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Fundamental Study

$L(A) = L(B)$? decidability results from complete formal systems

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Abstract

The equivalence problem for deterministic pushdown automata is shown to be decidable. We exhibit a *complete formal system* for deducing equivalent pairs of deterministic rational boolean series on the alphabet associated with a dpda $\mathscr{M}$. We then extend the result to deterministic pushdown transducers from a free monoid into an abelian group. A general algebraic and logical framework, inspired by Harrison et al. (Theoret. Comput. Sci. 9 (1979) 173–205), Conrcelle (Theoret. Comput. Sci. 6 (1978) 255–279) and Meitus (Kybernetika 5 (1989) 14–25 (in Russian)) is developed. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We solve here the *equivalence problem for deterministic pusdown automata* (see Section 1.2), which was an open question in *formal language* theory and also, undirectly, in *semantics* of programming languages (see Section 1.1). We use several ideas and techniques which appeared in previous works on the subject (see Section 1.3) and also introduce some new ones. We aim to give a self-contained and readable exposition of our solution (see Section 1.4).

1.1. Motivation

1.1.0.1. Origins The notion of *context-free grammar* and *context-free language* were introduced in the late 1950s [8], as a mathematical model for approximating natural languages. Their possible utilization for *defining* the syntax of programming languages was quickly recognized [30,50]. The notion of *pushdown automaton* was then devised in order to fit with this class of grammars [9,10]. Let us recall that this class of formal languages constitutes level 2 of Chomsky's celebrated hierarchy, whose level 1 is the class of *regular* languages (introduced in [41]). The efforts towards devising subclasses of context-free grammars allowing an *efficient* left-to-right parsing algorithm converged towards the definition of LR(k) grammars [42]; in the mean-time several definitions of the notion of *deterministic* pushdown automaton were given [29,31,64] which define the same class of languages [2] which is nothing else than the class of LR(k) languages. [3]

1.1.0.2. The equivalence problem for deterministic pushdown automata In [29] the authors investigate the mathematical properties of deterministic pushdown automata. After having proved some *positive* algorithmic results, for example:

- it is recursively solvable to determine for an arbitrary deterministic language L and a regular set R whether $L=R$ (Theorem 5.1, p. 645),

they show some negative results, for example:

- for arbitrary deterministic languages L, L', it is recursively unsolvable to determine whether $L \subseteq L'$ (Theorem 5.3, p. 646, point (b)).

They conclude their article by mentioning the question:

"is it recursively unsolvable to determine if $L_1 = L_2$ for arbitrary deterministic languages L_1 and L_2"?

[2] We do not know if a formal proof of this statement exists, but this is asserted in [29, p. 621, footnote 1] and believed by us.

[3] The fact that LR(k) grammars generate exactly the deterministic languages in the sense of [29] is formally proved in [42, Theorem, p. 628, Theorem p. 630].

From the beginning, the solvability of the problem when L_2 is rational (see above) and its unsolvability when the equality relation is replaced by the inclusion relation make this question likewise mysterious (hence mathematically attractive). [4]

1.1.0.3. Partial solutions This question has motivated a huge amount of works which, altogether, constructed increasing *subclasses* of dpda's where the answer was positive [5] and also increasingly sophisticated *methods* to prove so. The decidability of the above question (we name it the "equivalence problem for dpda's") was established for many subclasses. Let us mention some important such subclasses where the equivalence was proved decidable even though the inclusion problem was undecidable. [6]

- the subclass of dpda's with one state and no ε-transition [40],
- the case where one dpda has one state and no ε-transition and the other is general [34],
- the subclass of dpda's with only one stack symbol [80] (another related result is also [38],
- the subclass of LL(k) grammars [62], generalized by the subclass of non-singular dpda's [78],
- the subclass of finite-turn dpda's [4,79],
- the subclass of dpda's with no ε-transition, with empty final configurations [56,59] (a more general result implying this one is also [75]),
- the subclass of dpda's with no ε-transition, with arbitrary final configurations [55,60].

1.1.0.4. The equivalence problem for program schemes Let us say that two programs P, Q are *equivalent* iff, on every given input, either they both diverge or they both converge and compute the same result. It would be highly desirable to find an algorithm deciding this equivalence between programs since, if we consider that P is really a program and Q is a specification, this algorithm would be a "universal program-prover". Unfortunately, one can easily see that, as soon as the programs P, Q compute on a sufficiently rich structure (for example the ring of integers), this notion of equivalence is undecidable. Nevertheless, this seemingly hopeless dream lead many authors to analyze the *reason why* this problem is undecidable and the suitable *restrictions* (either on the shape of programs or on the meaning of the basic operations they can perform) which might make this equivalence decidable. [7] Informally, one can define an *interpretation* as an "universe of objects together with a certain definite meaning for each program primitive as a function on this universe" and a *program scheme*

[4] Even though, from this point of view, no practical application was expected.

[5] Let us call *positive* a proof of the *solvability* of the equivalence problem for some class of automata, i.e. we forget the *negative* way in which Ginsburg and Greibach raised the question.

[6] All the decidability results concerning classes of languages which are boolean algebras, are omitted in this introduction. They are, of course, interesting by themselves, but one can hardly hope to adapt the corresponding methods to the equivalence problem for dpda's.

[7] This second point of view appears to be quite *pragmatic*; even *approximate* decision procedures applicable to programs for which equivalence is undecidable were developed (see, for example [39]).

as a “program without interpretation” [49, p. 205, lines 5–13]. Several precise mathematical notions of “interpretation” and “program schemes” were given and studied ([12,13,24,28,36,39,44,49,52,57,61,63], see [19] for a survey). Many methods for either transforming programs or for proving properties of programs were established but, concerning the equivalence problem, the results turned out to be mostly negative: for example, in [44, p. 221, lines 24–26], the authors report that “for almost any reasonable notion of equivalence between computer programs, the two questions of equivalence and non-equivalence of pairs of schemas are *not* partially decidable”. Nevertheless, two kinds of program schemes survived all these studies:

- the *monadic recursion schemes*, where a special ternary function `if-then-else` has the fixed usual interpretation: in [28] the equivalence problem for such schemes is reduced to the equivalence problem for dpda’s and in [24] a reduction in the opposite direction is constructed;
- the *recursive polyadic program schemes*: in [12,13], following a representation principle introduced in [61], the equivalence problem for such schemes is reduced to the equivalence problem for dpda’s and conversely.

1.1.0.5. Other links Some other Turing-equivalent problems on *semi-Thue systems* were also found (see [67] for a survey) and formulations in terms of bisimulation equivalence of infinite *graphs* (or *processes*) have been found too (see [6] for a survey).

1.2. Results

We prove in this article the following results.

Theorem 87. *It is recursively* solvable *to determine if* $L(A) = L(B)$ *for arbitrary deterministic pushdown automata* A *and* B.

This theorem was exposed in [69,71] and proved in [70]. We give here a revised proof and a simplified “proof-system” for all the pairs of equivalent “deterministic rational boolean series” (see Section 1.3 and the system $\mathscr{D}_5$ in Section 10).

Corollary 180. *The equivalence problem for monadic recursion schemes* (*with interpreted* `if-then-else`), *is decidable.*

Corollary 181. *The equivalence problem for recursive polyadic program schemes* (*with completely uninterpreted function symbols*) *is decidable.*

Theorem 177. *It is recursively* solvable *to determine if* $T(A) = T(B)$ *for arbitrary deterministic pushdown transducers* A *and* B, *with outputs in an abelian group* H.

This theorem extends Theorem 87 which corresponds to the case where $H = \{1\}$.

1.3. Techniques

1.3.0.6. Deterministic series and matrices This idea appeared in [34]. One of the difficulties encountered in manipulating the *configurations* of a pushdown automaton (a configuration is a pair (state, stack-word)) is that no nice algebraic operation seems naturally defined on them. We propose to overcome this difficulty by *embedding* the set of configurations into a larger set: the set of *deterministic rational boolean series* which is endowed with a partial sum, a product, a partial star. This notion of *deterministic rational boolean series* is exactly the notion of *set of associates* defined in [34] and generalized here to infinite rational sets. In connection with the ideas of [47,48] we have then generalized this notion to vectors and to matrices. This allows then to define a notion of *linear combination* of deterministic rational boolean series.

1.3.0.7. Deterministic spaces The notion of *linear independence* of languages (and also of configurations) appeared in [47]. Let us sketch this idea for prefix languages. We recall that a language L is said to have the *prefix* property if, for every $u, v \in L$, if u is a prefix of v, then $u = v$. Similarly, we shall say that a vector of languages $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a prefix vector iff $\cup_{i=1}^{n} \alpha_i$ is prefix and for every $i \neq j$, $\alpha_i \cap \alpha_j = \emptyset$. Let $(L_1, L_2, \ldots, L_n)$ be a family of prefix languages:

(1) Either for every two prefix vectors $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, $(\beta_1, \beta_2, \ldots, \beta_n)$

$$\sum_{i=1}^{n} \alpha_i \cdot L_i = \sum_{i=1}^{n} \beta_i \cdot L_i \Rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \beta_2, \ldots, \beta_n)$$

(2) or, there exists some $i_0 \in [1, n]$, and a prefix vector $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, such that

$$L_{i_0} = \sum_{i=1}^{n} \gamma_i \cdot L_i \quad \text{where } \gamma_{i_0} = \emptyset$$

When (1) (resp. (2)) is true, the family $(L_1, L_2, \ldots, L_n)$ is said linearly independent (resp. linearly dependent). In other words, if (1) is not true, then (2) must be true. The adaptation of this idea to *equivalence of configurations* (instead of equality of languages) was technically non-obvious because, even when (1) is shown to be untrue by a pair of vectors α, β defined by configurations, the vector γ appearing in (2) need not be still defined by a configuration. But we prove that it always corresponds to a deterministic rational boolean vector (Lemma 30).

We are then naturally led to consider, for every given set of deterministic rational boolean series $\{U_i \mid i \in I\}$, the set of all deterministic rational linear combinations of these series. We call such a set the *deterministic space* generated by $\{U_i \mid i \in I\}$.

1.3.0.8. Deduction systems This idea appeared in full generality in [14]. We expose this idea in details in Section 4. A *deduction* system is a kind of *formal system*, i.e. a set of assertions together with a set of axioms and deduction rules. The originality of these systems stems in the fact that the notion of *proof* allows some *loops*. Such "looping proofs" are guaranteed to be correct as soon as some *cost*-function is strictly increasing at every deduction step. We introduce in Section 4.3 a deduction system $\mathscr{D}_0$ which treats assertions of the form $U \equiv V$, where U, V are "deterministic rational boolean series" and $U \equiv V$ means that U and V define the same deterministic, cf. language.

1.3.0.9. Strategies A *strategy* is a method allowing to *find* a proof of the fact that two configurations (or series) are equivalent. A basic step of all the usual strategies is to replace a pair

$$U \equiv V \tag{1}$$

by the finite set of all pairs obtained by letting one terminal letter $x \in X$ act on both sides:

$$\{U \odot x \equiv V \odot x,\ x \in X\}.$$

Such a step in the construction of a proof is called a T_A step.

In [78, p. 68], is introduced a second kind of step called a *replacement*, which introduces, from a pair (1), another finite set of pairs

$$U' \equiv V', \quad U'' \equiv V'' \tag{2}$$

such that

$$U \equiv V \Leftrightarrow (U' \equiv V' \text{ and } U'' \equiv V'').$$

In the case of finite-turn or real-time dpda's [4,55,78,79], the sequences of pairs (U_i, V_i) obtained by a suitable alternation of T_A steps and replacement steps are "smooth" in the sense that the lengths of both sides have similar variations.

We define here a kind of replacement called T_B (because it is also analogous with transformation T_B of [40]), which creates from two pairs

$$U \equiv V, \quad U' \equiv V' \tag{3}$$

a new pair

$$U'' \equiv V'' \tag{4}$$

such that

$$(U \equiv V \text{ and } U' \equiv V') \Leftrightarrow (U \equiv V \text{ and } U'' \equiv V'').$$

This transformation consists in replacing the pair $U' \equiv V'$ by the new pair $U'' \equiv V''$ *under the hypothesis* that $U \equiv V$. This type of replacements also leads to somewhat "smooth" sequences of pairs in an algebraic sense which is sketched below.

1.3.0.10. N-stacking sequences Let us call a $\mathscr{S}_{AB}$-tree the (possibly infinite) tree obtained from an initial true equation $U \equiv V$ by the above strategy. We show that this tree has "smooth branches" in the following sense: on every infinite branch $b=(x_i)_{0\leqslant i}$, there exists:

- a "short sequence" of nodes

$$(x_i)_{i_0\leqslant i\leqslant i_0+L_{d_0}+k_1}$$

(where L_{d_0}, k_1 are constants),

- a "small" generating set

$$\mathcal{G}_1 = \{U_i \mid 1\leqslant i\leqslant d_0\}$$

(where d_0 is a constant),

- and d_0 integers

$$\kappa_1, \kappa_2, \ldots, \kappa_{d_0} \in [i_0, i_0 + L_{d_0}]$$

such that all left- and right-hand sides of the equations at nodes $x_{\kappa_1}, x_{\kappa_2}, \ldots, x_{\kappa_{d_0}}$ belong to the deterministic space generated by $\mathcal{G}_1$ and have *small coefficients* on the generating set $\mathcal{G}_1$.

We are faced with a system of d_0 linear equations linking only d_0 different series. The "linear independence" idea (explained above) can then be applied to *cut* the branch b (we name T_C the precise tranformation allowing to cut a branch containing such a system of equations). At end, we obtain from the initial $\mathcal{S}_{AB}$-tree a *finite* $\mathcal{S}_{ABC}$-tree which is a proof in the formal system $\mathcal{D}_0$.

1.4. Organization of the paper

The overall organization of the paper should be clear from the table of contents. Let us give additional hints to the reader.

The main result, Theorem 87, is obtained in Section 9. In particular, Sections 10–13 are *not needed* to establish this theorem. The crucial part of the proof of Theorem 87 is Section 8.3, whose general idea is explained in the last paragraph of Section 1.3. The precise realization of this idea turns out to be complex and combines all the intermediate results of Sections 2.1–8.2. In some sense, everything in Sections 2.1–8.2 has been written in order to fit in some argument of Section 8.3.

Once this main result is established we give refinements, generalizations, examples, applications and perspectives (Sections 10–13):

- Section 10 is devoted to successive *simplifications* of the deduction system $\mathcal{D}_0$, so as to obtain a last system $\mathcal{D}_5$ which is quite simple and is still complete.
- In Section 11 we generalize the classical equivalence problem for "boolean dpda's" to "H-dpda's": these are dpda's whose transitions have outputs in some group H. We show that when H is abelian, the equivalence problem remains decidable (Theorem 177).
- Some examples of proofs in our formal systems are given in Section 12.

- We give Section 13 some immediate applications either to formal language theory or to other areas of theoretical computer science. We than sketch perspective of other applications or extensions of the methods and results.

2. Preliminaries

2.1. Pushdown automata

A *pushdown automaton* is a 6-tuple $\mathcal{M} = \langle X, Z, Q, \delta, q_0, z_0\rangle$ where X is the terminal alphabet, Z is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, z_0 is the initial stack-symbol and $\delta : QZ \times (X \cup \{\varepsilon\}) \to \mathcal{P}_f(QZ^*)$, is the transition mapping.

Let $q, q' \in Q, \omega, \omega' \in Z^*, z \in Z, f \in X^*$ and $a \in X \cup \{\varepsilon\}$; we note $(qz\omega, af) \mapsto_{\mathcal{M}} (q'\omega'\omega, f)$ if $q'\omega' \in \delta(qz, a)$. $\overset{*}{\mapsto}_{\mathcal{M}}$ is the reflexive and transitive closure of $\mapsto_{\mathcal{M}}$. For every $q\omega, q'\omega' \in QZ^*$ and $f \in X^*$, we note $q\omega \overset{f}{\mapsto}_{\mathcal{M}} q'\omega'$ iff $(q\omega, f) \overset{*}{\mapsto}_{\mathcal{M}} (q'\omega', \varepsilon)$. $\mathcal{M}$ is said *deterministic* iff, for every $z \in Z, q \in Q$:

$$\text{either } \mathrm{Card}(\delta(qz, \varepsilon)) = 1 \text{ and for every } x \in X,\ \mathrm{Card}(\delta(qz, x)) = 0, \tag{5}$$

$$\text{or } \mathrm{Card}(\delta(qz, \varepsilon)) = 0 \text{ and for every } x \in X,\ \mathrm{Card}(\delta(qz, x)) \leqslant 1. \tag{6}$$

$\mathcal{M}$ is said *real time* iff, for every $qz \in QZ$, $\mathrm{Card}(\delta(qz, \varepsilon)) = 0$. A dpda $\mathcal{M}$ is said *normalized* iff, for every $qz \in QZ$, $x \in X$:

$$q'\omega' \in \delta(qz, x) \Rightarrow |\omega'| \leqslant 2 \quad \text{and} \quad q'\omega' \in \delta(qz, \varepsilon) \Rightarrow |\omega'| = 0. \tag{7}$$

Given some finite set $F \subseteq QZ^*$ of configurations, the *language recognized by $\mathcal{M}$ with final configurations F* is defined by $\mathrm{L}(\mathcal{M}, F) = \{w \in X^* \mid \exists c \in F, q_0 z_0 \overset{w}{\longrightarrow}_{\mathcal{M}} c\}$.

2.2. Deterministic context-free grammars

Let $\mathcal{M}$ be some deterministic pushdown automaton (we suppose here that $\mathcal{M}$ is normalized). The *variable* alphabet $V_{\mathcal{M}}$ associated to $\mathcal{M}$ is defined as

$$V_{\mathcal{M}} = \{[p, z, q] \mid p, q \in Q, z \in Z\}.$$

The *context-free* grammar $G_{\mathcal{M}}$ associated to $\mathcal{M}$ is then

$$G_{\mathcal{M}} = \langle X, V, P\rangle,$$

where $V = V_{\mathcal{M}}$,

P is the set of all the pairs of one of the following forms:

$$([p, z, q], x[p', z_1, p''][p'', z_2, q]), \tag{8}$$

where $p, q, p', p'' \in Q, x \in X, p'z_1z_2 \in \delta(pz, x)$

$$([p, z, q], x[p', z', q]), \tag{9}$$

where $p, q, p' \in Q, x \in X, p'z' \in \delta(pz, x)$

$$([p, z, q], a), \tag{10}$$

where $p, q \in Q, a \in X \cup \{\varepsilon\}, q \in \delta(pz, a)$. $G_{\mathscr{M}}$ is a *strict-deterministic* grammar. (A general theory of this class of grammars is exposed in [33] and used in [34].)

We call *mode* every element of $QZ \cup \{\varepsilon\}$. For every $q \in Q$, $z \in Z$, qz is said ε-bound (resp. ε-free) iff condition (5) (resp. condition (6)) in the above definition of deterministic automata is realized. The mode ε is said ε-free. We define a mapping $\mu : V^* \to QZ \cup \{\varepsilon\}$ by

$$\mu(\varepsilon) = \varepsilon \quad \text{and} \quad \mu([p, z, q] \cdot \beta) = pz$$

for every $p, q \in Q$, $z \in Z$, $\beta \in V^*$. For every $w \in V^*$ we call $\mu(w)$ the *mode* of the word w.

For technical reasons (which will be made clear in Section 7), we suppose that Z contains a special symbol e such that, for every $q \in Q, \delta(qe, \varepsilon) = \{q\}$ and $\mathrm{im}(\delta) \subseteq \mathscr{P}_f(Q(Z - \{e\})^*)$. Equivalently,

$$\forall q \in Q, \quad ([q, e, q], \varepsilon) \in P \tag{11}$$

and

$$\forall (v, w) \in P, \quad w \in (V - \{[p, e, q] \mid p, q \in Q\})^*. \tag{12}$$

2.3. Free monoids acting on semi-rings

2.3.1. Semi-ring $\mathsf{K}\langle\langle W \rangle\rangle$

Let us consider a semi-ring $(\mathsf{K}, +, \cdot, 0_{\mathsf{K}}, 1_{\mathsf{K}})$ and an alphabet W. By $(\mathsf{K}\langle\langle W \rangle\rangle, +, \cdot, \emptyset, \varepsilon)$ we denote the semi-ring of *series* over the set of non-commutative undeterminates W, with coefficients in K:

the set $\mathsf{K}\langle\langle W \rangle\rangle$ is defined as K^{W^*}; the sum and product are defined by $\forall S, T \in \mathsf{K}^{W^*}$, $w \in W^*$,

$$(S + T)(w) = S(w) + T(w); \quad (S \cdot T)(w) = \sum_{w_1 \cdot w_2 = w} S(w_1) \cdot T(w_2).$$

Each word $w \in W^*$ can be identified with the element of K^{W^*} mapping the word w on 1_{K} and every other word $w' \neq w$ on 0_{K}; each scallar $k \in \mathsf{K}$ can be identified with the element of K^{W^*} mapping the word ε on k and every word $w' \neq \varepsilon$ on 0_{K}. A family of series $(S_i)_{i \in I}$ is said *locally finite* iff, for every $w \in W^*$, the set $\{i \in I \mid (S_i)(w) \neq 0\}$ is finite. The sum

$$\sum_{i \in I} S_i$$

of a locally finite family is defined as usual. Every series $S \in \mathsf{K}\langle\langle W \rangle\rangle$ can then be written in a unique way as

$$S = \sum_{w \in W^*} S_w \cdot w,$$

where, for every $w \in W^*$, $S_w \in \mathsf{K}$.

We recall that for every $S \in \mathsf{K}\langle\langle W \rangle\rangle$ such that $S_\varepsilon = 0$, $(S^n)_{n \in \mathbb{N}}$ is locally finite and S^* is the series defined by

$$S^* = \sum_{n \in \mathbb{N}} S^n. \tag{13}$$

The semi-rings K considered in this paper[8] are naturally endowed with a notion of sum

$$\sum_{i \in I} k_i$$

for every denumerable family $(k_i)_{i \in I}$ of elements of K which extends the notion of sum for locally finite families. Given two alphabets W, W' and two semi-rings K, K', a map $\psi : \mathsf{K}\langle\langle W \rangle\rangle \to \mathsf{K}'\langle\langle W' \rangle\rangle$ is said *σ-additive* iff it fulfills: for every denumerable family $(S_i)_{i \in I}$ of elements of $\mathsf{K}\langle\langle W \rangle\rangle$,

$$\psi\left(\sum_{i \in I} S_i\right) = \sum_{i \in I} \psi(S_i). \tag{14}$$

Let us denote by $\mathrm{C}(\mathsf{K}')$ the center of K', i.e. the set $\{k \in \mathsf{K}', \forall k' \in \mathsf{K}', k \cdot k' = k' \cdot k\}$. The following property of $\mathsf{K}\langle\langle W \rangle\rangle$ will be used in the sequel: for every semi-ring K', alphabet W', maps $\psi_{\mathsf{K}} : \mathsf{K} \to \mathsf{K}'$ which is a semi-ring homomorphism and $\psi_{\mathsf{K}}(\mathsf{K}) \subseteq \mathrm{C}(\mathsf{K}')$, $\psi_W : W \to \mathsf{K}'\langle\langle W' \rangle\rangle$, there exists a unique σ-additive semi-ring homomorphism

$$\tilde{\psi} : \mathsf{K}\langle\langle W \rangle\rangle \to \mathsf{K}'\langle\langle W' \rangle\rangle \text{ such that, } \forall k \in \mathsf{K}, \tilde{\psi}(k) = \psi_{\mathsf{K}}(k), \forall v \in W, \tilde{\psi}(v) = \psi_W(v) \tag{15}$$

A map $\psi : \mathsf{K}\langle\langle W \rangle\rangle \to \mathsf{K}\langle\langle W' \rangle\rangle$ which is a semi-ring homomorphism, a σ-additive map and which fixes every element of K, will be called a *substitution*. The *support* of S is the language

$$\mathrm{supp}(S) = \{w \in W^* \mid S_w \neq 0_{\mathsf{K}}\}.$$

A series S such that $\mathrm{supp}(S)$ is finite is called a *polynomial*. By $\mathsf{K}\langle W \rangle$ we denote the set of polynomials with coefficients in K and undeterminates in W. It is a sub-semi-ring of $\mathsf{K}\langle\langle W \rangle\rangle$.

[8] i.e. $\mathsf{K} = \mathsf{B}$ or $\mathsf{K} = \mathsf{B}\langle\langle V \rangle\rangle$ in Sections 2–10, $\mathsf{K} = \mathsf{B}\langle\langle H \rangle\rangle$ or $\mathsf{K} = \mathsf{B}\langle\langle H \rangle\rangle\langle\langle V \rangle\rangle$, where H is a group, in Section 11.

2.3.2. Semi-ring $\mathsf{B}\langle\langle W\rangle\rangle$

Let $(\mathsf{B}, +, \cdot, 0, 1)$ where $\mathsf{B} = \{0, 1\}$ denote the semi-ring of "booleans". In this particular case (which is the only case considered in Sections 2–10), we sometimes identify a series S with its support. The usual ordering $\leqslant$ on B extends to $\mathsf{B}\langle\langle W\rangle\rangle$ by

$$S \leqslant S' \text{ iff } \forall w \in W^*, S_w \leqslant S'_w.$$

2.3.3. Actions of monoids

Given a semi-ring $(\mathsf{S}, +, \cdot, 0, 1)$ and a monoid $(\mathsf{M}, \cdot, 1_M)$, a map $\circ : \mathsf{S} \times \mathsf{M} \to \mathsf{S}$ is called a *right-action* of the monoid M over the semi-ring S iff, for every $S, T \in \mathsf{S}, m, m' \in \mathsf{M}$:

$$0 \circ m = 0, \;\; S \circ 1_M = S, \;\; (S + T) \circ m = (S \circ m) + (T \circ m)$$

and

$$S \circ (m \cdot m') = (S \circ m) \circ m'. \tag{16}$$

A right-action $\circ$ is said to be a σ-right-action if it fulfills the additional property that, for every denumerable family $(S_i)_{i \in I}$ of elements of S and $m \in \mathsf{M}$:

$$\left(\sum_{i \in I} S_i\right) \circ m = \sum_{i \in I} (S_i \circ m). \tag{17}$$

2.3.4. The action of W^* *on* $\mathsf{B}\langle\langle W\rangle\rangle$

We recall the following classical σ-right-action $\bullet$ of the monoid W^* over the semi-ring $\mathsf{B}\langle\langle W\rangle\rangle$: for all $S \in \mathsf{B}\langle\langle W\rangle\rangle, u, w \in W^*$

$$(S \bullet u)_w = S_{u \cdot w}$$

(i.e. $S \bullet u$ is the *left-quotient* of S by u, or the *residual* of S by u). For every $S \in \mathsf{B}\langle\langle W\rangle\rangle$ we denote by $\mathsf{Q}(S)$ the set of residuals of S:

$$\mathsf{Q}(S) = \{S \bullet u \mid u \in W^*\}.$$

We recall that S is said *rational* iff the set $\mathsf{Q}(S)$ is *finite*. We define the *norm* of a series $S \in \mathsf{B}\langle\langle W\rangle\rangle$, denoted $\|S\|$ by

$$\|S\| = \mathrm{Card}(\mathsf{Q}(S)) \in \mathbb{N} \cup \{\infty\}.$$

2.3.5. The action of X^* *on* $\mathsf{B}\langle\langle V\rangle\rangle$

Let us fix now a deterministic (normalized) pda $\mathcal{M}$ and consider the associated grammar G. We define a σ-right-action $\otimes$ of the monoid $(X \cup \{e\})^*$ over the semi-ring $\mathsf{B}\langle\langle V\rangle\rangle$ by for every $p, q \in Q, \; z \in Z, \; \beta \in V^*, \; x \in X$

$$[p, z, q] \cdot \beta \otimes x = \left(\sum_{([p,z,q],m) \in P_{\mathcal{M}}} m \bullet x\right) \cdot \beta, \tag{18}$$

$$[p,z,q]\cdot\beta\otimes e=\beta \quad \text{iff } ([p,z,q],\varepsilon)\in P_{\mathscr{M}}, \tag{19}$$

$$[p,z,q]\cdot\beta\otimes e=\emptyset \quad \text{iff } (\{[p,z,q]\}\times V^*)\cap P_{\mathscr{M}}=\emptyset, \tag{20}$$

$$\varepsilon\otimes x=\emptyset,\ \varepsilon\otimes e=\emptyset. \tag{21}$$

A series $S\in \mathsf{B}\langle\langle V\rangle\rangle$ is said *ε-free* iff $\forall w\in V^*,\ S_w=1\Rightarrow \mu(w)$ is ε-free. We define the map

$$\rho_\varepsilon:\mathsf{B}\langle\langle V\rangle\rangle\to\mathsf{B}\langle\langle V\rangle\rangle$$

as the unique σ-additive map such that

$$\rho_\varepsilon(\emptyset)=\emptyset,\quad \rho_\varepsilon(\varepsilon)=\varepsilon$$

and for every $p\in Q,\ z\in Z,\ q\in Q,\ \beta\in V^*$,

$$\rho_\varepsilon([p,z,q]\cdot\beta)=\rho_\varepsilon(([p,z,q]\otimes e)\cdot\beta) \text{ if } pz \text{ is ε-bound and,}$$
$$\rho_\varepsilon([p,z,q]\cdot\beta)=[p,z,q]\cdot\beta \text{ if } pz \text{ is ε-free.}$$

The above definition is sound because, by hypothesis (7), every $[p,z,q]\otimes e$ is either the unit series ε or the empty series ∅. One can notice that for every $w\in V^*$, $\rho_\varepsilon(w)\in V^*\cup\{\emptyset\}$. We call ρ_ε the ε-reduction map. We then define $\odot$ as the unique σ-right-action of the monoid X^* over the semi-ring $\mathsf{B}\langle\langle V\rangle\rangle$ such that: for every $S\in\mathsf{B}\langle\langle V\rangle\rangle,\ x\in X$,

$$S\odot x=\rho_\varepsilon(\rho_\varepsilon(S)\otimes x).$$

One can notice that if $u\neq\varepsilon$, then $S\odot u$ is ε-free. Let us consider the unique substitution $\varphi:\mathsf{B}\langle\langle V\rangle\rangle\to\mathsf{B}\langle\langle X\rangle\rangle$ fulfilling: for every $p,q\in Q,\ z\in Z$,

$$\varphi([p,z,q])=\{u\in X^*\mid [p,z,q]\odot u=\varepsilon\}$$

(in other words, φ maps every subset $L\subseteq V^*$ on the language generated by the grammar G from the set of axioms L).

Lemma 1. *For every* $S\in\mathsf{B}\langle\langle V\rangle\rangle,\ u\in X^*$,
(1) $\varphi(S)=\varphi(\rho_\varepsilon(S))$,
(2) $\varphi(S\odot u)=\varphi(S)\bullet u$ (*i.e. φ is a morphism of right-actions*).

Proof. Let $p,q\in Q,\ z\in Z,\ \beta\in V^*,\ X\in X$. One can check on formulas (18–21) that

- if $[p,z,q]$ is ε-bound, then

$$\varphi(([p,z,q]\cdot\beta)\otimes e)=\varphi([p,z,q]\cdot\beta)$$

- if $[p,z,q]$ is ε-free, then

$$\varphi(([p,z,q]\cdot\beta)\otimes x)=\varphi([p,z,q]\cdot\beta)\bullet x.$$

By induction on $|w|$, it follows that, $\forall w \in V^*$,

$$\varphi(\rho_\varepsilon(w)) = \varphi(w), \quad \varphi(w \odot x) = \varphi(w) \bullet x.$$

By σ-additivity of φ and induction on $|u|$, the lemma follows. □

We denote by $\equiv$ the kernel of φ, i.e. for every $S, T \in \mathsf{B}\langle\langle V \rangle\rangle$,

$$S \equiv T \Leftrightarrow \varphi(S) = \varphi(T).$$

3. Series and matrices

3.1. Deterministic series and matrices

We introduce here a notion of *deterministic* series which, in the case of the alphabet V associated to a dpda $\mathcal{M}$, generalizes the classical notion of *configuration* of $\mathcal{M}$. The main advantage of this notion is that, unlike for configurations, it is possible to define *nice algebraic operations* on these series: a product, a partial sum and a kind of star operation.

Let us consider a pair $(W, \smile)$ where W is an alphabet and $\smile$ is an equivalence relation over W. We call $(W, \smile)$ a *structured* alphabet. The two examples we have in mind are:

- the case where $W = V$, the variable alphabet associated to $\mathcal{M}$ and $[p, A, q] \smile [p', A', q']$ iff $p = p'$ and $A = A'$ (see [33, Proof of Lemma 11.5.2])
- the case where $W = X$, the terminal alphabet of $\mathcal{M}$ and $x \smile y$ holds for every $x, y \in X$ (see [33, Proof of Lemma 11.5.2]).

3.1.1. Definitions

Definition 2. Let $S \in \mathsf{B}\langle\langle W \rangle\rangle$. S is said *left-deterministic* iff either
(1) $S = \emptyset$ or
(2) $S = \varepsilon$ or
(3) $\exists w_0 \in W^*$, $S_{w_0} = 1$ and $\forall w, w' \in W^*$,

$$S_w = S_{w'} = 1 \Rightarrow [\exists A, A' \in W, w_1, w_1' \in W^*, A \smile A', w = A \cdot w_1 \text{ and } w' = A' \cdot w_1'].$$

A left-deterministic series S is said to have the type $\emptyset$ (resp. ε, $[A]_\smile$) if case (1) (resp. (2), (3)) occurs.

Definition 3. Let $S \in \mathsf{B}\langle\langle W \rangle\rangle$. S is said *deterministic* iff, for every $u \in W^*$, $S \bullet u$ is left-deterministic.

This notion is the straighforward extension to the infinite case of the notion of (finite) *set of associates* defined in [34, Definition 3.2, p. 188].

We denote by $\mathsf{DB}\langle\langle W\rangle\rangle$ the subset of Deterministic Boolean series over W. Let us denote by $\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$ the set of (n,m)-matrices with entries in the semi-ring $\mathsf{B}\langle\langle W\rangle\rangle$.

Definition 4. Let $m\in\mathbb{N}$, $S\in\mathsf{B}_{1,m}\langle\langle W\rangle\rangle : S=(S_1,\dots,S_m)$. S is said *left-deterministic* iff either
(1) $\forall i\in[1,m],\ S_i=\emptyset$ or
(2) $\exists i_0\in[1,m], S_{i_0}=\varepsilon$ and $\forall i\neq i_0,\ S_i=\emptyset$ or
(3) $\exists i_0\in[1,m], S_{i_0}\neq\emptyset$ and, $\forall w,w'\in W^*$, $\forall i,j\in[1,m]$,
$(S_i)_w=(S_j)_{w'}=1\Rightarrow[\exists A,A'\in W, w_1,w_1'\in V^*, A\smile A', w=A\cdot w_1$ and $w'=A'\cdot w_1']$.

A left-deterministic row-vector S is said to have the type $\emptyset$ (resp. (ε,i_0), $[A]_\smile$) if case (1) (resp. (2), (3)) occurs.

Notice that $S=(S_1,\dots,S_m)$ is left-deterministic iff

$$[\forall i,j\in[1,m]^2,\ \mathrm{supp}(S_i)\cap\mathrm{supp}(S_j)\neq\emptyset\Rightarrow i=j] \text{ and}$$
$$\left[\sum_{j=1}^{m} S_j \text{ is left-deterministic}\right].$$

The right-action $\bullet$ on $\mathsf{B}\langle\langle W\rangle\rangle$ is extended componentwise to $\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$: for every $S=(s_{i,j})$, $u\in W^*$, the matrix $T=S\bullet u$ is defined by

$$t_{i,j}=s_{i,j}\bullet u.$$

The ordering $\leqslant$ on B is also extended componentwise to $\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$.

Definition 5. Let $S\in\mathsf{B}_{1,m}\langle\langle W\rangle\rangle$. S is said *deterministic* iff, for every $u\in W^*$, $S\bullet u$ is left-deterministic.

We denote by $\mathsf{DB}_{1,m}\langle\langle W\rangle\rangle$ the subset of deterministic row-vectors of dimension m over $\mathsf{B}\langle\langle W\rangle\rangle$.

Definition 6. Let $S\in\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$. S is said *deterministic* (resp. left-deterministic) iff, for every $i\in[1,n]$, $S_{i,*}$ is a deterministic (resp. left-deterministic) row-vector.

Let us notice first some easy facts about deterministic series.

Fact 7. *Let $S\in\mathsf{DB}\langle\langle W\rangle\rangle$. For every $T\in\mathsf{B}\langle\langle W\rangle\rangle, u\in W^*$*
(1) $T\leqslant S\Rightarrow T\in\mathsf{DB}\langle\langle W\rangle\rangle$,
(2) $S\bullet u\in\mathsf{DB}\langle\langle W\rangle\rangle$.

