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Folner Conditions, Nuclearity, and Subexponential Growth in C^*-Algebras

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Answering a first question of Voiculescu (On the existence of quasi-central approximate units relative to normed ideals, *J. Funct. Anal.* **91** (1990), 1–36) Vaillant

for C^*-algebras and asked about the relation to growth and nuclearity. In this work we clarify the relation among subexponential growth phenomena, this Følner condition suggested by Voiculescu, weak filtrability in the sense of Arveson and Bedos, and nuclearity in the C^*-algebra context.

MOTIVATION

The work of Gromov [14, 15] and Connes [10, 11] on discrete groups, growth, and Fredholm modules has inspired Voiculescu to initiate a research program on the structure of C^*-algebras along the lines of the analoguous study of discrete groups. In particular, Voiculescu pointed out the naturality of the concept of filtration in a C^*-algebra, filtrations which play an essential role in the study of C^*-growth and Folner type conditions for C^*-algebras and their relations to nuclearity.

Answering a first question of Voiculescu [24, Problem 5.9], we showed with Kirchberg that in the C^*-context subexponential growth implies nuclearity [19, Corollary 2.2]: if a unital C^*-algebra A admits a filtration $(A_n)_{n \in \mathbb{N}}$ such that the function $f(n) = \dim(A_n)$ satisfies

$$v[f] = \limsup_{n \to \infty} \frac{\ln \dim A_n}{n} = 0,$$

then A is a nuclear C^*-algebra.

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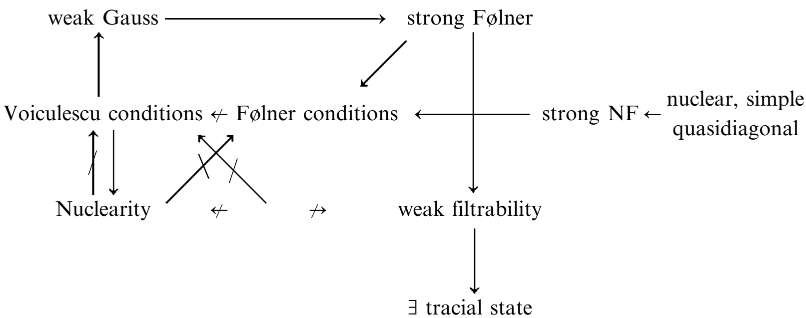
Voiculescu suggested then a Følner type condition for C^*-algebras (communicated to us by Kirchberg) and asked about the relation to subexponential growth and nuclearity. This work is devoted to the clarification of the relations between these concepts.

The first section is an exposition of our setting: the concepts of finitely generated C^*-algebra, filtration in a C^*-algebra, growth function, and degrees are introduced.

In the second section we give the relations between different subexponential growth conditions (Voiculescu condition and weak Gauss criterion) and the Følner–Voiculescu condition. We introduce a strong version of Voiculescu's Følner condition and show that if a C^*-algebra satisfies the weak Gauss criterion then it satisfies this strong Følner–Voiculescu condition. Furthermore, we show that the Følner–Voiculescu condition does not imply the Voiculescu condition.

In the third section we give the relations between the strong Følner–Voiculescu condition and an other Følner type condition introduced by Arveson (and Bedos) which he called weak filtrability and nuclearity.

The following diagram summarizes what we know about subexponential growth, Følner type conditions, and nuclearity in C^*-algebras:



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1. ON C^*-ALGEBRAS AND GROWTH FUNCTIONS

1.1. *Finitely generated C^*-Algebras and Filtrations*

1.1.1. Let A be a C^*-algebra. A is said to be finitely generated if it admits a finite-dimensional subspace $V = \mathrm{span}(X)$ such that $\bigcup_{n \in \mathbb{N}^*} \mathrm{span}(V^n)$ is dense in A; we call X a system of generators (we will suppose $X^* = X$).

If $\operatorname{span}(X^n)$ denotes the linear span of the words of length less or equal to n in the given generators, then we associate the non-negative growth function

$$f(n) = \dim(\operatorname{span} X^n)$$

to the generator system X. For a unital C^*-algebra, whenever X contains the identity element, then, taking $V^0 = \mathbb{C}1_A$, this associated function will be monotone increasing.

