Bitableau bases for Garsia–Haiman modules of hollow type

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Abstract

Garsia–Haiman modules $\mathbb{C}[X_n,Y_n]/I_\gamma$ are quotient rings in the variables $X_n = \{x_1, x_2, \ldots, x_n\}$ and $Y_n = \{y_1, y_2, \ldots, y_n\}$ that generalize the quotient ring $\mathbb{C}[X_n]/I$, where $I$ is the ideal generated by the elementary symmetric polynomials $e_j(X_n)$ for $1 \leq j \leq n$. A bitableau basis for the Garsia–Haiman modules of hollow type is constructed. Applications of this basis to representation theory and other related polynomial spaces are considered.

Keywords: Garsia–Haiman modules; Bitableau bases; Bipermanents; Bideterminants

1. Introduction

Let $X_n = \{x_1, x_2, \ldots, x_n\}$ and $Y_n = \{y_1, y_2, \ldots, y_n\}$ be sets of indeterminates. The main purpose of this paper is to give explicit combinatorial bases for certain quotients of the ring

$$\mathbb{C}[X_n, Y_n] = \mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$$

of polynomials in the variables $X_n$ and $Y_n$ with complex coefficients. In doing so, we give combinatorial interpretations for the corresponding Hilbert and Frobenius series. The ideals in the aforementioned quotients are defined via determinants as described below.

Throughout this paper, we will identify any element $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}) \in \mathbb{N}^2$ with the unit square in the first quadrant of the plane having $\alpha_i$ as its corner closest to the origin. A lattice diagram,
L[\alpha] = (\alpha_1, \alpha_2, \ldots, \alpha_n), is a sequence of such unit squares. To any lattice diagram \(L[\alpha]\) we associate a determinant

\[
\Delta_{L[\alpha]} = \Delta_{L[\alpha]}(X_n, Y_n) = \det \begin{pmatrix}
\delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,n} \\
\delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{n,1} & \delta_{n,2} & \cdots & \delta_{n,n}
\end{pmatrix},
\]

(2)

Given any polynomial \(P(X_n, Y_n) \in \mathbb{C}[X_n, Y_n]\), there is a corresponding polynomial of differential operators

\[
P(\partial_X, \partial_Y) = P(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}, \partial_{y_1}, \partial_{y_2}, \ldots, \partial_{y_n}).
\]

(We write \(\partial_{x_i}\) as shorthand for \(\partial/\partial x_i\).) With \(\alpha\) as above, define the ideal

\[
\mathcal{I}_{L[\alpha]} = \{ P(X_n, Y_n) \in \mathbb{C}[X_n, Y_n] : P(\partial_X, \partial_Y) \Delta_{L[\alpha]} = 0 \}.
\]

(4)

The quotients \(\mathbb{C}[X_n, Y_n]/\mathcal{I}_{L[\alpha]}\), known as Garsia–Haiman modules, were introduced by A. Garsia and M. Haiman in [1]. A good overview of the subject can be found in [2]. A. Garsia and M. Haiman introduced modules of this type to study the \(q,t\)-Kostka coefficients. This paper will henceforth concern itself only with Garsia–Haiman modules arising from hollow lattice diagrams. Roughly, a hollow lattice diagram is a subset of a hook shape obtained by removing a (perhaps trivial) contiguous region of squares from each of the arm and leg of the hook (see Fig. 1). More precisely we parametrize a hollow lattice diagram \(L[\alpha]\) by a sequence of three pairs \(\gamma = (m, k, p)\), with \(m = (m_1, m_2) \in \mathbb{Z}_{\geq 1}^2\), \(k = (k_1, k_2) \in \mathbb{N}^2\) and \(p = (p_1, p_2) \in \mathbb{N}^2\), by setting

\[
\alpha_\gamma = \begin{cases}
(0, m_2 + k + p - 1), & (0, m_2 + k_2 + p - 2), \ldots, (0, m_2 + k_2), \\
(0, m_2 - 1), & (0, m_2 - 2), \ldots, (0, 1), (0, 0), \\
(1, 0), & (2, 0), \ldots, (m_1 - 1, 0), \\
(m_1 + k_1, 0), & (m_1 + k_1 + 1, 0), \ldots, (m_1 + k_1 + p - 1, 0)).
\end{cases}
\]

(We only allow \(k_i = 0\) if \(p_i = 0\).) Unless otherwise noted, the number of squares in \(L[\alpha_\gamma]\) (namely, \(m_1 + p_1 + m_2 + p_2 - 1\)) will be denoted by \(n\).

Abusing notation slightly, we write \(\mathcal{I}_\gamma\) for \(\mathcal{I}_{L[\alpha_\gamma]}\) and \(\Delta_\gamma\) for \(\Delta_{L[\alpha_\gamma]}\). Our goal is to consider the combinatorics of the hollow Garsia–Haiman space \(\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma\). As is suggested by the previous terminology, the rings \(\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma\) carry symmetric-group representations: The symmetric group, \(S_n\), has a natural diagonal action on \(\mathbb{C}[X_n, Y_n]\) given by
\[ \sigma P(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}). \]  

(5)

This action passes through to an action on each \( \mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma \).

Let \( R \) be any \( S_n \)-module realized as a polynomial ring over \( X_n \) and \( Y_n \) (such as \( \mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma \)) and let \( R_{r,s} \) denote the subspace of \( R \) containing elements of total degree \( r \) in \( X_n \) and total degree \( s \) in \( Y_n \). We can decompose each \( R_{r,s} \) as

\[ R_{r,s} \cong \bigoplus_{\lambda \vdash n} c_{r,s}^{\lambda} S_\lambda \]

where each \( S_\lambda \) is an irreducible \( S_n \)-module (i.e., Specht module). Denote the Schur functions by \( s_\lambda \). The (bi-graded) character, Frobenius series and Hilbert series are then, respectively, given by

\[ \text{ch}(R) = \sum_{r,s} \left( \sum_{\lambda \vdash n} c_{r,s}^{\lambda} \chi_\lambda \right) t^r q^s, \]

\[ \mathcal{F}\text{ch}(R) = \sum_{r,s} \left( \sum_{\lambda \vdash n} c_{r,s}^{\lambda} s_\lambda \right) t^r q^s, \]

and

\[ \mathcal{H}(R) = \sum_{r,s} \dim(R_{r,s}) t^r q^s. \]  

(6)

Here \( \chi_\lambda \) denotes the character of \( S_\lambda \) and \( \lambda \vdash n \) signifies that \( \lambda \) is a partition of \( n \). The Frobenius series is the image of the graded character under the Frobenius map which sends \( \chi_\lambda \) to \( s_\lambda \). Note that the Hilbert series can be recovered from the Frobenius series by formally replacing each \( s_\lambda \) by the dimension of \( S_\lambda \).

By constructing an appropriate basis we will prove the following theorem. (The definition of a standard tableau will be given in Section 2; the cocharge statistics \( |X(rs(C_\gamma(T)))| \) and \( |Y(rs(C_\gamma(T)))| \) are defined in Section 5.)

**Theorem 1.** Let \( \mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma \) denote a hollow Garsia–Haiman module that is parametrized by \( \gamma = (m, k, p) \). The graded character, \( \text{ch}(\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma) \), of the quotient ring \( \mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma \) is given by

\[ \left[ \begin{array}{c} p_1 + k_1 \\ p_1 \\ \vdots \\ p_2 + k_2 \\ p_2 \end{array} \right] \sum_{\lambda \vdash n} \chi_\lambda \sum_{T \in \text{SYT}(\lambda)} t^{|X(rs(C_\gamma(T)))|} q^{|Y(rs(C_\gamma(T)))|}, \]  

(7)

where \( \text{SYT}(\lambda) \) denotes the collection of standard tableaux of shape \( \lambda \).

The modified Macdonald polynomials, \( \hat{H}_\mu(q,t) \), are a family of symmetric functions over the field of Laurent polynomials in two variables that specialize to many important classical symmetric functions; they are parametrized by partitions. An argument analogous to the one used to prove Theorem 2.1 in [3] can be used to deduce the following corollary from Theorem 1.

**Corollary 2.** The hollow Garsia–Haiman graded module \( \mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma \) has its Frobenius characteristic given by the polynomial

\[ \mathcal{F}\text{ch}(\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma) = \left[ \begin{array}{c} p_1 + k_1 \\ p_1 \\ \vdots \\ p_2 + k_2 \\ p_2 \end{array} \right] \hat{H}_{(m_2+p_2, 1^{m_1+p_1-1})}(q,t). \]  

(8)

To prove Theorem 1, we will define a sequence of ideals

\[ \mathcal{G}_\gamma(X_n, Y_n) \subset \mathcal{H}_\gamma(X_n, Y_n) \subset \mathcal{J}_\gamma(X_n, Y_n) \subset \mathcal{K}_\gamma(X_n, Y_n) \subset \mathcal{I}_\gamma(X_n, Y_n). \]  

(9)
For each ideal \( E = E(X_n, Y_n) \) in (9), we will define appropriate generators for \( E \), construct a base for the corresponding quotient space \( \mathbb{C}[X_n, Y_n]/E \), and compute the corresponding Hilbert series. To complete the proof, we will use a correspondence between our basis elements for \( \mathbb{C}[X_n, Y_n]/I_\gamma \) and the irreducible characters of \( S_n \). Sections 2–4 will introduce the necessary background and notation on tableaux, cocharge diagrams and symmetric polynomials, respectively. The following four sections consider the situations corresponding to \( G_\gamma, H_\gamma \), both \( J_\gamma \) and \( K_\gamma \), and both \( J_\gamma \) and \( I_\gamma \), respectively.

We note here that Garsia–Haiman modules corresponding to specific classes of lattice diagrams have been studied elsewhere. Periodic Garsia–Haiman modules were considered by the first author [4] and (in one variable) by H. Morita and H.-F. Yamada [5], R. Stanley [6] and J. Stembridge [7]. Dense Garsia–Haiman modules were investigated by the first author in [8]. F. Bergeron, A. Garsia and G. Tesler in [3] study several specific cases including arbitrary one-row diagrams and also hook diagrams plus an additional square. Those three authors, along with N. Bergeron and M. Haiman [9], studied the Garsia–Haiman modules corresponding to Ferrers diagrams with one square removed. J.-C. Aval [10] produces an explicit basis for hook shapes. He has studied certain sums of Garsia–Haiman modules in [11] and, with N. Bergeron [12], considered the action of certain partial differential operators on such spaces.

Finally, it should be noted that a conjecture has been announced by J. Haglund, M. Haiman, N. Loehr, J. Remmel, and A. Ulyanov [13] for a combinatorial formula for the character of the diagonal coinvariants of the symmetric group.

Note 1. There is a wide variety of indexing and notational conventions among papers in this field. Most obviously, when the primary lattice diagrams under consideration are partitions, the correspondence between \( \mathbb{N}^2 \) and first quadrant lattice points usually has the first index giving the \( y \)-coordinate. Also note that we here write \((m, k, p)\) for the tuple \([1^{m_1}, k_1 + 1, 1^{p_1 - 1}], [1^{m_2}, k_2 + 1, 1^{p_2 - 1}]\) of [8].

2. Tableaux

A partition \( \mu = (\mu_1, \mu_2, \ldots, \mu_j, \ldots) \) is a (possibly infinite) sequence of weakly decreasing integers with \( j \geq 1 \) nonzero terms. We will not distinguish between partitions with the same collection of nonzero terms. Write \( |\mu| = \mu_1 + \mu_2 + \cdots + \mu_j \) for the sum of the parts. If \( |\mu| = n \), then we say that \( \mu \) is a partition of \( n \) and write \( \mu \vdash n \). The length, \( j \), is denoted \( \ell(\mu) \). If \( i \) appears \( r_i \) times in \( \mu \) for each \( i \), then the tuple \((1^{r_1}, 2^{r_2}, \ldots)\) is called the type of \( \mu \). The transpose of \( \mu \) will be denoted by \( \mu^t \).

Let \( \mu, \lambda \vdash n \). We use \( \preceq_{\text{lex}} \) on partitions to denote the lexicographic order. A (French-style) Ferrers diagram of shape \( \mu \) is a collection of left-justified unit squares (“cells”) in the first quadrant with \( \mu_i \) squares in the \( i \)th row from the bottom. The shape of a Ferrers diagram \( D \), sh\( (D) \), is the partition obtained by listing the row lengths of \( D \). The notation \( \text{dg}(\mu) \) will be used to denote the canonical Ferrers diagram of shape \( \mu \).