3.1.2. Residuals

Lemma 8. *Let $S\in\mathsf{DB}\langle\langle W\rangle\rangle$, $T\in\mathsf{B}\langle\langle W\rangle\rangle$, $u\in W^*$. If $S\bullet u\neq\emptyset$ then $(S\cdot T)\bullet u=(S\bullet u)\cdot T$.*

Proof. Let $S \in \mathsf{DB}\langle\langle W\rangle\rangle$, $T \in \mathsf{B}\langle\langle W\rangle\rangle$, $u \in W^*$, such that $S \bullet u \neq \emptyset$. Let $u', u'' \in W^*$ such that $u = u' \cdot u'', u'' \neq \varepsilon$ and let $w \in \mathrm{supp}(S)$. If $w \bullet u' = \varepsilon$ then $S \bullet u' = \varepsilon$ (because $S \bullet u'$ is left-deterministic), hence $S \bullet u = \varepsilon \bullet u'' = \emptyset$, which would contradict the hypothesis. It follows that

$$\forall u' \prec u,\ \forall w \in \mathrm{supp}(S), \quad w \bullet u' \neq \varepsilon.$$

Hence,

$$\forall w_1 \in \mathrm{supp}(S), \forall w_2 \in \mathrm{supp}(T), \quad (w_1 \cdot w_2) \bullet u = (w_1 \bullet u) \cdot w_2.$$

This proves that $(S \cdot T) \bullet u = (S \bullet u) \cdot T$. □

Lemma 9. *Let $S \in \mathsf{DB}\langle\langle W\rangle\rangle$, $T \in \mathsf{B}\langle\langle W\rangle\rangle$, $u \in W^*$ and $U = S \cdot T$. Exactly one of the following cases is true:*

(1) $S \bullet u \neq \emptyset$;
in this case $U \bullet u = (S \bullet u) \cdot T$.
(2) $S \bullet u = \emptyset$, $\exists u', u''$, $u = u' \cdot u'', S \bullet u' = \varepsilon$;
in this case $U \bullet u = T \bullet u''$.
(3) $S \bullet u = \emptyset, \forall u' \leqslant u, S \bullet u' \neq \varepsilon$;
in this case $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

Proof. Clearly, one of the hypotheses (1)–(3) must occur. Let us examine each one of these cases.

In case (1), by Lemma 8, $U \bullet u = (S \bullet u) \cdot T$.

In case (2), $U \bullet u = (U \bullet u') \bullet u''$ and by case (1), $U \bullet u' = (S \bullet u') \cdot T$. It follows that $U \bullet u = T \bullet u''$.

In case (3), if $S = \emptyset$, the conclusion of lemma is clearly true. Let us suppose now that $S \neq \emptyset$ and let $u' \prec u$ be the maximum prefix of u such that $S \bullet u' \neq \emptyset$. Then, there exist some $A \in W, u'' \in W^*$ such that $u = u' \cdot A \cdot u''$ and there exist some $B_1, \ldots, B_q \in W$, $S_1, \ldots, S_q \in \mathsf{B}\langle\langle W\rangle\rangle - \{\emptyset\}$ such that $S \bullet u' = \sum_{1 \leqslant i \leqslant q} B_q \cdot S_q$ and $B_1 \smile \cdots \smile B_i \smile \cdots \smile B_q$ (because $S \bullet u'$ is left-deterministic). By maximality of u', A does not belong to $\{B_1, \ldots, B_q\}$, hence

$$U \bullet u = \left(\left(\sum_{1 \leqslant i \leqslant q} B_i \cdot S_i \cdot T\right) \bullet A\right) \bullet u'' = \emptyset \bullet u'' = \emptyset. \qquad \square$$

Lemma 10. *Let $S \in \mathsf{DB}_{1,m}\langle\langle W\rangle\rangle$, $T \in \mathsf{B}_{m,1}\langle\langle W\rangle\rangle$, $u \in W^*$ and $U = S \cdot T$. Exactly one of the following cases is true*:

(1) $\exists j, S_j \bullet u \notin \{\emptyset, \varepsilon\}$;
in the case $U \bullet u = (S \bullet u) \cdot T$.
(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', S_{j_0} \bullet u' = \varepsilon$;
in this case $U \bullet u' = T_{j_0} \bullet u''$.
(3) $\forall j, S_j \bullet u = \emptyset, \forall u' \leqslant u, S_j \bullet u' \neq \varepsilon$;
in this case $U \bullet u = \emptyset = (S \bullet u) \cdot T$.

Proof. Let us note $S=(S_j)_{1\leqslant j\leqslant m}, T=(T_j)_{1\leqslant j\leqslant m}$. Clearly, one of hypotheses (1)–(3) must occur. Let us examine each one of these cases.
In case (1), every 3-tuple (S_j, T_j, u) fulfills case (1) or (3) of Lemma 9, hence $(S_j \cdot T_j)\bullet u=(S_j\bullet u)\cdot T_j$. Hence,

$$U\bullet u=\sum_{1\leqslant j\leqslant m}(S_j\cdot T_j)\bullet u=\sum_{1\leqslant j\leqslant m}(S_j\bullet u)\cdot T_j=(S\bullet u)\cdot T.$$

In case (2), $S\bullet u'$ must be left-deterministic of type (ε, j_0), hence $\forall j\neq j_0, S_j\bullet u'=\emptyset$. It follows that

$$U\bullet u=T_{j_0}\bullet u''.$$

In case (3), every 3-tuple (S_j, T_j, u) fulfills case (3) of Lemma 9, hence $(S_j\cdot T_j)\bullet u=\emptyset =(S_j\bullet u)\cdot T_j$. It follows that

$$U\bullet u=\emptyset=(S\bullet u)\cdot T. \qquad \square$$

Lemma 11. *Let* $S\in \mathsf{DB}_{1,m}\langle\langle W\rangle\rangle, T\in \mathsf{B}_{m,s}\langle\langle W\rangle\rangle, u\in W^*$ *and* $U=S\cdot T$. *Exactly one of the following cases is true*:
(1) $\exists j,\ S_j\bullet u\notin\{\emptyset,\varepsilon\}$
in this case $U\bullet u=(S\bullet u)\cdot T$.
(2) $\exists j_0,\ \exists u', u'', u=u'\cdot u'', S_{j_0}\bullet u'=\varepsilon$;
in this case $U\bullet u=T_{j_0,*}\bullet u''$.
(3) $\forall j,\ \forall u'\leqslant u, S_j\bullet u=\emptyset, S_j\bullet u'\neq\varepsilon$;
in this case $U\bullet u=\emptyset^s=(S\bullet u)\cdot T$.

Proof. Let us notice that for every $k\in[1,s]$:

$$U_k=S\cdot T_{*,k} \tag{22}$$

and that the hypothesis of the 3 cases considered in Lemma 10 depend on the vector S and the word u only (but not on the integer $k\in[1,s]$). In case (1), by Lemma 10, $\forall k\in[1,s]$

$$U_k\bullet u=(S\bullet u)\cdot T_{*,k},$$

hence $U\bullet u=(S\bullet u)\cdot T$. Cases 2 and 3 can be treated in the same way. $\square$

Lemma 12. *For every* $S\in\mathsf{B}_{n,m}\langle\langle W\rangle\rangle, T\in\mathsf{B}_{m,s}\langle\langle W\rangle\rangle$, *if* S *and* T *are both left-deterministic, then* $S\cdot T$ *is left-deterministic.*

Lemma 13. *For every* $S\in\mathsf{DB}_{n,m}\langle\langle W\rangle\rangle, T\in\mathsf{DB}_{m,s}\langle\langle W\rangle\rangle, S\cdot T\in\mathsf{DB}_{n,s}\langle\langle W\rangle\rangle$.
(This statement appeared first in [34, Lemma 3.5*, p. 190] for* $n=s=1$.*)*

Proof. As the notion of deterministic matrix is defined row by row, it is sufficient to prove this lemma in the particular case where $n=1$. Let us note $U=S\cdot T$. Let $u\in W^*$.

Let us show that $U \bullet u$ is left-deterministic. Let us consider every one of the 3 cases considered in Lemma 11. In case (1) or (3),

$$U \bullet u = (S \bullet u) \cdot T$$

and in case (2),

$$U \bullet u = T \bullet u''.$$

In both cases, by Lemma 12, $U \bullet u$ is left-deterministic. □

3.1.3. Rational matrices, norm

Let us generalize the definition of *rationality* of series in $\mathsf{B}\langle\langle W \rangle\rangle$ to matrices. Given $M \in \mathsf{B}_{n,m}\langle\langle W \rangle\rangle$ we denote by $\mathsf{Q}(M)$ the set of *residuals* of M:

$$\mathsf{Q}(M) = \{M \bullet u \mid u \in W^*\}.$$

Similarly, we denote by $\mathsf{Q}_r(M)$ the set of *row-residuals* of M:

$$\mathsf{Q}_r(M) = \bigcup_{1 \leqslant i \leqslant n} \mathsf{Q}(M_{i,*}).$$

M is said *rational* iff the set $\mathsf{Q}(M)$ is finite. One can check that it is equivalent to the property that every coefficient $M_{i,j}$ is rational, or to the property that $\mathsf{Q}_r(M)$ is finite. We denote by $\mathsf{RB}_{n,m}\langle\langle W \rangle\rangle$ (resp. $\mathsf{DRB}_{n,m}\langle\langle W \rangle\rangle$) the set of rational (resp. deterministic, rational) matrices over $\mathsf{B}\langle\langle W \rangle\rangle$. For every $M \in \mathsf{RB}_{n,m}\langle\langle W \rangle\rangle$, we define the norm of M as

$$\|M\| = \mathrm{Card}(\mathsf{Q}_r(M)).$$

Lemma 14. *Let* $A \in \mathsf{DB}_{n,m}\langle\langle W \rangle\rangle, B \in \mathsf{B}_{m,s}\langle\langle W \rangle\rangle$. *Then* $\|A \cdot B\| \leqslant \|A\| + \|B\|$.

Proof. Let $A = (a_{i,k}), B = (b_{k,j}), C = A \cdot B, C = (c_{i,j})$. Let $1 \leqslant i \leqslant n, H \in \mathsf{Q}(C_{i,*})$. Let $u \in W^*$ such that

$$H = C_{i,*} \bullet u = (A_{i,*} \cdot B) \bullet u.$$

We apply Lemma 11 to $S = A_{i,*}$ and $T = B$. If case (1) or (3) of Lemma 11 is realized then

$$H = (A_{i,*} \bullet u) \cdot B.$$

If case (2) of Lemma 11 is realized then

$$H = B_{k_0,*} \bullet u''.$$

The number of residuals H obtained by case (1) is less or equal than $\|A\|$ and the number obtained by case (2) is less or equal than $\|B\|$. This proves the inequality. □

3.1.4. $W = V$

Let $(W, \smile)$ be the structured alphabet $(V, \smile)$ associated with $\mathcal{M}$ and let us consider a bijective numbering of the elements of Q: $(q_1, q_2, \ldots, q_{n_Q})$. Some particular "vectorial" notions turn out to be useful:

- we define a *Q-series* as a family $S = (S_q)_{q \in Q}$ such that the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ is deterministic,
- we define a *Q-form* as a family $\Phi = (\Phi_q)_{q \in Q}$ of deterministic series; more generally a *$Q - \lambda$-form* (where $\lambda \in \mathbb{N} - \{0\}$) is a family of deterministic row-vectors: $\Phi = (\Phi_q)_{q \in Q}$ with $\Phi_q \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$ for every $q \in Q$.

Given a Q-series S and a Q-λ-form Φ, their Q-product $S * \Phi$ is the deterministic row-vector defined by

$$S * \Phi = \sum_{q \in Q} S_q \cdot \Phi_q.$$

Given the above ordering of the elements of Q, one can identify the Q-series $(S_q)_{q \in Q}$ with the row-vector $(S_{q_1}, S_{q_2}, \ldots, S_{q_{n_Q}})$ and the Q-λ-form $(\Phi_q)_{q \in Q}$ with the n_Q-λ-matrix:

$$\begin{pmatrix} \Phi_{q_1} \\ \vdots \\ \Phi_{q_j} \\ \vdots \\ \Phi_{q_{n_Q}} \end{pmatrix}$$

The Q-product appears then to be just the ordinary product of matrices.

Let us define here handful notations for some particular row-vectors or Q-series. Let us use the *Kronecker symbol* $\delta_{i,j}$ meaning ε if $i = j$ and $\emptyset$ if $i \neq j$. For every $1 \leqslant n, 1 \leqslant i \leqslant n$, we define the row-vector ε_i^n as

$$\varepsilon_i^n = (\varepsilon_{i,j}^n)_{1 \leqslant j \leqslant n} \quad \text{where } \forall j, \varepsilon_{i,j}^n = \delta_{i,j}.$$

We call *unit row-vector* any vector of the form ε_i^n.
For every $1 \leqslant n, 1 \leqslant m$, we denote by $\emptyset^n \in \mathsf{DB}_{1,n}\langle\langle V \rangle\rangle$ the row-vector:

$$\emptyset^n = (\emptyset, \ldots, \emptyset)$$

and we denote by $\emptyset_m^n \in \mathsf{DB}_{m,n}\langle\langle V \rangle\rangle$ the matrix:

$$\begin{pmatrix} \emptyset^n \\ \vdots \\ \emptyset^n \\ \vdots \\ \emptyset^n \end{pmatrix}$$

For every $\omega \in Z^*$, $p, q \in Q$, $[p\omega q]$ is the deterministic series defined inductively by

$$[p\varepsilon q] = \emptyset \quad \text{if } p \neq q, \qquad [p\varepsilon q] = \varepsilon \quad \text{if } p = q,$$

$$[p\omega q] = \sum_{r\in Q} [p, A, r] \cdot [r\omega' q] \quad \text{if } \omega = A \cdot \omega' \text{ for some } A \in Z, \omega' \in Z^*.$$

(In particular, $[pAq] = [p, A, q]$.)
By $[p\omega]$ we denote the Q-series:

$$[p\omega] = ([p\omega q])_{q\in Q}.$$

(In particular $[q_i] = \varepsilon_i^{n_Q}$.) These Q-series represent faithfully the configurations of $\mathscr{M}$ in the sense that, for every ε-free configurations $q\omega, q'\omega'$ and word $f \in X^*$,

$$q\omega \xrightarrow{f}_{\mathscr{M}} q'\omega' \quad \text{iff } [q\omega] \odot f = [q'\omega']. \tag{23}$$

By $[\omega]$ we denote the Q–Q-matrix:

$$[\omega] = ([p\omega q])_{p\in Q, q\in Q}.$$

Here also we identify $[\omega]$ with the matrix $([q_i \omega q_j])_{1\leqslant i\leqslant n_Q, 1\leqslant j\leqslant n_Q} \in \mathsf{DB}_{n_Q,n_Q}\langle\langle V\rangle\rangle$.

Let us consider the componentwise extension of φ to row-vectors. For every $\lambda \in \mathbb{N} - \{0\}, S \in \mathsf{B}_{1,\lambda}\langle\langle V\rangle\rangle$ we define $\varphi(S) \in \mathsf{B}_{1,\lambda}\langle\langle X\rangle\rangle$ by

$$\forall j \in [1, \lambda], \quad \varphi(S)_{1,j} = \varphi(S_{1,j}).$$

We then extend $\equiv$ to $\bigcup_{1\leqslant\lambda} \mathsf{B}_{1,\lambda}\langle\langle V\rangle\rangle$ by: for every $\lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{B}_{1,\lambda}\langle\langle V\rangle\rangle$

$$S \equiv S' \Leftrightarrow \varphi(S) = \varphi(S'). \tag{24}$$

The next lemmas relate the mapping ρ_ε and right-action $\odot$ with the right-action $\bullet$.

Lemma 15. *Let* $\lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle$:

(1) *there exists* $v \in V^*$ *such that* $\rho_\varepsilon(S) = S \bullet v$,
(2) $\rho_\varepsilon(S) \equiv S$.

Proof. We treat first the case where $\lambda = 1$, i.e. S is a series.

If for every $w \in \mathrm{supp}(S), \rho_\varepsilon(w) = \emptyset$, then $\rho_\varepsilon(S) = \emptyset$, which is a residual of S, hence point (1) of the lemma is true. Moreover, in this case every $w \in \mathrm{supp}(S)$ contains a letter $[p, z, q]$ which is ε-bound and such that $[p, z, q] \otimes e = \emptyset$, hence $S \equiv \emptyset = \rho_\varepsilon(S)$, which establishes point (2) of the lemma.

Let us suppose now that there exists some $w_0 \in \mathrm{supp}(S)$ such that $\rho_\varepsilon(w_0) = w' \in V^*$. Then $w_0 = [p_1, z_1, q_1] \cdots [p_n, z_n, q_n] \cdot w'$, where $n \geqslant 0$ and for every $i \in [1, n]$, $[p_i, z_i, q_i] \otimes e = \varepsilon$. Let us set $v = [p_1, z_1, q_1] \cdots [p_n, z_n, q_n]$. We consider the set of words

$$D(v) = \{v' \cdot [p_{j+1}, z_{j+1}, q'_{j+1}],\ 0 \leqslant j \leqslant n-1,\ q'_{j+1} \in Q, v' = v(j), q'_{j+1} \neq q_{j+1}\},$$

where $v(j)$ denotes the prefix of v with length i.

We set

$$S' = \sum_{w \in D(v)} w \cdot (S \bullet w). \tag{25}$$

It is clear that, as S is deterministic:

$$S = v \cdot (S \bullet v) + S'. \tag{26}$$

Moreover, one can check that, for every $w \in D(v)$, $\rho_\varepsilon(w) = \emptyset$ (because the letters $[p_{j+1}, z_{j+1}, q'_{j+1}]$ fulfill $\rho_\varepsilon([p_{j+1}, z_{j+1}, q'_{j+1}]) = \emptyset$). Hence $\rho_\varepsilon(S') = \emptyset$. As S is deterministic, $S \bullet v$ must be left-deterministic of the same type as w', hence $S \bullet v$ is ε-free. Using now (26) we obtain

$$\rho_\varepsilon(S) = S \bullet v + \emptyset = S \bullet v. \tag{27}$$

Point (1) is then proved. Applying φ to the two members of Eq. (26) and using point (1) we obtain that

$$\varphi(S) = \varphi(v \cdot \rho_\varepsilon(S)) = \varphi(v) \cdot \varphi(\rho_\varepsilon(S)).$$

But, by the hypothesis on the letters $[p_i, z_i, q_i]$, $\varphi(v) = \varepsilon$. It follows that $\varphi(S) = \varphi(\rho_\varepsilon(S))$, i.e. point (2) is true.

Let us treate now the general case. Let $S = (S_1, \ldots, S_j, \ldots, S_\lambda)$. Let us consider $\bar{S} = \sum_{j=1}^{\lambda} S_j$. Let us apply the above arguments (and notations) on $\bar{S}$.

Case 1: $\forall w \in \mathrm{supp}(\bar{S})$, $\rho_\varepsilon(w) = \emptyset$. In that case $\rho_\varepsilon(S) = \emptyset^\lambda = S \bullet v$ (for some $v \in V^*$) and $S \equiv \rho_\varepsilon(S)$.

Case 2: $\exists w_0 \in \mathrm{supp}(\bar{S})$, $\rho_\varepsilon(w_0) = w' \in V^*$. For every $j \in [1, \lambda]$,

$$S_j = v \cdot (S_j \bullet v) + \sum_{w \in D(v)} w \cdot (S_j \bullet w). \tag{28}$$

where $\rho_\varepsilon(v) = \varepsilon, \rho_\varepsilon(\sum_{w \in D(v)} w \cdot (S_j \bullet w)) = \emptyset$ and $S_j \bullet v$ is ε-free. It follows that for every $j \in [1, \lambda]$, $\rho_\varepsilon(S_j) = S_j \bullet v$ hence

$$\rho_\varepsilon(S) = S \bullet v. \tag{29}$$

We also know that $\varphi(v) = \varepsilon, \varphi(\sum_{w \in D(v)} w \cdot (S_j \bullet w)) = \emptyset$, which together with (28) shows that:

$$\varphi(S) = \varphi(S \bullet v).$$

Hence points (1), (2) of the lemma are proved. □

Remark 16. Point (2) of the lemma is also a direct corollary of point (1) of Lemma 1. The proof given here for point (2) will be re-used in the proof of Lemma 111.

Corollary 17. (1) $\forall\lambda\in\mathbb{N}-\{0\}, \forall S\in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle, \rho_\varepsilon(S)\in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle$.
(2) $\forall\lambda\in\mathbb{N}-\{0\}, \forall S\in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle, \rho_\varepsilon(S)\in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$.

Lemma 18. *Let* $\lambda\in\mathbb{N}-\{0\}, S\in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle, u\in(X\cup\{e\})^*$. *One of the three following cases must occur*:
(1) $S\otimes u=\emptyset^\lambda$,
(2) $S\otimes u=\varepsilon_j^\lambda$ *for some* $j\in[1,\lambda]$,
(3) $\exists u_1,u_2\in(X\cup\{e\})^*, v_1\in V^*, p,q\in Q, A\in Z, \omega\in Z^*, \Phi$ *Q-λ-form such that*

$$u=u_1\cdot u_2, S\otimes u_1=S\bullet v_1=[qA]*\Phi,\quad S\otimes u=([qA]\otimes u_2)*\Phi, \textit{and}$$

$$[qA]\otimes u_2=[p\omega] \textit{ with } |\omega|\geqslant 1.$$

Proof. Let $u\in(X\cup\{e\})^*$. Let us prove the lemma by induction on $|u|$.
$u=\varepsilon$: If $S\in\emptyset^\lambda\cup\{\varepsilon_j^\lambda \mid 1\leqslant j\leqslant\lambda\}$ then clearly the conclusion of case (1) or (2) is realized. Otherwise, S has a decomposition as $S=[qA]*\Phi$ and the conclusion of case (3) is realized with $u_1=u_2=\varepsilon,\ v_1=\varepsilon,\ p=q,\ \omega=A$.
$u=u_0\cdot a, a\in X\cup\{e\}$: Let us consider the $u_1,\ u_2,\ v_1,\ p,\ q,\ A,\ \omega,\ \Phi$ given by the induction hypothesis on u_0.

$$(S\otimes u_0)\otimes a=(([qA]\otimes u_2)*\Phi)\otimes a$$

and

$$[qA]\otimes u_2=[p\omega],\quad |\omega|\geqslant 1.$$

Let $[p\omega]\otimes a=[p'\eta']$.
Case 1: $|\eta'|\geqslant 1$. Then $S\otimes ua=([qA]\otimes u_2a)*\Phi$. Hence conclusion (3) of the lemma is fulfilled by $u_1'=u_1, u_2'=u_2a, v_1'=v_1, q'=q, A'=A, \omega'=\eta', \Phi'=\Phi$.
Case 2: $|\eta'|=0$.

$$S\otimes u_0a=\Phi_r.$$

Subcase 1: $\Phi_r\in\{\emptyset^\lambda\}\cup\{\varepsilon_j^\lambda \mid 1\leqslant j\leqslant\lambda\}$. Conclusion (1) or (2) of the lemma is then realized.
Subcase 2: $\Phi_r=[r'B]*\Psi$ for some $r'\in Q, B\in Z, \Psi\in \mathsf{DB}_{Q,\lambda}\langle\langle V\rangle\rangle$.
Then

$$S\otimes ua=[r'B]*\Psi;\quad S\bullet(v_1[qAr])=\Phi_r=[r'B]*\Psi.$$

Conclusion (3) of the lemma is then realized by $u_1'=ua, u_2'=\varepsilon, v_1'=v_1[qAr], q'=r'$, $A'=B, \omega'=B, \Phi'=\Psi$. □

Lemma 19. *Let* $\lambda\in\mathbb{N}-\{0\}, S\in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle, u\in X^+$. *One of the three following cases must occur*:
(1) $S\odot u=\emptyset^\lambda$,
(2) $S\odot u=\varepsilon_j^\lambda$ *for some* $j\in[1,\lambda]$,

(3) $\exists u_1, u_2 \in X^*, v_1 \in V^*, q \in Q,\ A \in Z,\ \Phi Q$-$\lambda$-*form such that*

$$u = u_1 \cdot u_2, \rho_\varepsilon(S) \odot u_1 = S \bullet v_1 = [qA] * \Phi \text{ and } S \odot u = ([qA] \odot u_2) * \Phi.$$

Proof. Suppose that $u = x_1 \cdots x_l$ with $l \geqslant 1$. Let $u' = e^{n_0} x_1 e^{n_1} \cdots x_l e^{n_l}$ such that $S \odot u = S \otimes u'$. If the hypothesis of case (1) or (2) is realized, it is clear that the corresponding conclusion is realized. Otherwise

$$u' = u'_1 \cdot u'_2, \quad S \otimes u'_1 = S \bullet v_1 = [qA] * \Phi, \quad S \otimes u' = ([qA] \otimes u'_2) * \Phi$$

and

$$[qA] \otimes u'_2 = [p'\omega'], \quad |\omega'| \geqslant 1.$$

Let $u_1 = \Pi_X(u'_1), u_2 = \Pi_X(u'_2)$, (where $\Pi_X : (X \cup \{e\})^* \to X^*$ is the projection on the subalphabet X). If $u'_2 = \varepsilon$, then $S \odot u = [qA] * \Phi$ implies that $[qA]$ is ε-free; if $u'_2 \neq \varepsilon$, together with the condition $[qA] \otimes u'_2 \notin \{\emptyset^Q\} \cup \{\varepsilon_p^Q \mid p \in Q\}$ it implies that $[qA]$ is ε-free, hence that

$$S \otimes u'_1 = \rho_\varepsilon(S) \odot u_1.$$

The condition that $S \otimes u' = S \odot u, |u| \geqslant 1$ implies that $S \otimes u'$ is ε-free, hence that $[qA] \otimes u'_2$ is ε-free, so that

$$[qA] \otimes u'_2 = [qA] \odot u_2.$$

Hence point (3) of the lemma is realized. $\square$

Corollary 20. (1) $\forall S \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle, u \in X^*, S \odot u \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$.
(2) $\forall S \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle, u \in X^*, S \odot u \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$.

Proof. Let us consider case (3) of Lemma 19. Due to the form of the rules generating the right-action $\otimes$ (see Section 2.3), $[qA] \odot u_2$ is of the form $[p\omega]$ for some $p \in Q, \omega \in Z^*$. Hence $S \odot u$ is the Q-product of a Q-series by a Q-λ-form, which is a deterministic row-vector by Lemma 13. $\square$

We give now an adaptation of Lemma 11 to the actions $\otimes, \odot$ in place of $\bullet$.

Lemma 21. *Let* $S \in \mathsf{DB}_{1,m}\langle\langle V \rangle\rangle, T \in \mathsf{B}_{m,s}\langle\langle V \rangle\rangle, u \in (X \cup \{e\})^*$ *and* $U = S \cdot T$. *Exactly one of the following cases is true*:
(1) $S \otimes u \notin \{\emptyset^m\} \cup \{\varepsilon_j^m \mid 1 \leqslant j \leqslant m\}$
in this case $U \otimes u = (S \otimes u) \cdot T$.
(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', S \otimes u' = \varepsilon_{j_0}^m$;
in this case $U \otimes u = T_{j_0,*} \otimes u''$.
(3) $\forall j,\ \forall u' \preccurlyeq u, S \otimes u = \emptyset^m$ *and* $S \otimes u' \neq \varepsilon_j^m$;
in this case $U \otimes u = \emptyset^s = (S \otimes u) \cdot T$.

Proof. The arguments used in the proof of Lemmas (8)–(11), can be adapted to $\otimes$ in place of $\bullet$. The only non-trivial adaptation is that of lines 6–7 of the proof of Lemma 8: let us suppose that $u \in (X \cup \{e\})^*$ is such that

$$\forall u' \prec u,\ \forall w \in \text{supp}(S),\quad w \otimes u' \neq \varepsilon \tag{30}$$

and let us prove that

$$\forall w_1 \in \text{supp}(S), \forall w_2 \in \text{supp}(T),\quad (w_1 \cdot w_2) \otimes u = (w_1 \otimes u) \cdot w_2. \tag{31}$$

We prove by induction on $|u|$ that (30) implies (31).

$|u| = 0$: by definition of a right-action, $\forall w \in W^*, w \otimes \varepsilon = w$. Hence conclusion (31) is true.

$u = u_0 \cdot a$, where $u_0 \in (X \cup \{e\})^*, a \in X \cup \{e\}$:

Hypothesis (30) is fulfilled by u_0 too, hence, by induction hypothesis,

$$(w_1 \cdot w_2) \otimes u_0 = (w_1 \otimes u_0) \cdot w_2.$$

If $w_1 \otimes u_0 = \emptyset$, then, by the above equality $(w_1 \cdot w_2) \otimes u_0 = \emptyset$ too, hence

$$(w_1 \cdot w_2) \otimes u_0 a = \emptyset = (w_1 \otimes u_0 a) \cdot w_2,$$

hence (31) is true.

Otherwise, by hypothesis (30) $w_1 \otimes u_0 \notin \{\emptyset, \varepsilon\}$, hence there exists $p, q \in Q, A \in Z$ such that

$$w_1 \otimes u_0 = [p, A, q] \cdot w_3.$$

By definitions (18)–(20)

$$([p, A, q] \cdot w_3 w_2) \otimes a = ([p, A, q] \otimes a) \cdot w_3 w_2,$$

hence

$$(w_1 \cdot w_2) \otimes u_0 a = (w_1 \otimes u_0 a) \cdot w_2. \qquad \square$$

Lemma 22. *Let* $S \in \mathsf{DB}_{1,m}\langle\langle V \rangle\rangle, T \in \mathsf{B}_{m,s}\langle\langle V \rangle\rangle, u \in X^+$ *and* $U = S \cdot T$. *Exactly one of the following cases is true*:

(1) $S \odot u \notin \{\emptyset^m\} \cup \{\varepsilon_j^m \mid 1 \leqslant j \leqslant m\}$
in this case $U \odot u = (S \odot u) \cdot T$.

(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', \rho_\varepsilon(S \odot u') = \varepsilon_{j_0}^m$;
in this case $U \odot u = \rho_\varepsilon(T_{j_0,*} \odot u'')$.

(3) $\forall j,\ \forall u' \leqslant u, S \odot u = \emptyset^m$ *and* $\rho_\varepsilon(S \odot u') \neq \varepsilon_j^m$;
in this case $U \odot u = \emptyset^s = (S \odot u) \cdot T$.

Proof. Let $u = x_1 \cdots x_l$. Let us consider which case (as defined in the lemma) occurs.

Case 1: $S \odot u = S \otimes \bar{u}$ with $\bar{u} = e^{n_0} \cdot x_1 e^{n_1} \cdots x_l e^{n_l}$. By Lemma 21, $U \otimes \bar{u} = (S \otimes \bar{u}) \cdot T$, and, as $S \otimes \bar{u}$ is ε-free and $\notin \{\emptyset^m\} \cup \{\varepsilon_j^m \mid 1 \leqslant j \leqslant m\}$,

$$U \otimes \bar{u} = U \odot u, \quad S \otimes \bar{u} = S \odot u,$$

which shows that

$$U \odot u = (S \odot u) \cdot T.$$

Case 2: let $\bar{u}' = e^{n'_0} \cdot x_1 e^{n'_1} \cdots x_i e^{n'_i}$, with $0 \leqslant i \leqslant l$, such that

$$\rho_\varepsilon(S \odot u') = S \otimes \bar{u}'.$$

By Lemma 21, case (2), where $u'' = \varepsilon$

$$U \otimes \bar{u}' = T_{j_0,*},$$

hence $U \odot u' = \rho_\varepsilon(T_{j_0,*})$, hence

$$U \odot u = \rho_\varepsilon(T_{j_0,*}) \odot u'' = \rho_\varepsilon(T_{j_0,*} \odot u'').$$

Case 3: let $\bar{u} = e^{n_0} \cdot x_1 e^{n_1} \cdots x_l e^{n_l}$ such that

$$U \otimes \bar{u} = U \odot u.$$

The hypothesis of this case implies that, $\forall j$, $\forall u' \preccurlyeq u$,

$$S \otimes \bar{u} \neq \varepsilon_j^m$$

(because, as ε_j^m is ε-free, if $S \otimes \bar{u} = \varepsilon_j^m$ then $\rho_\varepsilon(S \odot u) = \varepsilon_j^m$ too). Hence, by Lemma 21

$$U \otimes \bar{u} = \emptyset^m.$$

Hence,

$$U \odot u = \emptyset^m = (S \odot u) \cdot T. \qquad \square$$

The particular letters $[p, e, q]$ for $p, q \in Q$ play a special role in Sections 7 and 8: we use them as *marks* in the series (somehow like the ceilings of [79]). We define below a map ρ_e which removes the marks in the series. Let us define $\rho_e : \mathsf{DB}\langle\langle V \rangle\rangle \to \mathsf{B}\langle\langle V \rangle\rangle$ as the unique substitution such that

$$\rho_e([p, e, q]) = \varepsilon \quad \text{if} \quad p = q, \qquad \rho_e([p, e, q]) = \emptyset \quad \text{if} \quad p \neq q,$$
$$\rho_e([p, A, q]) = [p, A, q] \quad \text{if } A \neq e. \tag{32}$$

We note $\bar{V}_e = \{[p, e, q] \mid p, q, \in Q\}$, $V_e = V - \bar{V}_e$. A deterministic series $S \in \mathsf{DB}\langle\langle V \rangle\rangle$ is said e-free iff its type is $(\emptyset)$ or (ε) or $([pA])$, with $A \neq e$.

Lemma 23. *For every* $S \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$
(1) $\rho_e(S) \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$,
(2) $\|\rho_e(S)\| \leqslant \|S\|$,
(3) $S \equiv \rho_e(S)$.

Sketch of proof. We establish first that, for every $u \in V_e^*, \exists u' \in V^*$ such that

$$\rho_e(S) \bullet u = \rho_e(S \bullet u') \text{ and } S \bullet u' \text{ is } e\text{-free.} \tag{33}$$

Let us prove (33) by induction on $|u|$.

$|u| = 0$: If $\rho_e(S) = \emptyset^\lambda$ then (33) is true: it suffices to choose some u' such that $S \bullet u' = \emptyset^\lambda$. Otherwise, $\rho_e(S) \neq \emptyset^\lambda$ and, using the determinism of S, one can show that there exists a maximal integer n such that

$$\exists (p_i)_{1 \leqslant i \leqslant n} \in Q^n, \quad S \bullet ([p_1, e, p_1] \cdots [p_n, e, p_n]) \neq \emptyset^\lambda.$$

Then $u' = [p_1, e, p_1] \cdots [p_n, e, p_n]$ (where $u' = \varepsilon$ when $n = 0$) satisfies (33).

$|u| = m + 1 : u = u_1 \cdot v_1$ where $u_1 \in V_e^*, |u_1| = m, v_1 \in V_e$. By induction hypothesis there exists $u_1' \in V^*$ such that

$$\rho_e(S) \bullet u_1 = \rho_e(S \bullet u_1') \text{ and } S \bullet u_1' \text{ is } e\text{-free.}$$

If $S \bullet u_1' \in \{\emptyset^\lambda, \varepsilon_1^\lambda, \ldots, \varepsilon_\lambda^\lambda\}$, then $u' = u_1' v_1$ satisfies (33). Otherwise let $([pA])$ be the type of $S \bullet u_1'$ $(p \in Q, A \neq e)$. We then have

$$\rho_e(S) \bullet u_1 = \rho_e(S \bullet u_1') \tag{34}$$

$$S \bullet u_1' = [pA] * \Phi \tag{35}$$

for some Q-λ-form Φ.

Subcase 1: $v_1 = [p, A, q_1]$ (for some $q_1 \in Q$). Let us consider the vector Φ_{q_1}: by induction hypothesis, there exists some $w_1' \in V^*$ such that

$$\rho_e(\Phi_{q_1}) \bullet \varepsilon = \rho_e(\Phi_{q_1} \bullet w_1') \text{ and } \Phi_{q_1} \bullet w_1' \text{ is } e\text{-free.} \tag{36}$$

Combining Eqs. (34) – (36) we see that $u' = u_1' \cdot [p, A, q_1] \cdot w_1'$ fulfills:

$$\begin{aligned}
\rho_e(S) \bullet (u_1 \cdot v_1) &= (\rho_e(S) \bullet u_1) \bullet v_1 \\
&= \rho_e(S \bullet u_1') \bullet [p, A, q_1] \\
&= \rho_e([pA] * \Phi) \bullet [p, A, q_1] \\
&= \rho_e(\Phi_{q_1}) \\
&= \rho_e(\Phi_{q_1} \bullet w_1')
\end{aligned}$$

where $S \bullet (u_1' \cdot [p, A, q_1] \cdot w_1') = \Phi_{q_1} \bullet w_1'$ and $\Phi_{q_1} \bullet w_1'$ is e-free. Hence (33) is fulfilled by our choice of u'.