1.1.2. Let A be a unital C^*-algebra, a filtration of A is a sequence $(A_n)_{n\in\mathbb{N}}$ of finite-dimensional linear subspaces of A such that

(i) $A_0 \subset A_1 \subset \cdots$,

(ii) $A_0 = \mathbb{C}1_A$, $A_n^* = A_n$, $A_n \cdot A_m \subset A_{n+m}$, $m, n \in \mathbb{N}$

(iii) $\bigcup_{n\in\mathbb{N}} A_n$ is dense in A,

where 1_A denotes the identity element of A and $\mathbb{C}$ the field of complex numbers.

To any filtration $(A_n)_{n\in\mathbb{N}}$ of a C^*-algebra one can associate a growth function, taking $f(n) = \dim A_n$, which is clearly non-negative monotone increasing.

Let A be a unital C^*-algebra with generating system X, then there exists a filtration to work with: the filtration $(A_n)_{n\in\mathbb{N}}$ given by $A_0 = \mathbb{C}1_A$ and $A_n = \operatorname{span}(X^n)$, which we called the natural filtration associated with the generating system X of A in [19].

1.2. *Growth Functions and Degrees*

Let $\mathscr{G}$ be the set of growth functions, i.e., the set of non-negative monotone increasing functions $f\colon \mathbb{N} \to \mathbb{R}_+$. Let $f, g \in \mathscr{G}$. One writes $f \preccurlyeq g$ if and only if there is a natural number m and a number $c > 0$ such that $f(n) \leqslant cg(mn)$ for all n; one says that f and g are equivalent, and writes $f \sim g$, if $f \preccurlyeq g$ and $g \preccurlyeq f$ (see [2]). The equivalence class containing f is called the growth of f and is denoted by $[f]$. The motivation is the following: consider a C^*-algebra A with system of generator X, the associated filtration (A_n) and growth function, then, if Y is another generating system such that $Y \subset A_i$ for some i, we do not want to make a difference between these two filtrations. In other words, it is just the asymptotic behaviors which play a role in our study, even if the growth of a C^*-algebra does not define an invariant and is not stable under perturbation in general.

One usually defines the order of exponential growth of a growth function f as

$$\rho[f] = \limsup_{n\to\infty} \frac{\ln \ln f(n)}{\ln n},$$

the growth function $f(n)=e^n$ having the first order of growth, and the corresponding class being called the exponential growth ([2], [21]). If $\rho[f]<1$ then the growth $[f(n)]$ is called subexponential. We will actually work with the two following conditions:

DEFINITIONS. A filtration $(A_n)_{n\in\mathbb{N}}$ of A satisfies the Voiculescu condition if

$$v[f]=\limsup_{n\to\infty}\frac{\ln f(n)}{n}=0.$$

A filtration $(A_n)_{n\in\mathbb{N}}$ of A satisfies the weak Gauss criterion if for every fixed $m\in\mathbb{N}$

$$\liminf_{n\to\infty}\frac{\dim A_{n+m}}{\dim A_n}=1.$$

We say that a C^*-algebra A satisfies the weak Gauss criterion if A admits a filtration $(A_n)_{n\in\mathbb{N}}$ satisfying it.

Remark that on one hand $\rho[f]<1$ implies $v[f]=0$, on the other hand, for $f(n)=\exp(n^{(1-(1/\sqrt{\ln n}))})$ we have $\rho[f]=1$ but $v[f]=0$.

LEMMA. *If a C^*-algebra satisfies the Voiculescu condition, then it satisfies the weak Gauss criterion.*

Proof. Let A be a C^*-algebra admitting a filtration $(A_n)_{n\in\mathbb{N}}$ which satisfies the Voiculescu condition.

Assume there exists $p>1$ and $m\in\mathbb{N}$ such that $\liminf_{n\to\infty}(\dim A_{n+m}/\mathrm{di}\, A_n)=p>1$, then for $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ and p' with $1<p-\varepsilon<p'\leqslant p$ such that $\dim(A_{n_0+(n+1)m})>p'\cdot\dim(A_{n_0+nm})$ for $n=0, 1, ...$ so that for $n\geqslant 1$

$$\dim A_{n_0+nm}>(p')^n\cdot\dim A_{n_0}$$

so that

$$v[(A_n)_{n\in\mathbb{N}}]=\limsup_{n\to\infty}\frac{\ln\dim A_{n_0+nm}}{n_0+nm}\geqslant\frac{\ln p'}{m}>0.$$

Thus we get a contradiction. ∎

2. GROWTH AND THE FØLNER–VOICULESCU CONDITION

2.0. An original criterion for the amenability of groups was found by Følner who showed that the amenability is equivalent to the existence of a sequence $\{F_n\}_{n=1}^{\infty}$ of finite sets satsifying the conditions