A \( \Sigma \)-filling \( f \) is a map \( f : \text{dg}(\mu) \to \Sigma \) from the cells of a Ferrers diagram to some totally ordered alphabet \( \Sigma \). We will consider fillings with three different alphabets:

(1) \( \mathcal{A}' = \{(a, b) : a, b \in \mathbb{N}\} \) ordered by \((a_1, b_1) <_{\mathcal{A}'} (a_2, b_2) \) whenever

(a) \( a_1 - b_1 < a_2 - b_2 \); or
(b) \( a_1 - b_1 = a_2 - b_2 \) and \( a_1 < a_2 \).
Geometrically, the order $<_A'$ can be visualized as listing the points in the first quadrant by reading down lines $y = x + c$ from left to right with successively smaller values of $c$.

(2) $\mathcal{A} = \{(a, b) \in \mathcal{A}' : a = 0 \text{ or } b = 0\}$ with the order $<_A$ induced by $<_A'$. Note that the elements of $\mathcal{A}$ index cells that can appear in a hollow lattice diagram. For brevity in formulas, we sometimes write $a$ for $(a, 0)$ and $b$ for $(0, b)$. The notation is meant to evoke positive and negative numbers, respectively, as this interpretation of the elements of $\mathcal{A}$ is consistent with $<_A$. 

(3) $\mathbb{N}$ ordered by $0 < 1 < 2 < 3 < \cdots$.

The picture (or pair $(f, dg(\mu))$) obtained by placing elements of $\Sigma$ in the cells of a Ferrers diagram of shape $\mu$ according to $f$ is a $\Sigma$-filled diagram. When $\Sigma$ is clear (or unimportant), we simply refer to filled diagrams. For a filled diagram $U = (f, dg(\mu))$ (with $f : dg(\mu) \to \Sigma$) and a map $g : \Sigma \to \Sigma$, we use the shorthand $g(U)$ for the new filled diagram $(g \circ f, dg(\mu))$.

A filled diagram is injective if the map $f$ is an injective map. When $\Sigma = \mathbb{Z}_{\geq 1} \subset \mathbb{N}$, we refer to a $\Sigma$-filled diagram as a tableau. A filled diagram of shape $\mu$ is said to be column strict if the entries increase weakly from left to right in each row and increase strictly in each column from bottom to top. We will denote the collection of column-strict tableaux for a given alphabet $\Sigma$ by $\text{CS}_\Sigma$ (and use $\text{CS}_{n, \Sigma}$ if we want to specify the number of cells). An injective column-strict tableau with distinct entries $\{1, 2, \ldots, n\}$ for some $n$ is often referred to as a standard tableau. Let $\text{SYT}$, $\text{SYT}_\lambda$, and $\text{SYT}_n$ denote the collections of standard tableaux, standard tableaux of shape $\lambda$, and standard tableaux with $n$ cells, respectively. In the context of filled diagrams, $T$ will be reserved for a standard tableau; $V$ for a column-strict tableau. Finally, for any filled diagram $U$, we define $U'$ to be the tableau obtained by reflecting $U$ along the line $y = x$. Fig. 2 illustrates an $\mathcal{A}$-filled diagram $U$ of shape $\text{sh}(U) = (3, 2)$ along with its transpose $U'$. Note that in this case, $U' \in \text{CS}_\mathcal{A}$.

Let $I$ denote an injective tableau of shape $\mu = (\mu_1, \mu_2, \ldots, \mu_j)$, $R_i \ (1 \leq i \leq j)$ denote the collection of integers in the $i$th row of $I$ and $D_i \ (1 \leq i \leq \mu_1)$ denote the collection of integers in the $i$th column of $I$. Set

$$R(I) = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_j}$$

and

$$D(I) = S_{D_1} \times S_{D_2} \times \cdots \times S_{D_{\mu_1}},$$

where $S_{R_i}$ and $S_{D_i}$ denote the symmetric group on the collections of elements $R_i$ and $D_i$, respectively. Define, in the group algebra $\mathbb{C}[S_n]$,

$$P(I) = \sum_{\sigma \in R(I)} \sigma \quad \text{and} \quad N(I) = \sum_{\sigma \in D(I)} \text{sgn}(\sigma) \sigma.$$
In this paper, a bitableau is a pair \((S, U)\) of filled diagrams of the same shape in which \(S\) is \(N\)-filled and \(U\) is \(A'\)-filled. A standard bitableau satisfies the additional stipulations that \(S\) is a standard tableau and \(U \in CS_{A'}\). The set of all standard bitableaux \((S, U)\) on \(n\) cells will be denoted \(\Theta_n\) if we restrict to \(U \in CS_n, A\) and \(\Theta_n'\) if we do not place this restriction.

Let \(U = (f, dg(\mu))\) be any column-strict \(\Sigma\)-filled diagram. Roughly, the standardization \(std(U) = (\xi_f, dg(\mu))\) of \(U\) is a standard tableau that retains the relative order of the entries in the cells of \(U\). When an entry occurs more than once in \(U\), we require that the corresponding entries in \(std(U)\) increase from top to bottom and from left to right. (An example can be found among the first two diagrams of Fig. 3.) More precisely, we set \(\xi_f\):

\[
\xi_f(c) = \bigcup_{d \in dg(\mu)} f(c) \leq f(d) \quad \text{implies} \quad \xi_f(c) < \xi_f(d).
\]

Similarly, the corresponding bipermanent \([S, U]_{\text{per}}\) is given by

\[
[S, U]_{\text{per}} = P(S)x_1^{u_1^S}y_1^{u_1^S}x_2^{u_2^S}y_2^{u_2^S} \cdots x_n^{u_n^S}y_n^{u_n^S} = \sum_{\sigma \in R(S)} x_{\sigma(1)}^{u_{\sigma(1)}^S}y_{\sigma(1)}^{u_{\sigma(1)}^S}x_{\sigma(2)}^{u_{\sigma(2)}^S}y_{\sigma(2)}^{u_{\sigma(2)}^S} \cdots x_{\sigma(n)}^{u_{\sigma(n)}^S}y_{\sigma(n)}^{u_{\sigma(n)}^S}. \tag{14}
\]

The following theorem is a special case of [14, Theorem 8] (cf. [8,15]). We suggest the reader work out some examples from the case \(n = 3\) by hand.
Lemma 3. The collections
\[ \mathcal{BP} = \left\{ (T, V)_{\text{per}} : (T, V) \in \Theta_n' \right\} \quad \text{and} \]
\[ \mathcal{BD} = \left\{ (T, V)_{\text{det}} : (T, V) \in \Theta_n' \right\} \]
are infinite bases for \( \mathbb{C}[X_n, Y_n] \).

For a \( \Sigma \)-filled diagram \( U \), the row sequence \( \text{rs}(U) \) is the sequence obtained by listing the entries of \( U \) in each row from left to right, starting with the bottom row. The column sequence \( \text{cs}(U) \) is found by listing the entries of \( U \) from bottom to top in each column, starting with the leftmost column. Finally, the content \( \kappa(U) \) is a rearrangement of the row sequence \( \text{rs}(U) \) of \( U \) into nondecreasing order with respect to \( <_{\Sigma} \).

Example 4. For \( U \) as in Fig. 2, we have
\[ \text{rs}(U) = ((0, 2), (0, 0), (3, 0), (0, 2), (4, 0)) = (\bar{2}, 0, 3, \bar{2}, 4), \]
\[ \text{cs}(U) = ((0, 2), (0, 2), (0, 0), (4, 0), (3, 0)) = (\bar{2}, \bar{2}, 0, 4, 3), \quad \text{and} \]
\[ \kappa(U) = ((0, 2), (0, 2), (0, 0), (3, 0), (4, 0)) = (\bar{2}, \bar{2}, 0, 3, 4). \]

There are two orderings of bitableaux that are particularly important when considering elements of \( \mathcal{BD} \) or \( \mathcal{BP} \). Let \( >_{\text{lex},(A')} \) denote the lexicographic order with respect to \( >_{A'} \). For diagrams \( S_1, U_1, S_2 \) and \( U_2 \) with \( \text{sh}(S_1) = \text{sh}(U_1) \) and \( \text{sh}(S_2) = \text{sh}(U_2) \), we will say that
\[ (S_1, U_1) <_{\text{det}} (S_2, U_2) \quad \text{(17)} \]
whenever

1. \( \text{sh}(S_1) <_{\text{lex}} \text{sh}(S_2) \);
2. if \( \text{sh}(S_1) = \text{sh}(S_2) \) then \( \kappa(U_1) >_{\text{lex},(A')} \kappa(U_2) \);
3. if \( \text{sh}(S_1) = \text{sh}(S_2) \) and \( \kappa(U_1) = \kappa(U_2) \) then
\[ \text{cs}(S_1) \text{cs}(U_1) >_{\text{lex},(A')} \text{cs}(S_2) \text{cs}(U_2), \quad \text{(18)} \]
where \( \text{cs}(S_i) \text{cs}(U_i) \) is the concatenation of \( \text{cs}(S_i) \) and \( \text{cs}(U_i) \) for \( i = 1, 2 \).

Example 5. Let
\[ (S_1, U_1) = \begin{pmatrix} 5 & 7 & 0 & 1 \\ 3 & 4 & 8 & \bar{1} \bar{0} \bar{1} \\ 1 & 2 & 6 & 2 1 1 \end{pmatrix} \quad \text{and} \quad (S_2, U_2) = \begin{pmatrix} 5 & 7 & 1 & 1 \\ 3 & 4 & 8 & \bar{1} \bar{0} \bar{1} \\ 1 & 2 & 6 & \bar{2} \bar{2} 1 \end{pmatrix}. \]

Certainly \( \text{sh}(S_1') = \text{sh}(S_2') \). However
\[ \kappa(U_1) = (\bar{2}, \bar{1}, \bar{1}, \bar{1}, 0, 0, 1, 1) >_{\text{lex},(A')} \kappa(U_2) = (\bar{2}, \bar{2}, \bar{1}, \bar{1}, 0, 1, 1, 1). \quad \text{(19)} \]
So \( (S_1, U_1) <_{\text{det}} (S_2, U_2) \).

Similarly, we will say that
\[ (S_1, U_1) <_{\text{per}} (S_2, U_2) \quad \text{(20)} \]
whenever
(1) \( \text{sh}(S_1) \leq_{\text{lex}(A)} \text{sh}(S_2) \);
(2) if \( \text{sh}(S_1) = \text{sh}(S_2) \) then \( \kappa(U_1) <_{\text{lex}(A')} \kappa(U_2) \);
(3) if \( \text{sh}(S_1) = \text{sh}(S_2) \) and \( \kappa(U_1) = \kappa(U_2) \) then
\[
\text{rs}(S_1) \text{rs}(U_1) >_{\text{lex}(A')} \text{rs}(S_2) \text{rs}(U_2) .
\]

\[ \text{rs}(S_i) \text{rs}(U_i) \text{ is the concatenation of } \text{rs}(S_i) \text{ and } \text{rs}(U_i) \text{ for } i = 1, 2. \]

**Theorem 6.** (See [8,14,15].) Let \([S, U]_{\text{det}}\) be a bitableau on \(n\) cells with \(U \ A'-\text{filled such that} \) either \(S\) is not standard or \(U\) is not column-strict. Then we can write
\[
[S, U]_{\text{det}} = \sum_i d_i [T_i, V_i]_{\text{det}}
\]
where, for each \(i\), it is true that \(d_i \in \mathbb{Z}, (T_i, V_i) \in \Theta'_n, (T_i, V_i) >_{\text{det}} (S, U), \kappa(T_i) = \kappa(S) \) and \(\kappa(V_i) = \kappa(U)\). The above statements hold, mutatis mutandi, for bipermanents and the order >_{\text{per}}.

**Example 7.**
\[
\begin{bmatrix}
3 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}
\]
\[
= (\epsilon - (2, 3)) x_1y_2x_3^2 = x_1y_2x_3^2 - x_1y_3x_2^2
\]
\[
= \begin{bmatrix}
3 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}
- \begin{bmatrix}
2 & 1 & 3 \\
2 & 1 & 1
\end{bmatrix}
- \begin{bmatrix}
3 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix}.
\]

### 3. Cocharge diagrams

The standardization map of the previous section associates to any column-strict, filled diagram a standard tableau. It does this in such a way as to retain as closely as possible the relative order of the entries. We now describe a more elaborate version of this association. Whereas the standardization map applies to a \(\Sigma\)-filled diagram for any \(\Sigma\), here we will only be interested in \(A\)-filled diagrams as inputs. And whereas the standardization map produces a \(N\)-filled diagram, our new map will return another \(A\)-filled diagram.

