^{\beta r} dr$$

provided $f(0) = 0$.

Now we state the following improvement of (1.1) as our main result.

**Theorem 1.1.** Let $0 \leq \alpha < \infty$, $0 < p < \infty$, and $p - \alpha > -2$. If $f$ is holomorphic in $D$ and if $\omega$ is a secure weight, then the following conditions are equivalent.

(i) $\|f\|_{p, \omega}^p < \infty$;

(ii) $\int_D |f(z)|^{p-\alpha} |\tilde{\nabla} f(z)|^\alpha \omega(z) dA(z) < \infty$;

(iii) $\int_0^1 M_p(r, f)^{p-\alpha} M_p(r, \tilde{\nabla} f)^\alpha \omega(r) r dr < \infty$.

Furthermore, if $f(0) = 0$ and $r_s = 0$, then three quantities on the left sides are mutually equivalent.

When $\omega(r) = (1 - r^2)^\alpha$, that (i) $\Leftrightarrow$ (ii) appeared in [8]. Theorem 1.1 will be proven in Sections 3–4 after considering special cases in Section 2. A relationship between secure weights and other classes of weights will be discussed in Section 5.

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2. Preliminary results and special cases

This section is a preparatory one to prove Theorem 1.1. We will prove special cases of Theorem 1.1 in Theorems 2.2 and 2.3.

**Lemma 2.1.** Let $0 < \alpha < \infty$. If $\omega$ is a secure weight, then the functions $L(z)$ and $K(z)$ defined by

$$L(z) = L_{\alpha, \beta}(z) := \int_{|z|}^1 \left( r \log \frac{r}{|z|} \right)^{\alpha-1} \omega(r) dr,$$

$$K(z) = K_{\alpha, \beta}(z) := \int_{|z|}^1 (r - |z|)^{\alpha-1} \omega(r) dr$$

satisfy
\( K(z) \approx (1 - |z|)^\alpha \omega(z) \) for \( r_s \leq |z| < 1 \), \hspace{1cm} (2.1)

and

\( L(z) \approx K(z) \) for \( \max \left\{ r_s, \frac{1}{2} \right\} \leq |z| < 1 \). \hspace{1cm} (2.2)

**Proof.** Let \( |z| \geq r_s \). By the almost monotonicity of \( \omega \) and (S3) it follows that

\( \omega(r) \approx \omega(z) \) for \( |z| < r < \frac{1 + |z|}{2} \)

and it simply follows that

\( (r - |z|)^{\alpha - 1} \approx (1 - |z|)^{\alpha - 1} \) for \( \frac{1 + |z|}{2} < r < 1 \).

Hence

\[
K(z) = \int_{|z|}^{1 + |z|/2} (r - |z|)^{\alpha - 1} \omega(r) \, dr \\
\approx \omega(z) \int_{|z|}^{1 + |z|/2} (r - |z|)^{\alpha - 1} \, dr + (1 - |z|)^{\alpha - 1} \int_{1 + |z|/2}^{1} \omega(r) \, dr \\
\approx (1 - |z|^2)^{\alpha} \omega(z).
\]

This gives (2.1).

From the inequality

\( x \log \frac{1}{x} \leq 1 - x^2 \leq 2 \log \frac{1}{x} \) for \( 0 < x < 1 \), \hspace{1cm} (2.3)

it follows that

\[
\frac{3}{4} (r - |z|) \leq r \log \frac{r}{|z|} \leq 3(r - |z|) \quad \text{for} \quad \frac{1}{2} \leq |z| \leq r < 1,
\]

which gives (2.2). \( \square \)

The following is a primitive form of our equivalence.

**Theorem 2.2.** Let \( \omega \) be a secure weight on \( D \) with \( r_s = 0 \). Then for \( f \) holomorphic in \( D \) with \( f(0) = 0 \) and for \( 0 < p < \infty \),

\[
\int_{D} |f(z)|^{p - 2} |\nabla f(z)|^2 \omega(z) \, dA(z) \approx \int_{D} |f(z)|^{p} \omega(z) \, dA(z).
\] \hspace{1cm} (2.4)