(1) $\bigcup_{n=1}^{\infty} F_n = G$,

(2) $F_n \subset F_{n+1}$,

(3) $\lim_{n \to \infty} |gF_n \Delta F_n|/|F_n| = 0$ for all $g \in G$,

where $E \Delta F$ denotes the symmetric difference of E and F and $|E|$ the cardinality of E. A sequence $\{F_n\}_{n=1}^{\infty}$ satisfying (1), (2), (3) is called a *Følner* sequence (see [20, 7, 15, 26] for historical developments).

This definition actually copies the notion of regular filtration for 2-dimensional Riemannian manifold (see [13] for the parallel). In the next subsection we study a definition for C^*-algebras suggested by Voiculescu (communicated to us by Kirchberg).

2.1. In the sequel the C^*-algebras are supposed to be unital and separable. Let $\mathscr{U} = \{u_i\}_{i=1}^{m}$ be a finite sequence of unitaries in A, X and Y be non-zero finite-dimensional linear subspaces in A. For $\varepsilon > 0$, we say that $u_i X$ is in the ε-neighbourhood of Y, and we note $u_i X \subseteq_\varepsilon Y$, if for $x \in X$, $\|x\| = 1$, there exists $u \in Y$ such that $\|u_i x - y\| \leqslant \varepsilon$.

DEFINITION. A unital separable infinite-dimensional C^*-algebra A is said to satisfy the *Følner–Voiculescu condition* if for every finite sequence $\mathscr{U} = \{u_i\}_{i=1}^{m}$ of unitaries in A and every $\varepsilon > 0$, there exist non-zero finite-dimensional linear subspaces X and Y in A such that

(1) $u_i X \subseteq_\varepsilon Y$, $1 \in X$, for all $u_i \in \mathscr{U}$,

(2) $|1 - (\dim Y/\dim X)| < \varepsilon$.

We say it satisfies the *strong Følner–Voiculescu condition* if there is a dense *-subalgebra B of A such that for every finite subset $\mathscr{U}_c = \{b_1, ..., b_n\}$ of contractions in B, there exist non-zero finite-dimensional linear subspaces X and Y in A with $1 \in X \subseteq Y \subseteq B$ such that one has $b_i X \subseteq X$, for all $b_i \in \mathscr{U}_c$, and (2) holds.

2.2. *Remarks.* (i) One sees easily that if a group G is amenable then $C^*(G)$ does satisfy the Følner–Voiculescu conditions:

(ii) About the condition "$1 \in X$":

Let A be a C^*-algebra such that $A = B \oplus C$ where $\nu_{\text{ess}}[C] \geqslant c > 0$, i.e., every filtration of C has exponential growth, and $\nu[B] = 0$, i.e., there exists a filtration $(B_n)_{n \in \mathbb{N}}$ of B satisfying the Voiculescu condition. Let $\mathscr{U}$ be a finite sequence of unitaries in A, then, since $A = B \oplus C$, $\mathscr{U} = \{(u_i, v_i) \mid u_i \in B, v_i \in C\}$. For $\varepsilon > 0$ there exists q such that $\mathscr{U} \subseteq_\varepsilon B_q$, take as (X, Y) the couple (B_p, B_{p+q}) in B for p sufficiently large (B satisfies the strong Følner–Voiculescu condition except for "$1 \in X$," since $v[B] = 0$ as we will see below) which satisfies conditions (1) and (2) of Definition 2.1 for A, except for "$1 \in X$." Hence A would give an example of a C^*-algebra which satifies the Følner–Voiculescu conditions except for "$1 \in X$" and does not satisfying the Voiculescu one. Concretely, $\mathbb{C} \oplus \mathcal{O}_2$ gives an example of such a C^*-algebra A, where $\mathcal{O}_2$ is the Cuntz algebra on two generators [12].

(iii) A NF (nuclear finite) algebra is a C^*-algebra A which can be written as the inductive limit of a generalized inductive system $(A_n, \phi_{k,n})$ of finite-dimensional C^*-algebras with completely positive asymptotically multiplicative contractive connecting maps (see [6] for a general study of these generalized inductive limits). In [23] we showed that a separable nuclear C^*-algebra has a quasidiagonal extension by the compact operators $\mathscr{K}$ iff it is a NF algebra. If each $\phi_{k,n}$ is a complete order embedding, the system is called a strong NF system and A is called a strong NF algebra. We will see that strong NF algebras satisfy the Følner–Voiculescu condition.