Our construction takes as input not only a standard tableau \(T = (f, \text{dg}(\mu))\), but also a hollow lattice diagram \(L[\alpha_y]\) (specifically, it requires the datum \(m_2 + p_2\) from \(\gamma\)). From \(T\) and \(\gamma\), we produce a **cocharge diagram** \(C_y(T) = (h_y \circ \pi_T \circ f, \text{dg}(\mu))\) in two steps.

The first step is to apply the usual **cocharge map** \(\pi_T : \{1, 2, \ldots, n\} \to \mathbb{N}\) to \(T\). This map is defined recursively as follows:
\[
\pi_T(i) = \begin{cases} 
0, & \text{if } i = 1, \\
\pi_T(i - 1), & \text{if } i > 1 \text{ occurs weakly southeast of } i - 1 \text{ in } T, \\
\pi_T(i - 1) + 1, & \text{if } i > 1 \text{ occurs weakly northwest of } i - 1 \text{ in } T.
\end{cases}
\]

The map \(\pi_T\) is well defined on standard diagrams. The essence of \(\pi_T\) is that it replaces a sequence of northwest-to-southeast string of entries \(i, i + 1, \ldots, j\) of \(T\) with the same value in \(\pi_T(T)\).
Lemma 8. Fix \( n \) and \( \gamma \). Define \( CO_{n,\gamma} = \{ C_{\gamma}(T) : T \in SYT_n \} \). There is a bijection between elements \( U \in CS_{n,\mathcal{A}} \) with \( u_{\text{std}(U)}^{m_2+p_2} = 0 \) and pairs \( (C, \alpha) \) with \( C \in CO_{n,\gamma} \) and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{A}^n \) such that \( \alpha_i \leq \alpha_{i+1} \) for \( 1 \leq i \leq n-1 \) and \( \alpha_{m_2+p_2} = (0,0) \).

For an \( \mathcal{A} \)-filled diagram as in Fig. 3, the sequence \( \alpha \) of Lemma 8 is

\[
\alpha = (0,5), (0,4), (0,1), (0,0), (0,0), (2,0), (2,0), (2,0), (4,0)
\]

Proof. Let \( T = \text{std}(U) \) and \( C = C_{\gamma}(T) \). Set \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) with \( \alpha_i = u_i^T - c_i^T \). Map \( U \) to \( (C, \alpha) \). By the definition of \( h_{\gamma}, c_{m_2+p_2}^T = 0 \). Combined with our requirement of \( u_{m_2+p_2}^{\text{std}(U)} = 0 \), it follows that \( \alpha_{m_2+p_2} = 0 \). By the definitions of \( h_{\gamma}, \pi_T \) and standardization, the sequences \( c_1^T, c_2^T, \ldots \) and \( u_1^T, u_2^T, \ldots \) are both weakly increasing. That the \( \alpha_i \) are weakly increasing then follows from the additional fact that \( U \) is column strict. \( \square \)

4. Some operations by symmetric polynomials

We now review some important definitions and results regarding symmetric polynomials. A standard reference for this material is [16]. We use the convention that tuples of elements of \( \mathcal{A}' \) or \( \mathcal{A} \) are written in boldface. In these definitions, let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots, \lambda_n) \) be a partition with \( j \leq n \) indexing the last nonzero part.

1. Define the monomial symmetric function

\[
m_{\lambda}(X_n) = \sum_{v=(v_1, v_2, \ldots, v_n)} x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n},
\]

where the sum is over all distinct permutations \( v \) of \( \lambda \).

2. For a sequence \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in (\mathcal{A}')^n \), define the MacMahon monomial symmetric function

\[
m_{\beta}(X_n, Y_n) = \sum_{\delta=(\delta_1, \delta_2, \ldots, \delta_n)} x_1^{\delta_{1,1}} y_1^{\delta_{1,2}} x_2^{\delta_{2,1}} y_2^{\delta_{2,2}} \cdots x_n^{\delta_{n,1}} y_n^{\delta_{n,2}},
\]

where the sum is over all distinct permutations \( \delta \) of \( \beta \).

3. For a positive integer \( r \), define the elementary symmetric function

\[
e_r(X_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}.
\]

Set \( e_0 = 1 \) and \( e_{\lambda} = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_j} \).
(4) For a positive integer \( r \), define the complete (or homogeneous) symmetric function

\[
h_r(X_n) = \sum_{\lambda \vdash r} m_{\lambda}(X_n).
\]

(27)

Set \( h_0 = 1 \) and \( h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_j} \) (we extend this definition in the obvious way to the case where \( \lambda \) is a \( j \)-tuple of nonnegative integers; i.e., not necessarily nonincreasing).

**Lemma 9.** (See [4, Theorem 5.4 and Corollary 5.5].) Suppose \( g \) is a map from \( CS_n,A' \) to \( CS_n,A' \) such that for all \( W \in CS_n,A' \), \( W \) and \( g(W) \) have the same standardization. Fix \( T \in SYT_n, V \in CS_n,A' \) and write \( U = g(V) \). Set \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) where \( \beta_i = v^{\text{std}}_i - u^{\text{std}}_i \). Suppose that \( \beta \in \mathcal{A}^0 \) and \( \beta_i \leq \mathcal{A} \beta_i + 1 \) for \( 1 \leq i \leq n - 1 \). Then

\[
m_\beta(X_n, Y_n)[T, U]_{\text{per}} = c_{T,V}[T, V]_{\text{per}} + \sum_{(S,W) \succ_{\text{per}} (T,V)} c_{S,W}[S, W]_{\text{per}}
\]

where \( c_{T,V} \neq 0 \) and \( \tilde{W} \) is \( A' \)-filled with at least one entry not in \( A \).

**Example 10.** For \( \beta = (2, 0, 3) \), \( U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \), \( T = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \) and \( V = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \), we have

\[
m_\beta(X_3, Y_3)[T, U]_{\text{per}} = 2[T, V]_{\text{per}} + 2 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}_{\text{per}} + 2 \begin{bmatrix} a & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}_{\text{per}},
\]

(29)

where \( a = (1, 2) \). We see that the second bipermanent is, in fact, larger than \( (T, V) \) in the order \( >_{\text{per}} \) (the content \( (2, 1, 3) \) is greater than that of \( V \) in the lexicographic order with respect to \( A \)).

We also have the following lemma (cf. [4, Theorem 5.2]).

**Lemma 11.** For \( \beta \in (A')^n \) and a lattice diagram \( L[\alpha] \) of \( n \) cells,

\[
m_\beta(\partial_X, \partial_Y) \triangle L[\alpha] = \sum_\delta c_\delta \triangle L[\alpha - \delta]
\]

(30)

for some constants \( c_\delta \in \mathbb{N} \). Here, the sum is over all distinct permutations \( \delta \) of \( \beta \). We use the convention that \( c_\delta = 0 \) if \( \alpha_i - \delta_i \notin A \) for some \( 1 \leq i \leq n \).

**Proof.** First note that \( m_\beta(\partial_X, \partial_Y) \) can be written as

\[
m_\beta(\partial_X, \partial_Y) = \sum_\delta \prod_{i=1}^n \partial_{\delta_i}^{\delta_i} \partial_{\delta_i}^{\delta_i} = K \sum_{v \in S_n} \prod_{i=1}^n \partial_{\delta_i(v)}^{\delta_i(v)} \partial_{\delta_i(v)}^{\delta_i(v)}
\]

for some constant \( K \in \mathbb{Q} \) dependent on the extent to which factors in \( \beta \) are repeated. Then, \( m_\beta(\partial_X, \partial_Y) \triangle L[\alpha] \) equals
\[
\left( \sum_{\nu \in S_n} K \prod_{i=1}^n \frac{\partial^{\nu_i}}{\partial x_i^{\nu_i}} \frac{\partial^{\nu_i}}{\partial y_i^{\nu_i}} \right) \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n x_i^{\alpha_{\sigma(i)} - 1} y_i^{\alpha_{\sigma(i)} - 2} = \sum_{\phi \in S_n} c_{\phi} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n x_i^{\alpha_{\sigma(i)} - 1} y_i^{\alpha_{\sigma(i)} - 2} = \sum_{\delta} c_{\delta} \Delta L[\alpha - \delta] \tag{31}
\]

for some constants \(c_{\delta}\). In the above, we consider the coefficient \(c_{\phi}\) to be zero if any of the exponents \(\alpha_{\sigma(i)} - 1 - \beta_{\phi(i)}\) are not in \(\mathcal{A}'\); \(c_{\delta}\) to be zero if \(\alpha - \delta \notin (\mathcal{A}')^n\). In addition, we let \(\delta\) run over all distinct permutations of \(\beta\). □

5. The ideal \(\mathcal{G}_\gamma(X_n, Y_n)\) and the ring \(\mathbb{C}[X_n, Y_n]/\mathcal{G}_\gamma\)

Let \(\mathcal{G}_\gamma(X_n, Y_n)\) be the ideal generated by the monomials in the collections

\[
\{ \prod_{i \in D} x_i \} \quad \text{and} \quad \{ \prod_{h \in E} y_h \} \quad \text{with} \quad |D| = m_1 + p_1 \quad \text{and} \quad |E| = m_2 + p_2. \tag{33}
\]

Note that each of the monomials in Eqs. (32) and (33) is in the ideal \(I_\gamma(X_n, Y_n)\). Thus \(\mathcal{G}_\gamma(X_n, Y_n) \subset I_\gamma(X_n, Y_n)\).

For a sequence \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{A}^n\), we define

\[
|X(\alpha)| = \alpha_{1,1} + \alpha_{2,1} + \cdots + \alpha_{n,1} \quad \text{and} \quad |Y(\alpha)| = \alpha_{1,2} + \alpha_{2,2} + \cdots + \alpha_{n,2}. \tag{34}
\]

For an arbitrary bipermanent \(b = [T, V]_{\text{per}}\), we write \(|X(b)|\) as shorthand for \(|X(rs(V))|\); similarly for \(|Y(b)|\). In addition we will write

\[
(q)_j = (1 - q)(1 - q^2) \cdots (1 - q^j) \quad \text{and} \quad (t)_j = (1 - t)(1 - t^2) \cdots (1 - t^j). \tag{35}
\]

The generating function for the sum

\[
\sum_{\alpha} t^{|X(\alpha)|} q^{|Y(\alpha)|}, \tag{36}
\]

subject to the constraints that \(\alpha \in \mathcal{A}^n\), \(\alpha_i \leq \mathcal{A} \alpha_{i+1}\) for \(1 \leq i \leq n - 1\) and \(\alpha_{m_2+p_2} = (0, 0)\), is given by

\[
\sum_{\alpha} t^{|X(\alpha)|} q^{|Y(\alpha)|} = \frac{1}{(t)_{m_1+p_1-1} (q)_{m_2+p_2-1}}. \tag{37}
\]

Define

\[
\mathcal{B}_\gamma = \{ [T, V]_{\text{per}} : T \in SYT_n, \ V \in CO_{n,\gamma} \}. \tag{38}
\]

**Theorem 12.** The Hilbert series of \(\mathbb{C}[X_n, Y_n]/\mathcal{G}_\gamma\) is given by

\[
\mathcal{H}(\mathbb{C}[X_n, Y_n]/\mathcal{G}_\gamma) = \frac{1}{(t)_{m_1+p_1-1} (q)_{m_2+p_2-1}} \sum_{b \in \mathcal{B}_\gamma} t^{|X(b)|} q^{|Y(b)|}. \tag{39}
\]
Proof. Suppose \((T, V) \in \Theta'_n\) and \(v_{m_2+p_2}^{\text{std}}(V) \neq (0, 0)\). Since \(V \in CS_{A'}\), either each of the monomials in \([T, V]_{\text{per}}\) has at least \(m_1 + p_1\) distinct \(x_i\)'s as factors, or each of the monomials has at least \(m_2 + p_2\) distinct \(y_i\)'s as factors. In either case, \([T, V]_{\text{per}} \in G'_\gamma(X_n, Y_n)\) as can be seen by examining the sets in (33). It follows that in looking for a basis of \(\mathbb{C}[X_n, Y_n]/G'_\gamma\), we can restrict our attention to those bipermants for which \(v_{m_2+p_2}^{\text{std}} = (0, 0)\).

Additionally, if \(v_i \notin A\) for some \(1 \leq i \leq n\), then each monomial of \([T, V]_{\text{per}}\) is divisible by some \(x_jy_j\). (Note that \(j\) need not equal \(i\) as \(\text{std}(T)\) need not equal \(\text{std}(V)\).) Examination of (32) then shows that \([T, V]_{\text{per}} \in G'_\gamma(X_n, Y_n)\) in this case as well.