Subcase 2: $v_1 \notin \{[pAq_1] | q_1 \in Q\}$

$$\rho_e(S) \bullet u_1 v_1 = ([pA] * \rho_e(\Phi)) \bullet v_1 = \emptyset^\lambda.$$

$$\rho_e(S \bullet u_1' v_1) = \rho_e(([pA] * \Phi) \bullet v_1) = \rho_e(\emptyset^\lambda) = \emptyset^\lambda.$$

Hence $u' = u_1' \cdot v_1$ satisfies (33).

Let us prove the lemma now. By (33) every residual $\rho_e(S) \bullet u$ is left-deterministic of the same type as $S \bullet u'$. Hence $\rho_e(S)$ is deterministic. Moreover formula (33) shows that $\|\rho_e(S)\| \leqslant \|S\|$.
Let us prove point (3) now. By definition (32), for every $v \in V$, $v \equiv \rho_e(v)$. As $\equiv$ is the kernel of a substitution, this is sufficient to ensure point (3). □

3.1.5. Equivalence on row-vectors

We give here some basic properties of the equivalence $\equiv$ over vectors (defined by (24)). Let us consider the structured alphabet $(X, \smile)$ where the equivalence $\smile$ is the coarsest one: $\forall x, y \in X, x \smile y$.

Lemma 24. *Let* $\lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle, j \in [1, \lambda]$. *Then*
(1) $\varphi(S) \in \mathsf{DB}_{1,\lambda}\langle\langle X \rangle\rangle$.
(2) $\varphi(S) = \varepsilon_j^\lambda \Leftrightarrow \rho_\varepsilon(S) = \varepsilon_j^\lambda$.

Proof. (1) Let us suppose that S is ε-free. Either S has type $\emptyset$ (resp. (ε, j)), and it is then clear that $\varphi(S)$ is left-deterministic of the same type, or S has type $[pzq]_\smile$ for some ε-free mode pz and then, $\varphi(S)$ is left-determinsitic of type $[x]_\smile$ (for any $x \in X$).
Let $S \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle, u \in X^*$. Let us show that

$$\varphi(S) \bullet u \text{ is left-deterministic.} \tag{37}$$

By Lemma 1 $\varphi(S) \bullet u = \varphi(\rho_\varepsilon(S \odot u))$. The vector $\rho_\varepsilon(S \odot u)$ is deterministic (by Corollaries 17, 20) and ε-free. Hence, by the ε-free case, $\varphi(\rho_\varepsilon(S \odot u))$, is left-deterministic. This proves (37), hence point (1) of the lemma.
(2) By Lemma 1, $\varphi(S) = \varphi(\rho_\varepsilon(S))$, and by the above arguments in the ε-free case: $\varphi(\rho_\varepsilon(S))$ has the type (ε, j) iff $\rho_\varepsilon(S)$ has the same type. This proves point (2). □

Lemma 25. *Let* $\lambda \in \mathbb{N} - \{0\}, j \in [1, \lambda]$, $S \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$. *Then* $\varphi(S)_{1,j} = \{u \in X^* \mid \rho_\varepsilon(S \odot u) = \varepsilon_j^\lambda\}$.

Proof. By Lemma 1,

$$u \in \varphi(S)_{1,j} \Leftrightarrow \varepsilon \in \varphi(S)_{1,j} \bullet u \Leftrightarrow \varepsilon \in \varphi(\rho_\varepsilon(S_{1,j} \odot u)).$$

By Lemma 24, point (1)

$$\varepsilon \in \varphi(\rho_\varepsilon(S_{1,j} \odot u) \Leftrightarrow \varphi(\rho_\varepsilon(S \odot u)) = \varepsilon_j^\lambda,$$

and by Lemma 24, point (2)

$$\varphi(\rho_\varepsilon(S \odot u)) = \varepsilon_j^\lambda \Leftrightarrow \rho_\varepsilon(S \odot u) = \varepsilon_j^\lambda.$$

The above equivalences prove the lemma. □

Corollary 26. *Let* $\lambda\in\mathbb{N}-\{0\}, S, S'\in \mathsf{DB}_{1,\lambda}\langle\langle V\rangle\rangle$. *Then* $S\equiv S'$ *if and only if*, $\forall u\in X^*$, $\forall j\in[1,\lambda]$,

$$\rho_\varepsilon(S\odot u)=\varepsilon_j^\lambda \Leftrightarrow \rho_\varepsilon(S'\odot u)=\varepsilon_j^\lambda.$$

Definition 27. For every $\lambda\in\mathbb{N}-\{0\}, S, S'\in \mathsf{B}_{1,\lambda}\langle\langle V\rangle\rangle$ we define: $\mathrm{Div}(S,S')=\inf\{|u|, u\in X^*, u\in\varphi(S)\triangle\varphi(S')\}$.

Where $S\triangle S'$ means $\sum_{1\leqslant j\leqslant\lambda} S_j\triangle S'_j$, the sum of the symmetric differences of the components of S and S'. We recall that for $\lambda=1$, $S\triangle S'=\sum_{u\in X^*, S_u\neq S'_u} u$. We recall also that $\inf(\emptyset)=\infty$. From Lemma 25 one can equivalently write

$$\mathrm{Div}(S,S')=\inf\{|u|, u\in X^*, \exists j\in[1,\lambda], (\rho_\varepsilon(S\odot u)=\varepsilon_j^\lambda)\Leftrightarrow(\rho_\varepsilon(S'\odot u)\neq\varepsilon_j^\lambda)\}. \tag{38}$$

For every $n\in\mathbb{N}\cup\{\infty\}$, we denote by $\equiv_n$ the following approximation of $\equiv$:

$$S\equiv_n S'\Leftrightarrow \varphi(S)\cap(X^{\leqslant n}\times\cdots\times X^{\leqslant n})=\varphi(S')\cap(X^{\leqslant n}\times\cdots\times X^{\leqslant n}).$$

(Hence $\equiv_\infty$ is just $\equiv$.) By Lemma 25 one can equivalently write

$$S\equiv_n S'\Leftrightarrow[\forall u\in X^{\leqslant n}, \forall j\in[1,\lambda], (\rho_\varepsilon(S\odot u)=\varepsilon_j^\lambda))\Leftrightarrow(\rho_\varepsilon(S'\odot u)=\varepsilon_j^\lambda)].$$

3.1.6. Operations on row-vectors

Let us introduce two new operations on row-vectors and prove some technical lemmas about them.

Given $A, B\in\mathsf{B}_{1,m}\langle\langle W\rangle\rangle$ and $1\leqslant j_0\leqslant m$ we define the vector $C=A\square_{j_0}B$ as follows:

if $A=(a_1,\ldots,a_j,\ldots,a_m), B=(b_1,\ldots,b_j,\ldots,b_m)$ then $C=(c_1,\ldots,c_j,\ldots,c_m)$, where

$$c_j=a_j+a_{j_0}\cdot b_j \text{ if } j\neq j_0 \quad c_j=\emptyset \text{ if } j=j_0.$$

Lemma 28. *Let* $A, B\in\mathsf{B}_{1,m}\langle\langle W\rangle\rangle$ *and* $1\leqslant j_0\leqslant m$:

(1) *if* A, B *are left-deterministic, then* $A\square_{j_0}B$ *is left-deterministic*,
(2) *if* A, B *are deterministic, then* $A\square_{j_0}B$ *is deterministic*,
(3) *if* A, B *are deterministic, then* $\|A\square_{j_0}B\|\leqslant\|A\|+\|B\|$.

Proof. Let $C=A\square_{j_0}B$.

(1) Let us prove first that if A, B are both left-deterministic, then C is left-deterministic too.

If A is left-deterministic of type $[pz]$, then C is left-deterministic of the same type.
If A is left-deterministic of type (ε, j_1) with $j_1\neq j_0$, then $C=A$, hence C is left-deterministic.
If A is left-deterministic of type (ε, j_0), then $C\leqslant B$, hence C is left-deterministic.
If A is left-deterministic of type $(\emptyset)$, then $C=\emptyset$, hence C is left-deterministic.

(2) Let us suppose now that A is deterministic and let us examine a residual $C \bullet u$, for some $u \in W^*$. Lemma 9 applies on $S = a_{j_0}$ and $T = b_j$ for every $j \neq j_0$. But the case of the lemma fulfilled by (S, T, u) depends on (S, u) only. Suppose $a_{j_0} \bullet u \neq \emptyset$ (case 1); in this case

$$C \bullet u = (A \bullet u) \; \square_{j_0} B. \tag{39}$$

Suppose $a_{j_0} \bullet u = \emptyset, \exists u', u'', u = u' \cdot u'', a_{j_0} \bullet u' = \varepsilon$ (case 2); in this case

$$C \bullet u = \langle B \bullet u'' | \emptyset^m_{j_0} \rangle, \tag{40}$$

where $\emptyset^m_{j_0}$ is the row vector $\varepsilon^m_{j_0}$ in which $\emptyset$ and ε have been exchanged and $\langle * | * \rangle$ is the "scalar product" defined by $\langle S, T \rangle = \sum_{j=1}^m S_j \cdot T_j$.

Suppose $a_{j_0} \bullet u = \emptyset$, $\forall u' \leqslant u, a_{j_0} \bullet u' \neq \varepsilon$ (case 3); in this case, Eq. (39) is true again. When Eq. (39) is true, $C \bullet u$ is left-deterministic by part (1) of this proof, and when Eq. (40) is true, $C \bullet u$ is left-deterministic because B is assumed deterministic. We have proved that $C \in \mathsf{DB}_{1,m}\langle\langle W \rangle\rangle$.

(3) The number of residuals of the form (39) is bounded above by $\|A\|$ and the number of residuals of the form (40) is bounded above by $\|B\|$. Hence $\|C\| \leqslant \|A\| + \|B\|$. $\square$

Given $A \in \mathsf{DB}_{1,m}\langle\langle W \rangle\rangle$ and $1 \leqslant j_0 \leqslant m$ we define the vector $A' = \square^*_{j_0}(A)$ as follows: if $A = (a_1, \ldots, a_j, \ldots, a_m)$ then $A' = (a'_1, \ldots, a'_j, \ldots, a'_m)$, where

$$a'_j = a^*_{j_0} \cdot a_j \quad \text{if } j \neq j_0, \quad a'_j = \emptyset \quad \text{if } j = j_0.$$

Lemma 29. *Let $A \in \mathsf{DB}_{1,m}\langle\langle W \rangle\rangle$ and $1 \leqslant j_0 \leqslant m$.*
Then $\square^*_{j_0}(A) \in \mathsf{DB}_{1,m}\langle\langle W \rangle\rangle$ and $\|\square^*_{j_0}(A)\| \leqslant \|A\|$.

Proof. Let us examine a residual $A' \bullet u$, for some $u \in W^*$. Let $u' = \max\{v \leqslant u \mid v \in a^*_{j_0}\}$. Let $u'' \in W^*$ such that $u = u' \cdot u''$. One can check that for every $S, T \in \mathsf{B}\langle\langle W \rangle\rangle$

$$(S \cdot T) \bullet u = (S \bullet u) \cdot T + \sum_{\substack{u = u_1 \cdot u_2, \\ \varepsilon \in S \bullet u_1}} T \bullet u_2.$$

Applying this formula to $S = a^*_{j_0}$ and $T = a_j$, with $j \neq j_0$ we obtain

$$a'_j \bullet u = (a^*_{j_0} \bullet u) \cdot a_j + \sum_{\substack{u = u_1 \cdot u_2, \\ \varepsilon \in a^*_{j_0} \bullet u_1}} a_j \bullet u_2. \tag{41}$$

Since a_{j_0} is deterministic and $a_{j_0} \bullet u'' \neq \varepsilon$ we get

$$a^*_{j_0} \bullet u = (a_{j_0} \bullet u'') \cdot a^*_{j_0}.$$

As A is deterministic, if u_2 has some prefix u_2' in a_{j_0}, then $a_j \bullet u_2' = \emptyset$ so that $a_j \bullet u_2 = \emptyset$. Hence

$$\sum_{\substack{u=u_1\cdot u_2,\\ \varepsilon\in a_{j_0}^*\bullet u_1}} a_j \bullet u_2 = a_j \bullet u''.$$

Plugging the two last equations into (41) we obtain

$$a_j' \bullet u = (a_{j_0} \bullet u'')\cdot a_{j_0}^* \cdot a_j + a_j \bullet u'' \text{ (for } j \neq j_0), \quad \text{and} \quad a_j' \bullet u = \emptyset \text{ (for } j = j_0)$$

which can be rewritten as

$$A' \bullet u = (A \bullet u'')\square_{j_0} A'. \tag{42}$$

Let us show that A' is left-deterministic. If A is left-deterministic of type $[pz]$, then A' is left-deterministic of the same type.

If A is left-deterministic of type (ε, j_1) with $j_1 \neq j_0$, then $A' = A$ (notice that $\emptyset^* = \varepsilon$), hence A' is left-deterministic.

If A is left-deterministic of type (ε, j_0) or $(\emptyset)$, then $A'=\emptyset$, hence A' is left-deterministic.

By point (1) of Lemma 28, the fact that $A \bullet u''$ and A' are both left-deterministic implies that $(A \bullet u'')\square_{j_0} A'$ is left-deterministic too. By formula (42), $A' \bullet u$ is left-deterministic. We have proved that $A' \in \mathsf{DB}_{1,m}\langle\langle W\rangle\rangle$.
Moreover, by formula (42), $\mathrm{Card}(\mathsf{Q}(A')) \leqslant \mathrm{Card}(\mathsf{Q}(A))$, i.e. $\|A'\| \leqslant \|A\|$. $\square$

3.2. Deterministic spaces

We adapt here the key idea of [47,48] to series.

3.2.1. Definitions

Let $(W, \sim)$ be some structured alphabet and let us consider the set $E = \mathsf{DRB}\langle\langle W\rangle\rangle$. A series $U = \sum_{i=1}^{n} \gamma_i \cdot U_i$ where $\vec{\gamma} \in \mathsf{DRB}_{1,n}\langle\langle W\rangle\rangle$, $U_i \in \mathsf{DRB}\langle\langle W\rangle\rangle$ is called a *linear combination* of the U_i's. We call *deterministic space* of rational series (d-space for short) any subset V of E which is closed under finite linear combinations. Given any set $\mathscr{G} = \{U_i \mid i \in I\}$, one can check that the set V of all (finite) linear combinations of elements of $\mathscr{G}$ is a d-space (by Lemma 13) and that it is the smallest d-space containing $\mathscr{G}$. Therefore, we call V the d-space *generated* by $\mathscr{G}$ and we call $\mathscr{G}$ a *generating set* of V (we note $\mathsf{V} = \mathsf{V}(\{U_i \mid i \in I\})$). (Similar definitions can be given for *families* of series.)

3.2.2. Linear independence

We let now $W = V$. Following an analogy with classical linear algebra, we develop now a notion corresponding to a kind of *linear independence* of the images by φ of the given series. Let us extend the equivalence relation $\equiv$ to d-spaces by: for every d-spaces $\mathsf{V}_1, \mathsf{V}_2$, $\mathsf{V}_1 \equiv \mathsf{V}_2 \Leftrightarrow \forall i, j \in \{1, 2\}, \forall S \in \mathsf{V}_i, \exists S' \in \mathsf{V}_j, S \equiv S'$.

Lemma 30. *Let* $S_1,\ldots,S_j,\ldots,S_m \in \mathsf{DRB}\langle\langle V\rangle\rangle$. *The following are equivalent*:
(1) $\exists\vec{\alpha},\vec{\beta}\in\mathsf{DRB}_{1,m}\langle\langle V\rangle\rangle, \vec{\alpha}\not\equiv\vec{\beta}$, *such that* $\sum_{1\leqslant j\leqslant m}\alpha_j\cdot S_j\equiv\sum_{1\leqslant j\leqslant m}\beta_j\cdot S_j$,
(2) $\exists j_0\in[1,m], \exists\vec{\gamma}\in\mathsf{DRB}_{1,m}\langle\langle V\rangle\rangle, \vec{\gamma}\not\equiv\varepsilon^m_{j_0}$, *such that* $S_{j_0}\equiv\sum_{1\leqslant j\leqslant m}\gamma_j\cdot S_j$,
(3) $\exists j_0\in[1,m], \exists\vec{\gamma}\,'\in\mathsf{DRB}_{1,m}\langle\langle V\rangle\rangle, \gamma'_{j_0}\equiv\emptyset$, *such that* $S_{j_0}\equiv\sum_{1\leqslant j\leqslant m}\gamma'_j\cdot S_j$,
(4) $\exists j_0\in[1,m]$, *such that* $\mathsf{V}((S_j)_{1\leqslant j\leqslant m})\equiv\mathsf{V}((S_j)_{1\leqslant j\leqslant m, j\neq j_0})$.

The equivalence between (1), (2) and (3) was first proved in [47, Lemma 11, p. 589 [9]], in the case where the S_j's are configurations $q_j\omega$, with the same ω.

Proof. Let us use the notation $S=(S_j)_{1\leqslant j\leqslant m}\in\mathsf{DRB}_{m,1}\langle\langle V\rangle\rangle$. In a first step, we assume that all the vectors $\vec{\alpha},\vec{\beta},S$ are ε-free.

(1) ⇒ (2): Let us consider

$$u=\min\{\varphi(\vec{\alpha})\Delta\varphi(\vec{\beta})\}.$$

By Lemma 25, under our ε-freeness assumption, $\exists j_0\in[1,m]$, such that

$$\vec{\alpha}\odot u=\varepsilon^m_{j_0}\Leftrightarrow\vec{\beta}\odot u\neq\varepsilon^m_{j_0}.$$

Let us suppose, for example, that $\vec{\alpha}\odot u=\varepsilon^m_{j_0}$ while $\vec{\beta}\odot u\neq\varepsilon^m_{j_0}$ and let $\vec{\gamma}=\vec{\beta}\odot u$. As $\equiv$ is preserved by the action $\odot$ (see Lemma 1):

$$(\vec{\alpha}\cdot S)\odot u\equiv(\vec{\beta}\cdot S)\odot u. \tag{43}$$

Using Lemma 22 we obtain

$$(\vec{\alpha}\cdot S)\odot u=S_{j_0}. \tag{44}$$

Let us examine now the righthand-side of equality (43). Let $u'\prec u$. By minimality of u, $\vec{\beta}\odot u'$ is a unit iff $\vec{\alpha}\odot u'$ is a unit. But if $\vec{\alpha}\odot u'$ is a unit, then $\vec{\alpha}\odot u=\emptyset^m$, which is false. Hence $\vec{\beta}\odot u'$ is not a unit. By Lemma 22

$$(\vec{\beta}\cdot S)\odot u=(\vec{\beta}\odot u)\cdot S. \tag{45}$$

Let us plug equalities (44) and (45) in equivalence (43) and let us define $\vec{\gamma}=\vec{\beta}\odot u$. We obtain

$$S_{j_0}\equiv\vec{\gamma}\cdot S,\ \text{where } \vec{\gamma}\not\equiv\varepsilon^m_{j_0}.$$

(2) ⇒ (3):

$$S_{j_0}\equiv\gamma_{j_0}\cdot S_{j_0}+\left(\sum_{j\neq j_0}\gamma_j\cdot S_j\right),\quad \gamma_{j_0}\not\equiv\varepsilon.$$

By the well-known Arden's lemma (see Corollary 55, point (C1)), we can deduce that

$$S_{j_0}\equiv\sum_{j\neq j_0}\gamma^*_{j_0}\gamma_j\cdot S_j=\square^*_{j_0}(\gamma)\cdot S.$$

[9] Numbering of the english version.

Taking $\gamma' = \square_{j_0}^*(\gamma)$ we obtain

$$S_{j_0} \equiv \gamma' \cdot S \quad \text{where } \gamma'_{j_0} = \emptyset.$$

(3) ⇒ (4): Let us denote by $\hat{S}$ the vector $(S_1, \ldots, S_{j_0-1}, \emptyset, S_{j_0+1} \ldots, S_m) \in \mathsf{DB}_{m,1}\langle\langle V \rangle\rangle$. If $T = \vec{\alpha} \cdot S$ then $T \equiv (\vec{\alpha} \square_{j_0} \vec{\gamma}') \cdot \hat{S}$.

(4) ⇒ (1): Let us suppose (4) is true for some integer j_0. The element S_{j_0} is clearly equivalent (mod $\equiv$) to two linear combinations of the S_j's with non-equivalent vectors of coefficients (mod $\equiv$). Hence (1) is true.

Let us consider now the general case where some vector $\vec{\alpha}, \vec{\beta}, S$ might be ε-bound. By Lemma 15, point (2),

$$\vec{\alpha} \equiv \rho_\varepsilon(\vec{\alpha}), \quad \vec{\beta} \equiv \rho_\varepsilon(\vec{\beta}), \quad S \equiv \rho_\varepsilon(S)$$

and by Corollary 17,

$$\rho_\varepsilon(\vec{\alpha}), \rho_\varepsilon(\vec{\beta}) \in \mathsf{DRB}_{1,d}\langle\langle V \rangle\rangle.$$

Hence the Lemma in the general case follows from the Lemma in the ε-free case. □

3.3. Height, defect and linearity

We define here notions of height and defect (for deterministic rational series) and a subsequent notion of (d, d')-linearity which will play a crucial role in Section 8. We then relate these "size notions" with the notion of norm.

3.3.1. Definitions

Let $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$. We call *linear decomposition of S* any pair $([p\omega], \Phi)$ where $p \in Q, \omega \in Z^*, \Phi \in \mathsf{DB}_{Q,1}\langle\langle V \rangle\rangle$ such that

$$S = [p\omega] * \Phi.$$

We denote by $\mathscr{D}(S)$ the set of all linear decompositions of S. We define the *right-defect* of S (rd(S) for short) by

$$\mathrm{rd}(S) = \min\{\|\Phi\| \mid \exists p \in Q, \omega \in Z^*, ([p\omega], \Phi) \in \mathscr{D}(S)\}.$$

One can easily see that the right-defect of S is finite and not greater than $\|S\|$. We call *minimal* decomposition of S the decomposition $([p_0\omega_0], \Phi_0)$ which makes the triple $(p, |\omega|, \|\Phi\|)$ minimal for the right-to-left lexicographic ordering in $Q \times \mathbb{N} \times \mathbb{N}$. We then define the *linear-height* of S (noted $|S|$) as the integer $|S| = |\omega_0|$ (and it is clear that $\mathrm{rd}(S) = \|\Phi_0\|$).

The height and right-defect of a Q-form are defined similarly.

Let $\Phi \in \mathsf{DRB}_{Q,1}\langle\langle V \rangle\rangle$. We call *linear decomposition of Φ* any pair $([\omega], \Psi)$ where $\omega \in Z^*, \Psi \in \mathsf{DB}_{Q,1}\langle\langle V \rangle\rangle$ such that

$$\Phi = [\omega] * \Psi.$$

$\mathscr{D}(\Phi)$ denotes the set of all linear decompositions of Φ. The *minimal* decomposition of Φ is the $([\omega_0], \Psi_0)$ which makes the pair $(|\omega|, \|\Psi\|)$ minimal for the right-to-left lexicographic ordering in $\mathbb{N} \times \mathbb{N}$. The integers $|\Phi|, \mathrm{rd}(\Phi)$ are then defined by

$$|\Phi| = |\omega_0|, \quad \mathrm{rd}(\Phi) = \|\Psi_0\|.$$

We say that S is *marked* iff S contains some occurence of some letter in $\{[p,e,q] \mid p,q \in q\} \subseteq V$ (we assumed the existence of such a "dummy" letter $e \in Z$ in Section 2.2).

Definition 31. Let $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$ and $d, d' \in \mathbb{N}, d \geqslant 1$.
(1) S is said $(0,d')$-linear iff $\mathrm{rd}(S) \leqslant d'$ and S is not marked,
(2) S is said (d,d')-linear iff, either it is $(0,d')$-linear or S has a decomposition $S = \sum_{q \in Q}[phq] \cdot [qeq] \cdot \Phi_q$ where every Φ_q is a $(0,d')$-linear series and $|h| \leqslant d$.

In case 2, we call the series Φ_q the d'-linear components of S. In case 1, we consider that S itself is the unique linear component of S. It should be clear that the set of d'-linear components are independent of the value of d (for d large enough) and that it is uniquely defined: it is empty if S is not (d,d')-linear for any d, otherwise it is equal to

$$\{S \bullet u \mid u \in V^*, S \bullet u \text{ is unmarked and, } \forall u' \prec u, S \bullet u' \text{ is marked}\}.$$

We denote by $\mathsf{DBlin}^{d'}\langle\langle V \rangle\rangle$ (resp. $\mathsf{DRBlin}^{d'}\langle\langle V \rangle\rangle$) the set of series in $\mathsf{DB}\langle\langle V \rangle\rangle$ (resp. $\mathsf{DRB}\langle\langle V \rangle\rangle$) which are $(0,d')$-linear.

Example. No series can be $(0,0)$-linear. S is $(0,1)$-linear iff $S = \emptyset$. S is $(0,2)$-linear iff there exists $\omega \in (Z - \{e\})^*, Q' \subseteq Q$, such that $S = \sum_{q \in Q'}[p\omega q]$.

Hence, one can view the $(0,d)$-linear series as series which have a structure "not too far" from the *linear* structure of the configurations of the initial dpda $\mathscr{M}$.

(We illustrate in Fig. 1 the above definitions.)

3.3.2. *Height and norm*

Let us define the integer constant $K_0 = |Q| + 1$.

(Here $|Q|$ denotes the cardinality of the set Q.)

Lemma 32. *Let* $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$, $x \in X, d, \ d' \in \mathbb{N}$,
(1) $\mathrm{rd}(S \odot x) \leqslant \mathrm{rd}(S)$,
(2) S *is* (d,d')*-linear* $\Rightarrow S \odot x$ *is* $(d+1,d')$*-linear*,
(3) $\|S \odot x\| \leqslant \|S\| + K_0$.

Sketch of proof. Point (1) follows from Lemmas 22 and 19. Points (2) and (3) follow from the hypothesis that the dpda $\mathscr{M}$ is normalized. □

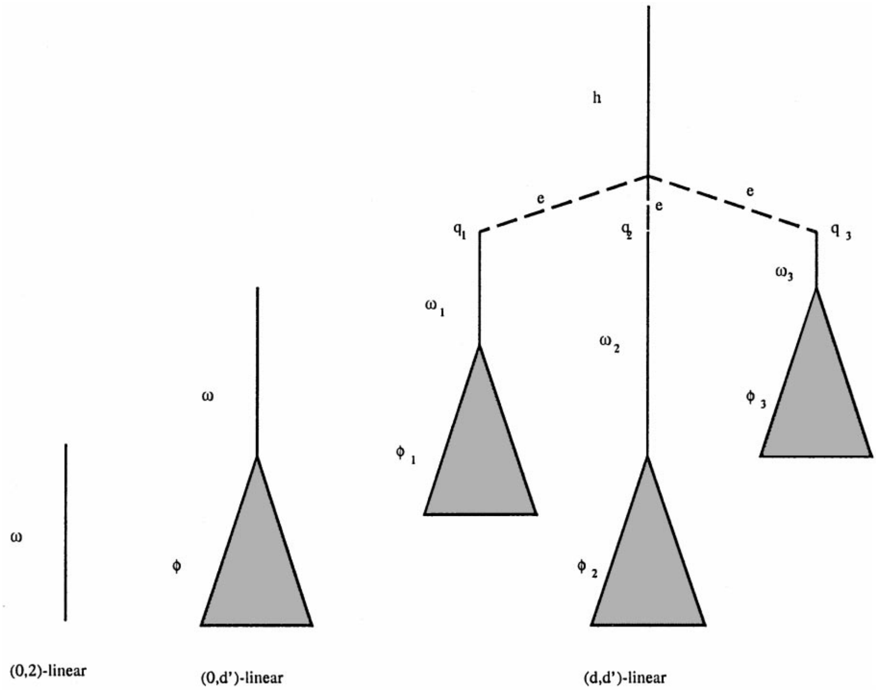

Fig. 1. (d,d')-linearity.

Lemma 33. *Let* $B,A \in Z$, $\Phi \in \mathsf{DB}_{Q,1}\langle\langle V \rangle\rangle$. *If* $\|[A] * \Phi\| > \|\Phi\|$ *then*, $\forall q \in Q, [qBA] * \Phi \notin \mathsf{Q}_r([A] * \Phi)$.

Proof. Suppose that $\exists q \in Q$, such that $[qBA] * \Phi \in \mathsf{Q}_r([A] * \Phi)$. Let us fix some $q \in Q$, $u_q \in V^*, q' \in Q$, fulfilling

$$[qBA] * \Phi = ([q'A] * \Phi) \bullet u_q.$$

Case 1: $u_q = \varepsilon$. Hence $q = q', B = A, [qB]*[A]*\Phi = [qB]*\Phi$. It follows that $[A]*\Phi = \Phi$ and finally $\|[A] * \Phi\| = \|\Phi\|$.

Case 2: $u_q \neq \varepsilon, \Phi \neq \emptyset^1_Q$.[10] Then $u_q = [q'Ar]v_q$ for some $r \in Q, v_q \in V^*$.

$$[qBA] * \Phi = (([q'A] * \Phi) \bullet [q'Ar]) \bullet v_q = \Phi_r \bullet v_q.$$

It follows that, $\forall p \in Q$,

$$[pA] * \Phi = \Phi_r \bullet v_q \bullet [qBp] \in \mathsf{Q}_r(\Phi),$$

hence $\|[A] * \Phi\| = \|\Phi\|$.

[10] By $\emptyset^1_Q$ we denote the $Q-1$-form which has all its entries equal to $\emptyset$.

Case 3: $u_q \neq \varepsilon, \Phi = \emptyset^1_Q$. Then $\|[A] * \Phi\| = \|\Phi\| = 1$.
In all cases the lemma is proved by contraposition. □

Lemma 34. *Let* $\omega \in Z^+, A', A \in Z, p \in Q$, $\Phi \in \mathsf{DB}_{Q,1}\langle\langle V \rangle\rangle$. *If* $\|[A] * \Phi\| > \|\Phi\|$, *then*
(1) $\|[\omega A] * \Phi\| = |Q| \cdot |\omega| + \|[A] * \Phi\|$,
(2) $\|[pA'\omega A] * \Phi\| = 1 + |Q| \cdot |\omega| + \|[A] * \Phi\|$.

Proof. Let us prove point (1) by induction on $n = |\omega|$.
$n = 0$: Formula (1) is obvious in this case.
$n = 1$: $\omega = B$. $\|[\omega A] * \Phi\| = \|[B] * ([A] * \Phi)\|$. By Lemma 33,

$$\mathsf{Q}_r([BA] * \Phi) = \{[qBA] * \Phi \mid q \in Q\} \dot{\cup} \mathsf{Q}_r([A] * \Phi),$$

(where $\dot{\cup}$ denotes a *disjoint* union) hence $\|[BA]*\Phi\| = |Q| + \|[A]*\Phi\|$. $n = m+1, m \geqslant 1$: $\omega = CB\omega'$ for some $C, B \in Z, \omega' \in Z^*$.
$\|[\omega A] * \Phi\| = \|[CB] * ([\omega' A] * \Phi)\|$. As a consequence of the induction hypothesis, $\|[B] * [\omega' A] * \Phi\| > \|[\omega' A] * \Phi)\|$, hence, by Lemma 33

$$\|[\omega A] * \Phi\| = |Q| + \|[B\omega' A] * \Phi\|.$$

By induction hypothesis, $\|[B\omega' A] * \Phi)\| = m \cdot |Q| + \|[A] * \Phi\|$, hence

$$\|[\omega A] * \Phi\| = (m+1) \cdot |Q| + \|[A] * \Phi\|.$$

Let us prove now point (2).
By the above induction, $\|[A'\omega A] * \Phi\| = |Q| + \|[\omega A] * \Phi\|$, i.e. $\forall q \in Q$, $[qA'\omega A] * \Phi \notin \mathsf{Q}_r([\omega A] * \Phi)$. Hence,

$$\mathsf{Q}_r([pA'\omega A] * \Phi) = \{[pA'\omega A] * \Phi\} \dot{\cup} \mathsf{Q}_r([\omega A] * \Phi).$$

It follows that

$$\|[pA'\omega A] * \Phi\| = 1 + \|[\omega A] * \Phi\| = 1 + |\omega| \cdot |Q| + \|[A] * \Phi\|. \qquad \square$$

Lemma 35. *Let* $\omega \in Z^*, B \in Z, p \in Q$, $\Phi \in \mathsf{DB}_{Q,1}\langle\langle V \rangle\rangle$. *Then* $\|[pB\omega] * \Phi\| \leqslant 1 + |Q| \cdot |\omega| + \|\Phi\|$.

Proof. One can notice that $\|[pB\omega]\| = 1 + |Q| \cdot |\omega|$. Hence, by Lemma 14,

$$\|[pB\omega] * \Phi\| \leqslant \|[pB\omega]\| + \|\Phi\| = 1 + |Q| \cdot |\omega| + \|\Phi\|. \qquad \square$$

Lemma 36. *Let* $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$. *Then* $1 + |Q| \cdot (|S| - 2) + \mathrm{rd}(S) \leqslant \|S\| \leqslant 1 + |Q| \cdot |S| + \mathrm{rd}(S)$.

Proof. The upper bound on $\|S\|$ follows from Lemma 35. The lower bound follows from Lemma 34 point (2) in the case where $|S| \geqslant 2$ and is clear for $|S| \leqslant 1$. □

The following lemma serves as a crucial technical argument in Section 8.

Lemma 37. *Let* $U=[ph]*\Phi, H=U\odot u$ *where* $p\in Q, h\in Z^*, |h|\geqslant 1, \Phi$ *is a* Q*-form,* $u\in X^*, |u|\leqslant k$. *Let us suppose that* $\|H\|\geqslant 1+k|Q|+\|\Phi\|$. *Then* $H=([ph]\odot u)*\Phi$ *where* $[ph]\odot u=[q\omega]$ *for some* $q\in Q, |\omega|\geqslant k$.

Intuitive meaning. If U is a deterministic rational series admitting a decomposition over the Q-form Φ and, $U\odot u$ has a sufficiently large norm (compared to $\|\Phi\|$ and $|u|$), then the action of u on U cannot have "touched" the form Φ.

Proof. If $|u|=0$ the conclusion of the lemma is clearly true. Let us suppose now that $|u|\geqslant 1$. By Lemma 22, one of the following 3 cases occurs.

Case 1: $\forall r\in Q, [phr]\odot u\notin\{\emptyset,\varepsilon\}$. Hence, $\exists\omega\in Z^*, \exists q\in Q, [ph]\odot u=[q\omega]$, $H=[q\omega]*\Phi$ and, by Lemma 35,

$$\|H\|\leqslant 1+|Q|(|\omega|-1)+\|\Phi\|.$$

The hypothesis about $\|H\|$ implies then: $|\omega|\geqslant k+1$. Hence $|\omega|\geqslant k$.

Case 2: $\exists q\in Q, \exists u', u''\in X^*, u=u'u'', \rho_\varepsilon([phq]\odot u')=\varepsilon$. Hence $H=\rho_\varepsilon(\Phi_q\odot u'')$ where $|u''|\leqslant k$.

Subcase 2.1: $u''=\varepsilon$. Then

$$\|H\|=\|\rho_\varepsilon(\Phi_q)\|\leqslant\|\Phi_q\|\leqslant\|\Phi\| < 1+k|Q|+\|\Phi\|.$$

Subcase 2.2: $u''\neq\varepsilon$. By Lemma 19, $\exists u_1'', u_2''\in X^*, u''=u_1''\cdot u_2''$, $\exists r\in Q$, $A\in Z, \Phi'$ Q-form such that

$$\rho_\varepsilon(\Phi_q\odot u_1'')=[rA]*\Phi'\in\mathsf{Q}_r(\Phi)\quad\text{and}\quad\Phi_q\odot u''=([rA]\odot u_2'')*\Phi'.$$

As $|u_2''|\leqslant|u''|\leqslant k$, we have

$$\begin{aligned}\|H\|=\|\Phi_q\odot u''\| &\leqslant \|[rA]*\Phi'\|+k|Q|\\ &\leqslant\|\Phi\|+k|Q|\\ &< 1+k|Q|+\|\Phi\|,\end{aligned}$$

contradicting the hypothesis about $\|H\|$. This case is impossible.

Case 3: $H=\emptyset$. This contradicts also the hypothesis. □

Lemma 38. *Let* $D\geqslant 0$. *Let* $\Phi=(\Phi_q)_{q\in Q}$ *be a* Q*-form and let* $S\in\mathsf{V}((\Phi_q)_{q\in Q})$ *such that*

(1) $\|\Phi\|\geqslant D+|Q|, |\Phi|\geqslant 2$,

(2) $\mathrm{rd}(S)\leqslant D$.

Then, $\exists\omega\in Z^*, \exists p\in Q, S=[p\omega]*\Phi$.