THEOREM 2.3. *If a C^*-algebra A admits a filtration $(A_n)_{n \in \mathbb{N}}$ which satisfies the weak Gauss criterion, then $\bigcup_{n \in \mathbb{N}} A_n$ satisfies the strong Følner–Voiculescu condition.*

Proof. Let $Z = \{b_1, ..., b_m\}$ a sequence of contractions in $\bigcup_{n \in \mathbb{N}} A_n$ and $\varepsilon > 0$. There exists an $n_0 \in \mathbb{N}$ such that $Z \subseteq A_{n_0}$.

For all fixed integer $n' \in \mathbb{N}$, we have

$$\lim_{n \to \infty} \left| 1 - \frac{\dim A_{n+n'}}{\dim A_n} \right| = 0.$$

There exists an $n_1 \in \mathbb{N}$ such that $1 - (\dim A_{n+n_0} / \dim A_n) < \varepsilon$ for $n > n_1$, then, with $X_n = A_n$ and $Y_n = A_{n+n_0}$, we have $|1 - (\dim Y_n / \dim X_n)| < \varepsilon$ for $n > n_1$.

Now for $x \in X_n$, since $Z = \{b_i\}_{i=1}^m \subseteq A_{n_0}$, we have $y = u_i x \in A_{n+n_0}$ for all $i = 1, ..., m$, hence $ZX_n \subseteq Y_n$ for $n > n_1$. Then, for $n > n_1$, the couple (X_n, Y_n) satisfies the required conditions. ∎

PROPOSITION–REMARK 2.4. *If a C^*-algebra satisfies the strong Følner–Voiculescu condition then it satisfies the Følner–Voiculescu condition.*

Proof. Let A be a unital separable infinite-dimensional C^*-algebra satisfying the strong Følner–Voiculescu condition; i.e., there is a dense $*$-subalgebra B of A such that for every finite subset $\mathscr{U}_c = \{b_1, ..., b_n\}$ of contractions in B, there exist non-zero finite-dimensional linear subspaces

X and Y in A with $1 \in X \subseteq Y \subseteq B$ such that one has $b_i X \subseteq Y$, for all $b_i \in \mathcal{U}_c$ and $|1-(\dim Y/\dim X)| < \varepsilon$.

Let $\mathcal{U} = (u_1, ..., u_m)$ be a sequence of unitaries in A and $\varepsilon > 0$.

Let $\mathcal{U}_c = (b_1, ..., b_m)$ be a sequence of contractions in B such that $\|u_i - b_i\| \leqslant \varepsilon$, $i \leqslant m$. There exist non-zero finite-dimensional linear subspaces X and Y in A with $1 \in X \subseteq Y \subseteq B \subset A$ such that $b_i X \subseteq Y$, for all $b_i \in \mathcal{U}_c$ and $|1-(\dim Y/\dim X)| < \varepsilon$.

Furthermore, for $u_i \in \mathcal{U}$ and $x \in X$, $\|x\| = 1$, taking $y = u_i x \in Y$ we have $\|u_i x - y\| = \|(u_i - b_i)\, x\| \leqslant \varepsilon$. ∎

PROPOSITION 2.5. *If a C^*-algebra A is strong NF then it satisfies the Følner condition.*

LEMMA 2.6. [6] *If A is a strong NF-algebra, then the identity map on A can be approximated in the point-norm topology by idempotent completely positive finite-rank contractions from A to A; i.e., given $x_1, ..., x_n \in A$ and $\varepsilon > 0$, there is an idempotent completely positive finite-rank contraction $P\colon A \to A$ with $\|x_i - P(x_i)\| < \varepsilon$ for $1 \leqslant i \leqslant n$.*

Proof of Proposition 2.5. Due to Lemma 2.6, if A is strong NF then the identity map on A can be approximated in the point-norm topology by idempotent completely positive finite-rank contractions from A to A.