On the other hand, it is easily seen that if \(v_{m_2+p_2}^{\text{std}} = (0, 0)\) and \(V \in CS_A\), then \([T, V]_{\text{per}} \notin G'_\gamma(X_n, Y_n)\). It follows from Lemma 3 that the set

\[
\left\{ [T, V]_{\text{per}} : (T, V) \in \Theta_n, \ v_{m_2+p_2}^{\text{std}} = (0, 0) \right\}
\]

is a basis for \(\mathbb{C}[X_n, Y_n]/G'_\gamma\).

The theorem then follows by combining the decomposition of Lemma 8 with the basis of (40) and the generating function of (37). \(\square\)

Although the above proof constructs a basis of \(\mathbb{C}[X_n, Y_n]/G'_\gamma\), it will be more useful in the next section to have the basis offered by Theorem 13.

**Theorem 13.** The collection

\[
\mathcal{EEB} = \left\{ \left( \prod_{i=1}^{m_1+p_1-1} e_i^{e_i}(X_n) \cdot \prod_{i=1}^{m_2+p_2-1} e_i^{b_i}(Y_n) \right) b : b \in B'_\gamma \right\}
\]

where the \(e_i\) and \(b_i\) are allowed to run over all nonnegative integers, is a basis for \(\mathbb{C}[X_n, Y_n]/G'_\gamma\).

**Proof.** For this proof, linearly extend the notation \(|X(p)|\) to apply to elements \(p \in \mathcal{EEB}\).

By the proof of Theorem 12, it suffices to consider a bideterminant \([T, V]_{\text{per}}\) where \((T, V) \in \Theta_n\) and \(v_{m_2+p_2}^{\text{std}}(V) = (0, 0)\). Set \(C = C'_\gamma(\text{std}(V))\). Note that \(c_{m_2+p_2}^{\text{std}}(V) = (0, 0)\) by construction.

If \(c_i^{\text{std}}(V) = v_i^{\text{std}}(V)\) for all \(1 \leq i \leq n\), then \(V \in CS_{n,\gamma}\); hence, by definition, \([T, V]_{\text{per}} \in B'_\gamma\). Assume not. We consider two possibilities.

Suppose there exists an index \(i\) with \(c_i^{\text{std}}(V) \neq v_i^{\text{std}}(V)\). Suppose the smallest such index \(i\) is greater than \(m_2 + p_2\). Let \(U\) denote the tableau for which \(u_j^{\text{std}}(V) = v_j^{\text{std}}(V)\) for \(1 \leq j < i\) and \(u_i^{\text{std}}(V) = v_i^{\text{std}}(V) - (1, 0)\) for \(i \leq j \leq n\). It follows from the proof of Lemma 8 that \(U \in CS_{n,\gamma}\).

Utilizing the identity \(e_{n-i+1}^{\text{std}}(X_n)[T, U]_{\text{per}} \equiv c_T, U^{\text{std}}(T, V)_{\text{per}} + \sum_{(S, W) > \per(T, V) \atop S \in SYT_n \atop W \in CS_{n,\gamma}} c_S, W^{\text{std}}(S, W)_{\text{per}} \mod (G'_\gamma)\),

(Here we have used the fact that the monomials in \(e_{n-i+1}^{\text{std}}(X_n)[T, U]_{\text{per}}\) arising in the third term in (28) all have a factor \(x_jy_j\) for some \(j\). But, as we see from (32), these monomials are in \(G'_\gamma\) by construction.) The only remaining possibility is that there exists such an index \(i\) less than \(m_2 + p_2\). Choose \(i\) to be the largest such index. Arguing as above, we obtain an equivalent expansion for \(e_i^{\text{std}}(Y_n)[T, U]_{\text{per}}\).
Eq. (42) expands the bipermanent \([T, V]_{\text{per}}\) in terms of the bipermanents \([T, U]_{\text{per}}\) and \([S, W]_{\text{per}}\). We wish to iterate this expansion in order to show that the collection \(\mathcal{EEB}\) spans \(\mathbb{C}[X_n, Y_n]/G_\gamma\). That this process ends depends on several observations. First, the total degree of the polynomial \([T, U]_{\text{per}}\) is strictly less than that of \([T, V]_{\text{per}}\). Second, the total degree of \([S, W]_{\text{per}}\) equals that of \([T, V]_{\text{per}}\), but \((S, W) >_{\text{per}} (T, V)\). Third, at most \(m_1 + p_1 - 1\) of the \(x_i\), and \(m_2 + p_2 - 1\) of the \(y_j\), can appear with positive degree in any of the monomials not in \(G_\gamma\). We conclude that

\[
\sum_{\mathbf{p} \in \mathcal{EEB}} t^{\sum_{i=1}^{m_1+1} p_1^{-1} i\epsilon_i} q^{\sum_{j=1}^{m_2+p_2-1} j\beta_j} = \sum_{\mathbf{b} \in \mathcal{B}_\gamma} t^{\sum_{i=1}^{m_1+1} p_1^{-1} i\epsilon_i} q^{\sum_{j=1}^{m_2+p_2-1} j\beta_j}.
\]

The fact that the collection \(\mathcal{EEB}\) spans \(\mathbb{C}[X_n, Y_n]/G_\gamma\) and yields the desired Hilbert series (see Theorem 12) implies that \(\mathcal{EEB}\) must be a basis for the quotient ring \(\mathbb{C}[X_n, Y_n]/G_\gamma\).

We have the following corollary.

**Corollary 14.** The collection

\[
\{ e_1(X_n), e_2(X_n), \ldots, e_{m_1+p_1-1}(X_n), e_1(Y_n), e_2(Y_n), \ldots, e_{m_2+p_2-1}(Y_n) \}
\]

is algebraically independent in the ring \(\mathbb{C}[X_n, Y_n]/G_\gamma\).

**Proof.** Any nontrivial algebraic dependence amongst the elements of (44) would yield a linear dependence amongst the elements of \(\mathcal{EEB}\), conflicting with its role as a basis for \(\mathbb{C}[X_n, Y_n]/G_\gamma\).

**6. The ideal \(\mathcal{H}_\gamma(X_n, Y_n)\) and the ring \(\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma\)**

Let \(\mathcal{H}_\gamma(X_n, Y_n)\) be the ideal in \(\mathbb{C}[X_n, Y_n]\) generated by the collections of monomials in Eqs. (32) and (33) as well as by the elementary symmetric polynomials in the collection

\[
\{ e_{p_1+1}(X_n), e_{p_1+2}(X_n), \ldots, e_{p_1+m_1-1}(X_n),
\]

\[
e_{p_2+1}(Y_n), e_{p_2+2}(Y_n), \ldots, e_{p_2+m_2-1}(Y_n) \}.
\]

Consider the action of an elementary symmetric differential operator \(e_j(\partial_{Y_n})\), such as is illustrated in Fig. 4 for \(j = 2\). This operator moves each of \(j\) distinct cells down by one place. Any configuration in which two cells end up in the same position or in which a cell moves to a position not indexed by an element of \(A\) contributes zero. (The action of an \(e_\ell(\partial_{X_n})\) is similar.) It follows that for any contributing monomial, the cells contiguous with \((0, 0)\) are not moved. But any \(e_{p_1+j}(\partial_X)\) for \(j > 0\) or \(e_{p_2+\ell}(\partial_Y)\) for \(\ell > 0\) must move one of these fixed cells. So \(\mathcal{H}_\gamma(X_n, Y_n) \subset \mathcal{I}_\gamma(X_n, Y_n)\). Note that by construction, \(\mathcal{H}_\gamma(X_n, Y_n) \supset G_\gamma(X_n, Y_n)\). Theorem 13 and Corollary 14 imply the following two corollaries.
Corollary 15. The set

$$\left\{ \left( \prod_{i=1}^{p_1} e_i^{e_i}(X_n) \cdot \prod_{i=1}^{p_2} e_i^{b_i}(Y_n) \right) b: b \in B_\gamma \right\},$$

(46)

is a basis for $\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma$. (Recall that $B_\gamma$ is defined in (38).)

Corollary 16. The Hilbert series of $\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma$ is given by

$$\mathcal{H}(\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma) = \frac{1}{(t)^{p_1} (q)^{p_2}} \sum_{b \in B_\gamma} t^{\lvert X(b) \rvert} q^{\lvert Y(b) \rvert}.$$

(47)

7. The ideals $J_\gamma(X_n, Y_n)$ and $K_\gamma(X_n, Y_n)$

In the previous section, we considered symmetric functions whose corresponding differential operators moved collections of cells, but for which each cell was only moved one place. Monomial symmetric functions yield operators that move cells farther. In this section we consider which ones will also annihilate $\Delta_{L[\alpha]}$. In fact, due to fortuitous cancellations, we can focus on the differential operators corresponding to the complete symmetric functions.

Fig. 5 illustrates the action of $h_2(\partial_X) = m_2(\partial_X) + m_{11}(\partial_X)$ on a given $\Delta_{L[\alpha]}$. Notice that the hollow lattice diagram corresponding to the triple $\tilde{\gamma} = ((1, 2), (2, 0), (2, 0))$ is obtained in two different ways: once through the action of $m_{11}(\partial_X)$ and once through $m_2(\partial_X)$. However, for $m_2(\partial_X)$ the cell that moves jumps over another cell. This leads to the introduction of a sign. Hence, the two $\Delta_{L[\gamma]}$ that appear cancel. In fact, as Lemma 17 shows, under the action of an $h_j$ on some $\Delta_{L[\alpha]}$, the only term that survives is that which moves the cell $(m_1 + k_1, 0)$ (or $(0, m_2 + k_2)$, as appropriate) $j$ spaces.
In particular, we wish to consider hollow diagrams modified by sliding certain cells closer to the origin.

Suppose we have a hollow lattice diagram \( \gamma \). The first-row cells not contiguous with \((0,0)\) (if any) occur in the positions \((m_1 + k_1,0),(m_1 + k_1 + 1,0),\ldots,(m_1 + k_1 + p_1 - 1)\). We will be interested in related diagrams in which these cells have been replaced by cells in positions \((m_1 + k_1 + 0 - a_0,0),(m_1 + k_1 + 1 - a_1,0),\ldots,(m_1 + k_1 + p_1 - 1 - a_{p_1 - 1},0)\), for some \(a_0 \geq a_1 \geq \cdots \geq a_{p_1 - 1} \geq 0\). (Note that some of these cells may overlap the cells \((0,0),(1,0),\ldots,(m_1 - 1,0)\) or even have negative coordinates. In such a case, the algebraic entity associated to the resulting lattice diagram will be trivial.) The new positions of the cells in the first column can be similarly recorded by a sequence \(b_0,b_1,\ldots,b_{p_2-1}\).

We can write this more formally as follows. For a hollow lattice diagram \(L[\alpha_\gamma]\) with

\[
\alpha_\gamma = (m_2 + k_2 + p_2 - 1, m_2 + k_2 + p_2 - 2, \ldots, m_2 + k_2, m_1 + k_1 + 1, m_1 + k_1 + p_1 - 1),
\]

we use \(\gamma[a_0,a_1,\ldots,a_{p_1-1};b_0,b_1,\ldots,b_{p_2-1}]\) to label the lattice diagram corresponding to the collection

\[
(m_2 + k_2 + p_2 - 1 - b_{p_2-1}, m_2 + k_2 + p_2 - 2 - b_{p_2-2}, \ldots, m_2 + k_2 + 0 - b_0, m_1 + k_1 + 0 - a_0, m_1 + k_1 + 1 - a_1, \ldots, m_1 + k_1 + p_1 - 1 - a_{p_1 - 1}).
\]

If \(a_\ell = 0\) for \(\ell > i\) and \(b_\ell = 0\) for \(\ell > j\), we sometimes abbreviate

\[
\gamma[a_0,a_1,\ldots,a_{p_1-1};b_0,b_1,\ldots,b_{p_2-1}] \quad \text{by} \quad \gamma[a_0,a_1,\ldots,a_i;b_0,b_1,\ldots,b_j].
\]

(While this notation is only lightly used in Lemma 17, it greatly simplifies the expression of Theorem 25.)

An example is illustrated in Fig. 6. In the figure, the left diagram is the hollow lattice diagram for \(\gamma = ((2,1),(5,2),(3,3))\), while the right diagram illustrates \(\gamma[4,4,3;2,1,0]\).