**Proof.** Using

\( \Delta |f|^p \approx |f|^{p - 2} |f'|^2 \)

off zeros of \( f \), Green’s theorem gives that

\[
\int_{0}^{2\pi} |f(re^{i\theta})|^p \, d\theta \approx \int_{|z| < r} |f(z)|^{p - 2} |f'(z)|^2 \log \frac{r}{|z|} \, dA(z).
\] \hspace{1cm} (2.5)
Taking the integration $\int_0^1 \omega(r)r\,dr$ on both sides,

$$
\int_D |f(z)|^p \omega(z)\,dA(z) \approx \int_0^1 \omega(r)r\,dr \int_{|z|<r} |f(z)|^{p-2} |f'(z)|^2 \log \frac{r}{|z|} \,dA(z)
$$

$$
= \int_D |f(z)|^{p-2} |f'(z)|^2 \left[ \int_1^1 \omega(r) \log \frac{r}{|z|} \,dr \right] \,dA(z)
$$

$$
\geq \int_D |f(z)|^{p-2} |f'(z)|^2 \left[ \int_1^{(1-|z|)} \omega(r) \,dr \right] \,dA(z)
$$

$$
\approx \int_D |f(z)|^{p-2} \nabla f(z)^2 \omega(z)\,dA(z),
$$

(2.6)

where we used the second inequality of (2.3) and the equivalence (2.1).

On the other hand, by taking the integration $\int_1^1 \omega(r)r\,dr$ on both sides of (2.5),

$$
\int_{|z|>\frac{1}{2}} |f(z)|^p \omega(z)\,dA(z) \approx \int_{|z|<r} \int_0^1 \omega(r)r\,dr \int_{|z|<r} |f(z)|^{p-2} |f'(z)|^2 \log \frac{r}{|z|} \,dA(z)
$$

$$
= \int_D |f(z)|^{p-2} |f'(z)|^2 \left[ \int_1^{\max\{1,|z|\}} \omega(r) \log \frac{r}{|z|} \,dr \right] \,dA(z)
$$

$$
\leq \int_D |f(z)|^{p-2} |f'(z)|^2 \left[ \int_{|z|}^{(1-|z|)} \omega(r) \,dr \right] \,dA(z)
$$

$$
\approx \int_D |f(z)|^{p-2} \nabla f(z)^2 \omega(z)\,dA(z),
$$

(2.7)

where we used (2.2) and (2.1).

Since $M_p(r, f)^p$ is increasing, by (S3) it follows that

$$
\int_D |f(z)|^p \omega(z)\,dA(z) \approx \int_{|z|>\frac{1}{2}} |f(z)|^p \omega(z)\,dA(z),
$$

so that (2.4) follows from (2.6) and (2.7). □

The following generalizes (1.1) on the settings of $\omega$.

**Theorem 2.3.** Let $\omega$ be a secure weight on $D$ with $r_2 = 0$. Then for $f$ holomorphic in $D$ with $f(0) = 0$ and for $0 < p < \infty$, we have

$$
\|f\|_{p,\omega}^p \approx \int_0^1 M_p(r, \nabla f)^p \omega(r) r\,dr.
$$

(2.8)

**Proof.** Let $g(z) = \frac{f(z)}{z}$. Then by the subharmonicity of $g$ and the monotonicity of $M_p(r, g)^p$
\[ |f'(0)|^p = |g(0)|^p \lesssim \int_{|w|<\frac{1}{6}} |g(w)|^p dA(w) \lesssim \int_{\frac{1}{12}<|w|<\frac{1}{6}} |g(w)|^p dA(w) \lesssim \int_{|w|<\frac{1}{6}} |f(w)|^p dA(w). \]

Replacing \( f \) by \( f \circ \varphi_z \),
\[ |\tilde{\nabla} f(z)|^p \lesssim \int_{|w|<\frac{1}{6}} |f \circ \varphi_z(w)|^p dA(w). \]

Thus,
\[ \int_D |\tilde{\nabla} f(z)|^p \omega(z) dA(z) \lesssim \int_D \int_{|w|<\frac{1}{6}} |f \circ \varphi_z(w)|^p \omega(z) dA(w) dA(z) \]
\[ = \int_D \omega(z) dA(z) \int_{|\varphi_z(u)|<\frac{1}{6}} |f(u)|^p |\varphi_z'(u)|^2 dA(u) \]
\[ = \int_D |f(u)|^p dA(u) \int_{|\varphi_z(u)|<\frac{1}{6}} \frac{(1-|z|^2)^2}{|1-\bar{z}u|^4} \omega(z) dA(z). \quad (2.9) \]