Let $u_1, ..., u_m$ a sequence of unitaries in A (in case $1 \in A$) and $\varepsilon > 0$, the there exist a contraction P having the previous mentioned properties such that

$$\|u_i - P(u_i)\| \leqslant \varepsilon,$$
$$\|(P(u_i^* u_i) - u_i^* u_i\| = \|P(1) - 1\| \leqslant \varepsilon,$$

so that we get

$$\begin{aligned}\|P(1) - P(u_i^*)\, P(u_i)\| &\leqslant \|P(1) - 1\| + \|1 - P(u_i^*)\, P(u_i)\| \\ &\leqslant \varepsilon + \|(u_i^* - P(u_i^*))\, P(u_i)\| + \|P(u_i^*)(u_i - P(u_i))\| \\ &\leqslant 3\varepsilon\end{aligned}$$

since P is a contraction.

But this implies that u is in the ε-multiplicative domain of P since

$$\begin{aligned}\|P(u_i^* z) - P(u_i^*)\, P(z)\|^2 &\leqslant \|P(1) - P(u_i^*)\, P(u_i)\| \cdot \|P(z^* z) - P(z^*)\, P(z)\| \\ &\leqslant 3\varepsilon \cdot 2\, \|z\|^2 \leqslant 6\varepsilon \cdot \|z\|^2,\end{aligned}$$

since P is a completely positive contraction.

Now let $X = Y = \mathrm{Im}(P) = P(A)$, then we have $\dim X = \dim Y < \infty$ and for any $x = P(z) \in X$, u_i, $i \leqslant m$, and $\eta = \varepsilon + \sqrt{6\varepsilon}$, we have, with $y = P(u_i z)$,

$$\begin{aligned} \|u_i x - y\| &= \|u_i P(z) - P(u_i z)\| \\ &\leqslant \|(u_i - P(u_i))\, P(z)\| + \|P(u_i)\, P(z) - P(u_i z)\| \\ &\leqslant (+ \sqrt{6\varepsilon}) \cdot \|x\|. \quad \blacksquare \end{aligned}$$

PROPOSITION 2.7. *The Følner–Voiculescu condition does not imply the Voiculescu condition.*

Proof. After Kirchberg [17] or since Blackadar [5] showed that $\mathcal{O}_2$ is a subquotient of $\mathcal{O}_2$, there exists a C^*-subalgebra A of the CAR algebra M_{2^∞} and an essential ideal J of A which is hereditary in M_{2^∞} such that $\mathcal{O}_2 \simeq A/J$ (or, even better, after the ICM communication of Kirchberg [18]). Blackadar and Kirchberg (see [6]) proved that A is strong NF. But after Proposition 2.5, strong NF satisfy the Følner–Voiculescu condition, this implies that A fullfills the Følner–Voiculescu condition.

On the other hand, let $(A_n)_{n \in \mathbb{N}}$ be a filtration of A and suppose it satisfies the Voiculescu condition. This would imply that there exists a filtration of $\mathcal{O}_2 \simeq A/J$ which satisfies the Voiculescu condition. But, using [24, Propositions 5.1 and 5.3] and [11, Theorem 8], we remarked in [19] that such filtrations do not exist.

Hence A gives an example of a C^*-algebra which satisfies the Følner–Voiculescu condition but does not admit any filtration satisfying the Voiculescu condition. $\blacksquare$

PROPOSITION 2.8. *Nuclearity does not imply the Følner–Voiculescu condition.*

Proof. Consider the Cuntz algebra $\mathcal{O}_2$ on two generators S_1, S_2 [12]. Consider a finite sequence $\mathcal{U}$ of unitaries describing $\{S_1, S_2\}$ (any element in a C^*-algebra can be written as the sum of four unitaries), then there is no couple (X, Y) of finite-dimensional linear subspaces in $\mathcal{O}_2$ satisfying the conditions (1) and (2) of the Definition 2.1. But $\mathcal{O}_2$ is nuclear. Hence the Følner–Voiculescu condition is not implied by nuclearity.

3. SUBEXPONENTIAL GROWTH AND WEAK FILTRATIONS

Let $\mathcal{H}$ denote a separable infinite-dimensional Hilbert space. A *filtration* of $\mathcal{H}$ is a sequence $\mathcal{H} = \{\mathcal{H}_n\}_{n \in \mathbb{N}}$ of finite-dimensional subspaces such that

$$\mathcal{H}_n \subseteq \mathcal{H}_{n+1} \qquad \text{and} \qquad \overline{\bigcup_{n \in \mathbb{N}} \mathcal{H}_n} = \mathcal{H}.$$

Let P_n denote the projection of $\mathcal{H}$ onto $\mathcal{H}_n$ and for $n \in \mathbb{N}$, define

$$C^*(\mathcal{F}) = \left\{ a \in B(\mathcal{H}) \mid \lim_{n \to \infty} \frac{\mathrm{Tr}(|P_n a - a P_n|)}{\mathrm{Tr}(P_n)} = 0 \right\}.$$

Remark that $C^*(\mathcal{F})$ is a unital C^*-subalgebra of $B(\mathcal{H})$ associated with the filtration $\mathcal{F}$ of $\mathcal{H}$.