**Lemma 17.** (Cf. [11, Proposition 2.6].) Let \(0 \leq j \leq k_1\) and \(0 \leq \ell \leq k_2\). Then

\[
h_j(\partial_X)\Delta_\gamma (X,Y) = c_j \Delta_{\gamma[j;0]} \quad \text{and} \quad h_\ell(\partial_Y)\Delta_\gamma (X,Y) = c_\ell \Delta_{\gamma[0;\ell]} \quad (48)
\]

for some constants \(c_j, c_\ell > 0\).
We now define $\Delta L$. The left-hand picture in Fig. 7 illustrates $\Delta L \nu$. We only prove the $\Delta L \nu$. Let us consider $\Delta_L(X, Y)$ for $L[\alpha]$ where
\[
\alpha = (0, 1, \ldots, m_1 - 1, m_1 + k_1, m_1 + k_1 + 1, \ldots, m_1 + k_1 + p_1 - 1, \ldots, m_2 - 1, m_2 + k_2, m_2 + k_2 + 1, \ldots, m_2 + k_2 + p_2 - 1).
\]
(Note the order in which we have listed the cells.) Let $\lambda (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be some permutation. Consider a particular permutation $\nu \lambda$. Assume that $\nu$ is a transposition. Now $\Delta L \lambda \nu \lambda$. The right-hand picture illustrates $\Delta L [\alpha - \nu]$. The left-hand picture in Fig. 7 illustrates $\Delta L [\alpha - \nu] \nu$. We now proceed to define an involution on our set of distinct permutations of $\nu$. As previously, we use the convention $c_\nu = 0$ if $\alpha \nu \lambda$. The indices $\ell \ < r$ to index the smallest two terms (with respect to < $\lambda$) of
\[
\alpha_m + 1 - \nu_{m_1 + 1}, \alpha_m + 2 - \nu_{m_1 + 2}, \ldots, \alpha_{m_1 + p_1} - \nu_{m_1 + p_1}.
\]
The indices $\ell \ < r$ index the two cells of $\Delta L [\alpha]$ not contiguous with the cell $(0, 0)$ that have moved farthest to the left upon subtraction of $\nu$. Set
\[
q = (\alpha_r - \nu_r) - (\alpha_\ell - \nu_\ell).
\]
We now define
\[
\nu' = g(\nu) = [\nu_1, \nu_2, \ldots, \nu_\ell-1, \nu_\ell - q, \nu_\ell+1, \nu_\ell+2, \ldots, \nu_r-1, \nu_r + q, \\
\nu_{r+1}, \nu_{r+2}, \ldots, \nu_n].
\]
The left-hand picture in Fig. 7 illustrates $\Delta_L (\alpha - \nu)$. The right-hand picture illustrates $\Delta_L [\alpha - \nu]$ for $\alpha = (0, 4, 5, 6, 1)$. The left-hand picture in Fig. 7 illustrates $\Delta_L (\alpha - \nu)$. The right-hand picture illustrates $\Delta_L [\alpha - \nu]$ for $\nu = (0, 3, 2, \hat{\nu}, 0)$ via the bottom triple of arrows ($\ell = 2, r = 3, \hat{\nu} = \alpha_2 - \nu_2 < \alpha_3 - \nu_3 = \hat{\nu}_2$, and $q = 2$) and $\Delta_L [\alpha - \nu']$ for $\nu' = (0, 1, 4, 2, 0)$ via the top triple of arrows. When viewed as sets, $\alpha - \nu$ equals $\alpha - \nu'$; they differ only in order.

We have that
\[
\Delta_L [\alpha - \nu] = -\Delta_L [\alpha - \nu'].
\]
Since $\alpha - \nu$ and $\alpha - \nu'$ differ by a transposition. Now $g(g(\nu)) = \nu$. As desired, this function $g$ yields a sign-reversing involution between all the terms $\Delta_L [\alpha - \nu]$ in Eq. (50) except for the unique
Lemma 20. It is not difficult to see that the coefficients $c_\varphi$ in the expansion of $m_\lambda (\partial X) \Delta_\gamma (X, Y)$ in terms of the $\Delta_{L(\alpha - \mu)}$ satisfy $c_\varphi = c_{\varphi'}$. Thus, all the terms in Eq. (50) cancel out except $\Delta_{L(\alpha - \mu)}$.

The coefficient $c_j$ equals $(m_1 + k_1)(m_1 + k_1 - 1) \cdots (m_1 + k_1 - j + 1)$. The positivity stems from the fact that the relative order of the cells does not change. This gives the lemma. \hfill \Box

Recall that $\Delta_\gamma (X, Y) = 0$ if two of the entries in $L(\alpha, \gamma)$ are identical. Thus, if $j > k_1$ or $\ell > k_2$, then $\Delta_{\gamma(j:0)} = 0$ or $\Delta_{\gamma(0:\ell)} = 0$, respectively. It follows that

Corollary 18. $h_{k_1+i}(X_n) \in I_\gamma(X_n, Y_n)$ for $i > 0$ and $h_{k_2+h}(Y_n) \in I_\gamma(X_n, Y_n)$ for $h > 0$.

We now utilize Corollary 18 to define a sub-ideal $K_\gamma(X_n, Y_n)$ of $I_\gamma(X_n, Y_n)$. Specifically, set $K_\gamma(X_n, Y_n)$ to be the ideal in $\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma$ generated by

$$\{h_{k_1+1}(X_n), h_{k_1+2}(X_n), \ldots \} \cup \{h_{k_2+1}(Y_n), h_{k_2+2}(Y_n), \ldots \}.$$  \hfill (55)

As it turns out, $K_\gamma(X_n, Y_n)$ is a finitely generated ideal in $\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma$. To this end, define $J_\gamma(X_n, Y_n)$ to be the sub-ideal of $K_\gamma(X_n, Y_n)$ in $\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma$ generated by

$$\{h_{k_1+1}(X_n), h_{k_1+2}(X_n), \ldots , h_{k_1+p_1}(X_n), h_{k_2+1}(Y_n), h_{k_2+2}(Y_n), \ldots , h_{k_2+p_2}(Y_n)\}.$$ \hfill (56)

It follows from Corollary 18 that $J_\gamma \subset K_\gamma \subset I_\gamma$.

Lemma 19. $K_\gamma(X_n, Y_n) \equiv J_\gamma(X_n, Y_n) \pmod{\mathcal{H}_\gamma(X_n, Y_n)}$.

Proof. A standard result (cf. [16, p. 21]) is that

$$\sum_{r=0}^{n} (-1)^r e_r(X_n) h_{n-r}(X_n) = 0.$$ \hfill (57)

We prove that $h_{k_1+p_1+a}(X_n) \equiv 0 \pmod{J_\gamma(X_n, Y_n)}$ for $a \geq 1$ by induction on $a$. For $a \geq 1$, we have

$$h_{k_1+p_1+a}(X_n) = \sum_{r=1}^{p_1} (-1)^{r+1} e_r(X_n) h_{k_1+p_1+a-r}(X_n)
+ \sum_{r=p_1+1}^{k_1+p_1+a} (-1)^{r+1} e_r(X_n) h_{k_1+p_1+a-r}(X_n).$$ \hfill (58)

Recall from (33) and (45) that $e_{p_1+j}(X_n) \in \mathcal{H}_\gamma(X_n, Y_n)$ for $j \geq 1$. So the second sum of (58) is in $\mathcal{H}_\gamma(X_n, Y_n) \subset J_\gamma(X_n, Y_n)$. On the other hand, by the definition of $J_\gamma(X_n, Y_n)$, $h_{k_1+i} \in J_\gamma(X_n, Y_n)$ for $1 \leq i \leq p_1$. Thus, for $a = 1$, the first sum of (58) is in $J_\gamma(X_n, Y_n)$ as well. This proves the claim for $a = 1$. The claim for $a > 1$ thereby follows by the obvious induction hypothesis. Similar arguments can be made about $h_{k_2+p_2+a}(Y_n)$ for $a \geq 1$. \hfill \Box

Our next goal is to show that the collection that generates $J_\gamma(X_n, Y_n)$ in Eq. (56) is itself algebraically independent in $\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma$. To prove this, we begin with the following lemma.

Lemma 20. The collections

$$\{h_{k_1+1}(X_n), h_{k_1+2}(X_n), \ldots , h_{k_1+p_1}(X_n)\} \quad \text{and} \quad (59)$$

$$\{h_{k_2+1}(Y_n), h_{k_2+2}(Y_n), \ldots , h_{k_2+p_2}(Y_n)\}$$ \hfill (60)

are each algebraically independent in the ring $\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma$.  


The reader is advised to follow Example 22 while reading the proof.

**Proof.** It suffices to show that there is no nontrivial polynomial \( Q \) such that
\[
Q(h_{k_1+1}(X_n), h_{k_1+2}(X_n), \ldots, h_{k_1+p_1}(X_n)) \in \mathcal{H}(X_n, Y_n).
\] (61)
The lemma is vacuous if \( p_1 = 0 \), so we assume \( p_1 > 0 \). So, to prove the theorem, we assume that such a \( Q \) does exist and obtain a contradiction. Specifically, we will show that if the \( h_i \) in question are algebraically dependent in \( \mathcal{H}(X_n, Y_n) \), then the \( e_i \) of (44) are algebraically dependent in \( \mathcal{G}(X_n, Y_n) \), in direct contradiction with Lemma 14.

A relation such as (61) can be rewritten as
\[
\sum \lambda c_\lambda h_\lambda(X_n) \in \mathcal{H}(X_n, Y_n),
\] (62)
where each \( \lambda \) is a partition with parts of length chosen from the collection \( \{k_1 + 1, k_1 + 2, \ldots, k_1 + p_1\} \). We assume that \( c_\lambda = 0 \) for any \( \lambda \) with \( h_\lambda \in \mathcal{H}(X_n, Y_n) \) and that there exists some \( c_\lambda \neq 0 \).

Since the elementary symmetric functions are a basis of the ring of symmetric functions, any such \( h_\lambda \) can be expanded in terms of the \( e_\mu \). In fact, this can be done explicitly as follows (cf. [17]).

Define a *domino* to be a set of horizontally consecutive squares in a Ferrers diagram. For \( \lambda, \mu \vdash n \), a domino tabloid of shape \( \lambda \) and type \( \mu \) is a tiling of \( \text{dg}(\lambda) \) with dominoes of length \( \mu_1, \mu_2, \ldots \). We consider dominoes of the same length to be indistinguishable. Let \( d_{\lambda, \mu} \) denote the number of domino tabloids of shape \( \lambda \) and type \( \mu \). Then we can write
\[
h_\lambda(X_n) = \sum \lambda (-1)^{|\mu| - \ell(\mu)} d_{\lambda, \mu} e_\mu(X_n).
\] (63)
Recall that \( e_\mu(X_n) \in \mathcal{G}(X_n, Y_n) \) (respectively, \( \mathcal{H}(X_n, Y_n) \)) whenever \( \mu \) has a part greater than or equal to \( m_1 + p_1 \) (respectively, \( p_1 + 1 \)). Combining (62) and (63), we find that
\[
\sum \mu (-1)^{|\mu| - \ell(\mu)} \left( \sum \lambda c_\lambda d_{\lambda, \mu} \right) e_\mu(X_n) \equiv 0 \pmod{\mathcal{H}(X_n, Y_n)},
\] (64)
or, equivalently, that
\[
\sum \mu (-1)^{|\mu| - \ell(\mu)} \left( \sum \lambda c_\lambda d_{\lambda, \mu} \right) e_\mu(X_n) - \sum \mu a_\mu e_\mu(X_n) \equiv 0 \pmod{\mathcal{G}(X_n, Y_n)}
\] (65)
for some constants \( a_\mu \). If one of the \( a_\mu \neq 0 \) or one of the \( \sum \lambda c_\lambda d_{\lambda, \mu} \neq 0 \) for some \( \mu \), then we have a contradiction of the algebraic independence in \( \mathbb{C}[X_n, Y_n] / \mathcal{G}(X_n, Y_n) \) of the set \( \{e_i(X_n)\}_{1 \leq i \leq p_1 + m_1 - 1} \) (asserted by Corollary 14). We will, in fact, show that at least one of the \( \sum \lambda c_\lambda d_{\lambda, \mu} \) is nonzero.

For a given \( \lambda \), there is a maximum (with respect to lexicographic order) \( \mu \) with \( d_{\lambda, \mu} \neq 0 \) and \( e_\mu \notin \mathcal{H}(X_n, Y_n) \). It is given by
\[
L(\lambda) = (p_1^{\alpha_{p_1}}, (p_1 - 1)^{\alpha_{p_1 - 1}}, \ldots, 1^{\alpha_1}),
\] (66)
where, for \( 1 \leq f < p_1 \), \( \alpha_f \) equals the number of entries in
\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})
\] (67)
The maximum element appearing in (63) with \( \alpha_{p_1} \equiv \sum_{r=1}^{\ell(\lambda)} \frac{\lambda_r}{p_1} \) (68)

([x] denotes the floor of x). In particular, modulo \( \mathcal{H}_\gamma(X_n, Y_n) \), we can rewrite (63) as

\[
h_{\lambda}(X_n) \equiv \pm d_{\lambda, L(\lambda)} e_{L(\lambda)}(X_n) + \sum_{\mu \prec_{lex} L(\lambda)} (-1)^{|\mu| - \ell(\mu)} d_{\lambda, \mu} e_{\mu}(X_n)
\]

(69)

with \( d_{\lambda, L(\lambda)} \neq 0 \).