An elementary calculation shows that
\[ |\varphi_z(u)| < \frac{1}{6} \implies |u| \leq \frac{1+6|z|}{6+|z|} \implies |u| \leq \frac{1+|z|}{2}. \quad (2.10) \]

Whence
\[ \int_{|\varphi_z(u)|<\frac{1}{6}} \frac{(1-|z|^2)^2}{|1-\bar{z}u|^4} \omega(z) dA(z) \lesssim \int_{|u| \leq \frac{1+|z|}{2}} \frac{(1-|z|^2)^2}{|1-\bar{z}u|^4} \omega(z) dA(z) \]
\[ \lesssim \int_{|u| \leq \frac{1+|z|}{2}} (1-r|u|)^{-1} \omega(r) r dr \]
\[ \lesssim (1-|u|)^{-1} \int_{|u| \leq \frac{1+|z|}{2}} \omega \left( \frac{1+r}{2} \right) r dr \]
\[ \lesssim (1-|u|)^{-1} \int_{|u|}^{1} \omega(s) ds \]
\[ \approx \omega(u), \quad (2.11) \]

where we have used (2.10), the well-known estimate [2, Lemma, p. 65]
\[ \int_0^{2\pi} \frac{(1-r^2)^2}{|1-re^{-i\theta}|^4} d\theta \lesssim (1-r|u|)^{-1}, \]

(S3) and (S4), in this order. Therefore by substituting (2.11) into (2.9), it follows that
\[ \|f\|_{p, \omega}^p \gtrsim \int_0^{2\pi} M_p(r, \tilde{\nabla} f)^p \omega(r) r dr. \]

For the converse direction, we consider the cases \( 0 < p \leq 1 \) and \( p \geq 1 \) separately. Suppose first \( 0 < p \leq 1 \). A well-known theorem of Littlewood–Paley [10,14] says that
\[
\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \lesssim \int_D (1 - |z|^2)^{p-1} |f'(z)|^p dA(z), \tag{2.12}
\]

where \( f_r(z) = f(rz) \). Taking \( \int_0^1 \omega(r)r \, dr \) after changing a variable on the right side of (2.12), it follows by (2.1) that

\[
\int_D |f(z)|^p \omega(z) dA(z) \lesssim \int_0^1 \omega(r) \left( \int_{|z|<r} (r-|z|)^{p-1} \omega(r) r \, dr \right) dA(z)
\]
\[
\lesssim \int_D |\tilde{\nabla} f(z)|^p \omega(z) dA(z). \tag{2.13}
\]

Next, suppose \( 1 \leq p < \infty \). Take a positive integer \( n \) such that \( 0 < p/n \leq 1 \). Set \( F = f^n \). Then \( F \) is holomorphic in \( D \) so that by the above case

\[
\int_D |F(z)|^{p/n} \omega(z) dA(z) \lesssim \int_D |\tilde{\nabla} F(z)|^{p/n} \omega(z) dA(z).
\]

Thus, it follows that

\[
\int_D |f(z)|^p \omega(z) dA(z) \lesssim \int_D |f(z)^{n-1} \tilde{\nabla} f(z)|^{p/n} \omega(z) dA(z).
\]

The last quantity is bounded by

\[
\left( \int_D |\tilde{\nabla} f(z)|^p \omega(z) dA(z) \right)^{1/n} \cdot \left( \int_D |f(z)|^p \omega(z) dA(z) \right)^{1-1/n}.
\]

Hence

\[
\|f\|_{p,\omega}^p \lesssim \int_0^1 M_p(r, \tilde{\nabla} f)^p \omega(r) r \, dr
\]

under the additional assumption \( \|f\|_{p,\omega}^p < \infty \). This assumption can be removed by a limiting process: use \( f_{\rho}(z) \) in place of \( f(z) \) and let \( \rho \to 1^- \). \( \square \)

3. Reducing to a decreasing weight case

We need the following technical lemma in reducing the proof of Theorem 1.1 to the case of secure weight with \( r_s = 0 \).

**Lemma 3.1.** Let \( \omega \) be a secure weight on \( D \). Then there is another secure weight \( \omega_1 \) on \( D \) with \( r_s = 0 \) such that \( \omega_1(r) = \omega(r) \) for \( r_s \leq r < 1 \).