DEFINITION 3.1. Let $A \subseteq B(\mathcal{H})$ be a C^*-algebra containing the identity operator I on $\mathcal{H}$. $\mathcal{F}$ is called a *A-filtration* if the *-subalgebra

$$\{ a \in A \mid \sup_{n \in \mathbb{N}} (\mathrm{rank}(P_n a - a P_n)) < \infty \}$$

is norm-dense in A. $\mathcal{F}$ is called a *weak A-filtration* if $A \subseteq C^*(\mathcal{F})$.

A unital C^*-algebra A is *weakly filtrable* if there exists a unital faithful *-representation π of A into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that there exists a weak $\pi(A)$-filtration of $\mathcal{H}$.

3.2. *Remarks and Examples.* (1) For a countably infinite discrete group G, the amenability of G is equivalent to the weak filtrability of $C_r^*(G)$ [3, Theorem 7]. Actually one could speak of “filtration of Følner type for A” instead of weak A-filtration.

One can observe this more precisely by comparing the definition of weak filtrability with the Følner condition of Connes for factors of type II_1 [9, Theorem 5.1 and Remark 5.35], and the Følner condition for unitary representations of locally compact groups of Bekka [4, Definition 6.1].

(2) A C^*-algebra A in $B(H)$ is called quasidiagonal [16, 25] if there exists an increasing sequence of finite rank self-adjoint projections $p_1 \leqslant p_2 \leqslant \cdots \in \mathcal{K}$ converging strongly to 1 (i.e., $\overline{\bigcup_i p_i H} = H$) such that

$$\lim_{i \to \infty} \| [p_i, a] \| = 0, \qquad \forall a \in A.$$

Hence, considering the filtration associated with the $\{p_i\}$, one has

$$\frac{\mathrm{Tr}\, |[p_i, a]|}{\mathrm{Tr}(p_i)} \leqslant 2 \, \| [p_i, a] \|.$$

Thus quasidiagonal C^*-algebras give us examples of a weakly filtrable C^*-algebras.

(3) See the list of examples after the Definition 4.1 in [1] for other examples of weak filtrable C^*-algebras, and the Section 2 of [3] for applications.

THEOREM 3.3. *If a C*-algebra satisfies the strong Følner–Voiculescu condition, then it is weakly filtrable.*

One introduces

$$A_0 = \left\{ a \in A \mid \lim_{n \to \infty} \frac{\operatorname{rank}(P_n a - aP_n)}{\operatorname{Tr} P_n} = 0 \right\}$$

which is a *-subalgebra of A such that $A_0 \subseteq C^*(\mathscr{F})$.

LEMMA 3.4. [3]. *If the *-algebra A_0 is norm-dense in A, then $\mathscr{F}$ is a weak A-filtration.*

Proof. Follows from the two following inequalities

$$\operatorname{Tr}(|F|) \leqslant \|F\| \cdot \operatorname{rank}(F), \qquad F \in \mathscr{R}(\mathscr{H}),$$

and

$$\begin{aligned} \operatorname{Tr}(|P_n a - aP_n|) &\leqslant \|P_n a - aP_n\| \cdot \operatorname{rank}(P_n a - aP_n) \\ &\leqslant 2 \cdot \|a\| \cdot \operatorname{rank}(P_n a - aP_n), \qquad A \in B(\mathscr{H}), \qquad n \geqslant 1. \end{aligned}$$

Proof of the Theorem. Let A be a C^*-algebra which satisfies the strong Følner–Voiculescu condition; i.e., there exists a dense *-subalgebra $\mathscr{A}$ of A such that for every finite subset $\mathscr{U}_c = \{b_1, ..., b_n\}$ of contractions in B, there exist $1 \in X \subseteq Y \subseteq B$ such that that one has $b_i X \subseteq Y$, for all $b_i \in \mathscr{U}_c$, and

$$\left| 1 - \frac{\dim Y}{\dim X} \right| < \varepsilon.$$

If A is finite-dimensional the result is immediate, so let us assume that A is infinite-dimensional. Let $D = \{1, d_1, d_2, ...\}$ a dense sequence of contractions in the unit-ball of A, e.g., $1 \in D \subset \mathscr{A}$ where $\mathscr{A}$ is dense in A, φ a faithful state on A and $(\pi_\varphi, H_\varphi, \eta_\varphi)$ the associated cyclic representation, $\{\varepsilon_n\}$ the decreasing sequence of real positive numbers $\{2^{-(n+1)}\}$.