Furthermore, given \( \mu \), there is a unique partition \( \tilde{\lambda} \) (possibly with \( c_{\tilde{\lambda}} = 0 \)) for which \( \mu \) is the maximum element appearing in (63) with \( d_{\tilde{\lambda}, \mu} \neq 0 \) for some \( \tilde{\lambda} \). To see this, note that each \( \alpha_f \) (when \( 1 \leq f < p_1 \)) gives the number of \( \tilde{\lambda}_j \) congruent to \( f \) (mod \( p_1 \)). Since \( k_1 + 1 \leq \tilde{\lambda}_j \leq k_1 + p_1 \) this uniquely identifies \( \tilde{\lambda}_j \). Each such \( \tilde{\lambda}_j \) requires \( \lfloor \frac{\tilde{\lambda}_j}{p_1} \rfloor \) dominoes of length \( p_1 \) to finish out the row. Any remaining length-\( p_1 \) dominoes will be used to construct the remaining rows of \( \tilde{\lambda} \).

To finish the proof, pick from (62) the partition \( \tilde{\lambda} \) occurring with \( c_{\tilde{\lambda}} \neq 0 \) for which \( L(\tilde{\lambda}) \) is maximal. Certainly \( d_{\tilde{\lambda}, L\tilde{\lambda}} \neq 0 \). However, by this choice of \( \tilde{\lambda} \), \( d_{\tilde{\lambda}, L\tilde{\lambda}} = 0 \) or \( c_{\lambda} = 0 \) for \( \lambda \neq \tilde{\lambda} \). Hence, the coefficient of \( e_{L\tilde{\lambda}} \) in (65) is nonzero as desired. \( \square \)

**Theorem 21.** The collection

\[
\{ h_{k_1+1}(X_n), h_{k_1+2}(X_n), \ldots, h_{k_1+p_1}(X_n), h_{k_2+1}(Y_n), h_{k_2+2}(Y_n), \ldots, h_{k_2+p_2}(Y_n) \}
\]

(70)

is algebraically independent in the ring \( \mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma \).

**Proof.** If \( p_1 = 0 \) or \( p_2 = 0 \), the statement of the theorem reduces to that of Lemma 20. If they are both zero, then the statement is vacuous. So assume \( p_1, p_2 > 0 \).

To show the algebraic independence of (70) in the ring \( \mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma \), we suppose on the contrary that there is a nontrivial polynomial \( P \) such that

\[
P(h_{k_1+1}(X_n), h_{k_1+2}(X_n), \ldots, h_{k_1+p_1}(X_n), h_{k_2+1}(Y_n), h_{k_2+2}(Y_n), \ldots, h_{k_2+p_2}(Y_n)) \in \mathcal{H}_\gamma(X_n, Y_n).
\]

(71)

By Lemma 20, the set \( \{ h_{k_1+1}(X_n), h_{k_1+2}(X_n), \ldots, h_{k_1+p_1}(X_n) \} \) is algebraically independent in \( \mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma \). This implies that in the expansion

\[
P(h_{k_1+1}(X_n), \ldots, h_{k_1+p_1}(X_n), h_{k_2+1}(Y_n), \ldots, h_{k_2+p_2}(Y_n))
\]

\[
= \sum_{\mathbf{e}=(e_1, e_2, \ldots, e_{p_1}) \in \mathbb{N}^{p_1}} Q_\mathbf{e}(h_{k_2+1}(Y_n), \ldots, h_{k_2+p_2}(Y_n)) \prod_{i=1}^{p_1} h_{k_1+e_i}(X_n),
\]

(72)

the polynomials \( Q_\mathbf{e} \) are uniquely determined. Furthermore, for the inclusion of (71) to hold, \( Q_\mathbf{e}(h_{k_2+1}(Y_n), h_{k_2+2}(Y_n), \ldots, h_{k_2+p_2}(Y_n)) \) must be in \( \mathcal{H}_\gamma(X_n, Y_n) \) for all \( \mathbf{e} \). But, by Lemma 20, this implies that each \( Q_\mathbf{e} \) is identically zero. In turn, \( P = 0 \). This yields our desired contradiction. \( \square \)

**Example 22.** Fix \( \gamma = ((4, 1), (6, 0), (3, 0)) \). In this example we consider the expansion of \( h_{\tilde{\lambda}} = h_{(9, 9, 9, 8, 7, 7)} \) in terms of the elementary symmetric functions modulo \( \mathcal{H}_\gamma(X_n, Y_n) \). In particular, we construct \( L(\lambda) \) and show that \( \lambda \) is recoverable from it.
From $p_1 = 3$ it follows that $L((9^3, 8, 7^2)) = (3^{15}, 2, 1^2)$. The dominoes on each of the rows of length 7 and 8 can be placed in three different ways, so we find $d_{(9^3, 8, 7^2), (3^{15}, 2, 1^2)} = 27$. Then, following (69), $h_{(9^3, 8, 7^2)}(X_n)$ is congruent to

$$-27e_{(3^{15}, 2, 1^2)}(X_n) + \sum_{\mu \in \text{lex}(3^{15}, 2, 1^2)} (-1)^{\mu} \ell(\mu) d_{(9^3, 8, 7^2), \mu} e_\mu(X_n)$$

modulo $\mathcal{H}_\gamma(X_n, Y_n)$. On the other hand, given $L(\lambda) = (3^{15}, 2, 1^2)$, we can recover $\lambda$ as follows. We know that the parts of $\lambda$ must all be between $k_1 + 1$ and $k_1 + p_1$; 7 and 9 in this case. The two 1’s in $L(\lambda)$ tell us that there must be two parts of $\lambda$ of length 1 modulo $p_1 = 3$ (i.e., two rows of length 7). Similarly, we compute that there is a unique row of length 8. These three rows account for six of the fifteen length 3 parts of $L(\lambda)$. The remaining nine length 3 parts must together comprise the parts of length 9. Hence there are three of them and we recover $\lambda = (9^3, 8, 7^2)$ as desired. This completes our example.

Theorem 21 implies the following.

**Corollary 23.** The Hilbert series of $\mathbb{C}[X_n, Y_n]/\mathcal{J}_\gamma$ is given by

$$\mathcal{H}(\mathbb{C}[X_n, Y_n]/\mathcal{J}_\gamma) = \frac{(t)_{k_1 + p_1}}{(t)_{k_1}(t)p_1} \frac{(q)_{k_2 + p_2}}{(q)_{k_2}(q)p_2} \sum_{b \in \mathcal{B}_\gamma} t^{|X(b)|} q^{|Y(b)|}$$

$$= \left[ \begin{array}{c} p_1 + k_1 \\ p_1 \end{array} \right] \left[ \begin{array}{c} p_2 + k_2 \\ p_2 \end{array} \right] \sum_{q \in \mathcal{B}_\gamma} t^{|X(b)|} q^{|Y(b)|}.$$ (74)

**8. The ideals $\mathcal{J}_\gamma(X_n, Y_n) = \mathcal{I}_\gamma(X_n, Y_n)$**

We need to identify the generators of $\mathcal{I}_\gamma(X_n, Y_n)$ (recall Eq. (4)). Specifically, we want to prove that it is finitely generated by a collection of complete symmetric functions in the ring $\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma$. The goal is to show that the ideals $\mathcal{J}_\gamma$ and $\mathcal{I}_\gamma$ are equal in $\mathbb{C}[X_n, Y_n]/\mathcal{H}_\gamma$. Since $\mathcal{J}_\gamma \subseteq \mathcal{I}_\gamma$, we will do this by constructing a linearly independent set in $\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma$ that gives the Hilbert series in Eq. (74). Let $Q_{k, p}$ denote the set of partitions that fit in a rectangle with $p$ rows of length $k$. Observe that $\binom{p+k}{p}$

We can now define the collection of polynomials $\mathcal{B}_\gamma$ that will turn out to be the required basis for $\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma$. (Recall that $\mathcal{B}_\gamma$ is defined in Eq. (38).)

$$\mathcal{B}_\gamma = \left\{ h_q(X_n)h_{q'}(Y_n)b : b \in \mathcal{B}_\gamma, q \in Q_{k_1, p_1}, q' \in Q_{k_2, p_2} \right\}.$$ (75)

It follows from the observation above that the number of elements in $\mathcal{B}_\gamma$ is

$$n! \binom{p_1 + k_1}{p_1} \binom{p_2 + k_2}{p_2}.$$ (76)

Furthermore, note that

$$\sum_{b \in \mathcal{B}_\gamma} t^{|X(b)|} q^{|Y(b)|} = \left[ \begin{array}{c} p_1 + k_1 \\ p_1 \end{array} \right] \left[ \begin{array}{c} p_2 + k_2 \\ p_2 \end{array} \right] \sum_{q \in \mathcal{B}_\gamma} t^{|X(b)|} q^{|Y(b)|},$$ (77)

which equals the summation found in Eq. (74).
In Theorem 25 we will consider the action of differential operators in the complete symmetric functions on the determinants $\Delta_\gamma$. This theorem generalizes Lemma 17. The below example gives a sample computation in this spirit.

**Example 24.** By Lemma 17, we have

$$h_{3,2}(\partial X) \Delta_L(\partial X, \partial Y) = h_{2}(\partial X)h_{3}(\partial X)\Delta_L(\partial X, \partial Y) = (m_2 + m_{1,1})(\partial X)(9 \cdot 8 \cdot 7)\Delta_L(\partial X, \partial Y)$$

(78)

equal to

$$\begin{align*}
\frac{11!}{6!} \Delta_L(\partial X, \partial Y) + \frac{11!}{6!} \Delta_L(\partial X, \partial Y) + 9 \cdot 10! \Delta_L(\partial X, \partial Y) \\
+ \frac{10!}{5!} \Delta_L(\partial X, \partial Y) + \frac{9!}{4!} \Delta_L(\partial X, \partial Y).
\end{align*}$$

(79)

(We have omitted those $\Delta_L(\alpha)$ that have repeated entries and are thus identically equal to zero. We have also included exact coefficients even those not given by Lemma 17.) Note that the first and second terms cancel as they differ by the transposition of adjacent elements. This completes our example.

Suppose the sequences $\beta$ and $\delta$ correspond to

$$\gamma' = \gamma[a_0, a_1, \ldots, a_{p_1-1}; b_0, b_1, \ldots, b_{p_2-1}]$$

(80)

and

$$\gamma'' = \gamma[a'_0, a'_1, \ldots, a'_{p_1-1}; b'_0, b'_1, \ldots, b'_{p_2-1}],$$

(81)

respectively, for some triple $\gamma$ and nonincreasing sequences $(a_\ell), (b_\ell), (a'_\ell)$ and $(b'_\ell)$.

(Recall that the notation for $\gamma'$ and $\gamma''$ is defined near the beginning of Section 7.) We write $\beta >_{\text{cont}} \delta$ (or $\gamma' >_{\text{cont}} \gamma''$) whenever

$$(a_0, a_1, \ldots, a_{p_1-1}, b_0, b_1, \ldots, b_{p_2-1}) >_{\text{lex}} (a'_0, a'_1, \ldots, a'_{p_1-1}, b'_0, b'_1, \ldots, b'_{p_2-1}).$$

(82)

**Theorem 25.** If

$$q = (q_0, q_1, \ldots, q_{p_1-1}) \in Q_{k_1, p_1} \quad \text{and} \quad q' = (q'_0, q'_1, \ldots, q'_{p_2-1}) \in Q_{k_2, p_2},$$

then

$$h_q(\partial X) h_{q'}(\partial Y) \Delta_\gamma(X, Y) = c_{\tilde{\gamma}} \Delta_{\tilde{\gamma}} + \sum_{\gamma'' >_{\text{cont}} \gamma'} c_{\gamma''} \Delta_{\gamma''},$$

(84)

where $c_{\tilde{\gamma}} > 0$ and

$$\tilde{\gamma} = \gamma[q_0, q_1, \ldots, q_{p_1-1}; q'_0, q'_1, \ldots, q'_{p_2-1}].$$

(85)

**Proof.** First note that the actions of $h_q(\partial X)$ and $h_{q'}(\partial Y)$ commute. It suffices to consider the action of $h_q(\partial X)$ in this proof.