**Proof.** If \( \omega(r_s) = 0 \), then \( \omega(r) = 0 \) for \( r \in [r_s, 1) \), so that we can take \( \omega_1 \equiv 0 \). Otherwise, we may suppose \( \omega(r_s) = 1 \). Take

\[
\omega_1(r) = \begin{cases} 
1, & \text{if } 0 \leq r \leq r_0, \\
\omega(r), & \text{if } r_s \leq r < 1.
\end{cases}
\]

Then \( \omega_1 \) satisfies (S1) and is (S2) for all \( z \in D \) and \( r \in [0, 1) \). We need to check that
\[ \omega_1(r) \approx \omega_1 \left( \frac{1 + r}{2} \right) \quad \text{for } r \in [0, 1), \] (3.1)

and that
\[ \int_r^1 \omega_1(\rho) \, d\rho \approx (1 - r) \omega_1(r) \quad \text{for } r \in [0, 1). \] (3.2)

By (S3) and (S4), these are obvious when \( r_s \leq r < 1 \).

To see (3.1) for \( r < r_s \), there are two cases: case \( r_s < \frac{1 + r}{2} < \frac{1 + r_s}{2} \) and case \( \frac{1 + r}{2} < r_s \). In the first case
\[ \omega_1(r) = \omega_1(r_s) \approx \omega_1 \left( \frac{1 + r_s}{2} \right) \approx \omega_1 \left( \frac{1 + r}{2} \right) \approx \omega_1(r_s), \]
and in the second case
\[ \omega_1(r) = \omega_1 \left( \frac{1 + r}{2} \right) = 1, \]
which gives (3.1).

To see (3.2) for \( r < r_s \), we note by (S4) that
\[ \frac{1}{C} (1 - r_s) \leq \int_{r_s}^1 \omega_1(\rho) \, d\rho \leq C (1 - r_s) \]
for some \( C > 1 \), from which it follows
\[ \frac{1}{C} (1 - r) \leq (r_s - r) + \frac{1}{C} (1 - r_s) \leq \int_r^{r_s} \omega_1(\rho) \, d\rho = \int_r^1 \omega_1(\rho) \, d\rho + \int_{r_s}^1 \omega_1(\rho) \, d\rho \]
\[ \leq (r_s - r) + C (1 - r_s) \leq C (1 - r). \]
This gives (3.2). \( \square \)

4. Proof of Theorem 1.1

With the help of Lemma 3.1, we are sufficient to prove the equivalence of the left-hand side quantities of (i), (ii), and (iii) under the condition \( r_s = 0 \) and \( f(0) = 0 \).

Fixing such a \( \omega \) and \( f \) throughout the proof, let us denote for simplicity
\[ I(p, \alpha; f) := \int_D \left| f(z) \right|^{p-\alpha} \left| \bar{\nabla} f(z) \right|^{\alpha} \omega(z) \, dA(z) \]
and
\[ J(p, \alpha; f) := 2\pi \int_0^1 M_p(r, f)^{p-\alpha} M_p(r, \bar{\nabla} f)^{\alpha} \omega(r) \, dr. \]

Then \( \log I(p, \alpha; f) \) and \( \log J(p, \alpha, f) \) are convex functions of \( \alpha \), that is,
\[ \log I(p, \alpha; f) \leq I(p, s; f)^{(t-\alpha)/(t-s)} \cdot I(p, t; f)^{1-(t-\alpha)/(t-s)} \] (4.1)
and
\[ \log J(p, \alpha; f) \leq J(p, s; f)^{(t-\alpha)/(t-s)} \cdot J(p, t; f)^{1-(t-\alpha)/(t-s)} \] (4.2)
if \( 0 \leq s < \alpha < t < \infty \). This can be easily checked by noting that
\[ \alpha = \frac{t - s}{t - s} + \frac{\alpha - t}{t - s} \]

and applying Hölder’s inequality with paring \((\frac{1}{t - s}, \frac{1}{\alpha - t})\).

We are going to prove that

\[ \|f\|_{p, \omega}^p \approx I(p, \alpha; f) \approx J(p, \alpha; f). \]

We will make use of Theorems 2.2 and 2.3 which can be summarized as

\[ \|f\|_{p, \omega} = J(p, 0; f) \approx I(p, 0; f) = I(p, 2; f) \approx I(p, p; f) = J(p, p; f). \]

(4.3)

Along with (4.3), the following result plays an essential role in this proof.