For $D_1 = \{1, d_1, d_1^*\}$ and ε_1, Let (X_1, Y_1) be the associated Følner couple so that we have

(i) $\dim X_1, \dim Y_1 < \infty$,

(ii) $d \cdot X_1 \subseteq Y_1$, for all $d \in D_1$,

(iii) $(\dim Y_1 / \dim X_1) - 1 < \varepsilon_1$,

(iv) $1 \in X_1$,

Let Z_1 be a finite family of contractions in Y_1 describing Y_1 linearly (i.e., $Y_1 \subseteq \operatorname{span} Z_1$) which exists since Y_1 is finite dimensional. Since Y_1 lies in $\mathscr{A}$, then Z_1 lies in $\mathscr{A}$. Let $D_2 = D_1^2 \cup \{d_2, d_2^*\} \cup Z_1$, again for D_2 and ε_2, there exists a Følner couple (X_2, Y_2) which satisfies conditions similar to (i)–(ii)–(iii)–(iv)–(v) above.

Recurrently we obtain a sequence $(D_n, \varepsilon_n, X_n, Y_n, Z_n)$ such that

(i) $D_n = D_{n-1}^2 \cup \{d_n, d_n^*\} \cup Z_n$

(ii) $\dim X_n, \dim Y_n < \infty$,

(iii) $d \cdot X_n \subseteq Y_n$, for all $d \in D_n$,

(iv) $(\dim Y_n / \dim X_n) - 1 < \varepsilon_n$,

(v) $1 \in X_n$ (then $D_n \subseteq Y_n$).

Let $H_n^{(X, D)}$ and $H_n^{(Y, D^2)}$ be the spaces defined by

$$H_n^{(X, D)} = \operatorname{span}(\pi_\varphi(X_n)\, \eta_\varphi, \pi_\varphi(D_n)\, \eta_\varphi),$$
$$H_n^{(Y, D^2)} = \operatorname{span}(\pi_\varphi(Y_n)\, \eta_\varphi, \pi_\varphi(D_n^2)\, \eta_\varphi),$$

then since $1 \in X_n \subseteq X_n$, $D_n \subseteq Y_n$, $Y_n \subseteq \operatorname{span} D_{n+1}$ we have

$$\cdots \subseteq H_n^{(X, D)} \subseteq H_n^{(Y, D^2)} \subseteq H_{n+1}^{(X, D)} \subseteq H_{n+1}^{(Y, D^2)} \subseteq \cdots.$$

Since D' is dense in the unit-ball of A and contained in $\bigcup_{n \geqslant 1} D_n$, and π is cyclic, $\bigcup_{n \geqslant 1} H_n^{(X, D)}$ is dense in H_φ.

Denote by p_n, q_n the orthogonal projections on $H_n^{(X, D)}$, $H_n^{(Y, D^2)}$, then, since for $d \in \pi(D_n)$

$$\begin{aligned}[]
[p_n, d] &= p_n d - d p_n = p_n d q_n - q_n d p_n \\
&= p_n d (q_n - p_n) + (p_n - q_n)\, d p_n
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\operatorname{Tr}(|[p_n, d]|)}{\operatorname{Tr} p_n} &= \frac{2}{\operatorname{Tr} p_n} \cdot \operatorname{Tr}(|p_n d(q_n - p_n)|) \\
&\leqslant \frac{2}{\dim H_n^{(X, D)}} \cdot \operatorname{Tr}(|q_n - p_n|) \\
&\leqslant \frac{2}{\dim H_n^{(X, D)}} \cdot (\dim H_n^{(Y, D^2)} - \dim H_n^{(X, D)}) \\
&\leqslant 2 \left(\left(\frac{\dim H_n^{(Y)}}{\dim H_n^{(X, D) - 1}} \right) + \frac{\dim H_n^{(D^2)}}{\dim H_n^{(X, D)}} \right) \\
&\leqslant (2 + \dim H_n^{(D^2)}) \cdot \varepsilon_n \to 0 \qquad \text{as} \quad n \to \infty,
\end{aligned}$$

since $(\dim Y_n / \dim X_n) - 1 < \varepsilon_n$ implies $\dim X_n > \varepsilon_n^{-1}$ for $X_n \neq Y_n$ (which we may assume without lost of generality).