By Lemma 17, the only surviving term in the expansion of $h_{q_0}(\partial X)\Delta_\gamma$, namely $\Delta_{\gamma[q_0; 0]}$, is attained by moving the cell at $(m_1 + k_1, 0)$ to the position labeled by $(m_1 + k_1 - q_0, 0)$. Suppose we then act on $\Delta_{\gamma[q_0; 0]}$ by $h_{q_1}(\partial X)$. If the cell at $(m_1 + k_1 - q_0, 0)$ is moved farther to the left, then any $\gamma''$ corresponding to such a configuration will be strictly greater than $\gamma[q_0, q_1; 0]$ in the $>_{\text{cont}}$ partial order. As such, we can restrict our attention to the movement of the cells.
in the positions \((m_1 + k_1 + \ell, 0)\) for \(1 \leq \ell < p_1\). Iterating this argument with \(j\) equal to \(q_\ell\) for \(\ell = 1, 2, \ldots, q_{p_1 - 1}\) in Lemma 17, we see that \(\Delta_\gamma[q_0, q_1, \ldots, q_{p_1 - 1}; 0]\) will be the smallest term in the expansion of \(h_q(\partial_X)\Delta_\gamma(X, Y)\). That it will have positive coefficients also follows from Lemma 17.

The primary goal of this paper is to prove the following theorem.

**Theorem 26.** The collection \(BB_\gamma\) is linearly independent in \(\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma\). Hence, by the equality of (74) and (77), \(BB_\gamma\) forms a basis for \(\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma\).

By [18, Theorem 2.1], the subspace of \(B_\gamma\) spanned by bipermanents \([T, V]_{\text{per}}\) of a given shape \(\lambda\) and given \(V\) carries a copy of the irreducible representation \(S^\lambda\). It is this fact that lets us conclude Theorem 1 from the Hilbert series (74) of Corollary 23.

Note that it is enough to show that the collection

\[
\{b(\partial_X, \partial_Y)\Delta_\gamma(X, Y): b \in BB_\gamma\}
\]

is linearly independent in \(\mathbb{C}[X_n, Y_n]\) since

\[
i \sum_{j=1}^i c_j b_j \equiv 0 \pmod{\mathcal{I}_\gamma(X_n, Y_n)} \iff \left(\sum_{j=1}^i c_j b_j\right)(\partial_X, \partial_Y)\Delta_\gamma(X, Y) = 0 \equiv \left(\sum_{j=1}^i (c_j b_j(\partial_X, \partial_Y)\Delta_\gamma(X, Y)) = 0.\right.
\]

To do this, however, we need the following lemma. A proof can be found in [4] (see Eqs. (6.5) and (6.6) in Theorem 6.2), however, there is a sign missing, so we include a proof here. The lemma lets us expand the action of a bipermanent on a determinant associated to a lattice diagram. To state the lemma succinctly, we first associate to the bipermanent in question a family of \(A\)-filled diagrams. So, consider a bitableau \((T, C) \in \Theta_n\) and set \(S = \text{std}(C)\). Write \(\iota_T, S\) for the map that takes \(i\) to \(s_T(i)\). For any permutation \(\phi \in S_n\), we then define a filled diagram \(E_\phi\) by placing \(\alpha_\phi(\iota_T, S(i)) - c_T\) in the cell containing \(i\) in \(T\).

**Lemma 27.** Let \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) and \((T, C) \in \Theta_n\). Then

\[
[T, C]_{\text{per}}(\partial_X, \partial_Y)\Delta_L[\alpha] = \text{sgn}(\iota_T, S) \sum_{\phi \in S_n} \text{sgn}(\phi)d_\phi[T', (E_\phi)^T]_{\text{det}},\]

for the \(A'\)-filled diagrams \(E_\phi^T\) defined above and integers \(d_\phi \geq 0\). We make the convention that \(d_\phi = 0\) if any entry of \(E_\phi^T\) is not in \(A\) or if \([T', (E_\phi)^T]_{\text{det}} = 0\); otherwise \(d_\phi > 0\).

Note that \([T', (E_\phi)^T]_{\text{det}} = 0\) when there is a repetition in some row of \(E_\phi^T\).

**Proof.** For the remainder of the paper we abbreviate the product \(x^{\alpha_1}_{j_1} y^{\alpha_2}_{j_2} \ldots z^{\alpha_n}_{j_n}\) by \(z_j^{\alpha_1} \ldots z_{j_n}^{\alpha_n}\). Define \(q_{T, C} = q_{T, C}(\partial_X, \partial_Y)\) to be the monomial \(\partial_{z_1}^{c_1} \partial_{z_2}^{c_2} \cdots \partial_{z_n}^{c_n}\). Then \([T, C]_{\text{per}}(\partial_X, \partial_Y)\) can be expanded as \(\sum_{\sigma \in R_T} \sigma(q_{T, C})\). Expanding \(\Delta_L[\alpha]\) as well, we have the following expression for the left-hand side of (88):
\[ [T, C]_{\text{per}}(\partial X, \partial Y) \Delta L_{[\alpha]} = \sum_{\sigma \in R_T} \sum_{\tau \in S_n} \text{sgn}(\tau) \sigma q_{T, C} \prod_{j=1}^n z_{j}^{\alpha_{\tau^{-1}(j)}} \]

\[ = \sum_{\sigma \in R_T} \sum_{\tau \in S_n} \text{sgn}(\tau) \sigma q_{T, C} \prod_{j=1}^n z_{j}^{\alpha_{\sigma(\tau^{-1})}} \]

\[ = \sum_{\phi \in S_n} \sum_{\sigma \in R_T} \text{sgn}(\tau) \sigma q_{T, C} \prod_{j=1}^n z_{j}^{\alpha_{\phi(\tau)}(\tau^{-1}(j))}. \] (89)

As an illustration of the equality between the first two lines, consider

\[ ((1, 5, 3) \partial^2_{x_5}) x_3^{\alpha_{3,1}} = \alpha_{3,1}(\alpha_{3,1} - 1)x_3^{\alpha_{3,1}-2} = (1, 5, 3)(\partial^2_{x_5}x_5^{\alpha_{3,1}}). \] (90)

In going from the second to the third lines, we set \( \phi = \sigma \circ \tau^{-1} \circ \iota_{T, S}^{-1} \), used the fact that the sign of any permutation is the sign of its inverse, and noted that as \( \tau \) runs over \( S_n \), so does \( \phi \).

For each \( \phi \), we would like to interpret the sum over \( \sigma \) as a bideterminant \( \left[ T \ t, (E^\alpha_{\phi})^t \right]_{\text{det}} \).

Formally this makes sense as bideterminants are signed sums over the elements in the column stabilizer of some filled diagram. Here we have a signed sum over a row stabilizer; hence we consider transposes. If \( q_{T, C}(\partial X, \partial Y) \) were not in the equation, we would use \( \alpha_{\phi(\tau)}(\tau^{-1}(i)) \) as our entry in \( E^\alpha_{\phi} \) corresponding to \( i \) in \( T \). (The \( \iota_{T, S} \) accounts for the fact that we are not assuming \( S = T \).) Up to the multiplicative constants \( d_{\phi} \), the action of \( q_{T, C}(\partial X, \partial Y) \) is to subtract \( c_i^T \) from the exponent of \( z_i \). This is consistent with the statement of the lemma. \( \square \)

**Example 28.** We illustrate Lemma 27 with the following simple computation for \( \alpha = (2, 0, 1, 3) \), \( T = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \) and \( C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). In this example, the identity is the only \( \phi \) for which the entries of \( E^\alpha_{\phi} \) are all in \( A \). Also, \( \iota_{T, S} \) is given in cycle notation by \( (2, 4, 3) \) which yields \( \text{sgn}(\iota_{T, S}) = 1 \).

\[
\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\( (\partial X, \partial Y) \Delta L_{[(2, 0, 1, 3)]} \) \( \text{per} \)

\[ = (\partial_{y_1} \partial^2_{x_2} \partial x_4 + \partial_{y_2} \partial^2_{x_1} \partial x_4) \text{det} \begin{bmatrix} y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ 1 & 1 & 1 & 1 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{bmatrix} \]

\[ = 12y_1x_2 - 12y_2x_1 = 12 \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{det} \]. (91)

Let \( (T_1, U_1), (T_2, U_2) \) be bitableaux. We define

\[ (T_1, U_1) \text{ < bitab } (T_2, U_2) \] (92)

according to the following tiebreakers (recall that \( \text{cs}(U) \) denotes the column sequence of \( U \); cf. Example 4):

1. \( \text{sh}(T_1^f) \text{ < lex } \text{sh}(T_2^f) \);
It is easily checked for this particular example that the entries of 
\((S,W) >_{\text{lex}} (A)\) with 
\(1150\ 
\)

The comparison of the \(cs(T_i)\) is not strictly necessary, however it serves to make \(<_{\text{bitab}}\) a total order.

We are now ready to prove that \(BB_\gamma\) is a basis for \(\mathbb{C}[X_n, Y_n]/I_\gamma\). As outlined in (87), it suffices to prove the following theorem.

**Theorem 29.** Let \(T, U \in SYT_n, \alpha \in A^\alpha\) correspond to some triple \(\gamma, q \in Q_{k_1, p_1}\) and \(q' \in Q_{k_2, p_2}\). Set \(C = C_\gamma(U)\). Then

\[
\begin{align*}
h_q(\partial X)h_{q'}(\partial Y)[T, C]_{\text{per}}(\partial X, \partial Y)\Delta_{\gamma} &= d[T', (E^\beta)']_{\det} + \sum_{(S, W)} d_{S, W}[S', W']_{\text{det}}, \\
\end{align*}
\]

where \(d \neq 0, \beta = (\beta_1, \beta_2, \ldots, \beta_n)\) with \(\beta_i = \alpha_{t, U(i)} - c_i\), and the sum is over all \((S, W) \in \Theta_n\) with \((S, W) >_{\text{bitab}} (T, E^\beta)\).

Before presenting the proof, we illustrate (93) with an explicit computation.

**Example 30.** Let \(\gamma\) derive from the collection of cells \(\alpha = (4, 3, 0, 3, 4, 5)\). Set \(T = \begin{pmatrix} 4 & 6 \\ 3 & 5 \\ 1 & 2 \end{pmatrix}\) and \(\iota_{T, U} = (2, 3)(4, 5)\). We consider the expansion of

\[
h_{1,1}(\partial X)h_{2,1}(\partial Y)[T, C]_{\text{per}}(\partial X, \partial Y)\Delta_{\gamma}
\]

\[
= [T, C]_{\text{per}}(\partial X, \partial Y)h_{1,1}(\partial X)h_{2,1}(\partial Y)\Delta_{\gamma}
\]

\[
= [T, C]_{\text{per}}(\partial X, \partial Y)\left(c_\beta \Delta_{L[\beta]} + \sum_{\delta > \text{cont} \beta} c_\delta \Delta_{L[\delta]}\right)
\]

\[
= c_\beta [T, C]_{\text{per}}(\partial X, \partial Y)\Delta_{L[\beta]} + \sum_{\delta > \text{cont} \beta} c_\delta [T, C]_{\text{per}}(\partial X, \partial Y)\Delta_{L[\delta]},
\]

where \(\beta = (3, 1, 0, 2, 3, 5)\). For \(\gamma\) associated to the \(\alpha\) above, \(m_2 + p_2 = 3\). In addition

\[
[T, C]_{\text{per}}(X_6, Y_6) = \sum_{\sigma \in S_{[1, 2]} \times S_{[3, 5]} \times S_{[4, 6]}} y_{\sigma(1)}x_{\sigma(4)}x_{\sigma(5)}x_{\sigma(6)}.
\]

It follows then that

\[
[T, C]_{\text{per}}(\partial X_6, \partial Y_6) = (\partial_{y_1} + \partial_{y_2})(\partial_{x_3} + \partial_{x_5})(2 \cdot \partial_{x_4}^2 \partial_{x_6}).
\]

By Lemma 27, \([T, C]_{\text{per}}(\partial X_6, \partial Y_6)\Delta_{L[\beta]}\) can be expanded as

\[
\text{sgn}(\iota_{T, U}) \sum_{\phi \in S_6} \text{sgn}(\phi) d_{\phi}[T', (E^\phi)']_{\text{det}}.
\]

It is easily checked for this particular example that the entries of \(E^\phi_{\phi}\) will all be in \(A\) only if \(\phi \in S_{[1, 2, 3]} \times S_{[4, 5, 6]}\) with \(\phi(1) \neq 3\). There are 24 such permutations. Table 1 gives these \(\phi\)
Table 1
Expansion of Example 30

<table>
<thead>
<tr>
<th>Case</th>
<th>φ</th>
<th>$d_\phi$</th>
<th>$(E^\beta_\phi)'$</th>
<th>(5,6)</th>
<th>$d_\phi$</th>
<th>$(E^\beta_\phi)'$</th>
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<td>720</td>
<td>0 1 2 3</td>
<td></td>
<td>720</td>
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<tr>
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<td></td>
<td></td>
<td>2 1 1</td>
<td></td>
<td></td>
<td>2 1 1</td>
</tr>
<tr>
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<td>(2,3)</td>
<td>720</td>
<td>1 1 2 3</td>
<td></td>
<td>720</td>
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<td></td>
<td>2 0 1</td>
</tr>
<tr>
<td>3</td>
<td>(4,5)</td>
<td>360</td>
<td>0 2 3</td>
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<td>180</td>
<td>0 4 1</td>
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<td></td>
<td></td>
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<td>2 1 0</td>
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<td></td>
<td>(2,3)(4,5)</td>
<td>360</td>
<td>1 2 3</td>
<td></td>
<td>180</td>
<td>1 4 0</td>
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<td>2 0 0</td>
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<td>120</td>
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<tr>
<td></td>
<td>(1,2)(4,6,5)</td>
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<td></td>
<td>0</td>
<td>0 2 0</td>
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<td>0 3 3</td>
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<td>0 3 3</td>
</tr>
</tbody>
</table>

along with the coefficients $d_\phi$ and diagrams $E^\beta_\phi$. We leave the expansion of the sum in (94) to the reader. This completes our example.