**Theorem A.** (See [9, Theorem C].) Let \(0 < p, \alpha < \infty\). Then

\[ \int_D |f(z)|^{p - \alpha} |f'(z)|^\alpha (1 - |z|)^{\alpha - 1} dA(z) \lesssim \sup_{0 < r < 1} M_p(r, f) \]

(4.4)

for all \(f\) in the Hardy space \(H^p\) if and only if \(2 \leq \alpha < p + 2\).

We now proceed the proof. Note by (4.3) that we may assume \(\alpha \neq 0\) and \(\alpha \neq p\). We consider the cases \(\alpha > p\) and \(\alpha < p\) separately.

**Case \(\alpha > p\).** We prove that (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (i).

(i) \(\Rightarrow\) (ii). If \(0 < \alpha \leq 2\), then by (4.1) and (4.3),

\[ I(p, \alpha; f) \leq I(p, 0; f)^{1 - \alpha/2} I(p, 2; f)^{\alpha/2} \approx I(p, 0; f) = \|f\|_{p, \omega}^p. \]

If \(2 < \alpha < \infty\), then by (4.4) we have

\[ \int_D |f(z)|^{p - \alpha} |f'(z)|^\alpha (1 - |z|)^{\alpha - 1} dA(z) \lesssim \int_0^{2\pi} |f(re^{i\theta})|^{p} d\theta \]

provided \(p - \alpha > -2\). Making a change of variables on the left side and taking \(\int_0^1 \omega(r)r dr\) on both sides,

\[ \int_0^1 \omega(r)r dr \int_{|z| < r} |f(z)|^{p - \alpha} |f'(z)|^\alpha (1 - \frac{|z|}{r})^{\alpha - 1} r^{\alpha - 2} dA(z) \lesssim \int_D |f(z)|^p \omega(z) dA(z). \]

(4.5)

The left side of (4.5) is

\[ \int_D |f(z)|^{p - \alpha} |f'(z)|^\alpha \left( \int_{|z|}^1 (r - |z|)^{\alpha - 1} \omega(r) dr \right) dA(z), \]

whence (2.1) gives

\[ I(p, \alpha; f) \lesssim I(p, 0; f) = \|f\|_{p, \omega}^p. \]

(ii) \(\Rightarrow\) (iii). Hölder’s inequality with the paring \((\frac{\alpha}{p}, \frac{\alpha}{\alpha - p})\),

\[ M_p(r, \nabla f)^p \leq \left( \int_0^{2\pi} |f(re^{i\theta})|^{p - \alpha} |\nabla f(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} \right)^{p/\alpha} M_p(r, f)^{p(1 - \alpha/p)}, \]

so that
\[ M_p(r, f)^{-p(1-p/\alpha)}M_p(r, \nabla f)^p \leq \left( \int_0^{2\pi} |f(re^{i\theta})|^{p-\alpha} |\nabla f(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} \right)^{p/\alpha}. \]

Taking \( f_0^1 \omega(r) \, dr \) on the \( \alpha \) power of both sides,

\[ J(p, \alpha; f) \lesssim I(p, \alpha; f). \]

(iii) \( \Rightarrow \) (i). Let \( 0 < \rho < 1 \). Taking \( f_0^\rho \omega(r) \, dr \) after making a change of variables on the right side of (2.12), it follows as in (2.13) that

\[
\int_{|z|<\rho} |f(z)|^p \omega(z) \, dA(z) \lesssim \int_0^\rho \omega(r) r \, dr \int_{|z|<r} (r - |z|)^{p-1} |f'(z)|^p \, dA(z)
\[
\leq \int_{|z|<\rho} |f'(z)|^p \left( \int_{|z|}^\rho (r - |z|)^{p-1} \omega(r) r \, dr \right) dA(z)
\[
\leq \int_{|z|<\rho} |\nabla f(z)|^p \omega(z) \, dA(z).
\]

Thus by applying Hölder’s inequality with the paring \((\alpha p, \alpha \alpha - \alpha)\),

\[
\int_0^\rho M_p(r, f)^p \omega(r) r \, dr \lesssim \int_0^\rho M_p(r, \nabla f)^p \omega(r) r \, dr
\[
\leq \left( \int_0^\rho M_p(r, f)^{p-\alpha} M_p(r, \nabla f)^\alpha \omega(r) r \, dr \right)^{p/\alpha} \cdot \left( \int_0^\rho M_p(r, f)^p \omega(r) r \, dr \right)^{1-p/\alpha},
\]

so that

\[
\int_0^\rho M_p(r, f)^p \omega(r) r \, dr \lesssim \int_0^\rho M_p(r, f)^{p-\alpha} M_p(r, \nabla f)^\alpha \omega(r) r \, dr \lesssim J(p, \alpha; f).
\]