Thus for the filtration $\mathscr{F} = \{H_1^{(X,D)} \subseteq H_2^{(X,D)} \subseteq \cdots\}$ of H_φ we have

$$\pi(\mathscr{A}) \subseteq C^*(\mathscr{F}) = \left\{a \in B(H_\varphi) \mid \lim_{n \to \infty} \frac{\mathrm{Tr}(|[p_n, a]|)}{\mathrm{Tr}(p_n)} = 0\right\}$$

furthermore $\mathscr{A}$ is dense in A and $C^*(\mathscr{F})$ is a norm closed *-subalgebra in $L(H_\varphi)$ so that $\mathscr{F}$ defines a $\pi_\varphi(A)$-weak filtration. But π_φ is faithful. ∎

PROPOSITION 3.5. *Nuclearity does not imply weak filtrability.*

LEMMA 3.6 [3, Proposition 3]. *Let $A \subseteq B(\mathscr{H})$ be a C^*-algebra containing the identity operator on $\mathscr{H}$ and suppose that $\mathscr{F} = \{\mathscr{H}_n\}_{n \geqslant 1}$ is filtration of $\mathscr{H}$ which is a weak A-filtration. For every $n \geqslant 1$, let ρ_n be the state of A defined by*

$$\rho_n(a) = \frac{1}{\mathrm{Tr}\, P_n} \mathrm{Tr}(P_n a)$$

*where the P_n are defined as before relative to $\mathscr{F}$. Let R_n be the weak *-closed convex hull of the set $\{\rho_n, \rho_{n+1}, \rho_{n+2}, ...\}$. Then $R_\infty = \bigcap_{n \geqslant 1} R_n$ is a non-empty set of traces on A.*

Proof of the Proposition. Follows from the last lemma and the fact that $\mathcal{O}_2$ has no tracial states. But $\mathcal{O}_2$ is nuclear. Hence weak filtrability is not implied by nuclearity. ∎

3.7. *Remarks.* The last Proposition gives an example of nuclear C^*-algebra A such that weak A-filtrations do not exist. An A-filtration is a weak $\mathscr{A}$-filtration and the result quotied as Lemma 3.6 is a generalization of [1, Proposition 4.4].

PROPOSITION 3.8. *Weak filtrability does not imply nuclearity.*

Proof. Choi [8, Theorem 7] showed that the full group C^*-algebra $C^*(F_2)$ on the free noncommutative group F_2 on two generators can be embedded into $\bigoplus_{n=1}^{\infty} M_{2n}$ as a C^*-subalgebra; hence it is (weakly) quasi-diagonal.

This implies that there exists ([16, section 4]) an increasing sequence of finite rank self-adjoint projections $p_1 \leqslant p_2 \leqslant \cdots \in \mathscr{K}$ converging strongly to 1 (i.e., $\overline{\bigcup_i p_i H} = H$) such that

$$\lim_{i \to \infty} \|[p_i, a]\| = 0, \qquad \forall a \in C^*(F_2).$$

Considering the filtration associated with the $\{p_i\}$, one has

$$\frac{\mathrm{Tr}\,|[p_i, a]|}{\mathrm{Tr}(p_i)} \leqslant 2\,\|[p_i, a]\|.$$

The algebra $C^*(F_2)$ gives thus an example of a nonnuclear C^*-algebra which is weakly filtrable. ∎

PROPOSITION 3.9. *Weak filtrability does not imply the Voiculescu condition.*

Proof. Consider for example $C^*(F_2)$ which is, as shown in the proof of Proposition 3.8, weakly filtrable, but on the other hand does not satisfy the condition of Voiculescu which implies nuclearity. But $C^*(F_2)$ is not nuclear. ∎

3.10. *Remark.* If there is no connection between nuclearity and weak filtrability in full generality, nevertheless it follows from [3, Theorem 7; 4], Remark 2.2 and Theorem 3.3 that, for a countably infinite discrete group G, the following are equivalent:

(i) G is amenable

(ii) $C_r^*(G)$ is nuclear

(iii) $C_r^*(G)$ satisfies the strong Følner–Voiculescu condition

(iv) $C_r^*(G)$ is weakly fitrable.

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