**Proof of Theorem 29.** Let $T, U \in SYT_n$. Using Theorem 25, we have that

$$h_q(\partial_X)h_{q'}(\partial_Y)[T, C]_{\text{per}}(\partial_X, \partial_Y)\Delta_Y(X, Y)$$

$$= [T, C]_{\text{per}}(\partial_X, \partial_Y)h_q(\partial_X)h_{q'}(\partial_Y)\Delta_Y(X, Y)$$
\[
[T, C]_{\text{per}}(\partial_X, \partial_Y) \left( c_\beta \Delta_{L[\beta]} + \sum_{\delta > \text{com}_{\partial_X}} c_\delta \Delta_{L[\delta]} \right) \\
= c_\beta [T, C]_{\text{per}}(\partial_X, \partial_Y) \Delta_{L[\beta]} + \sum_{\delta > \text{com}_{\partial_X}} c_\delta [T, C]_{\text{per}}(\partial_X, \partial_Y) \Delta_{L[\delta]},
\]

(98)

with \( c_\beta > 0 \). Now, Lemma 27 implies that

\[
[T, C]_{\text{per}}(\partial_X, \partial_Y) \Delta_{L[\beta]} = \text{sgn}(\iota_{T,U}) \sum_{\phi \in S_n} \text{sgn}(\phi) d_\phi[T', (E^{\beta}_\phi)^t]_{\text{det}}.
\]

(99)

Therefore,

\[
h_q(\partial_X) h_q'(\partial_Y) [T, C]_{\text{per}}(\partial_X, \partial_Y) \Delta_{L[\beta]}(X, Y) \\
= \text{sgn}(\iota_{T,U}) \sum_{\phi \in S_n} \text{sgn}(\phi) \left( c_\beta d_\phi [T', (E^{\beta}_\phi)^t]_{\text{det}} + \sum_{\delta > \text{com}_{\partial_X}} c_\delta d_\phi [T', (E^{\delta}_\phi)^t]_{\text{det}} \right).
\]

(100)

Our goal is to show that there is a minimum bitableau (with respect to \(<_{\text{bitab}}\)) occurring on the right-hand side of (100). First note that many of the bitableaux \((T', (E^{\beta}_\phi)^t)\) and \((T', (E^{\delta}_\phi)^t)\) are not standard. However, Theorem 6 tells us how the shapes and column sequences are affected by straightening.

Recall from (10) that \( S_{R(\text{std}(C))} \) represents the row stabilizer of the standard tableau \( \text{std}(C) \). We split into three cases dependent on the indexing permutations \( \phi \). (A proof of an argument with similar statements and complete details can be found in [4, Theorem 6.2].)

**Case 1.** \( \kappa(E^{\beta}_\phi) = \kappa(E^{\beta}_e) \) and \( \phi \in S_{R(\text{std}(C))} \).

Let \( A \subset S_n \) be the subset of such \( \phi \). For all \( \phi \in A \), \([T', (E^{\beta}_\phi)^t]_{\text{det}} = \text{sgn}(\phi)[T', (E^{\beta}_e)^t]_{\text{det}} \). So the contribution to (100) from \( \phi \in A \) is

\[
\text{sgn}(\iota_{T,U}) \sum_{\phi \in A} \text{sgn}(\phi) c_\beta d_\phi \text{sgn}(\phi)[T', (E^{\beta}_e)^t]_{\text{det}} \\
= \text{sgn}(\iota_{T,U}) \sum_{\phi \in A} c_\beta d_\phi [T', (E^{\beta}_e)^t]_{\text{det}}.
\]

(101)

Since each \( d_\phi > 0 \), there is no cancellation from such terms. We get the first term on the right-hand side of (93) with \( d = \text{sgn}(\iota_{T,U}) c_\beta \sum_{\phi \in A} d_\phi \neq 0 \).

**Case 2.** \( \kappa(E^{\beta}_\phi) = \kappa(E^{\beta}_e) \), but \( \phi \notin S_{R(\text{std}(C))} \).

It follows from the construction of \( C = C'_U(U) \) from \( U \) that \( \text{rs}((E^{\beta}_\phi)^t) = \text{cs}(E^{\beta}_\phi) >_{\text{lex}(A)} \text{cs}(E^{\beta}_e) = \text{rs}((E^{\beta}_e)^t) \). In fact, this inequality continues to hold even if we modify the column sequences by ordering the elements within each column from smallest to largest. As in Case 1, the bideterminants \([T', (E^{\beta}_\phi)^t]_{\text{det}}\) arising in this case equal, up to sign, standard bideterminants obtained by rearranging the entries in each column. It then follows that any bitableaux arising from this case is greater than \((T, V)\).

**Case 3.** \( \kappa(E^{\beta}_\phi) \neq \kappa(E^{\beta}_e) \).
The entries of $E^\beta_\phi$ are given by the multiset $\{\beta_{\phi(\ell)} - c^U_\ell | \ell \in \{1,2,\ldots,n\}\}$. Consider any pair $i < j$ for which $\phi(i) < \phi(j)$ and set $\tau = \phi_o (i, j)$. We claim that the entries of $\{\beta_{\tau(\ell)} - c^U_\ell | \ell \in \{1,2,\ldots,n\}\}$ when arranged in weakly increasing order will be lexicographically less than or equal to those of $\{\beta_{\phi(\ell)} - c^U_\ell | \ell \in \{1,2,\ldots,n\}\}$ arranged in weakly increasing order. Since every permutation has a reduced decomposition as a product of transpositions in the Bruhat order, it will follow from this claim that $\kappa(E^\beta_\phi) < \kappa(E^\beta_\tau)$ for $\phi$ arising in this case. Since straightening does not change the content, we conclude that $(S,W) >_{bitab} (T,V)$ for any bitableaux $(S,W)$ arising in this case.

We now prove the claim. Passing from $\phi$ to $\tau$ changes the multiset by removing $\beta_{\phi(i)} - c^U_i$ and $\beta_{\phi(j)} - c^U_j$ and replacing them with $\beta_{\tau(i)} - c^U_i$ and $\beta_{\tau(j)} - c^U_j$. Now, $\tau(i) = \phi(j)$ and $\tau(j) = \phi(i)$, so the new entries can be written $\beta_{\phi(j)} - c^U_j$ and $\beta_{\phi(i)} - c^U_i$. But $\beta_{\phi(i)} - c^U_i \leq \beta_{\phi(i)} - c^U_i$ since $c^U_j \geq c^U_i$ for any $j > i$. And $\beta_{\phi(i)} - c^U_i < \beta_{\phi(j)} - c^U_j$ by our assumption that $\phi(i) < \phi(j)$.

It is not difficult to see that the $E^\beta_\phi$ with $\delta >_{cont} \beta$ satisfy

$$(T, E^\beta_\phi) >_{bitab} (T, E^\beta_\tau).$$

Thus, combinations of $\phi$ and $\delta \neq \beta$ substituted into Eq. (100) (and using Theorem 6) yield bideterminants $[S,W]_\Delta$ with $(S,W) >_{bitab} (T, E^\beta_\tau)$. Thus, we have

$$h_q(\partial_X) h_q(\partial_Y)[T,C]_{per}(\partial_X, \partial_Y) \Delta_\gamma(X,Y) = d[T', (E^\beta_\tau)]_{det} + \sum_{(S,W) >_{bitab} (T,E^\beta_\tau)} d_{S,W}[S', W']_{det},$$

with $d \neq 0$. This proves the theorem.

Since we have that $J_\gamma(X_n,Y_n) \subset \mathcal{I}_\gamma(X_n,Y_n)$, the fact that the Hilbert series in Eq. (74) equals the summation in Eq. (77) implies that we must have $J_\gamma(X_n,Y_n) = \mathcal{I}_\gamma(X_n,Y_n)$. Thus, Eq. (74) must give the Hilbert series for $\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma$. Since the collection $BB_\gamma$ is linearly independent in $\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma$ and it gives the correct Hilbert series, we must have that $BB_\gamma$ is in fact a basis for $\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma$. The proof of Theorem 26 is now complete.

9. Some notes, applications and conjectures

**Note 2.** Suppose $D$ is any basis for $\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma$ where $\gamma = ((m_1, n - m_1 + 1), (0,0), (0,0))$ for some $m_1 > 0$. The basis $D$ can be substituted in the place of $B_\gamma$ in the definition of $BB_\gamma$ of Eq. (75) such that $BB_\gamma$ still yields a basis for $\mathbb{C}[X_n, Y_n]/\mathcal{I}_\gamma$. Examples of such bases $D$ include descent monomials (see [4,19] or [20]), Artin monomials (see [21]), Schubert monomials (see [22]) and higher Specht polynomials (see [5]).

**Note 3.** The ideas of the previous sections are easily extendable to the complex reflection groups $G(r,p,n)$. In this case, the lattice diagrams utilized are those of the form $L[\beta]$ where $L[\alpha]$ is a hollow lattice diagram and $\beta = \{(r\alpha_i,1, r\alpha_i,2): \alpha \in \alpha\}$. The resulting bases give representations of the complex reflection groups by ways of $m$-tableaux, standard $m$-tableaux and cocharge $m$-tableaux. See [4] to see how this translation is accomplished.

**Note 4.** Some of the results of this paper can be extended to diagonally symmetric and anti-symmetric rings in the four sets of variables $X_n, Y_n, Z_n$ and $W_n$. The diagonal action of $S_n$ on $\mathbb{C}[X_n, Y_n, Z_n, W_n]$ and the rings of symmetric polynomials and anti-symmetric polynomials
\[ \mathbb{C}^+\{X_n,Y_n,Z_n,W_n\} \] and \[ \mathbb{C}^-\{X_n,Y_n,Z_n,W_n\} \] are defined in the natural manner. For each of \(+\) and \(-\), define
\[ R_{\gamma_1,\gamma_2}^\pm \equiv \mathbb{C}^\pm\{X_n,Y_n,Z_n,W_n\}/\{P \in \mathbb{C}^\pm\{X_n,Y_n,Z_n,W_n\}: \]
\[ P(\partial X, \partial Y, \partial Z, \partial W) \Delta_{\gamma_1}(X_n,Y_n) \Delta_{\gamma_2}(Z_n,W_n) = 0 \}. \tag{104} \]
Furthermore, fixing an arbitrary standard tableau \(Q\), set
\[ [T_1,T_2]^+_{\text{per}} = \sum_{\sigma \in S_n} \sigma \left( [Q,T_1]_{\text{per}}(X_n,Y_n)[Q,T_2]_{\text{per}}(Z_n,W_n) \right), \tag{105} \]
\[ [T_1,T_2]^-_\text{per} = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \left( [Q,T_1]_{\text{per}}(X_n,Y_n)[Q,T_2]_{\text{per}}(\partial Z,\partial W) \Delta_{\gamma_2}(Z_n,W_n) \right). \tag{106} \]
Using the techniques found in [8], it should not be difficult to construct bases for \( R_{\gamma_1,\gamma_2}^\pm \) using variations of the polynomials in (105) and (106). These are generalizations of rings that were studied in [20,23].

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