Proof. Let S fulfilling hypotheses (1), (2). S can be written as

$$S=\sum_{q\in Q}\alpha_q\Phi_q=[p_0\omega_0]*\Psi,$$

where $([p_0\omega_0], \Psi)$ is the minimal decomposition of S. Let us suppose that

$$\exists q \in Q, \exists u \in \alpha_q, \exists u', u'' \in V^*, \quad u = u' \cdot u'' \text{ and } S \bullet u' = \Psi_q. \tag{46}$$

Then $\Psi_q \bullet u'' = \Phi_q$, hence $\mathsf{Q}_r(\Psi) \supseteq \mathsf{Q}_r(\Phi_q)$.
As $|\Phi| \geqslant 2$, we must have $\Phi = [A] * \Phi'$ for some $A \in Z, |\Phi'| \geqslant 1$, hence by Lemma 33

$$\begin{aligned}\mathsf{Q}_r(\Phi) &= \{[sA] * \Phi' \mid s \in Q\} \,\dot{\cup}\, \mathsf{Q}_r(\Phi') \\ &= \{[sA] * \Phi' \mid s \in Q - \{q\}\} \,\dot{\cup}\, \mathsf{Q}(\Phi_q)\end{aligned}$$

which shows that

$$\mathrm{Card}(\mathsf{Q}(\Phi_q)) = \mathrm{Card}(\mathsf{Q}_r(\Phi)) - |Q| + 1. \tag{47}$$

From (47) and the fact that $\mathsf{Q}_r(\Psi) \supseteq \mathsf{Q}(\Phi_q)$ we draw

$$\|\Psi\| \geqslant \|\Phi_q\| = \|\Phi\| - |Q| + 1 \geqslant D + 1.$$

But this contradicts the fact that $\mathrm{rd}(S) \leqslant D$. We have established that (46) is impossible. In other words, case (2) of Lemma 11 cannot occur in the action of a word $u \in \alpha_q$ on the linear combination $\sum_{q \in Q} \alpha_q \Phi_q$. Hence $\forall q \in Q,\ \forall u \in \alpha_q, \exists q_u \in Q, \omega_u \in Z^+$,

$$S \bullet u = ([p_0\omega_0] \bullet u) * \Psi = [q_u\omega_u] * \Psi. \tag{48}$$

Let us notice that ω_0, ω_u have the same rightmost letter A_0. For such q, u, by Eq. (47) we have the following equality

$$\|S \bullet u\| = \|\Phi_q\| = \|\Phi\| - |Q| + 1. \tag{49}$$

The minimality of decomposition $([p_0\omega_0], \Psi)$ implies that $\|[A_0] * \Psi\| > \|\Psi\|$, the hypothesis $|\Phi| \geqslant 2$ implies $|\Phi_q| \geqslant 2$ hence $|\omega_u| \geqslant 2$ and Lemma 34 point (2) gives the equality

$$\|S \bullet u\| = 1 + |Q|(|\omega_u| - 2) + \|[A_0] * \Psi\| \tag{50}$$

These two Eqs. (49), (50) show that there exists some unique integer $1 \leqslant n \leqslant |\omega|$, such that

$$\forall q \in Q, \forall u \in \alpha_q, \quad |\omega_u| = n.$$

But all the words ω_u are suffixes of ω_0, hence there exists some unique words $\omega_0', \omega_0'', \omega_0 = \omega_0' \cdot \omega_0''$ such that

$$\forall q \in Q, \forall u \in \alpha_q, \quad S \bullet u = [q\omega_0''] * \Psi = \Phi_q.$$

It follows that $[\omega_0''] * \Psi = \Phi$ and $S = [p_0\omega_0'] * \Phi$. $\square$

3.4. Derivations

3.4.1. Ordinary derivations

A sequence of deterministic series $S_0, S_1, \ldots, S_n$ is a *derivation* iff there exist $x_1, \ldots, x_n \in X$ such that $S_0 \odot x_1 = S_1, \ldots, S_{n-1} \odot x_n = S_n$. The *length* of this derivation is n. If $u = x_1 \cdot x_2 \cdot \cdots \cdot x_n$ we call $S_0, S_1, \ldots, S_n$ the derivation *associated* with (S_0, u). We denote this derivation by $S_0 \xrightarrow{u} S_n$.

A derivation $S_0, S_1, \ldots, S_n$ is said to be a *sub-derivation* of a derivation $S'_0, S'_1, \ldots, S'_m$ iff there exists some $i \in [0, m]$ such that, $\forall j \in [1, n]$, $S_j = S'_{i+j}$.

3.4.2. Stacking derivations

Let us adapt the usual notion of *stacking derivation* to derivations of series. For every $u \in X^*$ we define the binary relation $\uparrow(u)$ over $\mathsf{DB}\langle\langle V\rangle\rangle$ by for every $S, S' \in \mathsf{DB}\langle\langle V\rangle\rangle$, $S \uparrow(u) S' \Leftrightarrow \exists A \in Z, \omega \in Z^+, p, q \in Q, \Psi \in \mathsf{DB}_{Q,1}\langle\langle V\rangle\rangle$ such that

$$S = [pA] * \Psi, \quad [pA] \odot u = [q\omega], \quad S' = [q\omega] * \Psi.$$

It is clear that if $S \uparrow(u) S'$ then $S \odot u = S'$ and that the converse is not true in general. A derivation $S_0, S_1, \ldots, S_n$ is said to be *stacking* iff it is the derivation associated to a pair (S, u) such that $S = S_0$ and $S_0 \uparrow(u) S_n$.

Definition 39. A vector $S \in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$ is said *loop-free* if and only if for every $v \in V^+$, $S \bullet v \neq S$.

Let us notice that every polynomial is loop-free. The two following lemmas give other examples of loop-free vectors.

Lemma 40. *Let* $\alpha \in \mathsf{DB}_{1,n}\langle V\rangle, \Phi \in \mathsf{B}_{n,\lambda}\langle\langle V\rangle\rangle$, *such that* $\infty > \|\alpha \cdot \Phi\| > \|\Phi\|$. *Then* $\alpha \cdot \Phi$ *is loop-free.*

Proof. Let α, Φ fulfill the hypothesis of the lemma and suppose, for sake of contradiction, that there exists some $v \in V^+$ such that

$$(\alpha \cdot \Phi) \bullet v = \alpha \cdot \Phi.$$

By induction, for every $n \geqslant 0$,

$$(\alpha \cdot \Phi) \bullet v^n = \alpha \cdot \Phi. \tag{51}$$

As α is a polynomial, there exists some $n_0 \geqslant 0$ such that $|v^{n_0}|$ is greater than the greatest length of a monomial of α. Using Lemma 10, equality (51) for such an integer n_0 means that there exists some $k \in [1, n], v''$ suffix of v^{n_0} such that

$$\Phi_k \bullet v'' = \alpha \cdot \Phi. \tag{52}$$

Using the hypothesis of the lemma we conclude that

$$\|\Phi\| \geqslant \|\Phi_k \bullet v''\| = \|\alpha \cdot \Phi\| > \|\Phi\|$$

which is contradictory. □

Lemma 41. *Let* $S \in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle, u \in X^*$, *such that* $\|S \odot u\| > \|S\|$. *Then* $S \odot u$ *is loop-free.*

Proof. Let us consider S, u fulfilling the hypothesis of the lemma and let us consider the 3 possible forms of $S \odot u$ proposed by Lemma 19. Forms (1) or (2) are incompatible with the inequality $\|S \odot u\| > \|S\|$. Hence $S \odot u$ has the form (3)

$$u = u_1 \cdot u_2, \quad \rho_\varepsilon(S) \odot u_1 = S \bullet v_1 = [qA] \cdot \Phi, \quad S \odot u = ([qA] \odot u_2) \cdot \Phi$$

where

$$u_1, u_2 \in X^*, v_1 \in V^*, q \in Q, \ A \in Z.$$

Hence $S \odot u = \alpha \cdot \Phi$ for some polynomial $\alpha \in \mathsf{DRB}_{1,Q}\langle V\rangle$. As for every $r \in Q$, $\Phi_r = S \bullet (v_1[qAr])$, we obtain that $\|S\| \geqslant \|\Phi\|$. Finally,

$$\infty > \|S \odot u\| = \|\alpha \cdot \Phi\| > \|S\| \geqslant \|\Phi\|,$$

and by Lemma 40, $S \odot u$ is loop-free. □

Lemma 42. *Let* $S \in \mathsf{DRB}\langle\langle V\rangle\rangle, w \in X^*$, *such that*
(1) S *is* ε*-free and loop-free,*
(2) $\forall v \leqslant w, \|S \odot v\| \geqslant \|S\|$. *Then the derivation* $S \xrightarrow{w} S \odot w$ *is stacking.*

Proof. S is left-deterministic. If it has type $\emptyset$ or (ε, j), the lemma is trivially true. Otherwise

$$S = [qA] \cdot \Phi$$

for some $q \in Q$, $A \in Z$ and some matrix $\Phi \in \mathsf{DRB}_{Q,\lambda}\langle\langle V\rangle\rangle$. Suppose that for some prefix $u \leqslant w$ and $r \in Q$,

$$[qAr] \odot u = \varepsilon. \tag{53}$$

As S is ε-free, we must have $u \neq \varepsilon$.

Then, $S \odot u = \rho_\varepsilon(\Phi_r)$ so that

$$\|S \odot u\| \leqslant \|\rho_\varepsilon(\Phi_r)\| \leqslant \|\rho_\varepsilon(\Phi)\| \leqslant \|\Phi\| \leqslant \|S\|$$

which shows that $S = S \odot u$ while $u \neq \varepsilon$. This would contradict the hypothesis that S is loop-free, hence (53) is impossible.

Let us apply now Lemma 22 to the expression $([qA]\cdot\Phi)\odot w$: case (2) is impossible, hence

$$([qA]\cdot\Phi)\odot w = ([qA]\odot w)\cdot\Phi,$$

which is equivalent to

$$S\uparrow(w)S\odot w. \qquad \square$$

Lemma 43. *Let* $S,S'\in\mathsf{DRB}\langle\langle V\rangle\rangle, w\in X^*, k\in\mathbb{N}$, *such that* $S\odot w=S'$ *and* $\|S'\|\geqslant\|S\|+k\cdot K_0+1$. *Then the derivation* $S\xrightarrow{w}S'$ *contains some stacking sub-derivation of length* k.

Proof. Let $S=S_0,\ldots,S_i,\ldots,S_n$ be the derivation associated to (S,w). Let $i_0=\max\{i\in[0,n]\mid\|S_i\|=\min\{\|S_j\|\mid 0\leqslant j\leqslant n\}\}$ and $i_1=\max\{i\in[i_0+1,n]\mid\|S_i\|=\min\{\|S_j\|\mid i_0+1\leqslant j\leqslant n\}\}$. Let $w=w_0w_1w'$ where $|w_0|=i_0, |w_0w_1|=i_1$.

As $\|S\odot w_0w_1\|>\|S\odot w_0\|$, by Lemma 41, $S\odot w_0w_1=S_{i_1}$ is loop-free. Using Lemma 32, point (3):

$$\|S_n\|-\|S_{i_1}\|\geqslant\|S_n\|-\|S_{i_0}\|-(\|S_{i_1}\|-\|S_0\|)\geqslant(k-1)\cdot K_0+1.$$

Using Lemma 32, point (3), we must have $|w'|\geqslant k$. Let $w'=w_2w_3$ with $|w_2|=k$. By definition of i_1, $\forall i\in[i_1+1,i_1+k], \|S_i\|\geqslant\|S_{i_1}\|+1$.

By Lemma 42, the sub-derivation $S_{i_1},\ldots,S_{i_1+k}$ (associated to (S_{i_1},w_2)) is stacking. $\square$

Lemma 44. *Let* $S,S'\in\mathsf{DRB}\langle\langle V\rangle\rangle, w\in X^*, k,d,d'\in\mathbb{N}$, *such that* S *is* ε-*free*, (d,d')-*linear and*

(1) *the derivation* $S\xrightarrow{w}S'$ *contains no stacking sub-derivation of length* k.

(2) $|w|\geqslant d\cdot k$.

Then S' *is* $(0,d')$-*linear.*

Proof. If S is $(0,d')$-linear, then the result follows from Lemma 32, point (1). Otherwise,

$$S=\sum_{q\in Q}[p\omega q][qeq]T_q$$

for some $\omega\in Z^*, 1\leqslant|\omega|\leqslant d, (T_q)_{q\in Q}\in\mathsf{DRB}_{Q,1}\langle\langle V\rangle\rangle$ such that $\forall q\in Q, \mathrm{rd}(T_q)\leqslant d'$. Let $S\xrightarrow{w}S'=(S_0,\ldots,S_n)$. By induction on l, using hypothesis (1), one can show that: for every $l\in[0,|\omega|-1]$, either

$$\exists m\leqslant k\cdot l, \exists\omega_m\in Z^+, \quad |\omega_m|\leqslant|\omega|-l \text{ and } S_m=\sum_{q\in Q}[p_m\omega_m q][qeq]T_q$$

or

$$\exists m\leqslant k\cdot l, \exists q\in Q, \quad S_m=\rho_\varepsilon(T_q).$$

Similarly,

$$\exists m_0 \leqslant k \cdot |\omega|, \exists q \in Q, \quad S_{m_0} = \rho_\varepsilon(T_q).$$

Hence S_{m_0} is $(0,d')$-linear. Using Lemma 32 we obtain that $S_n = S'$ is $(0,d')$-linear too. $\square$

4. Deduction systems

4.1. General formal systems

We follow here the general philosophy of [14,34]. For any set $\mathscr{E}$, we denote by $\mathscr{P}(\mathscr{E})$ the set of its subsets and by $\mathscr{P}_f(\mathscr{E})$ the set of it *finite* subsets.

Let us call *formal system* any triple $\mathscr{D} = \langle \mathscr{A}, H, \mid\!\!-\!\!- \rangle$ where $\mathscr{A}$ is a denumerable set called the *set of assertions*, H, the *cost function* a mapping $\mathscr{A} \to \mathbb{N} \cup \{\infty\}$ and $\mid\!\!-\!\!-$, the *deduction relation* is a subset of $\mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$.

$\mathscr{A}$ is given with a fixed bijection with $\mathbb{N}$ (an "encoding" or "Gödel numbering") so that the notions of recursive subset, recursively enumerable subset, recursive function, … over $\mathscr{A}, \mathscr{P}_f(\mathscr{A}), \ldots$ are defined, up to this fixed bijection; we assume that $\mathscr{D}$ satisfies the following axiom:

(A1) $\forall (P,A) \in \mid\!\!-\!\!-$, $(\min\{H(p), p \in P\} < H(A))$ or $(H(A) = \infty)$.
(We let $\min(\emptyset) = \infty$.) We call $\mathscr{D}$ a *deduction system* iff $\mathscr{D}$ is a formal system satisfying the additional axiom:

(A2) $\mid\!\!-\!\!-$ is recursively enumerable.

In the sequel, we use the notation $P \mid\!\!-\!\!- A$ for $(P,A) \in \mid\!\!-\!\!-$. We call *proof* in the system $\mathscr{D}$, *relative to the set of hypotheses* $\mathscr{H} \subseteq \mathscr{A}$, any subset $P \subseteq \mathscr{A}$ fulfilling:

$$\forall p \in P, (\exists Q \subseteq P, Q \mid\!\!-\!\!- p) \text{ or } (p \in \mathscr{H}).$$

We call P a *proof* iff

$$\forall p \in P, (\exists Q \subseteq P, Q \mid\!\!-\!\!- p)$$

(i.e. iff P is a proof relative to $\emptyset$).

Let us define the total map $\chi : \mathscr{A} \to \{0,1\}$ and the partial map $\overline{\chi} : \mathscr{A} \to \{0,1\}$ by

$$\chi(A) = 1 \text{ if } H(A) = \infty, \chi(A) = 0 \text{ if } H(A) < \infty,$$

$$\overline{\chi}(A) = 1 \text{ if } H(A) = \infty, \overline{\chi} \text{ is undefined if } H(A) < \infty.$$

(χ is the "truth-value function", $\overline{\chi}$ is the "1-value function".)

Lemma 45. *Let P be a proof relative to $\mathscr{H} \subseteq H^{-1}(\infty)$ and $A \in P$. Then $\chi(A) = 1$.*

In other words, if an assertion is provable from true hypotheses, then it is true.

Proof. Let P be a proof. We prove by induction on n that

$$\mathscr{P}(n): \forall p \in P, H(p) \geqslant n.$$

It is clear that, $\forall p \in P, H(p) \geqslant 0$. Suppose that $\mathscr{P}(n)$ is true. Let $p \in P - \mathscr{H}: \exists Q \subseteq P$, $Q \vdash\!\!- p$. By induction hypothesis, $\forall q \in Q, H(q) \geqslant n$ and by (A1), $H(p) \geqslant n+1$. It follows that: $\forall p \in P - \mathscr{H}, H(p) = \infty$. But by hypothesis, $\forall p \in \mathscr{H}, H(p) = \infty$. □

A formal system $\mathscr{D}$ will be said *complete* iff, conversely, $\forall A \in \mathscr{A}, \chi(A) = 1 \Rightarrow$ there exists some *finite* proof P such that $A \in P$. (In other words, $\mathscr{D}$ is complete iff every true assertion is "finitely" provable.)

Lemma 46. *If $\mathscr{D}$ is a complete deduction system, $\overline{\chi}$ is a recursive partial map.*

Proof. Let $i \mapsto P_i$ be some recursive function whose domain is $\mathbb{N}$ and whose image is $\mathscr{P}_f(\mathscr{A})$. Let $h: (\mathscr{P}_f(\mathscr{A}) \times \mathscr{A} \times \mathbb{N}) \to \{0,1\}$ be a total recursive function such that

$$P \vdash\!\!- A \text{ iff } \exists n \in \mathbb{N}, \quad h(P, A, n) = 1$$

(such an h exists, because the r.e. sets are the projections of the recursive sets, see [58]).

The following (informal) semi-algorithm computes $\overline{\chi}$ on the assertion A:

(1) i := 0; n := 0; $s := i + n$;
(2) $P := P_i$;
(3) $b := \min_{p \in P}\{\max_{Q \subseteq P}\{h(Q, p, n)\}\}$;
(4) $c := (A \in P)$;
(5) `if` $(b \wedge c)$ `then` ($\overline{\chi}(A) = 1$; `stop`);
(6) `if` $i = 0$ `then` ($i := s + 1$; $n := 0$; $s := i + n$)
    `else` ($i := i - 1$; $n := n + 1$);
(7) `goto 2;` □

In words, the property "$H(A) = \infty$" is semi-decidable just because the property "there exists a finite P such that P is a $\mathscr{D}$-proof and $A \in P$" is semi-decidable too.

In order to define deduction relations from more elementary ones, we set the following definitions.

Let $\vdash\!\!- \subseteq \mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$. For every $P, Q \in \mathscr{P}_f(\mathscr{A})$ we set

- $P \overset{[0]}{\vdash\!\!-} Q$ iff $P \supseteq Q$,
- $P \overset{[1]}{\vdash\!\!-} Q$ iff $\forall q \in Q, \exists R \subseteq P, R \vdash\!\!- q$,
- $P \overset{\langle 0 \rangle}{\vdash\!\!-} Q$ iff $P \overset{[0]}{\vdash\!\!-} Q$,
- $P \overset{\langle 1 \rangle}{\vdash\!\!-} Q$ iff $\forall q \in Q, (\exists R \subseteq P, R \vdash\!\!- q)$ or $(q \in P)$,
- $P \overset{\langle n+1 \rangle}{\vdash\!\!-} Q$ iff $\exists R \in \mathscr{P}_f(\mathscr{A}), P \overset{\langle 1 \rangle}{\vdash\!\!-} R$ and $R \overset{\langle n \rangle}{\vdash\!\!-} Q$ (for every $n \geqslant 0$),
- $\overset{\langle * \rangle}{\vdash\!\!-} = \bigcup_{n \geqslant 0} \overset{\langle n \rangle}{\vdash\!\!-}$.

Given $\vdash\!\!-_1, \vdash\!\!-_2 \subseteq \mathscr{P}_f(\mathscr{A}) \times \mathscr{P}_f(\mathscr{A})$, for every $P, Q \in \mathscr{P}_f(\mathscr{A})$ we set

$$P(\vdash\!\!-_1 \circ \vdash\!\!-_2)Q \text{ iff } \exists R \subseteq \mathscr{A}, (P \vdash\!\!-_1 R) \wedge (R \vdash\!\!-_2 Q).$$

The particular deduction systems $\mathscr{D}_i = \langle \mathscr{A}_i, H_i, \vdash\!\!-_{\mathscr{D}_i} \rangle$ ($i \in [0,5]$), that we shall introduce in Sections 4.3 and 10, will always be defined from simpler binary relations $\Vdash\!\!-_j$ by means of the above constructions.

The key statement of this work is that a particular deduction system, $\mathscr{D}_0$ (defined in Section 4.3), is complete (Theorem 86). We prove this completeness result by exhibiting a "strategy" $\mathscr{S}$ which, for every true assertion constructs a finite $\mathscr{D}_0$-proof of this assertion. Notice that, by Lemma 46 we do not need to prove that $\mathscr{S}$ is computable in any sense to establish that $\overline{\chi}$ is partial-recursive.

4.2. *Strategies*

Let $\mathscr{D} = \langle \mathscr{A}, H, \vdash\!\!- \rangle$ be a deduction system. We call a *strategy* for $\mathscr{D}$ any partial map $\mathscr{S} : \mathscr{A}^+ \to \mathscr{A}^*$ such that

(S1) if $\mathscr{S}(A_1A_2\cdots A_n) = B_1\cdots B_m$ then $\exists Q \subseteq \{A_i \mid 1 \leqslant i \leqslant n-1\}$ such that

$$\{B_j \mid 1 \leqslant j \leqslant m\} \cup Q \vdash\!\!- A_n,$$

(S2) if $\mathscr{S}(A_1A_2\cdots A_n) = B_1\cdots B_m$ then

$$\min\{H(A_i) \mid 1 \leqslant i \leqslant n\} = \infty \Rightarrow \min\{H(B_j) \mid 1 \leqslant j \leqslant m\} = \infty.$$

Remark 47. Axiom (A1) on systems is similar to the "monotonicity" condition of [34] or axiom (2.4.2′) of [14].

Axiom (S2) on strategies is similar to the "validity" condition of [34] or property (2.4.1′) of [14]. Notice that (S2) is imposed on the strategy $\mathscr{S}$ only, but not on the inverse of the deduction relation $(\vdash\!\!-)^{-1}$. The trick is that $(\vdash\!\!-)^{-1}$ is not valid in general (see the rules R2, $R''6$ of $\mathscr{D}_0$, Section 4.3) but is computable while $\mathscr{S}$ is valid but is *not required to be computable* in general. In the case of $\mathscr{D}_0$, the strategy $\mathscr{S}_{ABC}$ defined below (Section 7) turns out to be computable but we shall not be in position to show this computability property before knowing that $\mathscr{D}_0$ is complete (i.e. Theorem 86).

Given a strategy $\mathscr{S}$, we define $\mathscr{T}(\mathscr{S}, A)$, the proof-tree associated to the strategy $\mathscr{S}$ and the assertion A as the unique tree t such that

$$\varepsilon \in \text{dom}(t),\ \ t(\varepsilon) = A,$$

and, for every path $x_0x_1, \ldots, x_{n-1}$ in t, with labels $t(x_i) = A_{i+1}$ (for $0 \leqslant i \leqslant n-1$) if x_{n-1} has m sons $x_{n-1}\cdot 1, \ldots, x_{n-1}\cdot m \in \text{dom}(t)$ with labels $t(x_{n-1}\cdot j) = B_j$ (for $1 \leqslant j \leqslant m$) then

$$(\forall i \in [1, n-1], A_i \neq A_n \text{ and } \mathscr{S}(A_1\cdots A_n) = B_1\cdots B_m)$$

or

$$(\exists i \in [1, n-1], A_i = A_n \text{ and } m = 0)$$

or

$$(A_1 \cdots A_n \notin \mathrm{dom}(\mathscr{S}) \text{ and } m = 0). \tag{54}$$

Notice that x_{n-1} is a leaf (i.e. $m=0$) iff:

$$(\mathscr{S}(A_1 \cdots A_n) = \varepsilon) \text{ or } (\exists i \in [1, n-1], A_i = A_n) \text{ or } (A_1 \cdots A_n \notin \mathrm{dom}(\mathscr{S})). \tag{55}$$

Let us say that $\mathscr{S}$ *terminates* iff, $\forall A \in \chi^{-1}(1), \mathscr{T}(\mathscr{S}, A)$ is finite; $\mathscr{S}$ is said *closed* iff, $\forall W \in (\chi^{-1}(1))^+, W \in \mathrm{dom}(\mathscr{S})$ (i.e. $\mathscr{S}$ is defined on every non-empty sequence of true assertions). For every tree t let us define

$$\mathscr{L}(t) = \{t(x) | \forall y \in \mathrm{dom}(t), x \preceq Y \Rightarrow x = y\},$$

$$\mathscr{I}(t) = \{t(x) \mid \exists y \in \mathrm{dom}(t), x \prec y\}.$$

(Here $\mathscr{L}$ stands for “leaves” and $\mathscr{I}$ stands for “internal labels”.)

Lemma 48. *If $\mathscr{S}$ is a strategy for the deduction-system $\mathscr{D}$ then, for every true assertion A*

(1) *the set of labels of $\mathscr{T}(\mathscr{S}, A)$ is a $\mathscr{D}$-proof, relative to the set $\mathscr{L}(\mathscr{T}(\mathscr{S}, A)) - \mathscr{I}(\mathscr{T}(\mathscr{S}, A))$.*

(2) *every label of a leaf is true.*

Proof. Let us suppose that $H(A) = \infty$. Let $t = \mathscr{T}(\mathscr{S}, A)$, $P = \mathrm{im}(t)$ (the set of labels of t), $\mathscr{H} = \mathscr{L}(\mathscr{T}(\mathscr{S}, A)) - \mathscr{I}(\mathscr{T}(\mathscr{S}, A))$.

Using (S2), one can prove by induction on the depth of $x \in \mathrm{dom}(t)$ that, $H(t(x)) = \infty$. Point (2) is then proved. Let x be an internal node of t, with sons $x \cdot 1, x \cdot 2, \ldots, x \cdot m$ ($m \geqslant 0$), and with ancestors $y_1, y_2, \ldots, y_{n-1}, y_n = x$ ($n \geqslant 1$), such that

$$t(y_1) \cdots t(y_n) = A_1 \cdots A_n, \quad t(x \cdot 1) t(x \cdot 2) \cdots t(x \cdot m) = B_1 \cdot B_2 \cdots B_m.$$

By definition of $\mathscr{T}(\mathscr{S}, A)$,

$$\mathscr{S}(A_1 \cdots A_n) = B_1 \cdots B_m$$

and by condition (S1):

$$\exists Q \subseteq \{A_i \mid 1 \leqslant i \leqslant n-1\}, \text{ such that } \{B_j \mid 1 \leqslant j \leqslant m\} \cup Q \mid\!\!-\!\!- A_n.$$

It follows that for every $p \notin \mathscr{H}, \exists R \subseteq P, R \mid\!\!-\!\!- p$, hence

$$\forall p \in P, (\exists R \subseteq P, R \mid\!\!-\!\!- p) \text{ or } p \in \mathscr{H}.$$

Point (1) is proved. □

Lemma 49. *If $\mathscr{S}$ is a closed strategy for $\mathscr{D}$, then, for every true assertion A, the set of labels of $\mathscr{T}(\mathscr{S}, A)$ is a $\mathscr{D}$-proof.*

Proof. Let us suppose that $H(A)=\infty$. Let $t=\mathscr{T}(\mathscr{S},A)$ and let $P,\mathscr{H}$ be defined as above. By Lemma 48, P is a $\mathscr{D}$-proof relative to $\mathscr{H}$. By Lemma 48 point (2) and Lemma 45, every label of a node of t is true. By the definition of a closed strategy, if $p\in\mathscr{H}$ and x is a leaf of t such that $p=t(x)$ then, the only possible true assertion in clause (55) is "$\mathscr{S}(A_1\cdots A_n)=\varepsilon$", which implies that

$$\exists Q\subseteq P,\quad Q\mid\!\!-\!\!- t(x).$$

Lemma 48 point (1) and this fact show that P is a proof. □

Lemma 50. *If $\mathscr{D}$ admits some terminating, closed strategy then $\mathscr{D}$ is complete.*

Proof. Clear from Lemma 49. □

Remark 51. By the same arguments, if $\mathscr{D}$ admits some closed (but not necessarily terminating) strategy then $\mathscr{D}$ is ∞-complete in the sense that every true assertion has a $\mathscr{D}$-proof. This might be helpful only in cases where the proof-trees and the associated proofs are *regular* in a reasonable sense. This point of view will not be developed here. The comparison algorithms based on Valiant's methods of alternate-staking or parallel-stacking, might be seen in this way (this idea is due to B. Courcelle, thanks to him and to M. Oyamaguchi for discussions on this subject).

4.3. System $\mathscr{D}_0$

Let us define here a particular deduction system $\mathscr{D}_0$ "Taylored for the equivalence problem for dpda's".

Given a fixed dpda $\mathscr{M}$ over the terminal alphabet X, we consider the variable alphabet V associated to $\mathscr{M}$ (see Section 3.1) and the set $\mathsf{DRB}\langle\langle V\rangle\rangle$ (the set of Deterministic Rational Boolean series over V^*). The set of assertions is defined by

$$\mathscr{A}=\mathbb{N}\times\mathsf{DRB}\langle\langle V\rangle\rangle\times\mathsf{DRB}\langle\langle V\rangle\rangle$$

i.e. an assertion is here a *weighted equation* over $\mathsf{DRB}\langle\langle V\rangle\rangle$.

The "cost-function" $H:\mathscr{A}\to\mathbb{N}\cup\{\infty\}$ is defined by

$$H(n,S,S')=n+2\cdot\mathrm{Div}(S,S').$$

We recall $\mathrm{Div}(S,S')$, the *divergence* between S and S', is defined by

$$\mathrm{Div}(S,S')=\inf\{|u|\mid u\in\varphi(S)\triangle\varphi(S')\}$$

(See Definition 27).

Let us notice that here

$$\chi(n,S,S')=1\Leftrightarrow S\equiv S'.$$

We define a binary relation $\Vdash\!\!-\; \subset \mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:
(R0)

$$\{(p,S,T)\} \Vdash\!\!- (p+1,S,T)$$

for $p \in \mathbb{N}, S, T \in \mathsf{DRB}\langle\langle V \rangle\rangle$,
(R1)

$$\{(p,S,T)\} \Vdash\!\!- (p,T,S)$$

for $p \in \mathbb{N}, S, T \in \mathsf{DRB}\langle\langle V \rangle\rangle$,
(R2)

$$\{(p,S,S'),(p,S',S'')\} \Vdash\!\!- (p,S,S'')$$

for $p \in \mathbb{N},\ S, S', S'' \in \mathsf{DRB}\langle\langle V \rangle\rangle$,
(R3)

$$\emptyset \Vdash\!\!- (0,S,S)$$

for $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$,
(R′3)

$$\emptyset \Vdash\!\!- (0,S,T)$$

for $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$, $T \in \{\emptyset, \varepsilon\}$, $S \equiv T$,
(R4)

$$\{(p+1, S \odot x, T \odot x) \mid x \in X\} \Vdash\!\!- (p,S,T)$$

for $p \in \mathbb{N}, S, T \in \mathsf{DRB}\langle\langle V \rangle\rangle$, $(S \not\equiv \varepsilon \wedge T \not\equiv \varepsilon)$,
(R5)

$$\{(p,S,S')\} \Vdash\!\!- (p+2, S \odot x, S' \odot x)$$

for $p \in \mathbb{N}, S, T \in \mathsf{DRB}\langle\langle V \rangle\rangle$, $x \in X$,
(R6)

$$\{(p, S \cdot T' + S', T')\} \Vdash\!\!- (p, S^* \cdot S', T')$$

for $p \in \mathbb{N}, (S,S') \in \mathsf{DRB}_{1,2}\langle\langle V \rangle\rangle, T' \in \mathsf{DRB}\langle\langle V \rangle\rangle$, $S \not\equiv \varepsilon$,
(R7)

$$\{(p,S,S'),(p,T,T')\} \Vdash\!\!- (p, S+T, S'+T')$$

for $p \in \mathbb{N}, (S,T),(S',T') \in \mathsf{DRB}_{1,2}\langle\langle V \rangle\rangle$,
(R8)

$$\{(p,S,S')\} \Vdash\!\!- (p, S \cdot T, S' \cdot T)$$

for $p \in \mathbb{N}, S, S', T \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

(R9)

$$\{(p, T, T')\} \Vdash\!\!- (p, S \cdot T, S \cdot T')$$

for $p \in \mathbb{N}, S, T, T' \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

(R10)

$$\emptyset \Vdash\!\!- (0, S, \rho_\varepsilon(S))$$

for $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

(R11)

$$\emptyset \Vdash\!\!- (0, S, \rho_e(S))$$

for $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$.

Remark 52. (1) We do not claim that this system is *minimal*. This system is devised so as to simplify (as much as we can) the *proof* of completeness. Successive simplifications of the system itself will be achieved later on, in Section 10.

(2) One can check that, by the results of Section 3, the above rules really belong to $\mathcal{P}_f(\mathcal{A}) \times \mathcal{A}$.

Lemma 53. *Let* $P \in \mathcal{P}_f(\mathcal{A}), A \in \mathcal{A}$ *such that* $P \Vdash\!\!- A$. *Then* $\min\{H(p) \mid p \in P\} \leqslant H(A)$.

Proof. Let us check this property for every type of rule.

R0. $p + 2 \cdot \mathrm{Div}(S, T) \leqslant p + 1 + 2 \cdot \mathrm{Div}(S, T)$.

R1. $p + 2 \cdot \mathrm{Div}(S, T) = p + 2 \cdot \mathrm{Div}(T, S)$.

R2. as the weight p is the same in all the considered equations, we are reduced to prove that

$$\forall n \in \mathbb{N},\ S \equiv_n S' \wedge S' \equiv_n S'' \Rightarrow S \equiv_n S'' \quad \text{(obvious)}$$

R3, R′3. $\infty = \mathrm{Div}(S, S)$.

R4. Let $S, T \in \mathsf{DRB}\langle\langle V \rangle\rangle, S \not\equiv \varepsilon, T \not\equiv \varepsilon$. If $\mathrm{Div}(S, T) = \infty$ the required inequality is true. If $\mathrm{Div}(S, T) = n \in \mathbb{N}$, let us consider some $u \in \varphi(S) \triangle \varphi(T), |u| = n$. We can suppose, for example, that $S \odot u = \varepsilon, T \odot u \neq \varepsilon$. As $S \not\equiv \varepsilon, \exists x_0 \in X, \exists v \in X^*$, $u = x_0 \cdot v$. Hence $(S \odot x_0) \odot v = \varepsilon, (T \odot x_0) \odot v \neq \varepsilon, \mathrm{Div}(S \odot x_0, T \odot x_0) \leqslant |v|$. Hence we have

$$\begin{aligned}
\min\{H(p+1, S \odot x, T \odot x) \mid x \in X)\} &\leqslant H(p+1, S \odot x_0, T \odot x_0) \\
&\leqslant (p+1) + 2|v| \\
&\leqslant (p+1) + 2 \cdot \mathrm{Div}(S, T) - 2 \\
&< H(p, S, T).
\end{aligned}$$

R5. Let us suppose $H(p+2, S\odot x, S'\odot x)\neq\infty$. Let $\mathrm{Div}(S\odot x, S'\odot x)=n, v\in(\varphi(S\odot x)\triangle\varphi(S'\odot x))$. As $xv\in\varphi(S)\triangle\varphi(S')$, $\mathrm{Div}(S,S')\leqslant n+1$. Hence,

$$H(p,S,S')\leqslant p+2(n+1)=(p+2)+2\cdot\mathrm{Div}(S\odot x, S'\odot x)$$

$$=H(p+2, S\odot x, S'\odot x).$$

R7. $S\equiv_n S', T\equiv_n T'\Rightarrow S+T\equiv_n S'+T'$ (obvious).

R8. $S\equiv_n S'\Rightarrow S\cdot T\equiv_n S'\cdot T$ (clear because, every prefix of a word of length $\leqslant n$ has length $\leqslant n$).

R9. $T\equiv_n T'\Rightarrow S\cdot T\equiv_n S\cdot T'$ (analogous to R8).