By letting \( \rho \to 1^- \), we obtain

\[ \| f \|_{p, \omega}^p = J(p, 0; f) \lesssim J(p, \alpha; f). \]

Case \( \alpha < p \). We prove that (i) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (iii). By (4.2) and (4.3),

\[ J(p, \alpha; f) \leq J(p, 0; f)^{1-\alpha/p} J(p, p; f)^{\alpha/p} \approx I(p, 0; f) = \| f \|_{p, \omega}^p. \]

(iii) \( \Rightarrow \) (ii). By Hölder’s inequality with the paring \((\alpha p - \alpha, \frac{p}{\alpha - \alpha})\),

\[
\int_0^{2\pi} |f(re^{i\theta})|^{p-\alpha} |\nabla f(re^{i\theta})|^\alpha \frac{d\theta}{2\pi} \leq M_p(r, f)^{p-\alpha} M_p(r, \nabla f)^\alpha,
\]

so that the left side of (ii) is bounded by a constant times the left side of (iii).

(ii) \( \Rightarrow \) (i). By (4.3), we may assume \( \alpha \neq 2 \). Then (4.3) and (4.1) gives the following:

\[ \| f \|_{p, \omega}^p \approx I(p, p; f) \lesssim I(p, \alpha; f)^{(2-p)/(2-\alpha)} I(p, 2; f)^{(p-\alpha)/(2-\alpha)} \quad \text{if } 0 < \alpha < p \leq 2; \]

\[ \| f \|_{p, \omega}^p \approx I(p, 2; f) \lesssim I(p, \alpha; f)^{(p-2)/(p-\alpha)} I(p, p; f)^{(2-\alpha)/(p-\alpha)} \quad \text{if } 0 < \alpha < 2 < p; \]

\[ \| f \|_{p, \omega}^p \approx I(p, 2; f) \lesssim I(p, 0; f)^{1-2/\alpha} I(p, \alpha; f)^{2/\alpha} \quad \text{if } 2 < \alpha < p. \]
Therefore
\[ \|f\|_{p,\omega}^p \lesssim I(p, \alpha; f) \quad (4.6) \]
by (4.3) once more under the additional assumption \( I(p, 0; f) < \infty \). But since \( p > \alpha \), \(|f|^{p-\alpha} |f'|^\alpha\) as well as \(|f|^p\) are subharmonic, so that (4.6) follows by applying \( f_\rho(z) = f(\rho z) \), \( 0 < \rho < 1 \), in place of \( f(z) \) and letting \( \rho \to 1^- \).

5. A comparison of secure weights with admissible weights and Majorants

We are going to consider relationships with other terminologies related to weights.

A function \( \phi : [0, \pi] \to [0, \infty) \) is called an “admissible weights” in the sense of [7] if the followings are satisfied:

(AW1) \( \phi \) is continuous and increasing.

(AW2) \( \phi(0) = 0 \) and \( \phi(t) > 0 \) if \( t > 0 \).

(AW3) \[ \int_0^\delta \frac{\phi(t)}{t} dt \lesssim \phi(\delta) \quad \text{for } 0 < \delta < 1. \]

(AW4) \[ \int_\delta^\pi \frac{\phi(t)}{t^2} dt \lesssim \frac{\phi(\delta)}{\delta} \quad \text{for } 0 < \delta < 1. \]

A function \( m : [0, 2] \to [0, \infty) \) is called a “majorant” in the sense of [3,4,11] if the following are satisfied:

(M1) \( m \) is continuous and increasing.

(M2) \( m(0) = 0 \) and \( \frac{m(t)}{t} \) is decreasing.

A majorant \( m \) is called “regular majorant” if

(R) \[ \int_0^x \frac{m(t)}{t} dt + \int_x^2 \frac{m(t)}{t^2} dt \lesssim m(x) \quad \text{for } 0 < x < 2. \]

We call \( m : [0, 2] \to [0, \infty) \) an “almost majorant” if (M1) and (M2) are satisfied with almost increasing and almost decreasing respectively in the place of increasing and decreasing. Also we call \( m \) an “almost regular majorant” if \( m \) is an almost majorant satisfying (R). It was observed in [6, Lemma 1] that (AW1) and (AW4) implies \( \frac{\phi(t)}{t} \) is almost decreasing on \((0, 1] \).