R6. We are reduced to prove that, for every $n\geqslant 0, (S,S')\in\mathsf{DRB}_{1,2}\langle\langle V\rangle\rangle, T'\in\mathsf{DRB}\langle\langle V\rangle\rangle, S\not\equiv\varepsilon$,

$$S\cdot T'+S'\equiv_n T'\Rightarrow S^*\cdot S'\equiv_n T'. \tag{56}$$

Let us suppose that $S\cdot T'+S'\equiv_n T'$. By definition of the star operation:

$$S^{n+1}\cdot S^*\cdot S'+\sum_{k=0}^{n} S^k\cdot S'=S^*\cdot S'. \tag{57}$$

And by the properties established in the treatment of (R7), (R9):

$$S^{n+1}\cdot T'+\sum_{k=0}^{n} S^k\cdot S'\equiv_n T'. \tag{58}$$

Let $u\in X^{\leqslant n}$, $u\neq\varepsilon$. As $S\not\equiv\varepsilon, \forall u'\preccurlyeq u$, $\rho_\varepsilon(S^{n+1}\odot u')\neq\varepsilon$. By Lemma 22, for every $U\in\mathsf{B}\langle\langle V\rangle\rangle$,

$$(S^{n+1}\cdot U)\odot u=(S^{n+1}\odot u)\cdot U\neq\varepsilon.$$

Using now Eqs. (57), (58) we obtain that

$$S^*\cdot S'\odot u=\varepsilon\Leftrightarrow\sum_{k=0}^{n} S^k\cdot S'\odot u=\varepsilon\Leftrightarrow T'\odot u=\varepsilon.$$

As well

$$\rho_\varepsilon(S^*\cdot S')=\varepsilon\Leftrightarrow\rho_\varepsilon\left(\sum_{k=0}^{n} S^k\cdot S'\right)=\varepsilon\Leftrightarrow\rho_\varepsilon(T')=\varepsilon.$$

At last,

$$\{u\in X^{\leqslant n}\mid\rho_\varepsilon(S^*\cdot S'\odot u)=\varepsilon\}=\{u\in X^{\leqslant n}\mid\rho_\varepsilon(T'\odot u)=\varepsilon\},$$

which, by property (38), shows that $S^* \cdot S' \equiv_n T'$. This ends the proof of implication (56).

R10. By Lemma 15, point (2), $\mathrm{Div}(S, \rho_\varepsilon(S)) = \infty$.

R11. By Lemma 23, $\mathrm{Div}(S, \rho_e(S)) = \infty$. $\square$

Let us define $\models\!\!-$ by for every $P \in \mathscr{P}_f(\mathscr{A})$, $A \in \mathscr{A}$,

$$P \models\!\!- A \Leftrightarrow P \overset{\langle * \rangle}{|\!\models\!\!-} \circ \underset{0,3,4,10,11}{\overset{[1]}{|\!\models\!\!-}} \circ \overset{\langle * \rangle}{|\!\models\!\!-} \{A\}.$$

where $|\!\models\!\!-_{0,3,4,10,11}$ is the relation defined by R0, R3, R′3, R4, R10, R11 only. We let

$$\mathscr{D}_0 = \langle \mathscr{A}, H, \models\!\!- \rangle.$$

Lemma 54. *$\mathscr{D}_0$ is a deduction system.*

Proof. It should be clear, from the well-known decidability properties of finite automata, that $|\!\models\!\!-$ is recursively enumerable. Using Lemma 53, one can show by induction on n that

$$P \overset{\langle n \rangle}{|\!\models\!\!-} Q \Rightarrow \forall q \in Q, \quad \min\{H(A) | A \in P\} \leqslant H(q).$$

The proof of Lemma 53 also reveals that

$$P \, |\!\models\!\!-_{\{0,3,4,10,11\}} q \Rightarrow (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

It follows that, for every $m, n \geqslant 0$,

$$P \overset{\langle n \rangle}{|\!\models\!\!-} Q \underset{0,3,4,10,11}{\overset{[1]}{|\!\models\!\!-}} R \overset{\langle m \rangle}{|\!\models\!\!-} q \Rightarrow (\min\{H(p) \mid p \in P\} < H(q)) \text{ or } H(q) = \infty.$$

Both axioms (A1), (A2) are fulfilled. $\square$

Let us remark the following algebraic corollaries of Lemma 53.

Corollary 55. (C1) $\forall (S, S') \in \mathsf{DRB}_{1,2}\langle\langle V \rangle\rangle, T' \in \mathsf{DRB}\langle\langle V \rangle\rangle, S \not\equiv \varepsilon,$

$$S \cdot T' + S' \equiv T' \Rightarrow S^* \cdot S' \equiv T'$$

(C2) $\forall S, S' \in \mathsf{DRB}\langle\langle V \rangle\rangle, T \in \mathsf{DRB}\langle\langle V \rangle\rangle,$

$$[S \cdot T \equiv S' \cdot T \text{ and } T \not\equiv \emptyset] \Rightarrow S \equiv S'$$

Proof. Statement (C1) is a direct corollary of the fact that the value of H at the left-hand side of rule (R6) is smaller or equal to the value of H at the right-hand side of rule (R6). Let us prove (C2): let us consider $S, S' \in \mathsf{DRB}\langle\langle V \rangle\rangle, T \in \mathsf{DRB}\langle\langle V \rangle\rangle,$

such that

$$S \cdot T \equiv S' \cdot T \quad \text{and} \quad S \not\equiv S'. \tag{59}$$

Let

$$u = \min\{v \in X^* \mid (\rho_\varepsilon(S \odot v) = \varepsilon) \Leftrightarrow (\rho_\varepsilon(S' \odot v) \neq \varepsilon)\}.$$

From the hypothesis that $S \cdot T \equiv S' \cdot T$, we get that, for every $v \in X^*$,

$$(S \cdot T) \odot v \equiv (S' \cdot T) \odot v$$

and by the choice of u we obtain that

$$T \equiv (S' \odot u) \cdot T \quad \text{or} \quad (S \odot u) \cdot T \equiv T,$$

which, by (C1), implies

$$T \equiv (S' \odot u)^* \cdot \emptyset \quad \text{or} \quad (S \odot u)^* \cdot \emptyset \equiv T,$$

i.e.

$$T \equiv \emptyset. \tag{60}$$

We have proved that (59) implies (60), hence (C2). □

4.4. Congruence closure: definition

Let us consider the subset $\mathscr{C}$ of the rules of $\mathscr{D}_0$, consisting of all the instances of the metarules R0–R3, R′3, R6–R11. We also denote by $||\!\!-\!\!-_{\mathscr{C}} \subseteq \mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$ the set of all instances of these meta-rules. This subset will be used in Section 7. The non-obvious properties of this sytem will be needed in Section 10 only. Therefore the study of $\mathscr{C}$ postponed to Section 10.1.

5. Triangulations

Let $S_1, S_2, \ldots, S_d$ be a family of deterministic rational boolean series over the structured alphabet V (i.e. $S_i \in \mathsf{DRB}\langle\langle V \rangle\rangle$). We recall V is the alphabet associated with some dpda $\mathscr{M}$ as defined in Section 2.2.

Let us consider a sequence $\mathscr{S}$ of n "weighted" linear equations

$$(\mathscr{E}_i)\colon\ p_i, \sum_{j=1}^{d} \alpha_{i,j} S_j, \sum_{j=1}^{d} \beta_{i,j} S_j \tag{61}$$

where $p_i \in \mathbb{N} - \{0\}$, and $A = (\alpha_{i,j})$, $B = (\beta_{i,j})$ are deterministic rational matrices of dimension (n, d), with indices $m \leqslant i \leqslant m + n - 1$, $1 \leqslant j \leqslant d$.

For any weighted equation, $\mathscr{E} = (p, S, S')$, we recall the "cost" of this equation is $H(\mathscr{E}) = p + 2\ \mathrm{Div}(S, S')$.

We associate to every system (61) another system of weighted equations, INV($\mathscr{S}$), which "translates the equations of $\mathscr{S}$ into equations over the coefficients $(\alpha_{i,j}, \beta_{i,j})$ only".[11] The general idea of the construction of INV consists in iterating the transformation used in the proof of (1) ⇒ (2) ⇒ (3) in Lemma 30, i.e. the classical idea of *triangulating* a system of linear equations. Of course we must deal with the weights and relate the construction with the deduction system $\mathscr{D}_0$.

5.1. Restricted systems

We assume here that

$$\forall j \in [1,d], \quad S_j \not\equiv \emptyset \tag{62}$$

and

$$\forall i \in [m, m+n-1], \forall j \in [1,d], \alpha_{i,j}, \beta_{i,j} \text{ are } \varepsilon\text{-free.} \tag{63}$$

A system $\mathscr{S}$ fulfilling both hypotheses (62), (63) will be called a *restricted* system of weighted linear equations.

Let us define INV($\mathscr{S}$), W($\mathscr{S}$) $\in \mathbb{N} \cup \{\bot\}$, D($\mathscr{S}$) $\in \mathbb{N}$, by induction on n. W($\mathscr{S}$) is the *weight* of $\mathscr{S}$. D($\mathscr{S}$) is the *weak codimension* of $\mathscr{S}$.

Case 1: $\alpha_{m,*} \equiv \beta_{m,*}$

$$\mathrm{INV}(\mathscr{S}) = ((\mathrm{W}(\mathscr{S}), \alpha_{m,j}, \beta_{m,j}))_{1 \leqslant j \leqslant d}, \mathrm{W}(\mathscr{S}) = p_m - 1, D(\mathscr{S}) = 0.$$

Case 2: $\alpha_{m,*} \not\equiv \beta_{m,*}, n \geqslant 2, p_{m+1} - p_m \geqslant 2 \cdot \mathrm{Div}(\alpha_{m,*}, \beta_{m,*}) + 1$. Let us consider

$$u = \min\{v \in X^* \mid \exists j \in [1,d], (\alpha_{m,*} \odot v = \varepsilon_j^d) \Leftrightarrow (\beta_{m,*} \odot v \neq \varepsilon_j^d)\}. \tag{64}$$

(Lemma 25 and the ε-freeness assumption (63) ensure the existence of such a word u.) Let $j_0 \in [1,n]$ such that $(\alpha_{m,*} \odot u = \varepsilon_{j_0}^d) \Leftrightarrow (\beta_{m,*} \odot u \neq \varepsilon_{j_0}^d)$.

Subcase 1: $\alpha_{m,j_0} \odot u = \varepsilon, \beta_{m,j_0} \odot u \neq \varepsilon$. Let us consider the equation

$$(\mathscr{E}'_m)\colon\ p_m + 2 \cdot |u|, S_{j_0} \sum_{\substack{j=1 \\ j \neq j_0}}^{d} (\beta_{m,j_0} \odot u)^* (\beta_{m,j} \odot u) S_j$$

and define a new system of weighted equations $\mathscr{S}' = (\mathscr{E}'_i)_{m+1 \leqslant i \leqslant m+n-1}$ by

$$(\mathscr{E}'_i)\colon\ p_i, \sum_{j \neq j_0} [\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j_0} \odot u)^* (\beta_{m,j} \odot u)] \cdot S_j,$$
$$\sum_{j \neq j_0} [\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j_0} \odot u)^* (\beta_{m,j} \odot u)] \cdot S_j.$$

[11] This function INV is an "elaborated version" of the *inverse* systems defined in [47, Eq. (2.8), p. 586, English version] or [48, Eq. (2.8), p. 677, English version] in the case of a single equation.

(The above equation is seen as an equation between two linear combinations of the S_i's, $1 \leqslant i \leqslant d$, where the j_0th coefficient is $\emptyset$ on both sides.) We then define

$$\mathrm{INV}(\mathscr{S}) = \mathrm{INV}(\mathscr{S}')\mathrm{W}(\mathscr{S}) = \mathrm{W}(\mathscr{S}')\mathrm{D}(\mathscr{S}) = \mathrm{D}(\mathscr{S}') + 1.$$

Subcase 2: $\alpha_{m,j_0} \odot u \neq \varepsilon, \beta_{m,j_0} \odot u = \varepsilon$ (analogous to subcase 1).
Case 3: $\alpha_{m,*} \not\equiv \beta_{m,*}, n = 1$. We then define

$$\mathrm{INV}(\mathscr{S}) = \bot, \quad \mathrm{W}(\mathscr{S}) = \bot, \quad \mathrm{D}(\mathscr{S}) = 0,$$

where $\bot$ is a special symbol which can be understood as meaning "undefined".
Case 4: $\alpha_{m,*} \not\equiv \beta_{m,*} n \geqslant 2$, $p_{m+1} - p_m \leqslant 2 \cdot \mathrm{Div}(\alpha_{m,*}, \beta_{m,*})$. We then define

$$\mathrm{INV}(\mathscr{S}) = \bot, \quad \mathrm{W}(\mathscr{S}) = \bot, \quad \mathrm{D}(\mathscr{S}) = 0.$$

Lemma 56. *Let $\mathscr{S}$ be a restricted system of weighted linear equations with deterministic rational coefficients. If* $\mathrm{INV}(\mathscr{S}) \neq \bot$ *then*, $\mathrm{INV}(\mathscr{S})$ *is a system of weighted linear equations with deterministic rational coefficients.*

Proof. Follows from Lemmas 28, 29 and the formula defining $\mathscr{S}'$ from $\mathscr{S}$. □

From now on, and up to the end of this section, we simply write "linear equation" to mean weighted linear equation with deterministic rational coefficients.

Lemma 57. *Let $\mathscr{S}$ be a system of linear equations. If* $\mathrm{INV}(\mathscr{S}) \neq \bot$ *then* $\mathrm{INV}(\mathscr{S}) = (\bar{\mathscr{E}}_j)_{1 \leqslant j \leqslant d}$ *fulfills*
(1) $\{\bar{\mathscr{E}}_j | 1 \leqslant j \leqslant d\} \cup \{\mathscr{E}_i | m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S}) - 1\} \mid\!\!-\!\!- \mathscr{E}_{m+\mathrm{D}(\mathscr{S})}$,
(2) $\min\{H(\mathscr{E}_i) \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S})\} = \infty \Rightarrow \min\{H(\bar{\mathscr{E}}_j) \mid 1 \leqslant j \leqslant d\} = \infty$.

In what follows, we sometimes write $\mathrm{INV}(\mathscr{S})$ to mean the set $\{\bar{\mathscr{E}}_j \mid 1 \leqslant j \leqslant d\}$ (i.e. we do not distinguish between the family of equations $\mathrm{INV}(\mathscr{S})$ and the corresponding set of equations). We also denote by $H(\mathrm{INV}(\mathscr{S}))$ the element $\min\{H(\bar{\mathscr{E}}_j) | 1 \leqslant j \leqslant d\} \in \mathbb{N} \cup \infty$.

Proof. See in Fig. 2 the "graph of the deductions" we use for proving point (1). Let us prove by induction on $\mathrm{D}(\mathscr{S})$ the following strengthened version of point (1):

$$\mathrm{INV}(\mathscr{S}) \cup \{\mathscr{E}_i \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S}) - 1\} \; \overset{\langle * \rangle}{\Vert\!\!-\!\!-} \; \tau_{-1}(\mathscr{E}_{m+\mathrm{D}(\mathscr{S})}), \tag{65}$$

where for every integer $k \in \mathbb{Z}$, $\tau_k : \{(p, S, S') \in \mathscr{A} \mid p \geqslant -k\} \to \mathscr{A}$ is the *translation* map on the weights: $\tau_k(p, S, S') = (p+k, S, S')$. **if** $\mathrm{D}(\mathscr{S}) = 0$: as $\mathrm{INV}(\mathscr{S}) \neq \bot, \mathscr{S}$ must fulfill the hypothesis of case 1:

$$\mathscr{E}_m = \left(p_m, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j \right) = \mathscr{E}_{m+\mathrm{D}(\mathscr{S})}$$

$$\mathrm{INV}(\mathscr{S}) = ((p_m - 1, \alpha_{m,j}, \beta_{m,j}))_{1 \leqslant j \leqslant d}.$$

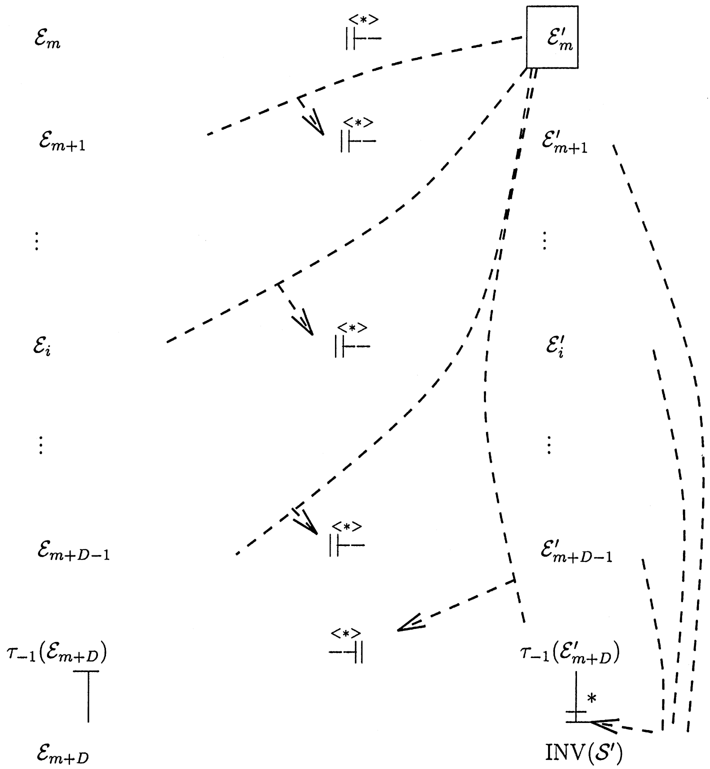

Fig. 2. Proof of Lemma 5.2.

Using rules (R7), (R8) we obtain

$$\mathrm{INV}(\mathscr{S}) \overset{\langle * \rangle}{||\!\!-\!\!-} \left(p_m - 1, \sum_{j=1}^{d} \alpha_{m,j} S_j, \sum_{j=1}^{d} \beta_{m,j} S_j\right) = \tau_{-1}(\mathscr{E}_m).$$

if $\mathrm{D}(\mathscr{S}) = n+1, n \geqslant 0$: $\mathscr{S}$ must fulfill case 2.

Suppose case 2, subcase 1 occurs.

Using $|u|$ times (R5) and then (R6) (this is possible because $\beta_{m,j_0} \odot u \not\equiv \varepsilon$), we obtain a deduction

$$\mathscr{E}_m \overset{\langle 2\cdot|u|+1 \rangle}{||\!\!-\!\!-} \mathscr{E}'_m. \tag{66}$$

Using (R7)–(R9) we get that, for every $i \in [m+1, m+\mathrm{D}(\mathscr{S})]$,

$$\{\mathscr{E}_i, \mathscr{E}'_m\} \overset{\langle * \rangle}{||\!\!-\!\!-} \Bigg(\max\{p_i, p_m + 2|u|\}, \sum_{j \neq j_0} (\alpha_{i,j} + \alpha_{i,j_0}(\beta_{m,j_0} \odot u)^*(\beta_{m,j} \odot u)) \cdot S_j,$$

$$\sum_{j \neq j_0} (\beta_{i,j} + \beta_{i,j_0}(\beta_{m,j_0} \odot u)^*(\beta_{m,j} \odot u)) \cdot S_j\Bigg).$$

The hypothesis of case 2 implies that $\max\{p_{m+1}, p_m + 2|u|\} = p_{m+1}$ and the fact that $\mathrm{INV}(\mathscr{S}')$ is defined implies that $\forall i \in [m+1, m+\mathrm{D}(\mathscr{S})]$, $p_i \geqslant p_{m+1}$, hence, $\max\{p_i, p_m + 2|u|\} = p_i$ and the right-hand side of the above deduction is exactly $\mathscr{E}'_i$. Hence,

$$\forall i \in [m+1, m+\mathrm{D}(\mathscr{S})], \{\mathscr{E}_i, \mathscr{E}'_m\} \overset{\langle * \rangle}{\Vdash\!\!-} \mathscr{E}'_i. \tag{67}$$

Using deductions (66) and (67), we obtain that

$$\{\mathscr{E}_i \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S}) - 1\} \overset{\langle * \rangle}{\Vdash\!\!-} \{\mathscr{E}'_i \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S}) - 1\}. \tag{68}$$

By induction hypothesis

$$\mathrm{INV}(\mathscr{S}') \cup \{\mathscr{E}'_i \mid m+1 \leqslant i \leqslant m+1+\mathrm{D}(\mathscr{S}') - 1\} \overset{\langle * \rangle}{\Vdash\!\!-} \tau_{-1}(\mathscr{E}'_{m+1+\mathrm{D}(\mathscr{S}')})$$

which is equivalent to

$$\mathrm{INV}(\mathscr{S}) \cup \{\mathscr{E}'_i \mid m+1 \leqslant i \leqslant m + \mathrm{D}(\mathscr{S}) - 1\} \overset{\langle * \rangle}{\Vdash\!\!-} \tau_{-1}(\mathscr{E}'_{m+\mathrm{D}(\mathscr{S})}). \tag{69}$$

As $p_m + 2 \cdot |u| \leqslant p_{m+1} - 1 \leqslant p_{m+\mathrm{D}(\mathscr{S})} - 1$, we have also the following inverse deduction (which is similar to deduction (67)):

$$\{\mathscr{E}'_m, \tau_{-1}(\mathscr{E}'_{m+\mathrm{D}(\mathscr{S})})\} \overset{\langle * \rangle}{\Vdash\!\!-} \tau_{-1}(\mathscr{E}_{m+\mathrm{D}(\mathscr{S})}). \tag{70}$$

Combining together deductions (68)–(70), we have proved (65). Using rule (R0), this last deduction leads to point (1) of the lemma.

Suppose that case 2, subcase 2 occurs: This case can be treated in the same way as subcase 1 just by exchanging the roles of $\vec{\alpha}, \vec{\beta}$.

Let us prove statement (2) of the lemma.

We prove by induction on $\mathrm{D}(\mathscr{S})$ the statement:

$$\min\{H(\mathscr{E}_i) \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S})\} = \infty \Rightarrow H(\mathrm{INV}(\mathscr{S})) = \infty. \tag{71}$$

if $\mathrm{D}(\mathscr{S}) = 0$: As $\mathrm{INV}(\mathscr{S}) \neq \perp$, case 1 must occur. $\alpha_{m,*} \equiv \beta_{m,*}$ implies that $H(\mathrm{INV}$ $(\mathscr{S}) = \infty$, hence the statement is true.

if $\mathrm{D}(\mathscr{S}) = p+1, p \geqslant 0$: As $\mathrm{D}(\mathscr{S}) \geqslant 1$ and $\mathrm{INV}(\mathscr{S}) \neq \perp$, case 2 must occur. Using deductions (66) and (67) established above we obtain that

$$\{\mathscr{E}_i \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S})\} \overset{\langle * \rangle}{\Vdash\!\!-} \{\mathscr{E}'_i \mid m+1 \leqslant i \leqslant m+1+\mathrm{D}(\mathscr{S}')\},$$

which proves that

$$\min\{H(\mathscr{E}_i) \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S})\} \leqslant \min\{H(\mathscr{E}'_i) \mid m+1 \leqslant i \leqslant m+1+\mathrm{D}(\mathscr{S}')\}. \tag{72}$$

As $D(\mathscr{S}') = D(\mathscr{S}) - 1$, we can use the induction hypothesis

$$\min\{H(\mathscr{E}'_i) \mid m+1 \leqslant i \leqslant m+1+D(\mathscr{S}')\} = \infty \Rightarrow H(\mathrm{INV}(\mathscr{S}')) = \infty. \tag{73}$$

As $\mathrm{INV}(\mathscr{S}) = \mathrm{INV}(\mathscr{S}')$, (72) and (73) imply statement (71). □

Lemma 58. *Let $\mathscr{S}$ be a restricted system of linear equations satisfying the hypothesis of case 2. Then,* $\forall i \in [m+1, m+n-1]$, $\|\alpha'_{i,*}\| \leqslant \|\alpha_{i,*}\| + \|\beta_{m,*}\| + K_0|u|$, $\|\beta'_{i,*}\| \leqslant \|\beta_{i,*}\| + \|\beta_{m,*}\| + K_0|u|$.

Proof. The formula defining $\mathscr{S}'$ from $\mathscr{S}$ show that

$$\alpha'_{i,*} = \alpha_{i,*}\square_{j_0}(\square^*_{j_0}\beta_{m,*}); \quad \beta'_{i,*} = \beta_{i,*}\square_{j_0}(\square^*_{j_0}\beta_{m,*}).$$

From these equalities and Lemmas 28, 29, 32, the inequalities on the norm follow. □

Let us consider the function F defined by

$$F(d,n) = \max\{\mathrm{Div}(A,B) \mid A,B \in \mathsf{DRB}_{1,d}\langle\langle V\rangle\rangle, \|A\| \leqslant n, \|B\| \leqslant n, A \not\equiv B\}. \tag{74}$$

For every integer parameters $K_0, K_1, K_2, K_3, K_4 \in \mathbb{N} - \{0\}$, we define integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leqslant i \leqslant m+n-1}$ by

$$\delta_m = 0,\ \ell_m = 0,\ L_m = K_2,\ s_m = K_3 \cdot K_2 + K_4,\ S_m = 0,\ \Sigma_m = 0, \tag{75}$$

$$\begin{aligned}
&\delta_{i+1} = 2 \cdot F(d, s_i + \Sigma_i) + 1,\\
&\ell_{i+1} = 2 \cdot \delta_{i+1} + 3,\\
&L_{i+1} = K_1 \cdot (L_i + \ell_{i+1}) + K_2,\\
&s_{i+1} = K_3 \cdot L_{i+1} + K_4,\\
&S_{i+1} = s_i + \Sigma_i + K_0 F(d, s_i + \Sigma_i),\\
&\Sigma_{i+1} = \Sigma_i + S_{i+1}
\end{aligned} \tag{76}$$

for $m \leqslant i \leqslant m+n-2$.

These sequences are intended to have the following meanings when K_0, K_1, K_2, K_3, K_4 are chosen to be the constants defined in Section 6 and equations $(\mathscr{E}_i)$ are labelling nodes of a N-stacking sequence (see Section 8.3):

$\delta_{i+1} \leqslant$ increase of weight between $\mathscr{E}_i$, $\mathscr{E}_{i+1}$,
$\ell_{i+1} \geqslant$ increase of depth between $\mathscr{E}_i$, $\mathscr{E}_{i+1}$,
$L_{i+1} \geqslant$ increase of depth between $\mathscr{E}_m, \mathscr{E}_{i+1}$,
$s_{i+1} \geqslant$ size of the coefficients of $\mathscr{E}_{i+1}$,
$S_{i+1} \geqslant$ size of the coefficients of $\mathscr{E}^{(i+1-m)}_{i+1}$ (these systems are introduced below in the proof of Lemma 59),
$\Sigma_{i+1} \geqslant$ increase of the coefficients between $\mathscr{E}^{(i-m)}_k$, $\mathscr{E}^{(i+1-m)}_k$ (for $k \geqslant i+1$).

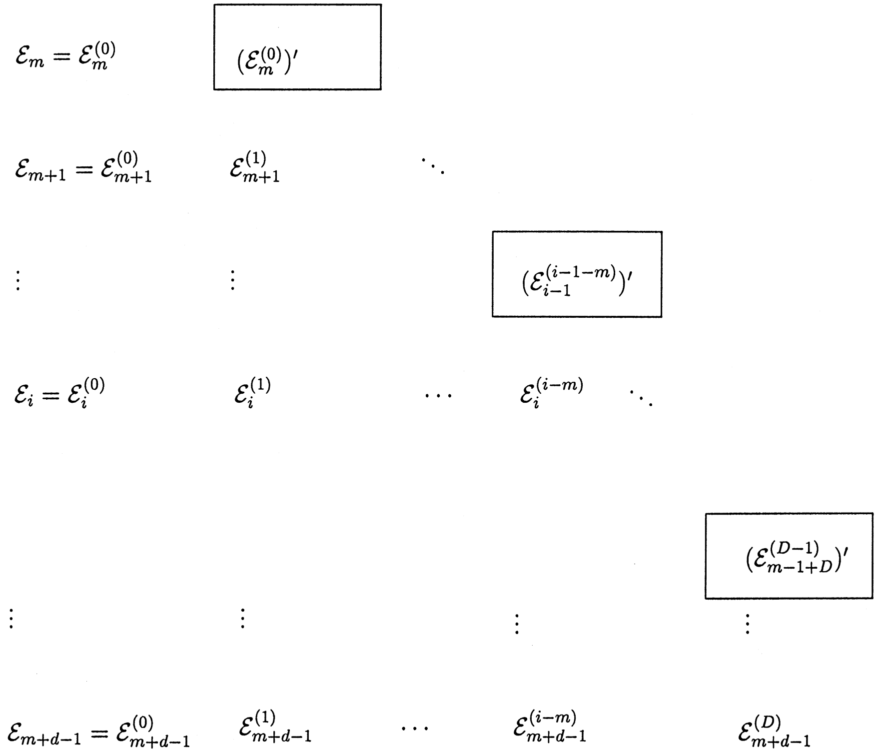

Fig. 3. Proof of Lemma 5.4.

For every linear equation $\mathscr{E} = (p, \sum_{j=1}^{d} \alpha_j S_j \sum_{j=1}^{d} \beta_j S_j)$, we define

$$|||\mathscr{E}||| = \max\{\|(\alpha_1, \ldots, \alpha_d)\|, \|(\beta_1, \ldots, \beta_d)\|\}.$$

Lemma 59. *Let* $\mathscr{S} = (\mathscr{E}_i)_{m \leqslant i \leqslant m+d-1}$ *be a restricted system of* d *linear equations such that* $H(\mathscr{E}_i) = \infty$ *(for every* i*) and*
(1) $\forall i \in [m, m+d-1], |||\mathscr{E}_i||| \leqslant s_i$,
(2) $\forall i \in [m, m+d-2], \mathrm{W}(\mathscr{E}_{i+1}) - \mathrm{W}(\mathscr{E}_i) \geqslant \delta_{i+1}$.
Then $\mathrm{INV}(\mathscr{S}) \neq \bot, \mathrm{D}(\mathscr{S}) \leqslant d-1, \forall \mathscr{E} \in \mathrm{INV}(\mathscr{S}), |||\mathscr{E}||| \leqslant \Sigma_{m+\mathrm{D}(\mathscr{S})} + s_{m+\mathrm{D}(\mathscr{S})}$.

Proof. (Fig. 3 *might help the reader to follow the definitions below*). Let us define a sequence of systems $\mathscr{S}^{(i-m)} = (\mathscr{E}_k^{(i-m)})_{m \leqslant i \leqslant k \leqslant m+d-1}$, where $i \in [m, m+\mathrm{D}(\mathscr{S})]$, by induction

- $\mathscr{E}_k^{(0)} = \mathscr{E}_k$ for $m \leqslant k \leqslant m+d-1$,
- if case 1 or case 3 or case 4 is realized, $\mathrm{D}(\mathscr{S}) = 0$, hence $\mathscr{S}^{(i-m)}$ is well-defined for $m \leqslant i \leqslant m+\mathrm{D}(\mathscr{S})$
- if case 2 is realized then we set : $\forall i \geqslant m+1, \mathscr{E}_k^{(i-m)} = (\mathscr{E}'_k)^{(i-m-1)}$, for $m+1 \leqslant k \leqslant m+d-1$.

Let us prove by induction on $i \in [m, m + \mathrm{D}(\mathscr{S})]$ that, $\forall k \in [i, m+d-1]$:

$$|||\mathscr{E}_k^{(i-m)}||| \leqslant s_k + \Sigma_i. \tag{77}$$

$i = m$: In this case

$$|||\mathscr{E}_k^{(i-m)}||| = |||\mathscr{E}_k||| \leqslant s_k = s_k + \Sigma_m.$$

$i + 1 \leqslant m + \mathrm{D}(\mathscr{S})$: In this case, by Lemma 58,

$$|||\mathscr{E}_k^{(i+1-m)}||| \leqslant |||\mathscr{E}_k^{(i-m)}||| + |||\mathscr{E}_i^{(i-m)}||| + K_0|u_i|$$

where

$$u_i = \min\{v \in X^* \mid \exists j \in [1,d], (\alpha_{i,*}^{(i-m)} \odot v = \varepsilon_j^d) \Leftrightarrow (\beta_{i,*}^{(i-m)} \odot v \neq \varepsilon_j^d)\}. \tag{78}$$

By definition of F and the induction hypothesis

$$|u_i| \leqslant F(d, |||\mathscr{E}_i^{(i-m)}|||) \leqslant F(d, s_i + \Sigma_i).$$

Hence,

$$|||\mathscr{E}_k^{(i+1-m)}||| \leqslant (s_k + \Sigma_i) + (s_i + \Sigma_i) + K_0 F(d, s_i + \Sigma_i) = (s_k + \Sigma_i) + S_{i+1}$$
$$= s_k + \Sigma_{i+1}.$$

Let us notice that $\mathrm{D}(\mathscr{S})$ is always an integer and that this proof is valid for $m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S})$, $i \leqslant k \leqslant m + d - 1$.

Let us prove now that $\mathrm{INV}(\mathscr{S}) \neq \perp$. Let us consider the system $(\mathscr{E}_k^{(\mathrm{D}(\mathscr{S}))})_{m+\mathrm{D}(\mathscr{S}) \leqslant k \leqslant m+d-1}$. *If* $\mathrm{D}(\mathscr{S}) = d - 1$, as the system $(\mathscr{E}_{\mathrm{D}(\mathscr{S})}^{(\mathrm{D}(\mathscr{S}))})$ consists of a single equation, it must fulfill either case 1 or case 3 of the definition of INV.

Using the successive deductions (66) and (67) established in the proof of Lemma 57, we get

$$\{\mathscr{E}_i | m \leqslant i \leqslant m + d - 1\} \overset{\langle * \rangle}{|\!|\!\!-\!\!-} \{\mathscr{E}_{m+d-1}^{(d-1)}\}.$$

Using now the hypothesis that $H(\mathscr{E}_i) = \infty$ (for $m \leqslant i \leqslant m + d - 1$), we obtain

$$H(\mathscr{E}_{m+d-1}^{(d-1)}) = \infty. \tag{79}$$

For any system of equations $\mathscr{S}$, let us define the *column-support* of the system as

$$\mathrm{csupp}(\mathscr{S}) = \left\{ j \in [1,d] \middle| \sum_{i=m}^{m+n-1} \alpha_{i,j} + \beta_{i,j} \neq \emptyset \right\}.$$

Let us consider $\delta = \mathrm{Card}(\mathrm{csupp}(\mathscr{S}^{(d-1)})$. One can prove by induction on i that

$$\mathrm{Card}(\mathrm{csupp}(\mathscr{S}^{(i-m)}) \leqslant d - i + m,$$

hence

$$\delta = \mathrm{Card}(\mathrm{csupp}(\mathscr{S}^{(d-1)}) \leqslant d - (d-1) = 1.$$

- If $\delta = 1, \mathrm{csupp}(\mathscr{S}^{(d-1)}) = \{j_0\}$, for some $j_0 \in [1,d]$.

By Corollary 55, point (C2), and hypothesis (62), the implication

$$[\alpha^{(d-1)}_{m+d-1,j_0} S_{j_0} \equiv \beta^{(d-1)}_{m+d-1,j_0} S_{j_0}] \Rightarrow \alpha^{(d-1)}_{m+d-1,j_0} \equiv \beta^{(d-1)}_{m+d-1,j_0}$$

holds. Hence, by (79), $\alpha^{(d-1)}_{m+d-1,j_0} \equiv \beta^{(d-1)}_{m+d-1,j_0}$, i.e. $\mathscr{S}^{(d-1)}$ fulfills case 1, so that

$$\mathrm{INV}(\mathscr{S}) = \mathrm{INV}(\mathscr{S}^{(d-1)}) \neq \perp .$$

- If $\delta = 0, \mathrm{csupp}(\mathscr{S}) = \emptyset$.
  Then $\alpha^{(d-1)}_{m+d-1,*} = \beta^{(d-1)}_{m+d-1,*} = \emptyset^d$. Here also $\mathscr{S}^{(d-1)}$ fulfills case 1.

If $\mathrm{D}(\mathscr{S}) < d-1$, by hypothesis

$$\mathrm{W}(\mathscr{E}_{m+\mathrm{D}(\mathscr{S})+1}) - \mathrm{W}(\mathscr{E}_{m+\mathrm{D}(\mathscr{S})}) \geqslant \delta_{m+\mathrm{D}(\mathscr{S})+1} = 2F(d, s_{m+\mathrm{D}(\mathscr{S})} + \Sigma_{m+\mathrm{D}(\mathscr{S})}) + 1.$$

If $\alpha^{\mathrm{D}(\mathscr{S})}_{m+\mathrm{D}(\mathscr{S}),*} \equiv \beta^{\mathrm{D}(\mathscr{S})}_{m+\mathrm{D}(\mathscr{S}),*}$, then $\mathscr{E}^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S})}$ fulfills case 1 of the definition of INV, hence $\mathrm{INV}(\mathscr{S}) \neq \perp$.