Lemma 5.1. Let \( \psi : [0, 1) \to [0, \infty) \) is locally integrable. If either

(AM) \( \psi \) is almost increasing and \( \frac{\psi(t)}{t} \) is almost decreasing or if

\( \psi \) is almost decreasing and \( \frac{\psi(t)}{t} \) is almost increasing,

then \( \omega : D \to [0, \infty) \) defined by \( \omega(z) = \psi(1 - |z|) \) is a secure weight.

Proof. Consider the first case (AM). (S1) and (S2) are obvious. The assumption gives the doubling property of \( \psi \):

\[ \psi(t) \lesssim \psi(2t) = 2t \frac{\psi(2t)}{2t} \lesssim 2t \frac{\psi(t)}{t} = 2\psi(t) \]

and (1.2)
\[ \psi(r) = \frac{2}{r^2} \int_0^r t \, dt \lesssim \frac{1}{r} \int_0^r \frac{\psi(t)}{t} \, dt = \frac{1}{r} \int_0^r \psi(t) \, dt \lesssim \psi(r) \frac{1}{r} \int_0^r \, dt = \psi(r) \]

which are equivalent to (S3) and (S4), respectively. The second case is similar. \( \square \)

**Proposition 5.2.**

(i) If \( \phi \) is an admissible weight, then \( \omega : D \to [0, \infty) \) defined either by

\[ \omega(z) = \phi(1 - |z|) \quad \text{or by} \quad \omega(z) = \frac{\phi(1 - |z|)}{1 - |z|} \]

are secure weights.

(ii) If \( m \) is an almost majorant, then \( \omega : D \to [0, \infty) \) defined by \( \omega(z) = m(1 - |z|) \) is a secure weight.

(iii) If \( m \) is an almost regular majorant, then \( \omega : D \to [0, \infty) \) defined by

\[ \omega(z) = \frac{m(1 - |z|)}{1 - |z|} \]

is a secure weight.

**Proof.** (i) \( \phi \) is increasing by (AW1) and \( \frac{\phi(t)}{t} \) is almost decreasing by [6, Lemma 1], so that by lemma \( \omega(z) = \phi(1 - |z|) \) is secure. For the second case, let

\[ \omega(r) = \frac{\phi(1 - r)}{1 - r}, \quad 0 \leq r < 1. \]

Then \( \omega \) is almost increasing, so (S2) follows. Almost increasing property of \( \omega \) and (AW1) gives

\[ \omega(r) = \frac{\phi(1 - r)}{1 - r} \gtrsim \frac{\phi(\frac{1-r}{2})}{1 - r} = 2\omega \left( \frac{1 + r}{2} \right) \gtrsim 2\omega(r), \]  \hspace{1cm} (5.1)

so that (S3) follows. (S4) follows by using (AW3):

\[ (1 - r)\omega(r) \lesssim \int_r^1 \omega(\rho) \, d\rho = \int_0^{1-r} \frac{\phi(t)}{t} \, dt \lesssim \phi(1 - r) = (1 - r)\omega(r). \]  \hspace{1cm} (5.2)

(ii) (M1) and (M2) gives (AM), so that by lemma the result follows.

(iii) Since \( \omega \) is increasing, (5.1) with \( m \) in place of \( \phi \) holds, which gives (S3). By use of (R), (3.2) also holds with \( m \) in place of \( \phi \), which gives (S4). \( \square \)

**Example 5.3.** Let \( b < -1 \). It follows simply that

\[ \int_r^1 (1 - \rho)^{-1} \left( 1 + \log \frac{1}{1 - \rho} \right)^b \, d\rho = \frac{-1}{b + 1} \left( 1 + \log \frac{1}{1 - r} \right)^{b+1}. \]

Whence by (S4) the integrand is not a secure weight. The functions of the form

\[ t^{-1} \left( 1 + \log \frac{1}{t} \right)^b \]

is neither an admissible weight nor a majorant. The functions of the form

\[ \left( 1 + \log \frac{1}{t} \right)^b \]

is neither an admissible weight nor a regular majorant.
References