Otherwise, let us consider

$$u = \min\{v \in X^* \mid \exists j \in [1,d], (\alpha^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S}),*} \odot v = \varepsilon^d_j) \Leftrightarrow (\beta^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S}),*} \odot v \neq \varepsilon^d_j)\}. \quad (80)$$

By definition of F and inequality (77),

$$|u| \leqslant F(d, \|\|\mathscr{E}^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S})}\|\|) \leqslant F(d, s_{m+\mathrm{D}(\mathscr{S})} + \Sigma_{m+\mathrm{D}(\mathscr{S})}).$$

Hence $p_{m+\mathrm{D}(\mathscr{S})+1} - p_{m+\mathrm{D}(\mathscr{S})} \geqslant 2\,|u| + 1$, i.e. the hypothesis of case 2 is realized. This proves that $\mathrm{D}(\mathscr{S}^{(\mathrm{D}(\mathscr{S}))}) \geqslant 1$ while in fact, $\mathrm{D}(\mathscr{S}^{(\mathrm{D}(\mathscr{S}))}) = 0$. This contradiction shows that this last case ($\mathrm{D}(\mathscr{S}) < d-1$ and $\mathscr{E}^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S})}$ not fulfilling case 1 of definition of INV) is impossible. We have proved point (2) of the lemma. □

5.2. General systems

We consider now the general case where assumptions (62) and (63) are removed. Let us suppose that

$$\exists d_1 \in [1,d], \quad S_{d_1} \not\equiv \emptyset. \quad (81)$$

Up to some permutation of the column indices (such a permutation leaves function H invariant), we can suppose that there exists $\hat{d} \in [1,d]$ such that

$$\forall j \in [1,\hat{d}],\ S_j \not\equiv \emptyset, \quad \forall j \in [\hat{d}+1,d],\ S_j \equiv \emptyset. \quad (82)$$

We then associate to the original system $\mathscr{S}$ a new system $\hat{\mathscr{S}}$ of n linear equations:

$$(\hat{\mathscr{E}}_i):\ p_i, \sum_{j=1}^{\hat{d}} \rho_\varepsilon(\alpha_{i,j}) \cdot S_j, \quad \sum_{j=1}^{\hat{d}} \rho_\varepsilon(\beta_{i,j}) \cdot S_j,$$

where $m \leqslant i \leqslant m+n-1$.

We then define

$$\mathrm{INV}(\mathscr{S}) = \mathrm{INV}(\hat{\mathscr{S}}), \quad \mathrm{W}(\mathscr{S}) = \mathrm{W}(\hat{\mathscr{S}}), \quad \mathrm{D}(\mathscr{S}) = \mathrm{D}(\hat{\mathscr{S}}).$$

Let us show that Lemmas 56, 57 and 59 remain true in the general case.

Lemma 60 (Preliminary lemma). *For every* $i \in [m, m+n-1]$

(1) $\hat{\mathscr{E}}_i \overset{\langle * \rangle}{||\!\!\vdash\!\!-} \mathscr{E}_i$,

(2) $\mathscr{E}_i \overset{\langle * \rangle}{||\!\!\vdash\!\!-} \hat{\mathscr{E}}_i$.

Proof. By (R11), $\forall i \in [m, m+n-1]$ $\forall j \in [1, d]$,

$$\emptyset \overset{\langle * \rangle}{||\!\!\vdash\!\!-} (0, \alpha_{i,j}, \rho_\varepsilon(\alpha_{i,j})),$$

whose combination with (R7), (R8) gives, $\forall i \in [m, m+n-1]$:

$$\emptyset \overset{\langle * \rangle}{||\!\!\vdash\!\!-} \left(0, \sum_{j=1}^{d} \alpha_{i,j} \cdot S_j, \sum_{j=1}^{d} \rho_\varepsilon(\alpha_{i,j}) \cdot S_j\right). \tag{83}$$

Using rule (R3) (for all the triples $(0, S_j, S_j)$, $j \in [1, \hat{d}]$) and rule (R$'$3) (for all the triples $(0, S_j, \emptyset), j \in [\hat{d}+1, d]$), combined with rules (R7), (R8) we get, $\forall i \in [m, m+n-1]$,

$$\emptyset \overset{\langle * \rangle}{||\!\!\vdash\!\!-} \left(0, \sum_{j=1}^{d} \rho_\varepsilon(\alpha_{i,j}) \cdot S_j, \sum_{j=1}^{\hat{d}} \rho_\varepsilon(\alpha_{i,j}) \cdot S_j\right). \tag{84}$$

Using then rules (R1), (R2), deductions (83), (84) and their analogues for the right-hand sides, we obtain points (1) and (2) of the lemma. □

Lemma 61. *Let* $\mathscr{S}$ *be a system of linear equations. If* $\mathrm{INV}(\mathscr{S}) \neq \bot$ *then,* $\forall \mathscr{E} \in \mathrm{INV}(\mathscr{S}), \mathscr{E}$ *is a linear equation.*

Proof. As $\hat{\mathscr{S}}$ is a restricted system, this follows from Lemma 56. □

Lemma 62. *Let* $\mathscr{S}$ *be a system of linear equations. If* $\mathrm{INV}(\mathscr{S}) \neq \bot$ *then*

(1) $\mathrm{INV}(\mathscr{S}) \cup \{\mathscr{E}_i \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S}) - 1\} \vdash\!\!- \mathscr{E}_{m+\mathrm{D}(\mathscr{S})}$,

(2) $\min\{H(\mathscr{E}_i) \mid m \leqslant i \leqslant m + \mathrm{D}(\mathscr{S})\} = \infty \Rightarrow H(\mathrm{INV}(\mathscr{S})) = \infty$.

Proof. Point (1) follows from Lemma 57 point (1) and from Lemma 60. Point (2) follows from Lemma 57 point (2) and from Lemma 60, point (2). □

Lemma 63. *Let* $\mathscr{S} = (\mathscr{E}_i)_{m \leqslant i \leqslant m+d-1}$ *be a system of* d *linear equations such that* $H(\mathscr{E}_i) = \infty$ *(for every* i*) and*

(0) $\exists j \in [1, d], S_j \not\equiv \emptyset$,
(1) $\forall i \in [m, m+d-1], |||\mathscr{E}_i||| \leqslant s_i$,
(2) $\forall i \in [m, m+d-2], \mathrm{W}(\mathscr{E}_{i+1}) - \mathrm{W}(\mathscr{E}_i) \geqslant \delta_{i+1}$.

Then $\mathrm{INV}(\mathscr{S}) \neq \perp, \mathrm{D}(\mathscr{S}) \leqslant d-1, \forall \mathscr{E} \in \mathrm{INV}(\mathscr{S}), |||\mathscr{E}||| \leqslant \Sigma_{m+\mathrm{D}(\mathscr{S})} + s_{m+\mathrm{D}(\mathscr{S})}$.

Proof. By hypothesis (0), $\hat{\mathscr{S}}$ is defined and is a restricted system of linear equations. Moreover, using Lemma 15 for every $i \in [m, m+d-1]$, $|||\hat{\mathscr{E}}_i||| \leqslant |||\mathscr{E}_i|||$. Hence $\hat{\mathscr{S}}$ fulfills the hypothesis of Lemma 59 and the conclusion of this previous lemma applied on the system $\hat{\mathscr{S}}$ gives

$$\mathrm{INV}(\mathscr{S}) \neq \perp,\ \mathrm{D}(\mathscr{S}) \leqslant \hat{d} - 1,\ \forall \mathscr{E} \in \mathrm{INV}(\mathscr{S}),\ |||\mathscr{E}||| \leqslant \Sigma_{m+\mathrm{D}(\mathscr{S})} + s_{m+\mathrm{D}(\mathscr{S})}. \qquad \Box$$

6. Constants

Let us *fix* a normalized dpda $\mathscr{M}$ and an initial equation $A_0 = (\Pi_0, S_0^-, S_0^+) \in \mathbb{N} \times \mathsf{DRB}\langle\langle V \rangle\rangle \times \mathsf{DRB}\langle\langle V \rangle\rangle$ in the corresponding set of assertions. This short section is devoted to the definition of some integer *constants*: these integers are constant in the sense that they are depending only on this dpda $\mathscr{M}$ and initial equation A_0. The *motivation* of each of these definitions will appear later on, in different places for the different constants. The equations below provide merely an overview of the dependencies between these constants and allow to check that the definitions are sound (i.e. there is no hidden loop in the dependencies).

Definition 64. For every series $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$, we define the *valuation* of S, $v(S)$ by $v(S) = \inf\{\, |u| \mid u \in X^*, S \odot u = \varepsilon\}$:

$$k_0 = \max\{v([pzq]) \mid p, q \in Q, z \in Z, [pzq] \not\equiv \emptyset\}, \quad k_1 = \max\{2k_0 + 1, 3\}, \tag{85}$$

$$D_1 = 4 \cdot k_0 + 3, \quad k_2 = (D_1 + 5) \cdot k_1 + k_0 + 1. \tag{86}$$

k_1 is used in the definition of strategy T_B (Section 7), D_1 appears as an upper bound on the left-defect of series in Lemma 72 and k_2 is used in the definition of a “security band” before Lemma 78.

$$K_0 = |Q| + 1. \tag{87}$$

This constant appeared in Lemma 32.

$$K_1 = k_1 \cdot K_0 + 1, \quad K_2 = 6 \cdot D_1 \cdot k_1^2 \cdot K_0. \tag{88}$$

These constants K_1, K_2 appear in Lemma 81.

$$K_3 = 2k_0 K_0^2, \quad K_4 = (2k_2 + k_1 + 3) \cdot K_0^2 + (k_1 + 2) \cdot K_0 + 2. \tag{89}$$

These constants K_3, K_4 appear in Lemma 82.

$$d_0 = 2 \cdot |Q| \cdot (\mathrm{Card}(X^{\leqslant k_1}) + 1). \tag{90}$$

d_0 appears as an upper bound on the dimension of the d-space V_1 defined by Eq. (127) and used in Lemma 81. We consider now the integer sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \Sigma_i)_{m \leqslant i \leqslant m+n-1}$ defined by relations (76) of Section 5 where the parameters $K_0, \ldots, K_4$ are chosen to be the above constants and $m = 1, n = d = d_0$. Equivalently, they are defined by

$$\delta_1 = 0,\ \ell_1 = 0,\ L_1 = K_2,\ s_1 = K_3 \cdot K_2 + K_4,\ S_1 = 0,\ \Sigma_1 = 0, \tag{91}$$

$$\delta_{i+1} = 2 \cdot F(d_0, s_i + \Sigma_i) + 1,$$

$$\ell_{i+1} = 2 \cdot \delta_{i+1} + 3,$$

$$L_{i+1} = K_1 \cdot (L_i + \ell_{i+1}) + K_2,$$

$$s_{i+1} = K_3 \cdot L_{i+1} + K_4,$$

$$S_{i+1} = s_i + \Sigma_i + K_0 \cdot F(d_0, s_i + \Sigma_i),$$

$$\Sigma_{i+1} = \Sigma_i + S_{i+1} \tag{92}$$

for $1 \leqslant i \leqslant d_0 - 1$.

$$D_2 = \max\left\{\Sigma_{d_0} + s_{d_0}, \| S_0^- \|, \| S_0^+ \|\right\}, \tag{93}$$

$\Sigma_{d_0} + s_{d_0}$ appears in the conclusion of Lemma 63 when we take $d = d_0$ in the hypothesis and suppose that $\mathrm{D}(\mathscr{S})$ has its maximal possible value, i.e. $\mathrm{D}(\mathscr{S}) = d_0 - 1$. It is used as an upper bound on the right-defect in the definition of the trees τ analyzed in Section 8 (inequation (107)).

$$N_0 = 1 + (k_2 + 2)K_0 + D_2. \tag{94}$$

N_0 appears as a lower bound for the norm in the definition of a N-stacking sequence (Section 8.3, condition (113)).

$$C_2 = \mathrm{Card}\{U \in \mathsf{DRB}\langle\langle V \rangle\rangle, \|U\| \leqslant D_2\}, \tag{95}$$

$$K_6 = 6 \cdot [(C_2 \cdot |Q||Z|^{k_2 + D_2 + 3})^{|Q|} \cdot |Q||Z|^{D_1}]^2, \quad K_5 = (K_6 + 1) \cdot k_0 \cdot K_0. \tag{96}$$

K_5, K_6 appear in Lemma 84 and C_2 is used in the proof of Lemma 84.

7. Strategies for $\mathscr{D}_0$

Let us define strategies for the particular system $\mathscr{D}_0$. We define first auxiliary strategies $T_{cut}, T_\emptyset, T_\varepsilon, T_A, T_B, T_C$ and then derive some closed strategies from them. Let us fix here some total ordering on X: $x_1 < x_2 < \cdots < x_\alpha$ and also some total ordering $\leqslant$ of type ω on $\mathscr{A}$ (inherited from the usual well-ordering of $\mathbb{N}$ by the fixed encoding). From these orderings one can construct in the usual way an ordering of type ω on the sets $X^*, \mathscr{A}^*$ and $\mathbb{N}^* \times (\mathsf{DRB}\langle\langle V\rangle\rangle)^*$:

T_{cut}: $T_{\mathrm{cut}}(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $\exists i \in [1, n-1], \exists S, T,$

$$A_i = (p_i, S, T), \quad A_n = (p_n, S, T), \quad p_i < p_n \quad \text{and} \quad m = 0$$

$T_\emptyset$: $T_\emptyset(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ iff $\exists S, T, A_n = (p, S, T),\ p \geqslant 0,\ S \equiv T \equiv \emptyset$ and $m = 0$

T_ε: $T_\varepsilon(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $A_n = (p, S, T),\ p \geqslant 0,\ S \equiv T \equiv \varepsilon$ and $m = 0$

T_A: $T_A(A_1 \cdots A_n) = B_1 \cdots B_m$ iff

$$A_n = (p, S, T), \quad m = |X|,$$
$$B_1 = (p + 1, S \odot x_1, T \odot x_1), \ldots, B_m = (p + 1, S \odot x_m, T \odot x_m),$$

where $S \not\equiv \varepsilon,\ T \not\equiv \varepsilon$

T_B^+: $T_B^+(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $n \geqslant k_1 + 1,\ A_{n-k_1} = (\pi, \bar{U}, U')$, (where $\bar{U}$ is unmarked)

$$U' = \sum_{q \in Q} [\bar{p} z q] \cdot V_q \quad (\text{for some } \bar{p} \in Q, z \in Z, V_q \in \mathsf{DRB}\langle\langle V\rangle\rangle)$$

$A_i = (\pi + k_1 + i - n, U_i, U_i')$ for $n - k_1 \leqslant i \leqslant n$, $(U_i)_{n-k_1 \leqslant i \leqslant n}$ is a derivation, $(U_i')_{n-k_1 \leqslant i \leqslant n}$ is a "stacking derivation" (see definitions in Section 3.4),

$$U_n' = \sum_{q \in Q} [p \tau q] \cdot V_q \quad \text{for some } p \in Q,\ \tau \in Z^+,$$

$m = 1,\ B_1 = (\pi + k_1 - 1, V, V'), V = U_n,\ V' = \sum_{q \in Q'} [p \tau q] \cdot [qeq] \cdot (\bar{U} \odot u_q)$, where $Q' = \{q \in Q \mid [\bar{p} z q] \not\equiv \emptyset\}, \forall q \in Q', u_q = \min(\varphi([\bar{p} z q]))$.

T_B^-: T_B^- is defined in the same way as T_B^+ by exchanging the left series (S^-) and right series (S^+) in every assertion (p, S^-, S^+).

T_C: $T_C(A_1 \cdots A_n) = B_1 \cdots B_m$ iff there exists $d \in [1, d_0], D \in [0, d-1], S_1, S_2, \ldots, S_d \in \mathsf{DRB}\langle\langle V\rangle\rangle, 1 \leqslant \kappa_1 < \kappa_2 < \cdots < \kappa_{D+1} = n$, such that,

(C1) Every equation $\mathscr{E}_i = A_{\kappa_i} = (p_{\kappa_i}, S_{\kappa_i}^-, S_{\kappa_i}^+)$, for $1 \leqslant i \leqslant D+1$, is a weighted equation over $S_1, S_2, \ldots, S_d$.

(C2) $\mathscr{S} = (\mathscr{E}_i)_{1 \leqslant i \leqslant D+1}$ is such that, $\mathrm{INV}(\mathscr{S}) \neq \perp, \mathrm{D}(\mathscr{S}) = D$ and $|||\mathscr{S}||| \leqslant s_{d_0}$,

(C3) $(\kappa_1, \kappa_2, \ldots, \kappa_{D+1}, S_1, \ldots, S_d) \in \mathbb{N}^* \times (\mathsf{DRB}\langle\langle V\rangle\rangle)^*$ is the minimal vector satisfying conditions (C1) and (C2) for the given sequence $(A_1 \cdots A_n)$.

(C4) $B_1 \cdots B_m = \rho_e(\mathrm{INV}(\mathscr{S}))$ (where ρ_e is the obvious extension of ρ_e to pairs of series and then to sequences of weighted equations; in other words, the result of T_C is $\mathrm{INV}(\mathscr{S})$ where the marks have been removed).

Lemma 65. *$T_{\mathrm{cut}}, T_\emptyset, T_\varepsilon, T_A$ are $\mathscr{D}_0$-strategies.*

Proof. T_{cut}: (S1) is true by rule (R0). (S2) is trivially true.
$T_\emptyset$: (S1) is true by rule (R′3). (S2) is trivially true.

T_ε: (S1) is true by rule (R′3). (S2) is trivially true.

T_A: by rule (R4), $\{B_j \mid 1 \leqslant j \leqslant m\} \mid\!\!\vdash\!\!-_4 A_n$, which proves (S1). Suppose $H(A_n) = \infty$, i.e. $S \equiv T$. Then, $\forall j \in [1,m]$, $S \odot x_j \equiv T \odot x'_j$, so that $\min\{H(B_j) \mid 1 \leqslant j \leqslant m\} = \infty$. (S2) is proved. □

Lemma 66. *T_B^+, T_B^- are $\mathscr{D}_0$-strategies.*

Prooof. Let us show that T_B^+ is a $\mathscr{D}_0$-strategy. Let us use the notation of the definition of T_B^+. Let $\mathscr{H} = \{(\pi, \bar{U}, U'), (\pi + k_1 - 1, V, V')\}$. Let us show that

$$\mathscr{H} \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathscr{D}_0} (\pi + k_1 - 1, U_n, U'_n). \tag{97}$$

Using rule (R5) we obtain $\forall q \in Q'$,

$$\begin{aligned} \{(\pi, \bar{U}, U')\} = \left\{ (\pi, \bar{U}, \sum_{r \in Q} [\bar{p}zr] \cdot V_r) \right\} &\overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{R5} (\pi + 2 \cdot |u_q|, \bar{U} \odot u_q, U' \odot u'_q) \\ &\overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{R0} (\pi + 2 \cdot k_0, \bar{U} \odot u_q, U' \odot u'_q) \\ &= (\pi + 2 \cdot k_0, \bar{U} \odot u_q, V_q). \end{aligned} \tag{98}$$

By rule (R′3), for every q such that $[\bar{p}zq] \equiv \emptyset$,

$$\emptyset \vdash\!\!- (0, [\bar{p}zq], \emptyset). \tag{99}$$

Let us show that, for every q such that $[\bar{p}zq] \equiv \emptyset$,

$$\emptyset \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathscr{C}} (0, [p\tau q], \emptyset). \tag{100}$$

From the equations $[\bar{p}zq] \equiv \emptyset$ and $[\bar{p}zq] \odot u = [p\tau q]$ (for some $u \in X^{k_1}$) we get that $[p\tau q] \equiv \emptyset$. Hence, by rule (R′3), (100) is true. From this deduction, we obtain

$$\emptyset \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathscr{C}} \left(\pi + 2k_0, U'_n, \sum_{[\bar{p}zq] \not\equiv \emptyset} [p\tau q] \cdot V_q \right). \tag{101}$$

Using rule (R′3), for every $q \in Q$,

$$\emptyset \mid\!\!\vdash\!\!-_{R'3} (0, [qeq], \varepsilon). \tag{102}$$

Using (102), (98)and (R7)–(R9) we obtain

$$\{(\pi, \bar{U}, U')\} \overset{\langle * \rangle}{\mid\!\!\vdash\!\!-}_{\mathscr{C}} \left(\pi + 2k_0, \sum_{[\bar{p}Aq] \not\equiv \emptyset} [p\tau q] \cdot V_q, \sum_{[\bar{p}Aq] \not\equiv \emptyset} [p\tau q][qeq] \cdot (\bar{U} \odot u_q) \right). \tag{103}$$

By (101), (103) and (R0), (R2) we get

$$\{(\pi, \bar{U}, U')\} \overset{\langle * \rangle}{\Vdash\!\!-} (\pi + 2k_0, U'_n, V'). \tag{104}$$

Let us recall that $U_n = V$. Hence, by (R1), (R2)

$$\{(\pi + k_1 - 1, V, V'), (\pi + 2k_0, U'_n, V')\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} (\pi + k_1 - 1, U_n, U'_n). \tag{105}$$

By (104), (105) and (97) is proved. Using now (97) and rule (R0), we obtain

$$\mathscr{H} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} (\pi + k_1 - 1, U_n, U'_n) \vdash\!\!-_{R0} (\pi + k_1, U_n, U'_n) \tag{106}$$

i.e. T_B^+ fulfills (S1).

Let us suppose now that $\forall i \in [n - k_1, n], U_i \equiv U'_i$. Then, by (104), $U'_n \equiv V'$ and by hypothesis $V = U_n \equiv U'_n$. Hence $V \equiv V'$. This shows that T_B^+ fulfills (S2).

An analogous proof can obviously be written for T_B^-. □

Lemma 67. *Let (p, S, S') be a weighted equation, i.e. $p \in \mathbb{N}, S, S' \in \mathsf{DRB}\langle\langle V \rangle\rangle$. Then $\{(p, S, S')\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} \{(p, \rho_e(S), \rho_e(S'))\}$ and $\{(p, \rho_e(S), \rho_e(S'))\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} \{(p, S, S')\}$.*

Proof. Follows easily from rules (R1), (R2), (R11). □

Lemma 68. *T_C is a $\mathscr{D}_0$-strategy.*

Proof. By Lemma 62, point (1), combined with Lemma 67, (S1) is proved. By Lemma 62, point (2), combined with Lemma 67, (S2) is proved. □

Let us define the strategy $\mathscr{S}_{AB}$ by: for every $W = A_1 A_2 \cdots A_n$,

(0) if $W \in \mathrm{dom}(T_{\mathrm{cut}})$, then $\mathscr{S}_{AB}(W) = T_{\mathrm{cut}}(W)$,
(1) elsif $W \in \mathrm{dom}(T_\emptyset)$, then $\mathscr{S}_{AB}(W) = T_\emptyset(W)$,
(2) elsif $W \in \mathrm{dom}(T_\varepsilon)$, then $\mathscr{S}_{AB}(W) = T_\varepsilon(W)$,
(4) elsif $W \in \mathrm{dom}(T_B^+)$, then $\mathscr{S}_{AB}(W) = T_B^+(W)$,
(5) elsif $W \in \mathrm{dom}(T_B^-)$, then $\mathscr{S}_{AB}(W) = T_B^-(W)$,
(6) elsif $W \in \mathrm{dom}(T_A)$, then $\mathscr{S}_{AB}(W) = T_A(W)$,
(7) else $\mathscr{S}_{AB}(W)$ is undefined.

The strategy $\mathscr{S}_{ABC}$ is obtained by inserting "(3) elsif $W \in \mathrm{dom}(T_C)$, then $\mathscr{S}_{ABC}(W) = T_C(W)$" in the above list of cases.

Lemma 69. *$\mathscr{S}_{ABC}, \mathscr{S}_{AB}$ are closed.*

Proof. Given any true assertion $A_n = (\pi, S, T)$ and any word $W = A_1 \cdots A_n$, at least one of T_ε, T_A is defined on W. $\square$

8. Tree analysis

This section is devoted to the analysis of the proof-trees τ produced by the strategy $\mathcal{S}_{AB}$ defined in Section 7. The main results are Lemmas 83 and 84 whose combination asserts that if some path (from a node x to a node y) of τ is such that its origin has "small defect and large norm" and its length is "large", then there exists some ancestor of y at which T_C has a non-empty value. This key technical result will ensure termination of the strategy $\mathcal{S}_{ABC}$ (see Section 9).

8.1. Depth and weight

In this section we show that the *weight* and the *depth* of a given node are closely related. Let us say that the strategy T "occurs at" node x iff

$$T(\tau(x[0]) \cdot \tau(x[1]) \cdots \tau(x[|x| - 1])) = \tau(x),$$

i.e. the image of the path from ε (included) to x (excluded) by the strategy T, is equal to the label of x.

For short, we say that T_B occurs at x iff T_B^+ or T_B^- occurs at x.

Lemma 70. *Let $\alpha \in \{-, +\}, A_1, \ldots, A_n \in \mathcal{A}$ such that $T_B^\alpha(A_1 \cdots A_n)$ is defined. Then, $\forall i \in [n - k_1 + 1, n]$, $\forall \alpha' \in \{+, -\}$, $A_i \neq T_B^{\alpha'}(A_1 \cdots A_{i-1})$.*

In other words: if T_B occurs at node x of τ, it cannot occur at any of its k_1 above immediate ancestors.

Proof. Suppose that $\exists i \in [n - k_1 + 1, n]$, $\alpha' \in \{+, -\}$, $A_i = T_B^{\alpha'}(A_1 \cdots A_{i-1})$. Hence $\pi_i = \pi_{i-1} - 1 < \pi_{n-k_1} + i$, contradicting one of the hypothesis under which $T_B^\alpha(A_1 \cdots A_n)$ is defined. $\square$

Lemma 70 ensures that, in every branch $(x_i)_{i \in I}$ and for every interval $[n + 1, n + 4] \subseteq I$, at most one integer j is such that T_B occurs at j.

Lemma 71. *Let τ be a proof-tree associated to the strategy $\mathcal{S}_{AB}$. Let $x, x' \in \mathrm{dom}(\tau)$, $x \leqslant x'$. Then $|W(x') - W(x)| \leqslant |x'| - |x| \leqslant 2 \cdot (W(x') - W(x)) + 3$.*

(We recall the *depth* of a node x is just its length $|x|$.) We denote by $W(x)$ the weight of x which we define as the first component of $\tau(x)$, i.e. the weight of the equation labelling x.)

Proof. Let x, x' be such that $|x'| = |x| + 1$. Then $W(x') - W(x) \in \{-1, +1\}$ hence the inequality $|W(x') - W(x)| \leqslant |x'| - |x|$ is fulfilled such nodes. The general case follows by induction on $(|x'| - |x|)$.

Let us prove now the other inequality. We distinguish two cases.

Case 1: $|x'| - |x| \leqslant 3$. Then $|x'| - |x| \leqslant 2 \cdot (W(x') - W(x)) + 3$ (because there is at most one T_B step in a sequence of length $\leqslant 3$).

Case 2: $|x'| - |x| \geqslant 4$. Let $x = x_0, x_1, \dots, x_q, x'$ be the sequence of nodes such that $|x'| - |x| = 4 \cdot q + r$, $0 \leqslant r < 4$ and $\forall i \in [0, q-1], |x_{i+1}| - |x_i| = 4$.

By Lemma 70, in every set $\{y \in \mathrm{dom}(\tau) \mid x_i \prec y \leqslant x_{i+1}\}$ at most one node z is such that T_B occurs at z. Hence $W(x_{i+1}) - W(x_i) \geqslant 2$.

It follows that

$$\begin{aligned}
|x'| - |x| &= \sum_{i=0}^{q-1} [|x_{i+1}| - |x_i|] + |x'| - |x_q| \\
&\leqslant \sum_{i=0}^{q-1} 2(W(x_{i+1}) - W(x_i)) + |x'| - |x_q| \\
&\leqslant 2(W(x_q) - W(x)) + 2(W(x') - W(x_q)) + 3 \quad \text{(by the first case)} \\
&\leqslant 2(W(x') - W(x)) + 3. \qquad \square
\end{aligned}$$

We recall that (Π_0, S_0^+, S_0^-) is an initial assertion which has been fixed in Section 6. We recall the definitions of some constants (defined in Section 6):

$$k_0 = \max\{v([pzq]) \mid p, q \in Q, z \in Z, [pAq] \not\equiv \emptyset\}, \quad k_1 = \max\{2k_0 + 1, 3\},$$

$$D_1 = 4k_0 + 3, \quad k_2 = (D_1 + 4) \cdot k_1 + k_0 + 1,$$

$$\begin{aligned}
&d_0 = 2 \cdot |Q| \cdot (\mathrm{Card}(X^{\leqslant k_1}) + 1), \quad D_2 = \max\{\Sigma_{d_0} + s_{d_0}, \|S_0^-\|, \|S_0^+\|\}, \\
&N_0 = 1 + (k_2 + 2)K_0 + D_2.
\end{aligned}$$

We fix throughout the remaining of this section a tree $\tau = \mathscr{T}(\mathscr{S}_{AB}, (\pi_0, U_0^-, U_0^+))$ (i.e. τ is the proof tree associated to the assertion (π_0, U_0^-, U_0^+) by the strategy $\mathscr{S}_{AB}$). We suppose that

$$\mathrm{rd}(U_0^-) \leqslant D_2, \quad \mathrm{rd}(U_0^+) \leqslant D_2, \quad U_0^-, U_0^+ \text{ are both unmarked}, \tag{107}$$

$$U_0^- \equiv U_0^+. \tag{108}$$

We recall that, formally, τ is a map $\mathrm{dom}(\tau) \to \mathbb{N} \times \mathsf{DRB}\langle\langle V \rangle\rangle \times \mathsf{DRB}\langle\langle V \rangle\rangle$ such that $\mathrm{dom}(\tau) \subseteq \{1, \dots, |X|\}^*$ is closed under prefix and under "left-brother" (i.e. $w \cdot (i+1) \in \mathrm{dom}(\tau) \Rightarrow w \cdot i \in \mathrm{dom}(\tau)$). We denote by $pr_{2,3} : \mathbb{N} \times \mathsf{DRB}\langle\langle V \rangle\rangle \times \mathsf{DRB}\langle\langle V \rangle\rangle \to \mathsf{DRB}\langle\langle V \rangle\rangle \times \mathsf{DRB}\langle\langle V \rangle\rangle$ the projection $(\pi, U, U') \mapsto (U, U')$. By τ_s we denote the tree obtained from τ by forgetting the weights: $\tau_s = \tau \circ pr_{2,3}$.

8.2. Linearity

Lemma 72. *For every label* (π, U^-, U^+) *of* τ,
(1) $\forall\alpha \in \{-,+\}$, U^α *is* (D_1, D_2)*-linear*,
(2) *if* U^α *is unmarked, then* U^α *is* $(0, D_2)$*-linear*,
(3) $\exists\alpha \in \{-,+\}$, U^α *is unmarked.*

Proof. Let $x \in \mathrm{dom}(\tau), \tau(x) = (\pi, U^-, U^+)$. We denote by $x(i)$ the prefix of length i of the node x. For every $0 \leqslant i \leqslant |x|$ we note $\tau(x(i)) = (\pi_i, U_i^-, U_i^+) = A_i$. (Hence $(A_i)_{0\leqslant i\leqslant |x|}$ is the sequence of labels on the path from ε to x.) We prove points (1)–(3) by induction on $|x|$ (the depth of node x).

$|x| = 0$: By hypothesis (107) points (1)–(3) are true.

$|x| = n+1$:

Case 1: (π, U^-, U^+) is the result of the application of T_B^α on $(\pi_i, U_i^-, U_i^+), (\pi_n, U_n^-, U_n^+)$ where $i = n - k_1$.

Then $U_i^{-\alpha}$ is unmarked (by definition of T_B^α) and $(0, D_2)$-linear (by induction hypothesis). It follows that U^α is $(2k_0+2, D_2)$-linear. Moreover, by definition of $\mathscr{S}_{AB}$, $U_n^\alpha \not\equiv \emptyset$, hence U^α is marked. On the other hand, there exists some $w \in X^{k_1}$ such that $U^{-\alpha} = U_i^{-\alpha} \odot w$. By induction hypothesis $\mathrm{rd}(U_i^{-\alpha}) \leqslant D_2$ and by Lemma 32 point (1), $\mathrm{rd}(U^{-\alpha}) \leqslant D_2$. Moreover, as no letter $[qeq]$ can be introduced by the action $\odot$ (see hypothesis (12) in Section 2.2), $U^{-\alpha}$ is unmarked.

Case 2: $(\pi, U^-, U^+) = (\pi_n+1, U_n^- \odot x, U_n^+ \odot x)$ for some $x \in X$. If for every $1 \leqslant i \leqslant n$, (π_i, U_i^-, U_i^+) is not the result of an application of T_B, then U_{n+1}^-, U_{n+1}^+ are both $(0, D_2)$-linear. Otherwise, there is a maximal integer, m_0, such that $(\pi_{m_0}, U_{m_0}^-, U_{m_0}^+)$ is the result of an application of T_B. Let $\alpha \in \{-,+\}$ such that T_B^α occurs at m_0. We have

$$U_{m_0}^{-\alpha} \text{ is unmarked and } (0, D_2)\text{-linear}, \tag{109}$$

$$U_{m_0}^{+\alpha} = \sum_{q\in Q} [p\omega q][qeq]T_q \tag{110}$$

for some $\omega \in Z^+$, $1 \leqslant |\omega| \leqslant 2k_0+2$, $(T_q)_{q\in Q} \in \mathsf{DRB}_{Q,1}\langle\langle V\rangle\rangle$, where every T_q is unmarked and $(0, D_2)$-linear. As $U^{-\alpha} = U_{m_0}^{-\alpha} \odot w$ for some $w \in X^{n+1-m_0}$, property (109) implies that $U^{-\alpha}$ is unmarked and $(0, D_2)$-linear.

If there exists some $i \in [m_0, n+1]$ such that $U_i^{+\alpha} = \rho_\varepsilon(T_q)$ then, by hypothesis $U_i^{+\alpha}$ is unmarked and $(0, D_2)$-linear. It follows that in this case $U^{+\alpha}$ is unmarked and $(0, D_2)$-linear.

If there exists no $i \in [m_0, n+1]$ such that $U_i^{+\alpha} = \rho_\varepsilon(T_q)$ then

$$\forall i \in [m_0, n+1],\ \exists p_i \in Q,\ \omega_i \in Z^+, \quad U_i^{+\alpha} = \sum_{q\in Q} [p_i\omega_i q][qeq]T_q.$$

Let $m_0 \leqslant i_0 \leqslant n+1$ such that $|\omega_{i_0}|$ is minimal. If $|\omega_{n+1}| - |\omega_{i_0}| > 2k_0+1$, then $T_B^\alpha(A_0 \ldots A_i)$ would be non-empty for some $i_0 < i < n+1$, contradicting the maximality of m_0. Hence $|\omega_{n+1}| - |\omega_{i_0}| \leqslant 2k_0+1$, and since $|\omega_{i_0}| \leqslant |\omega_{m_0}| \leqslant 2k_0+2$ we have $|\omega_{n+1}| \leqslant 4k_0+3$. Hence $U^{+\alpha}$ is (D_1, D_2)-linear. □

8.3. N-stacking sequences

We show here that the tree τ is somewhat "smooth" in the sense that its labels cannot be varying in a too chaotic way along a given branch. We shall establish that every "sufficiently long" branch must contain a "reasonably short factor" (a "N-stacking sequence") where at least d_0 labels (U, U') are belonging to the same d-space V of dimension $\leqslant d_0$ with coordinates not greater than s_{d_0} (over some fixed generating family of cardinality $\leqslant d_0$). Let us define now this notion of stacking sequence which is, roughly speaking, an extension to sequences of *pairs* (U_i, U_i') appearing in τ_s, of the notion of stacking derivation (see Section 3.4.2).

For every $U, U' \in \mathsf{DRB}\langle\langle V\rangle\rangle$ we set

$$|||U||| = \max\{\|\tilde{U}\|,\ \tilde{U} \in \mathsf{Q}(U) \text{ and } \tilde{U} \text{ is unmarked}\},$$

$$|||(U, U')||| = \max\{|||U|||, |||U'|||\}.$$

Let $x \in \mathrm{dom}(\tau)$. We define now a kind of norm on nodes which in some sense "erases" the short-term variations of $|||(*, *)|||$:

$$\mathrm{N}(x) = \max\{|||\tau_s(x(j))|||, j \in \mathbb{N}, |x| - k_1 \leqslant j \leqslant |x|\}.$$

Let us say that a function $f : \mathbb{N} \to \mathbb{N}$ is k-up Lipschitz iff

$$\forall i, j \in \mathrm{dom}(f), \quad i \leqslant j \Rightarrow f(j) - f(i) \leqslant k(j - i). \tag{111}$$

Lemma 73. *Let $(x_i)_{i\in I}$ (where I is a beginning section of $\mathbb{N}$) be some branch of τ.*
(1) *The function $\mathrm{N} : I \to \mathbb{N}$ is $(k_0 \cdot K_0)$-up-Lipschitz.*
(2) *The restriction $\mathrm{N} : J \to \mathbb{N}$ to any interval $J \subseteq I$ such that neither T_B^-, nor T_B^+ occur in J, is K_0-up-Lipschitz.*

(In the above lemma and in the proof below we use the simplified notation $\mathrm{N}(i)$ for the integer $\mathrm{N}(x_i)$.)

Proof. Let us prove that for every i such that $i + 1 \in I$,

$$\mathrm{N}(i + 1) \leqslant \mathrm{N}(i) + k_0 K_0. \tag{112}$$

Let $\alpha \in \{-, +\}$. Let us consider four cases.

Case 1: $U_{i+1}^{\alpha} = U_i^{\alpha} \odot x$ (for some $x \in X$, U_i^{α} unmarked). Then, by Lemma 32, $\|U_{i+1}^{\alpha}\| \leqslant \|U_i^{\alpha}\| + K_0 \leqslant \mathrm{N}(i) + K_0$.

Case 2: U_{i+1}^{α} is obtained by T_B^{α} transformation

$$U_{i+1}^{\alpha} = \sum_{q \in Q'} [p_i \omega_i q][q e q](U_{i-k_1}^{-\alpha} \odot u_q),$$

where $Q' \subseteq Q, 1 \leqslant |u_q| \leqslant k_0$. Then, by Lemma 32 we have

$$|||U^{\alpha}_{i+1}||| = \max_{q \in Q'} \{\|U^{-\alpha}_{i-k_1} \odot u_q\|\} \leqslant \|U^{-\alpha}_{i-k_1}\| + k_0K_0$$
$$\leqslant \mathrm{N}(i) + k_0K_0.$$

Case 3: $U^{\alpha}_{i+1} = U^{\alpha}_i \odot x$ (for some $x \in X, U^{\alpha}_i, U^{\alpha}_{i+1}$ are marked).

$$U^{\alpha}_i = \sum_{q \in Q} [p_i\omega_i q][qeq]V_q$$

where the V_q are unmarked

$$U^{\alpha}_{i+1} = \sum_{q \in Q} ([p_i\omega_i q] \odot x)[qeq]V_q,$$

hence, $|||U^{\alpha}_{i+1}||| = \max_{q \in Q} \{\|V_q\|\} = |||U^{\alpha}_i||| \leqslant \mathrm{N}(i)$.

Case 4: $U^{\alpha}_{i+1} = U^{\alpha}_i \odot x$ (for some $x \in X, U^{\alpha}_i$ marked, U^{α}_{i+1} unmarked).

$$U^{\alpha}_i = \sum_{q \in Q} [p_i\omega_i q][qeq]V_q,$$

where the V_q are unmarked and ε-free $U^{\alpha}_{i+1} = V_{q_0}$, for some $q_0 \in Q$, hence

$$|||U^{\alpha}_{i+1}||| = \|U^{\alpha}_{i+1}\| = \|V_{q_0}\| \leqslant \max_{q \in Q}\{\|V_q\|\} = |||U^{\alpha}_i||| \leqslant \mathrm{N}(i).$$

As in every case, $|||U^{\alpha}_{i+1}||| \leqslant \mathrm{N}(i) + k_0K_0$, inequality (112) and point (1) of the lemma are proved.

The discussion above also shows that, if T_B does not occur in J, then for every J such that $j \in J, j+1 \in J$, $\mathrm{N}(j+1) \leqslant \mathrm{N}(j) + K_0$, which proves point (2) of the lemma.

□

Let $\sigma = (x_i)_{i \in I}$ be a path in τ, where $I \subseteq \mathbb{N}$ is a non-empty interval and $i_0 = \min(I)$. We call σ a *N-stacking sequence* iff:

$$\forall i \in I, \quad \mathrm{N}(x_i) \geqslant \mathrm{N}(x_{i_0}) \quad \text{and} \quad \mathrm{N}(x_{i_0}) \geqslant N_0. \tag{113}$$

From now on and until Lemma 84, we fix a N-stacking sequence $\sigma = (x_i)_{i \in I}$. We call Card(I)-1 the *length* of σ (denoted $|\sigma|$). Let us use the simplified notation $\mathrm{N}(i)$ for $\mathrm{N}(x_i)$ and let us note $\tau_s(x_i) = (U^-_i, U^+_i)$.

Lemma 74. *$\forall i \in \mathbb{N}$, if $i + k_1 \in I$, then $\exists j \in [i, i+k_1], |||(U^-_j, U^+_j)||| \geqslant \mathrm{N}(i_0)$.*

Proof. This follows from the fact that $\mathrm{N}(i+k_1) \geqslant \mathrm{N}(i_0)$. □

Lemma 75. *Let $i \in \mathbb{N}$, such that $i + k_1 + 1 \in I$. If T_B occurs at node x_{i+1} then $|||(U^-_{i+1}, U^+_{i+1})||| \geqslant \mathrm{N}(i_0) - k_1K_0$.*

Proof. Suppose that $|||(U_{i+1}^-, U_{i+1}^+)||| \leqslant \mathrm{N}(i_0) - k_1K_0 - 1$. Then, T_B cannot occur inside $[i+1, i+1+k_1]$ (see Lemma 70). Hence, by Lemma 73 $\forall j \in [i+1, i+1+k_1]$, $|||(U_j^-, U_j^+)||| \leqslant \mathrm{N}(i_0) - 1$. Finally, $\mathrm{N}(i+1+k_1) \leqslant \mathrm{N}(i_0) - 1$, contradicting the fact that σ is N-stacking. □

Lemma 76. *Suppose that T_B^α occurs at node x_{i+1} where $i+k_1+1 \in I$. Then $U_{i+1}^\alpha = \sum_{q \in Q'} [p_i\mu_i q][qeq](\overline{U_{i+1}^\alpha} \odot u_q)$, for some $p_i \in Q, Q' \subseteq Q, \mu_i \in (Z - \{e\})^*, 1 \leqslant |\mu_i|, |u_q| \leqslant k_0, \|\overline{U_{i+1}^\alpha}\| \geqslant \mathrm{N}(i_0) - 2k_1K_0$.*

Proof. We distinguish two cases.

Case 1: $|||U_{i+1}^\alpha||| = \max\{|||U_{i+1}^-|||, |||U_{i+1}^+|||\}$. By lemma 75, $|||U_{i+1}^\alpha||| \geqslant \mathrm{N}(i_0) - k_1K_0$. So, $\exists q \in Q', \|\overline{U_{i+1}^\alpha} \odot u_q\| \geqslant \mathrm{N}(i_0) - k_1K_0$, hence $\|\overline{U_{i+1}^\alpha}\| \geqslant \mathrm{N}(i_0) - (k_1 + k_0)K_0 \geqslant \mathrm{N}(i_0) - 2k_1K_0$.

Case 2: $|||U_{i+1}^{-\alpha}||| = \max\{|||U_{i+1}^-|||, |||U_{i+1}^+|||\}$. Hence $\|U_{i-k_1}^{-\alpha}\| \geqslant \|U_{i+1}^{-\alpha}\| - k_1K_0 = |||U_{i+1}^{-\alpha}||| - k_1K_0 \geqslant \mathrm{N}(i_0) - 2k_1K_0$. As $\overline{U_{i+1}^\alpha} = U_{i-k_1}^{-\alpha}$, the lemma is proved. □

We define an integer i_1 by

$$i_1 = \min\{i \in [i_0 - k_1, i_0] \cap \mathbb{N},\ |||(U_i^-, U_i^+)||| \geqslant \mathrm{N}(i_0) - (k_2 - k_0)K_0\}, \tag{114}$$

if T_B does not occur in $[i_0 - k_1, i_0]$;

$$i_1 \text{ is the unique element of } [i_0 - k_1, i_0] \cap \mathbb{N} \text{ where } T_B \text{ occurs}, \tag{115}$$

if T_B occurs in $[i_0 - k_1, i_0]$.

Let us notice that, by Lemma 74, i_1 is always defined (i.e. the set used in the r.h.s. of definition (114) cannot be empty).

For every $\alpha \in \{-, +\}$, we define a Q-series $[p^\alpha \omega^\alpha]$ and a Q-form Φ^α as follows.

Case 1: $U_{i_1}^\alpha$ is unmarked and $\|U_{i_1}^\alpha\| \geqslant \mathrm{N}(i_0) - (k_2 - k_0)K_0$. As $\mathrm{rd}(U_{i_1}^\alpha) \leqslant D_2$, $U_{i_1}^\alpha$ has a minimal decomposition

$$U_{i_1}^\alpha = [p^\alpha \omega_1^\alpha] * \Phi_1^\alpha \text{ with } \|\Phi_1^\alpha\| \leqslant D_2.$$

Using the inequality

$$\|U_{i_1}^\alpha\| \geqslant \mathrm{N}(i_0) - (k_2 + 1)K_0 \geqslant D_2,$$

we conclude that ω_1^α admits a decomposition $\omega_1^\alpha = \omega^\alpha \omega_2^\alpha$ such that

$$U_{i_1}^\alpha = [p^\alpha \omega^\alpha] * \Phi^\alpha, \quad \Phi^\alpha = [\omega_2^\alpha] * \Phi_1^\alpha \tag{116}$$

with

$$\mathrm{N}(i_0) - (k_2 + 1)K_0 \leqslant \|\Phi^\alpha\| < \mathrm{N}(i_0) - k_2K_0, \quad \|\Phi^\alpha\| \geqslant D_2 + K_0, \ |\Phi^\alpha| \geqslant 1. \tag{117}$$

Case 2: $U^{\alpha}_{i_1}$ is marked and $|||U^{\alpha}_{i_1}||| \geqslant \mathrm{N}(i_0) - (k_2 - k_0)K_0$. Hence, $U^{\alpha}_{i_1}$ can be written as

$$U^{\alpha}_{i_1} = \sum_{q \in Q'} [p^{\alpha}_1 \mu^{\alpha}_1 q][qeq](\overline{U^{\alpha}_{i_1}} \odot u_q)$$

for some $p^{\alpha}_1 \in Q$, $Q' \subseteq Q, 1 \leqslant |\mu^{\alpha}_1| \leqslant D_1$, $|u_q| \leqslant k_0$, $\overline{U^{\alpha}_{i_1}} = U^{-\alpha}_j$, $j < i_1$.

Let $q \in Q'$ such that $|||U^{\alpha}_{i_1}||| = \|\overline{U^{\alpha}_{i_1}} \odot u_q\|$. We have

$$\|\overline{U^{\alpha}_{i_1}}\| \geqslant \|\overline{U^{\alpha}_{i_1}} \odot u_q\| - |u_q|K_0 \geqslant \mathrm{N}(i_0) - (k_2 - k_0)K_0 - k_0 K_0 = \mathrm{N}(i_0) - k_2 K_0.$$

From this inequality we conclude, as in case 1, that $\overline{U^{\alpha}_{i_1}}$ has a decomposition

$$\overline{U^{\alpha}_{i_1}} = [p^{\alpha}\omega^{\alpha}] * \Phi^{\alpha} \tag{118}$$

with

$$\mathrm{N}(i_0) - (k_2 + 1)K_0 \leqslant \|\Phi^{\alpha}\| < \mathrm{N}(i_0) - k_2 K_0, \quad \|\Phi^{\alpha}\| \geqslant D_2 + K_0, \ |\Phi^{\alpha}| \geqslant 1. \tag{119}$$

Case 3: $|||U^{\alpha}_{i_1}||| < \mathrm{N}(i_0) - (k_2 - k_0)K_0$. In this case, $[p^{\alpha}\omega^{\alpha}], \Phi^{\alpha}$ are both undefined.

Remark 77. If definition (114) applies then, the inequality $|||(U^{-}_{i_1}, U^{+}_{i_1})||| \geqslant \mathrm{N}(i_0) - (k_2 - k_0)K_0$ implies that there exists at least one $\alpha \in \{-, +\}$ such that $[p^{\alpha}\omega^{\alpha}], \Phi^{\alpha}$ are both defined. If definition (115) applies then, Lemma 75 implies the same result.

Let us define now the following families of series and d-spaces:

$$\mathscr{G}^{\alpha}_0 = \{\Phi^{\alpha}_q \odot u \mid q \in Q, u \in X^{\leqslant k_0}\} \cup \{\rho_{\varepsilon}(\Phi^{\alpha}_q) \mid q \in Q\}, \tag{120}$$

$$W^{\alpha} = \mathsf{V}((\Phi^{\alpha}_q)_{q \in Q}), \quad V^{\alpha} = \mathsf{V}(\mathscr{G}^{\alpha}_0) \tag{121}$$

for every $\alpha \in \{-, +\}$;

$$\tilde{W} = W^{-} \cup W^{+}, \quad \tilde{V} = V^{-} \cup V^{+}, \quad \bar{V} = V^{-} + V^{+}, \tag{122}$$

where, for every d-spaces $V_1, V_2 \subseteq \mathsf{DRB}\langle\langle V \rangle\rangle$, $V_1 + V_2$ denotes the smallest d-space containing $V_1 \cup V_2$. We illustrate in Fig. 4 the above definitions. The band $\{U \in \mathsf{DRB}\langle\langle V \rangle\rangle \mid \mathrm{N}(i_0) - 2k_1 K_0 \leqslant |||U|||\}$ is a *full security band* in the sense that every U^{α}_i in this band *belongs to* $\bar{V}$ (Lemma 79). The band $\{U \in \mathsf{DRB}\langle\langle V \rangle\rangle \mid \mathrm{N}(i_0) - (k_2 - k_0)K_0 \leqslant |||U||| < \mathrm{N}(i_0)\}$ is a *security band* in the sense that if [U^{α}_j is in this band, $j < i$, U^{α}_i belongs to the full security band, and $U^{\alpha}_j \to U^{\alpha}_i$], then U^{α}_j *belongs to* $\bar{V}$ (property $\mathscr{P}(\alpha, i, j)$ established in the proof of Lemma 79).

Lemma 78. *Let* $\alpha \in \{-, +\}$, $i \in I$, $j \in \mathbb{N}$, $i_1 \leqslant j \leqslant i$ *such that* $|||U^{\alpha}_i||| \geqslant \mathrm{N}(i_0) - 2 \cdot k_1 \cdot K_0$ *and* T^{α}_B *does not occur in* $[j+1, i]$. *Then* $|||U^{\alpha}_j||| \geqslant \mathrm{N}(i_0) - (k_2 - k_0)K_0$.

Proof. Let us suppose α, i, j fulfill the hypothesis of the lemma and let us suppose that the following inequality is realized:

$$|||U^{\alpha}_j||| < \mathrm{N}(i_0) - (k_2 - k_0)K_0. \tag{123}$$

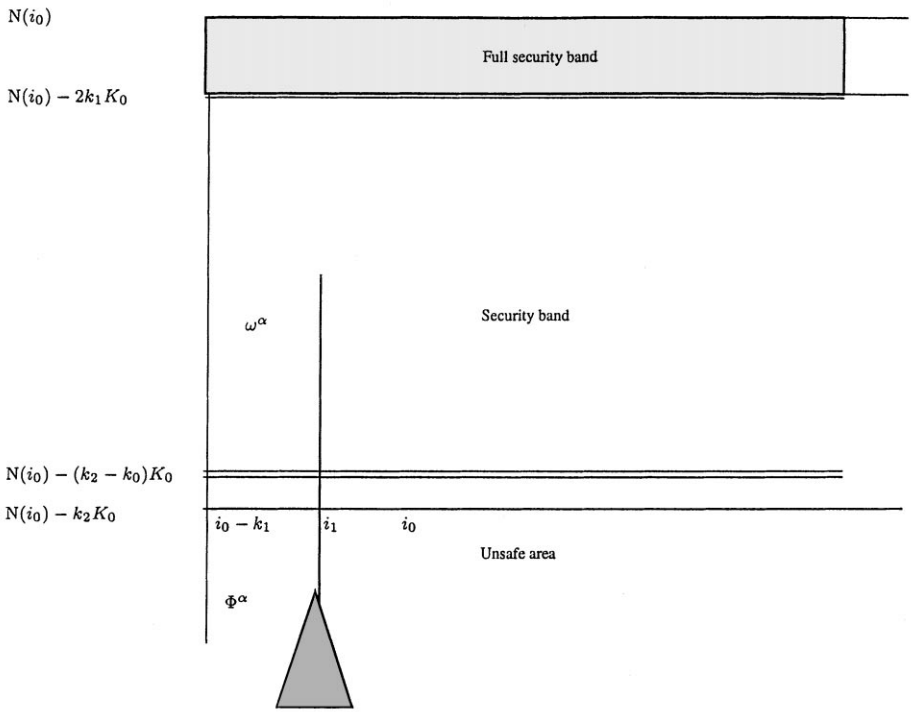

Fig. 4. The generating family.

As $|||U_i^\alpha||| \geqslant \mathrm{N}(i_0) - 2k_1K_0$, the following integers are well defined:

$$i_2 = -1 + \min\{j' \in [j + k_1, i] \mid \mathrm{N}(i_0) - 4k_1K_0 \leqslant |||U_{j'}^\alpha||| < \mathrm{N}(i_0) - (4k_1 - 1)K_0\},$$

$$i_3 = -1 + \min\{j' \in [i_2, i] \mid \mathrm{N}(i_0) - 2k_1K_0 \leqslant |||U_{j'}^\alpha|||\}.$$

As for every $j' \in [j, i_2 - k_1]$, $|||U_{j'}^\alpha||| < \mathrm{N}(i_0) - 4k_1K_0$, by Lemma 76, $T_B^{-\alpha}$ does not occur in $[j + k_1 + 1, i_2 + 1]$. But $|||U_{i_2}^\alpha||| - |||U_j^\alpha||| \geqslant (k_2 - k_0 - 4k_1 - 1)K_0 = (D_1 + 1) \cdot k_1 \cdot K_0$, hence $i_2 - j \geqslant (D_1 + 1) \cdot k_1$ so that the interval $[j + k_1, i_2]$ has a length greater or equal to $D_1 \cdot k_1$. Applying Lemma 44 we conclude that $U_{i_2}^{-\alpha}$ is unmarked. (Let us notice that $U_{i_2}^\alpha$ is unmarked too, just because $|||U_{i_2}^\alpha||| > |||U_j^\alpha|||$ while $U_{i_2}^\alpha = U_j^\alpha \odot w$ for some $w \in X^{i_2 - j}$.)

As $\|U_{i_3}^\alpha\| \geqslant \|U_{i_2}^\alpha\| + k_1 \cdot K_0 + 1$, by Lemma 43 the derivation $U_{i_2}^\alpha \to U_{i_3}^\alpha$ must contain a stacking subderivation

$$U_{i_2+k}^\alpha \uparrow (u) U_{i_2+k+k_1}^\alpha$$

for some $u \in X^{k_1}$. As $U_{i_2}^{-\alpha}$ is unmarked, there exists some $j' \in [i_2 + k_1 + 1, i_3 + 1]$ such that either T_B^α occurs at j', which contradicts the hypothesis of the lemma, or $T_B^{-\alpha}$ occurs at j' which contradicts Lemma 76. (We illustrate our argument in Fig. 5.) □

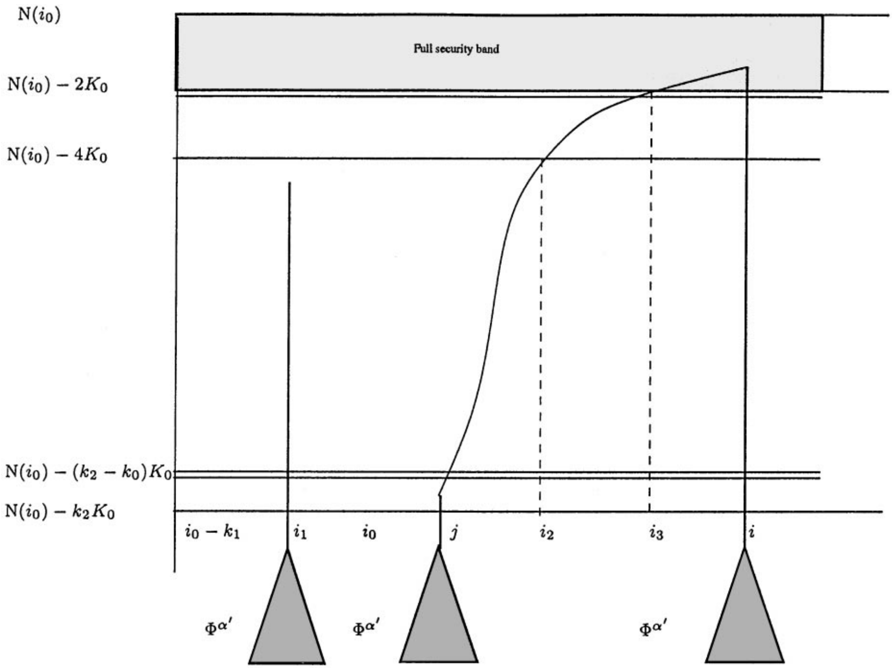

Fig. 5. U_j^α out of security band is impossible.

Lemma 79. *Let* $i \geqslant i_1$, $i + k_1 \in I$, $\alpha \in \{-,+\}$ *such that* $|||U_i^\alpha||| \geqslant \mathrm{N}(i_0) - 2k_1K_0$. *Then* $U_i^\alpha \in \bar{V}$.

Proof. Let us consider the following property $\mathscr{P}(\alpha, i, j)$:

$$\{j \leqslant i \text{ and } |||U_i^\alpha||| \geqslant \mathrm{N}(i_0) - 2k_1K_0 \text{ and no } T_B^\alpha \text{ occurs in } [j+1, i]\} \tag{124}$$

$$\Rightarrow \{(\text{if } U_j^\alpha \text{ is unmarked then } U_j^\alpha \in \tilde{W}) \text{ and } (\text{if } U_j^\alpha \text{ is marked then, there exists some } \alpha' \in \{-,+\} \text{ such that, every linear component of } U_j^\alpha \text{ is in } (W^{\alpha'} \cap \mathsf{DRBlin}^{D_2}\langle\langle V \rangle\rangle) \odot X^{\langle 1,k_0\rangle}\}. \tag{125}$$

(Here we denote by $X^{\langle m,m'\rangle}$ the set $\{u \in X^*, m \leqslant |u| \leqslant m'\}$.)

We prove by lexicographic induction on the pair (i, j) that

$$\forall i \geqslant i_1, \forall j \geqslant i_1, \ [\forall \alpha \in \{-,+\}, \mathscr{P}(\alpha, i, j)]. \tag{126}$$

Let us consider a pair $(i, j) \in \mathbb{N} \times \mathbb{N}$, $i \geqslant i_1$, $j \geqslant i_1$ and some $\alpha \in \{-,+\}$, fulfilling the left-hand side of the implication $\mathscr{P}(\alpha, i, j)$ (i.e. fulfilling condition (124)).

Case 1: $j = i_1$. If U_j^α is unmarked, by Lemma 78 $|||U_j^\alpha||| \geqslant \mathrm{N}(i_0) - (k_2 - k_0)K_0$. Hence $U_j^\alpha \in W^\alpha$ (by case 1 of the definition of Φ^α).

If U_j^α is marked, then, by case 2 of the definition of Φ^α, every linear component of U_j^α is in $(W^\alpha \cap \mathsf{DRBlin}^{D_2}\langle\langle V \rangle\rangle) \odot X^{\langle 1,k_0\rangle}$.

Case 2: $j > i_1$ and T_B^α occurs at j:

$$U_j^\alpha = \sum_{q \in Q'} [p_j \omega_j q][qeq](\overline{U_j^\alpha} \odot u_q)$$

for some $Q' \subseteq Q$, $\omega_j \in (Z - \{e\})^+$, $p_j \in Q$, $|u_q| \leqslant k_0$, and $\overline{U_j^\alpha} = U_{j-k_1-1}^{-\alpha}$.
By Lemma 76

$$|||\overline{U_j^\alpha}||| \geqslant \mathrm{N}(i_0) - 2 \cdot k_1 \cdot K_0.$$

It follows that $j - k_1 - 1 \geqslant i_1$ (by minimality of i_1 in the case where T_B does not occur in $[i_0 - k_1, i_0] \cap \mathbb{N}$ and because two successive occurrences of T_B must be at distance $\geqslant k_1 + 1$ in the case where T_B occurs at i_1). As $(j - k_1 - 1, j - k_1 - 1) < (i, j)$, by induction hypothesis, $\mathscr{P}(\alpha, j - k_1 - 1, j - k_1 - 1)$ is true: $U_{j-k_1-1}^{-\alpha}$ is unmarked (by definition of T_B^α) and $|||U_{j-k_1-1}^{-\alpha}||| \geqslant \mathrm{N}(i_0) - 2 \cdot k_1 \cdot K_0$, hence $U_{j-k_1-1}^{-\alpha} \in W^{\alpha'}$ (for some α'). It follows that

$$U_j^\alpha \text{ is marked and for every } q \in Q',\ \overline{U_j^\alpha} \odot u_q \in (W^{\alpha'} \cap \mathsf{DRBlin}^{D_2}\langle\langle V \rangle\rangle) \odot X^{\langle 1, k_0 \rangle}.$$

Case 3: $j > i_1$ and T_B^α does not occur at j.
Subcase 1: $U_{j-1}^\alpha, U_j^\alpha$ are both marked. $U_{j-1}^\alpha \odot x = U_j^\alpha$ for some $x \in X$. By Lemma 72,

$$U_{j-1}^\alpha = \sum_{q \in Q'} [p_{j-1} \omega_{j-1} q][qeq] T_q$$

for some $\omega_{j-1} \in (Z - \{e\})^+, 1 \leqslant |\omega_{j-1}| \leqslant D_1, (T_q)_{q \in Q} \in \mathsf{DRB}_{Q,1}\langle\langle V \rangle\rangle$, where every T_q is unmarked and $(0, D_2)$-linear. It follows that

$$U_j^\alpha = \sum_{q \in Q'} ([p_{j-1} \omega_{j-1} q] \odot x)[qeq] T_q.$$

By the induction hypothesis $\mathscr{P}(\alpha, i, j-1)$ we get there exists $\alpha' \in \{-, +\}$ such that $\forall q \in Q', T_q \in (W^{\alpha'} \cap \mathsf{DRBlin}^{D_2}\langle\langle V \rangle\rangle) \odot X^{\langle 1, k_0 \rangle}$. Hence (125) is true for (α, i, j).
Subcase 2: $U_{j-1}^\alpha, U_j^\alpha$ are both unmarked. By induction hypothesis $U_{j-1}^\alpha \in \tilde{W}$, i.e. $U_{j-1}^\alpha \in \mathsf{V}(\{\Phi_q^{\alpha'} \mid q \in Q\})$ for some $\alpha' \in \{-, +\}$. By definition of $\Phi^{\alpha'}$ we have
(1) $\|\Phi^{\alpha'}\| \geqslant D_2 + K_0, \quad |\Phi^{\alpha'}| \geqslant 1$,
(2) $\mathrm{rd}(U_{j-1}^\alpha) \leqslant D_2$.
Hence, by Lemma 38, $\exists \omega_{j-1} \in Z^*$, $\exists p_{j-1} \in Q$, $U_{j-1}^\alpha = [p_{j-1} \omega_{j-1}] * \Phi^{\alpha'}$. As $\|U_{j-1}^\alpha\| > \|\Phi^{\alpha'}\|$, we must have $|\omega_{j-1}| \geqslant 1$. Let us apply Lemma 37 on $U = U_{j-1}^\alpha, \Phi = \Phi^{\alpha'}, H = U_j^\alpha$, $u = x, k = 1$. One can check that, by Lemma 78 the hypothesis

$$\|H\| \geqslant 1 + k \cdot |Q| + \|\Phi\|$$

is fulfilled. Hence $U_j^\alpha = ([p_{j-1} \omega_{j-1}] \odot x) * \Phi^{\alpha'}$, which proves that $U_j^\alpha \in \tilde{W}$.
Subcase 3: U_{j-1}^α is marked while U_j^α is unmarked.

By Lemma 72

$$U^{\alpha}_{j-1} = \sum_{q \in Q} [p_{j-1}\omega_{j-1}q][qeq]T_q$$

for some $\omega_{j-1} \in (Z - \{e\})^+, 1 \leqslant |\omega_{j-1}| \leqslant D_1, (T_q)_{q \in Q} \in \mathsf{DRB}_{Q,1}$, where every T_q is unmarked and $(0, D_2)$-linear. It follows that

$$U^{\alpha}_{j} = \rho_{\varepsilon}(T_{q_0})$$

for some $q_0 \in Q$. By induction-hypothesis there exists $\alpha' \in \{-,+\}$ such that $T_{q_0} \in (W^{\alpha'} \cap \mathsf{DRBlin}^{D_2}\langle\langle V \rangle\rangle) \odot X^{\langle 1,k_0 \rangle}$, i.e. there exist $\overline{T_{q_0}} \in W^{\alpha'} \cap \mathsf{DRBlin}^{D_2}\langle\langle V \rangle\rangle, u_{q_0} \in X^{\langle 1,k_0 \rangle}$ such that

$$T_{q_0} = \overline{T_{q_0}} \odot u_{q_0}.$$

(Let us notice that this equality implies that $T_{q_0} = \rho_{\varepsilon}(T_{q_0})$.)
$\overline{T_{q_0}} \in \mathsf{V}((\Phi^{\alpha'}_q)_{q \in Q})$, and by definition of $\Phi^{\alpha'}$ we have
(3) $\|\Phi^{\alpha'}\| \geqslant D_2 + K_0, |\Phi^{\alpha'}| \geqslant 1$,
(4) $\mathrm{rd}(\overline{T_{q_0}}) \leqslant D_2$.
Applying Lemma 38, we obtain that

$$\exists \omega_0 \in Z^*,\ \exists p_0 \in Q,\ \overline{T_{q_0}} = [p_0\omega_0] * \Phi^{\alpha'}.$$

Using the inequality given by Lemma 78 (for U^{α}_j) and inequality (117) or (119) we obtain that

$$\begin{aligned} \|U^{\alpha}_j\| \geqslant \mathrm{N}(i_0) - (k_2 - k_0)K_0 &\geqslant [\mathrm{N}(i_0) - (k_2 - k_0)K_0 - 1 - k_0K_0] + 1 + k_0 \cdot K_0 \\ &\geqslant \|\Phi^{\alpha'}\| + 1 + k_0 \cdot |Q|, \end{aligned}$$

which is equivalent to

$$\|\overline{T_{q_0}} \odot u_{q_0}\| \geqslant \|\Phi^{\alpha'}\| + 1 + k_0|Q|.$$

Hence the hypothesis of Lemma 37 are met by $U = \overline{T_{q_0}}, H = T_{q_0},\ \ p = p_0,\ \ h = \omega_0,\ \Phi = \Phi^{\alpha'},\ u = u_{q_0},\ k = k_0$. Lemma 37 then concludes that

$$T_{q_0} = ([p_0\omega_0] \odot u_{q_0}) * \Phi^{\alpha'} \quad \text{where } [p_0\omega_0] \odot u_{q_0} = [q'\omega'],$$
$$\text{for some } q' \in Q, |\omega'| \geqslant k_0.$$

As $U^{\alpha}_j = T_{q_0}$, this establishes that $U^{\alpha}_j \in W^{\alpha'}$, hence that $\mathscr{P}(\alpha, i, j)$ is true. Let us observe finally that $\mathscr{P}(\alpha, i, i)$ shows that, if U^{α}_i is unmarked then $U^{\alpha}_i \in \tilde{W} \subseteq \bar{V}$ else $U^{\alpha}_i \in V^{\alpha'} \subseteq \bar{V}$. Hence the lemma is proved. □

Lemma 80. *Suppose* $i \geqslant i_1,\ \ i + k_1 \in I,\ \ \|U^{\alpha}_i\| \geqslant \mathrm{N}(i_0) - 2k_1K_0,\ \ U^{\alpha}_i$ *unmarked. Then* $\exists \alpha' \in \{-,+\}, p \in Q, \omega \in Z^*, U^{\alpha}_i = [p\omega] * \Phi^{\alpha'}$ *with* $1 \leqslant |\omega| \leqslant 2k_0(i - i_0) + 2(k_2 + k_1k_0 + 2)$.

Proof. By property $\mathscr{P}(\alpha, i, i)$ established in the proof of Lemma 79, $\exists \alpha' \in \{-,+\}$, $p \in Q, \omega \in Z^*$,

$$U_i^\alpha = [p\omega] * \Phi^{\alpha'}.$$

By Lemma 34 point (2), as $|\Phi^{\alpha'}| \geqslant 1$,

$$\|U_i^\alpha\| = 1 + (|\omega| - 1)|Q| + \|\Phi^{\alpha'}\|.$$

If $i \geqslant i_0$, by Lemma 73

$$\|U_i^\alpha\| \leqslant \mathrm{N}(i) \leqslant (i - i_0)k_0K_0 + \mathrm{N}(i_0) \leqslant (i + k_1 - i_0)k_0K_0 + \mathrm{N}(i_0).$$

If $i \in [i_1, i_0]$,

$$\|U_i^\alpha\| \leqslant \mathrm{N}(i_0) \leqslant (i + k_1 - i_0)k_0K_0 + \mathrm{N}(i_0).$$

By inequality (117) or (119),

$$\|\Phi^{\alpha'}\| \geqslant \mathrm{N}(i_0) - (k_2 + 1)K_0.$$

Hence, we get

$$\begin{aligned}(|\omega| - 1)|Q| &\leqslant 1 + (|\omega| - 1)|Q| \\ &= \|U_i^\alpha\| - \|\Phi^{\alpha'}\| \\ &\leqslant (i + k_1 - i_0)k_0K_0 + \mathrm{N}(i_0) - (\mathrm{N}(i_0) - (k_2 + 1)K_0) \\ &= (i + k_1 - i_0)k_0K_0 + (k_2 + 1)K_0.\end{aligned}$$

Hence $|\omega| \leqslant (i - i_0)k_0 \cdot K_0/|Q| + (k_2 + k_1k_0 + 2) \cdot K_0/|Q| \leqslant (i - i_0) \cdot 2k_0 + (2k_2 + 2k_1k_0 + 4)$ (because $K_0 \leqslant 2|Q|$). □

Let us define some additional spaces of series

$$\begin{aligned}\mathscr{G}_1^\alpha &= \{\Phi_q^\alpha \odot u \mid q \in Q, u \in X^{\leqslant k_1}\} \cup \{\rho_\varepsilon(\Phi_q^\alpha) \mid q \in Q\}, \\ V_1^\alpha &= \mathsf{V}(\mathscr{G}_1^\alpha), \quad \bar{V_1} = V_1^- + V_1^+.\end{aligned} \tag{127}$$

We recall that

$$K_1 = k_1K_0 + 1, \quad K_2 = 6D_1k_1^2K_0.$$

Lemma 81. *Let* $L \geqslant 0$ *such that* $(i_0 + K_1 \cdot L + K_2) + k_1 \in I$. *Then, there exists* $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ *such that,* $U_i^- \in \bar{V}_1, U_i^+ \in \bar{V}_1$.

Proof. Let us establish that

$$\exists i \in [i_0 + L, i_0 + K_1 \cdot L + K_2], \exists \alpha \in \{-,+\}, T_B^\alpha \text{ occurs at } i. \tag{128}$$

If $\exists \alpha \in \{-,+\}, T_B^\alpha$ occurs in $[i_0 + L, i_0 + L + D_1k_1]$ then (128) is true.

Otherwise, by Lemmas 72 and 44, both $U^{\alpha}_{i_0+L+D_1\cdot k_1}$ are unmarked. If some α-stacking sequence of length k_1 occurs in the interval $I(i_0,L)=[i_0+L+D_1k_1, i_0+K_1L+K_2-1]$, then some $T_B^{\alpha'}$ occurs at some $i\in[i_0+L+D_1k_1+1, i_0+K_1L+K_2]$ and (128) is true.

Let us suppose now that, $\forall\alpha\in\{-,+\}$, no α-stacking sequence of length $\geqslant k_1$ occurs in the interval $I(i_0,L)$.

By Lemma 73 we must then have

$$|||(U^{-}_{i_0+L+D_1k_1}, U^{+}_{i_0+L+D_1k_1})|||\leqslant \mathrm{N}(i_0)+(L+D_1k_1)K_0.$$

For each $\alpha\in\{-,+\}$ we distinguish two cases.

Case 1: $\exists i\in[i_0+L+D_1k_1, i_0+K_1L+K_2-k_1]$, $\|U_i^{\alpha}\|\leqslant\mathrm{N}(i_0)-k_2K_0$. By Lemma 43, as there is no α-stacking sequence of length k_1 in $I(i_0,L)$:

$$\|U^{\alpha}_{i_0+K_1L+K_2-k_1}\|\leqslant\mathrm{N}(i_0)-k_2K_0+k_1K_0\leqslant\mathrm{N}(i_0)-2k_1K_0. \tag{129}$$

Case 2: $\forall i\in[i_0+L+D_1k_1, i_0+K_1L+K_2-k_1]$, $\|U_i^{\alpha}\|>\mathrm{N}(i_0)-k_2K_0$. By Lemma 72, for every $i\in[i_0+L+D_1k_1, i_0+K_1L+K_2]$, U_i^{α} is $(0,D_2)$-linear.

Using Lemma 40 it follows that all these U_i^{α} are loop-free.

By Lemma 42, for every $\ell\leqslant(L+D_1k_1+3k_1)K_0$, there exists some $k^{\alpha}\leqslant(L+D_1k_1)+(L+D_1k_1+3k_1)K_0k_1$ such that

$$\|U^{\alpha}_{i_0+k^{\alpha}}\|\leqslant\|U^{\alpha}_{i_0+L+D_1k_1}\|-\ell.$$

Taking $\bar{\ell}=(L+D_1k_1+3k_1)K_0$ we obtain an integer

$$k^{\alpha}\leqslant(L+D_1k_1)+(L+D_1k_1+3k_1)K_0k_1\leqslant K_1L+K_2-k_1 \text{ such that}$$

$$\begin{aligned}\|U^{\alpha}_{i_0+k^{\alpha}}\| &\leqslant \|U^{\alpha}_{i_0+L+D_1k_1}\|-\bar{\ell}\\ &\leqslant \mathrm{N}(i_0)+(L+D_1k_1)K_0-\bar{\ell}\\ &= \mathrm{N}(i_0)-3k_1K_0.\end{aligned}$$

By Lemma 43, it follows that

$$\|U^{\alpha}_{i_0+K_1L+K_2-k_1}\|\leqslant(\mathrm{N}(i_0)-3k_1K_0)+k_1K_0=\mathrm{N}(i_0)-2k_1K_0. \tag{130}$$

Inequalities (129) and (130) show that

$$|||U^{-}_{i_0+K_1L+K_2-k_1},\ U^{+}_{i_0+K_1L+K_2-k_1}|||\leqslant\mathrm{N}(i_0)-2k_1K_0$$

and, finally,

$$\mathrm{N}(i_0+K_1L+K_2)\leqslant\mathrm{N}(i_0)-k_1K_0.$$

As this inequality would contradict the hypothesis that σ is N-stacking, assertion (128) is proved.

Let us consider an integer i satisfying (128). Suppose that T_B^{α} occurs at i. By Lemma 76, $\|\overline{U_i^{\alpha}}\|\geqslant\mathrm{N}(i_0)-2k_1|Q|$. One can check that, by the same argument as in the proof of Lemma 79, case 2, necessarily $i - k_1 - 1 \geqslant i_1$. By the property $\mathscr{P}(\alpha, i', j')$ (where we choose $i' = j' = i - k_1 - 1$), established in the proof of Lemma 79, $\overline{U_i^\alpha} \in \tilde{W}$. Hence $U_i^\alpha \in \tilde{V} \subseteq \overline{V_1}$ and $U_i^{-\alpha} \in \overline{V_1}$. $\square$

Let us give now a stronger version of Lemma 81 where we analyze the *size of the coefficients* of the linear combinations whose existence is proved in Lemma 81. We recall that

$$K_3 = 2k_0|Q|^2, \quad K_4 = (2k_2 + k_1 + 3) \cdot K_0^2 + (k_1 + 2) \cdot K_0 + 2.$$

Let us fix a total ordering on $\mathscr{G}_1 = \mathscr{G}_1^- \cup \mathscr{G}_1^+$:

$$\mathscr{G}_1 = \{\theta_1, \theta_2, \dots, \theta_d\}, \quad \text{where } d = \text{Card}(\mathscr{G}_1).$$

Let us remark that $d \leqslant 2 \cdot |Q| \cdot (\text{Card}(X^{\leqslant k_1}) + 1) = d_0$.

Lemma 82. *Let $L \geqslant 0$ such that $(i_0 + K_1 L + K_2) + k_1 \in I$. There exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ and, for every $\alpha \in \{-, +\}$, there exists a deterministic rational family $(\beta_{i,j}^\alpha)_{1 \leqslant j \leqslant d}$ fulfilling*
(1) $U_i^\alpha = \sum_{j=1}^d \beta_{i,j}^\alpha \cdot \theta_j$,
(2) $\|\beta_{i,*}^\alpha\| \leqslant K_3 \cdot (i - i_0) + K_4$.

Proof. We have already established property (128), i.e. $\exists i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$, $\exists \alpha \in \{-, +\}$, T_B^α occurs at i. Then

$$U_i^{-\alpha} = \bar{U} \odot u, \quad \text{for some } u \in X^{k_1},$$

$$U_i^{+\alpha} = \sum_{q \in Q'} [phq][qeq](\bar{U} \odot u_q), \quad \text{for some } Q' \subseteq Q,\ h \in Z^{\langle 1,k_1 \rangle}, u_q \in X^{\langle 1,k_0 \rangle},$$

$$\bar{U} = U_{i-k_1-1}^{-\alpha} = [r\omega] * \Phi^{\alpha'}, \quad \text{for some } r \in Q,\ \omega \in Z^*,\ \alpha' \in \{-, +\}$$

and by Lemmas 76 and 80

$$|\omega| \leqslant 2k_0(i - i_0 - k_1 - 1) + 2(k_2 + k_1 k_0 + 2). \tag{131}$$

Coefficients of $U_i^{-\alpha}$. Let us analyze the coefficients of $U_i^{-\alpha}$ expressed as a linear combination of the $\{\Phi_q^{\alpha'} \odot w \mid 0 \leqslant |w| \leqslant k_1\} \cup \{\rho_\varepsilon(\Phi_q^{\alpha'}) \mid q \in Q\}$.

Either $U_i^{-\alpha} = ([r\omega] \odot u) * \Phi^{\alpha'}$ and then

$$\begin{aligned} \|[r\omega] \odot u\| \leqslant \|[r\omega]\| + K_0|u| &\leqslant 1 + (|\omega| - 1)|Q| + |u|K_0 \\ &\leqslant 1 + [2k_0(i - i_0 - k_1 - 1) \\ &\quad + 2(k_2 + k_1 k_0 + 1) + k_1] \cdot K_0 \\ &\leqslant 2k_0 K_0(i - i_0) + (2k_2 + k_1 + 3)K_0, \end{aligned}$$

or $U_i^{-\alpha} = \rho_\varepsilon(\Phi_q^{\alpha'} \odot u'')$ for some $q \in Q$, u'' suffix of u and then

$$\|(\emptyset, \dots, \emptyset, \varepsilon, \emptyset, \dots, \emptyset)\| = 2 \leqslant k_0 K_0(i - i_0) + (2k_2 + k_1 + 3)K_0.$$

Coefficients of $U_i^{+\alpha}$. Replacing u by u_q in the above analysis, we obtain

$$\forall q \in Q', \quad \bar{U} \odot u_q = \sum_{j=1}^{d} \gamma_{q,j} \cdot \theta_j,$$

where $\|\gamma_{q,*}\| \leqslant 2k_0K_0(i-i_0)+(2k_2+k_1+3)K_0$. We can then decompose

$$U_i^{\alpha} = \tau \cdot \gamma \cdot \theta,$$

where τ is the deterministic row-vector $([phq_1][q_1eq_1],\ldots,[phq_n][q_neq_n])$, γ is a deterministic matrix of dimension $(|Q'|,d)$, θ is a row-vector of dimension $(d,1)$ whose components are the elements of $\mathscr{G}_1$.

Let us choose $\beta_{i,*}^{\alpha} = \tau \cdot \gamma$.

We obtain the upper bound

$$\begin{aligned}
\|\beta_{i,*}^{\alpha}\| &\leqslant \|\tau\| + \|\gamma\| \\
&\leqslant \|([phq_1],\ldots,[phq_n]\| + \|D\| + \|\gamma\| \\
&\qquad \text{(where } D \text{ is the diagonal matrix with diagonal coefficients } [q_ieq_i]) \\
&\leqslant \|[ph]\| + \|D\| + \|\gamma\| \\
&\leqslant (k_1|Q|+1) + (2|Q|+1) + 2k_0K_0|Q|(i-i_0) + (2k_2+k_1+3)K_0|Q| \\
&\leqslant K_3(i-i_0)+K_4. \qquad \square
\end{aligned}$$

Lemma 83. *Let us suppose that* $|\sigma| \geqslant L_d + k_1$. *Then, there exists* $i_0 \leqslant \kappa_1 < \kappa_2 < \cdots < \kappa_d$ *and deterministic rational vectors* $(\beta_{i,j}^{\alpha})_{1\leqslant j\leqslant d}$ *(for every* $i \in [1,d]$*) such that*

(0) $W(\kappa_1) \geqslant 1$,

(1) $\forall i, \ \forall \alpha, \ U_{\kappa_i}^{\alpha} = \sum_{j=1}^{d} \beta_{i,j}^{\alpha}\theta_j \in \overline{V_1}$,

(2) $\forall i, \ \forall \alpha, \ \|\beta_{i,*}^{\alpha}\| \leqslant s_i$,

(3) $\forall i, \ W(\kappa_{i+1}) - W(\kappa_i) \geqslant \delta_{i+1}$,

where the sequences $(\delta_i, \ell_i, L_i, s_i, S_i, \sigma_i)$ *are those defined by relations* (91) *and* (92) *in Section 6.*

Proof. Let us consider the additional property

(4) $\kappa_i - i_0 \leqslant L_i$.

We prove by induction on i the conjunction $(1)\wedge(2)\wedge(3)\wedge(4)$.

$i=1$: By Lemma (82), there exists $\kappa_1 \in [i_0, i_0+K_2]$ such that $\forall \alpha \in \{-,+\}, \exists$ a deterministic vector $(\beta_{1,j}^{\alpha})_{1\leqslant j\leqslant d}$, such that

$$U_{\kappa_1}^{\alpha} = \sum_{j=1}^{d} \beta_{1,j}^{\alpha}\theta_j$$

and in addition $\|\beta_{1,*}^{\alpha}\| \leqslant K_3K_2 + K_4 = s_1$.

$i \to i+1$: Suppose that $\kappa_1 < \kappa_2 < \cdots < \kappa_i$ are fulfilling $(1)\wedge(2)\wedge(3)\wedge(4)$. By Lemma 82, there exists $\kappa_{i+1} \in [i_0 + L_i + \ell_{i+1}, i_0 + K_1(L_i + \ell_{i+1}) + K_2]$ such that

$\forall\alpha \in \{-,+\}, \exists$ a deterministic vector $(\beta^{\alpha}_{i+1,j})_{1\leqslant j\leqslant d}$, such that

$$U^{\alpha}_{\kappa_{i+1}} = \sum_{j=1}^{d} \beta^{\alpha}_{i+1,j}\theta_j \tag{132}$$

and, in addition,

$$\begin{aligned}\|\beta^{\alpha}_{i+1,*}\| \leqslant K_3(K_1(L_i + \ell_{i+1}) + K_2) + K_4 &= K_3 L_{i+1} + K_4 \\ &= s_{i+1}.\end{aligned} \tag{133}$$

By Lemma 71

$$2(W(\kappa_{i+1}) - W(\kappa_i)) + 3 \geqslant \kappa_{i+1} - \kappa_i \geqslant \ell_{i+1} = 2\delta_{i+1} + 3,$$

hence,

$$W(\kappa_{i+1}) - W(\kappa_i) \geqslant \delta_{i+1}. \tag{134}$$

At last

$$\kappa_{i+1} - i_0 \leqslant K_1(L_i + l_{i+1}) + K_2 = L_{i+1}. \tag{135}$$

The above properties (132)–(135) prove the required conjunction. It remains to prove point (0): the integer κ_1 introduced by Lemma 82 is such that T_B occurs at κ_1, hence

$$\begin{aligned} W(\kappa_1) &= W(\kappa_1 - k_1 - 1) + k_1 - 1 \\ &\geqslant W(\kappa_1 - k_1 - 1) + 2 \geqslant 1. \quad \square \end{aligned}$$

We recall that

$$C_2 = \mathrm{Card}\{U \in \mathsf{DRB}\langle\langle V \rangle\rangle, \|U\| \leqslant D_2\},$$

$$K_6 = 5 \cdot [(C_2 \cdot |Q||Z|^{k_2+D_2+3})^{|Q|} \cdot |Q||Z|^{D_1}]^2, \quad K_5 = (K_6 + 1) \cdot k_0 \cdot K_0.$$

Lemma 84. *Let $(x_i)_{i\in I}$ be a path in τ (we suppose $I \subseteq \mathbb{N}$ is a non-empty interval). Let $L > 0$. One of the following cases is true*:

(0) $N(i_0) \geqslant N_0$, *where* $i_0 = \min(I)$,
(1) $|I| \leqslant K_5 \cdot L + K_6$,
(2) $(x_i)_{i\in I}$ *contains a N-stacking sequence of length* $\geqslant L$.

Proof. Suppose that neither (0) nor (2) is realized. The set $\{\tau_s(x_i) \mid i \in I\}$ can contain at most $[(C_2 \cdot |Q||Z|^{k_2+D_2+3})^{|Q|} \cdot |Q||Z|^{D_1}]^2$ pairs (U^-, U^+) such that $|||(U^-, U^+)||| < N_0$ (because, by Lemma 72, they are (D_1, D_2)-linear). But no pair can appear more than 6 times on a given path, due to the inequality of Lemma 71 and to the rule T_{cut}. Hence,

$$\mathrm{Card}\{i \in I | \mathrm{N}(i) < N_0\} \leqslant K_6.$$

Let us consider now an interval $J=[i,i+\ell]\subseteq I$ which is a maximal sub-interval of I on which N takes values in $[N_0,+\infty[$. As N is k_0K_0-up-Lipschitz, and either $i=i_0$ or $\mathrm{N}(i-1)<N_0$ we have

$$\mathrm{N}(i) < N_0 + k_0K_0.$$

As J does not contain any N-stacking sequence of length L,

$$\ell \leqslant L\cdot k_0\cdot K_0 - 1.$$

Finally, I contains at most (K_6+1) maximal sub-intervals J on which N takes values in $[N_0,+\infty[$. It follows that

$$|I| \leqslant \mathrm{Card}(I) \leqslant K_6 + (K_6+1)\cdot L\cdot k_0\cdot K_0 = K_5\cdot L + K_6,$$

i.e. property (1) is realized. □

9. Completeness of $\mathscr{D}_0$

We show that, up to some slight details, $\mathscr{S}_{ABC}$ is terminating.

Lemma 85. *Let A_0 be some true assertion which is supposed unmarked. Then the tree $\mathscr{T}(\mathscr{S}_{ABC},A_0)$ is finite.*

Proof. Suppose $A_0=(\Pi_0,S_0^-,S_0^+)$ is true, unmarked and $t=\mathscr{T}(\mathscr{S}_{ABC},A_0)$ is infinite.

By Koenig's lemma, t contains an infinite branch whose (infinite) labelling word is $A_0A_1\cdots A_n\cdots$. Lemma 63 applied to $m=1$ and $d\leqslant d_0$ and combined with Lemma 23 shows that the equations $B_j=(\pi_j,T_j,U_j)$ produced by T_C have size $\max\{\|T_j\|,\|U_j\|\}\leqslant D_2$, hence that the number of possible results of T_C is *finite*. Hence T_C occurs only a finite number of times on this branch (otherwise T_{cut} would occur on this branch, which is impossible on an infinite branch). Let n_0 be the last point where T_C occurs or $n_0=0$ if T_C does not occur on this branch. $(A_{n_0+i})_{i\geqslant 0}$ is a branch of the tree $t'=\mathscr{T}(\mathscr{S}_{AB},A_{n_0})$. As every result of T_C is a weighted linear equation which has size bounded by D_2 (by the above arguments) and which is unmarked (by point (C4) in the definition of T_C), if $n_0\neq 0$ then t' fulfills hypothesis (107) assumed in Section 8.3.

As A_0 is true and the strategies T_A,T_B,T_C preserve truth, A_{n_0} is also true. Hence t' fulfills hypothesis (108) assumed in Section 8.3.

If $n_0=0$, by definition of D_2 (see Section 6) $rd(S_0^-)\leqslant D_2$, $rd(S_0^+)\leqslant D_2$ and by hypothesis S_0^-,S_0^+ are unmarked. Hence, in this case too, t' fulfills hypothesis (107) assumed in Section 8.3. As A_0 is true, hypothesis (107) is fulfilled. Let us now apply the results of Section 8.3.

By Lemma 84, the branch $(A_{n_0+i})_{i\geqslant 0}$ must contain an N-stacking sequence σ with length $|\sigma|\geqslant L_{d_0}+k_1$. Let us remark that, as $T_\emptyset$ does not occur (otherwise the branch would be finite) every equation (π,U^-,U^+) labelling this branch is such that $U^-\neq\emptyset$, $U^+\neq\emptyset$. By Lemma 83 such an N-stacking sequence contains a subsequence

$(A_{\kappa_1}, A_{\kappa_2}, \ldots, A_{\kappa_i}, \ldots, A_{\kappa_d})$ with $d \leqslant d_0$, fulfilling hypotheses (1), (2) of Lemma 63, and by the above remark it fulfills hypothesis (81) of Section 5 too. Hence some finite prefix of $(A_{n_0+i})_{i\geqslant 0}$ belongs to $\mathrm{dom}(T_C)$. The priority ordering given in the definition of $\mathscr{S}_{ABC}$ then implies that either $T_{\mathrm{cut}}, T_{\emptyset}, T_{\varepsilon}$ or T_C occurs at some $n_0 + i + 1$. But the three first cases cannot occur on an infinite branch and the fourth one contradicts the maximality of n_0. □

Theorem 86. *The system $\mathscr{D}_0$ is complete.*

Proof. By Lemma 65 $\mathscr{S}_{ABC}$ is a strategy for $\mathscr{D}_0$, by Lemma 69 $\mathscr{S}_{ABC}$ is closed and by Lemma 85 $\mathscr{S}_{ABC}$ is terminating on every unmarked true assertion. By a slight variant of Lemma 50, every unmarked true assertion has a $\mathscr{D}_0$-proof. But for every $A\in\mathscr{A}$, there exists a finite $\mathscr{D}_0$-proof of $(0, A, \rho_e(A))$. It follows that every true assertion A has a $\mathscr{D}_0$-proof. □

Theorem 87. *The equivalence problem for deterministic pushdown automata is decidable.*

Proof. Let $\mathscr{M}$ be some dpda. The equivalence relation $\equiv$ on $\mathsf{DRB}\langle\langle V\rangle\rangle$ (where V is the structured alphabet associated to the given $\mathscr{M}$) has a recursively enumerable complement (this is well known). By Theorem 86 and Lemma 46 $\equiv$ is recursively enumerable too. Hence $\equiv$ is recursive. In addition, the system $\mathscr{D}_0$ associated with $\mathscr{M}$ is computable from $\mathscr{M}$, hence the theorem follows. □

10. Elimination

The aim of this section is to simplify, as much as we can, the deduction system $\mathscr{D}_0$. We introduce some technical tools (in Sections 10.1, 10.4) and perform successive simplifications (in Sections 10.2, 10.3, 10.5–10.7).

10.1. Congruence closure: properties

Let us study the subset $\mathscr{C}$ of the rules of $\mathscr{D}_0$, defined in Section 4.4. We recall it consists of all the instances of the meta-rules R0–R3, R′3, R6–R11. We also denote by $\Vdash\!\!-_{\mathscr{C}} \subseteq \mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$ the set of all instances of these meta-rules. We use here (and later on, in Section 10.2.2) the following notation: for every $p, n \in \mathbb{N}, S, S' \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

$$[p, S, S', n] = \{(p + |u|, S \odot u, S' \odot u) \mid u \in X^{\leqslant n}\}. \tag{136}$$

Next lemma expresses the fact that the “congruence closure”operation commutes with the right-action $\odot$.

Lemma 88. (1) Symmetry: *For every* $p, n \in \mathbb{N}$, $S, S' \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

$$[p, S, S', n] \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [p, S', S, n].$$

(2) Composition: *For every* $p, n \in \mathbb{N}, S, T \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

$$[p, S, S', n] \cup [p, S', S'', n] \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [p, S, S'', n].$$

(6) Star: *For every* $p, n \in \mathbb{N}, (S, S') \in \mathsf{DRB}_{1,2}\langle\langle V \rangle\rangle, T' \in \mathsf{DRB}\langle\langle V \rangle\rangle, S \not\equiv \varepsilon$,

$$[p, S \cdot T' + S', T', n] \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [p, S^* \cdot S', T', n].$$

(7) Sum: *For every* $p, n \in \mathbb{N}, (S, T), (S', T') \in \mathsf{DRB}_{1,2}\langle\langle V \rangle\rangle$,

$$[p, S, S', n] \cup [p, T, T', n] \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [p, S + T, S' + T', n].$$

(8) Right-product: *For every* $p, n \in \mathbb{N}, S, S', T \in \mathsf{DRB}\langle\langle V \rangle\rangle$, *if* $S \equiv_n S'$ *then*

$$[p, S, S', n] \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [p, S \cdot T, S' \cdot T, n].$$

(9) Left-product: *For every* $p, n \in \mathbb{N}, S, T, T' \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

$$[p, T, T', n] \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [p, S \cdot T, S \cdot T', n].$$

(10) ε-Reduction: *For every* $p, n \in \mathbb{N}, S, \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

$$\emptyset \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [p, S, \rho_\varepsilon(S), n].$$

(11) *e*-Reduction: *For every* $p, n \in \mathbb{N}, S, \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

$$\emptyset \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [p, S, \rho_e(S), n].$$

Sketch of proof. Points (1), (2), (7), (9), (10) can be checked easily.

(8) *Right-product*: Let $u \in X^*, |u| \leqslant n$. Let us use Lemma 22.

Case 1: $\forall u' \leqslant u, \rho_\varepsilon(S \odot u') \neq \varepsilon$. Then

$$\begin{aligned}[p, S, S', n] \supseteq (p + |u|, S \odot u, S' \odot u) &\overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} (p + |u|, (S \odot u) \cdot T, (S' \odot u) \cdot T) \\ &= (p + |u|, (S \cdot T) \odot u, (S' \cdot T) \odot u).\end{aligned}$$

Case 2: $\exists u', u''$ such that $u = u' \cdot u'', \rho_\varepsilon(S \odot u') = \varepsilon$. As $|u'| \leqslant |u| \leqslant n$ and $S \equiv_n S'$, $\rho_\varepsilon(S' \odot u') = \varepsilon$ too. Then

$$\begin{aligned}\emptyset &\overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} (p + |u|, \rho_\varepsilon(T \odot u''), \rho_\varepsilon(T \odot u'')) \\ &= (p + |u|, (S \cdot T) \odot u, (S' \cdot T) \odot u).\end{aligned}$$

(6) *Star*: Let $u \in X^*, |u| \leqslant n$. We use the same type of arguments as in the proof of Lemma 53, point (R6). We remark first that

$$[p, S \cdot T' + S', T', n] \supseteq \left(p + |u|, \left(S^{n+1} \cdot T' + \sum_{k=0}^{n} S^k \cdot S' \right) \odot u, T' \odot u \right)$$
$$= \left(p + |u|, (S^{n+1} \odot u) \cdot T' + \left(\sum_{k=0}^{n} S^k \cdot S' \right) \odot u, T' \odot u \right). \tag{137}$$

Using at first (R6) and afterwards (R9), (R7) and (R0), we have

$$(p, S \cdot T' + S', T') \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} (p, S^* \cdot S', T')$$
$$\overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} \left(p + |u|, (S^{n+1} \odot u) \cdot S^* \cdot S' + \left(\sum_{k=0}^{n} S^k \cdot S' \right) \odot u, (S^{n+1} \odot u) \cdot T' \right.$$
$$\left. + \left(\sum_{k=0}^{n} S^k \cdot S' \right) \odot u \right)$$
$$= \left(p + |u|, (S^* \cdot S') \odot u, (S^{n+1} \odot u) \cdot T' + \left(\sum_{k=0}^{n} S^k \cdot S' \right) \odot u \right). \tag{138}$$

Composing (138) and (137) by (R2), we obtain

$$[p, S \cdot T' + S', T', n] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} (p + |u|, (S^* \cdot S') \odot u, T' \odot u).$$

(11) *e-Reduction*: Let $u \in X^*, |u| \leqslant n$. Let us use Lemma 19.

Case 1: $S \odot u \in \{\emptyset, \varepsilon\}$. By Lemma 23 $S \equiv \rho_e(S)$ and by Lemma 1 $(S \odot u) \equiv (\rho_e(S) \odot u)$. Hence, by rules, (R′3), (R0)

$$\emptyset \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} (p + |u|, S \odot u, \rho_e(S) \odot u).$$

Case 2: $S \odot u = ([qz] \odot u'') * \Phi$, where $u', u'' \in X^*$, $p, q, r \in Q$, $\omega \in Z^*$, $\eta \in Z^+$, $z \in Z$, Φ Q–form such that

$$S = [p\omega] * \Phi, \; u = u' \cdot u'', \; \rho_\varepsilon([p\omega] \odot u') = [qz] * \Phi \quad \text{and} \quad \rho_\varepsilon([qz] \odot u'') = [r\eta].$$

Using the technical hypothesis (12) we obtain

$$S \odot u = [r\eta] * \Phi \quad \text{while } \rho_e(S) \odot u = [r\eta] * \rho_e(\Phi).$$

Hence, using (R11) and (R7), (R9),

$$\emptyset \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} \{(0, \Phi_s, \rho_e(\Phi_s)) \; s \in Q\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} (0, [r\eta] * \Phi, [r\eta] * \rho_e(\Phi))$$
$$= (0, S \odot u, \rho_e(S) \odot u) \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}} (p + |u|, S \odot u, \rho_e(S) \odot u). \qquad \square$$

Given a subset $P \in \mathscr{P}_f(\mathscr{A})$, we call *congruence closure* of P, denoted by Cong(P), the set

$$\mathrm{Cong}(P) = \{A \in \mathscr{A} \mid P \overset{\langle * \rangle}{||\!\!\!\vdash\!\!-}_{\mathscr{C}} \{A\}\}. \tag{139}$$

As well, for every integer $q \geqslant 0$ we define

$$\mathrm{Cong}_q(P) = \{A \in \mathscr{A} \mid P \overset{\langle q \rangle}{||\!\!\!\vdash\!\!-}_{\mathscr{C}} \{A\}\}. \tag{140}$$

10.2. System $\mathscr{D}_1$

We prove here that the new formal system $\mathscr{D}_1$ obtained by *elimination* of meta-rule (R5) in $\mathscr{D}_0$ is recursively enumerable and *complete*.

10.2.1. Rules

Let $\mathscr{D}_1 = \langle \mathscr{A}_1, H_1, \vdash\!\!-_{\mathscr{D}_1} \rangle$ where $\mathscr{A}_1 = \mathscr{A}, H_1 = H$, are the same as in $\mathscr{D}_0$, but the *elementary deduction relation* $||\!\!\!\vdash\!\!-_{\mathscr{D}_1}$ is the relation generated by the subset of meta-rules R0–R3, R′3, R4–R11, i.e. all the meta-rules of $\mathscr{B}_0$ except R5. The deduction relation $\vdash\!\!-_{\mathscr{D}_1}$ is now defined by

$$\vdash\!\!-_{\mathscr{D}_1} = \overset{\langle * \rangle}{||\!\!\!\vdash\!\!-}_{\mathscr{D}_1} \circ \overset{[1]}{||\!\!\!\vdash\!\!-}_{\mathrm{R0,R3,R'3,R4,R10,R11}} \circ \overset{\langle * \rangle}{||\!\!\!\vdash\!\!-}_{\mathscr{D}_1}.$$

Lemma 89. *$\mathscr{D}_1$ is a deduction system.*

Proof. As $\vdash\!\!-_{\mathscr{D}_1} \subseteq \vdash\!\!-_{\mathscr{D}_0}$, $\mathscr{D}_1$ must fulfill axiom (A1). As every meta-rule of $\mathscr{D}_0$ is recursively enumerable, this is also true for $\mathscr{D}_1$, hence $\mathscr{D}_1$ fulfills axiom (A2). □

10.2.2. Completeness

Definition 90. Let P be a subset of $\mathscr{A}$. P is said consistent iff, $\forall n \in \mathbb{N}, \forall \pi \in \mathbb{N}, \forall S, S' \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

$$(\pi, S, S') \in \mathrm{Cong}(P) \Rightarrow [\pi, S, S', n] \subseteq \mathrm{Cong}(P).$$

Lemma 91. *Let $A_0 \in \mathscr{A}$ be some true assertion. Let us consider the tree $t = \mathscr{T}(\mathscr{S}_{ABC}, A_0)$. Then,* im($t$) *is consistent.*

Proof. Let us note $P = \mathrm{im}(t)$ and let us consider the following property $\mathscr{Q}(\pi, n, p)$:

$$\forall S, S' \in \mathsf{DRB}\langle\langle V \rangle\rangle, \quad (\pi, S, S') \in \mathrm{Cong}_p(P) \Rightarrow [\pi, S, S', n] \subseteq \mathrm{Cong}(P). \tag{141}$$

We prove by lexicographic induction over $(\pi + n, n, p)$ that, for every triple of integers (π, n, p), $\mathscr{Q}(\pi, n, p)$ is true. Let $(\pi, n, p) \in \mathbb{N}^3$. Let us suppose that

$$\forall (\pi', n', p') \in \mathbb{N}^3, \quad (\pi' + n', n', p') < (\pi + n, n, p) \Rightarrow \mathscr{Q}(\pi', n', p'). \tag{142}$$

Case 1: $p \geqslant 1$. There exists a subset $Q \subseteq \mathscr{P}_f(\mathscr{A})$, such that

$$P \overset{\langle p-1\rangle}{||\!\!-\!\!-}_{\mathscr{C}} Q \quad \text{and} \quad Q \overset{\langle 1\rangle}{||\!\!-\!\!-}_{\mathscr{C}} \{(\pi, S, S')\}.$$

As every rule of $\mathscr{C}$ does not decrease the weight, every assertion of Q has a weight $\leqslant \pi$. Hence, by induction hypothesis

$$\forall (\pi', T, T') \in Q, \quad [\pi', T, T', n] \subseteq \text{Cong}(P). \tag{143}$$

Let us consider the type of rule used in the last step, $Q \overset{\langle 1\rangle}{||\!\!-\!\!-}_{\mathscr{C}} \{(\pi, S, S')\}$, of the above deduction.

R0. $(\pi - 1, S, S') \in Q$.

By (143), $[\pi - 1, S, S', \mathscr{R}_n] \subseteq \text{Cong}(P)$. As $[\pi - 1, S, S', n] \overset{\langle 1\rangle}{||\!\!-\!\!-}_{\mathscr{C}} [\pi, S, S', n]$,

$$[\pi, S, S', n] \subseteq \text{Cong}(P).$$

R1. $(\pi, S', S) \in Q$.

(analogous to the above case)

R2. $(\pi, S, T), (\pi, T, S') \in Q$.

By (143), $[\pi, S, T, n] \subseteq \text{Cong}(P)$ and $[\pi, T, S', n] \subseteq \text{Cong}(P)$. Using then Lemma 88, we get that

$$[\pi, S, S', n] \subseteq \text{Cong}(P).$$

R3, R$'$3.

In this case,

$$[\pi, S, S', n] \subseteq \text{Cong}(\emptyset) \subseteq \text{Cong}(P).$$

R6. $(\pi, S_1 \cdot S' + T, S') \in Q, S = S_1^* \cdot T$.

By (143), $[\pi, S_1 \cdot S' + T, S', n] \subseteq \text{Cong}(P)$. Using then Lemma 88 we get

$$\begin{aligned}[\pi, S, S', n] = [\pi, S_1^* \cdot T, S', n] &\subseteq \text{Cong}\ [\pi, S_1 \cdot S' + T, S', n] \\ &\subseteq \text{Cong}\ (P).\end{aligned}$$

R7. $(\pi, S_1, S_1') \in Q, (\pi, T, T') \in Q,\ S = S_1 + T,\ S' = S_1' + T'$ where $(S_1, T), (S_1', T') \in \mathsf{DRB}_{1,2}\langle\langle V \rangle\rangle$.

By (143),

$$[\pi, S_1, S_1', n] \cup [\pi, T, T', n] \subseteq \text{Cong}(P). \tag{144}$$

Combining Lemma 88 with (144) we get

$$\begin{aligned}[\pi, S, S', n] = [\pi, S_1 + T, S_1' + T', n] &\subseteq \text{Cong}([\pi, S_1, S_1', n] \cup [\pi, T, T', n]) \\ &\subseteq \text{Cong}(P).\end{aligned}$$

R8. $(\pi, S_1, S_1') \in Q,\ S = S_1 \cdot T,\ S' = S_1' \cdot T.$

By (143), $[\pi, S_1, S_1', n] \subseteq \mathrm{Cong}(P)$. Using then Lemma 88 we get

$$[\pi, S, S', n] = [\pi, S_1 \cdot T, S_1' \cdot T, n] \subseteq \mathrm{Cong}([\pi, S_1, S_1', n]) \subseteq \mathrm{Cong}(P).$$

R9. $(\pi, T, T') \in Q,\ S = S_1 \cdot T,\ S' = S_1 \cdot T'.$
By (143), $[\pi, T, T', n] \subseteq \mathrm{Cong}(P)$. Using Lemma 88, we get

$$[\pi, S, S', n] = [\pi, S_1 \cdot T, S_1 \cdot T', n] \subseteq \mathrm{Cong}([\pi, T, T', n]) \subseteq \mathrm{Cong}(P).$$

R10. $S' = \rho_\varepsilon(S)$.
By Lemma 88 (ε-reduction), $\emptyset \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [\pi, S, \rho_\varepsilon(S), n]$, hence

$$[\pi, S, S', n] = [\pi, S, \rho_\varepsilon(S), n] \subseteq \mathrm{Cong}(\emptyset) \subseteq \mathrm{Cong}(P).$$

R11. $S' = \rho_e(S)$.
By Lemma 88 (e-reduction), $\emptyset \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{C}} [\pi, S, \rho_e(S), n]$, hence

$$[\pi, S, S', n] = [\pi, S, \rho_e(S), n] \subseteq \mathrm{Cong}(\emptyset) \subseteq \mathrm{Cong}(P).$$

In all cases $\mathscr{Q}(\pi, n, p)$ has been established.

Case 2: $n = 0, p = 0$. In this case, for every π, property $\mathscr{Q}(\pi, n, p)$ is trivially true.

Case 3: $n = 1, p = 0$. Given a node $x \in \mathrm{dom}(t)$, let us define the sequence of assertions W_x by

$$W_x = t(x(0)) \cdot t(x(1)) \cdots t(x(|x|)).$$

(Here $x(k)$ denotes the prefix of length k of the word x, equivalently, it is the ancestor of x which has depth k.) Let us say that the strategy T "applies on" node x iff, $W_x \in \mathrm{dom}(T)$, x has exactly μ sons (with $\mu \geqslant 0$) $x \cdot 1, x \cdot 2, \dots, x \cdot \mu$ and

$$T(W_x) = t(x \cdot 1) \cdot t(x \cdot 2) \cdots t(x \cdot \mu),$$

i.e. the strategy T maps the path W_x on the word consisting of the labels of the sons of x.

In the sequel, we consider a node $x \in \mathrm{dom}(t)$ such that $t(x) = (\pi, S, S')$ and which is minimal among the set of nodes having the same label:

$$x = \min_{\preceq} \{ y \in \mathrm{dom}(t) \mid t(y) = t(x) \}. \tag{145}$$

As x is minimal by (145) and the strategy $\mathscr{S}_{ABC}$ is closed, $\mathscr{S}_{ABC}$ must apply on x.

Subcase 1: T_{cut} applies on x. There exists $x' \in \mathrm{dom}(t), \exists \pi' \in \mathbb{N}$, such that

$$t(x') = (\pi', S, S') \quad \text{and} \quad \pi' < \pi.$$

As $(\pi'+1,1,0)<(\pi+1,1,0)$, by induction hypothesis we have

$$[\pi', S, S', 1] \subseteq \mathrm{Cong}(\mathrm{im}(t))$$

and by means of rule R0:

$$[\pi, S, S', 1] \subseteq \mathrm{Cong}([\pi', S, S', 1]).$$

Hence, the right-hand side of implication (141) is true.

Subcase 2: $T_\emptyset$ to T_ε applies on x. By rules R1–R′3,

$$[\pi, S, S', 1] \subseteq \mathrm{Cong}(\emptyset) \subseteq \mathrm{Cong}(\mathrm{im}(t)).$$

Subcase 3: T_A applies on x. Then

$$[\pi, S, S', 1] \subseteq \mathrm{im}(t) \subseteq \mathrm{Cong}(\mathrm{im}(t)).$$

Subcase 4: T_B^α applies on x (for some $\alpha \in \{-,+\}$). Let us suppose $\alpha = +$. Let $x' = x(|x| - k_1)$ (the prefix of x having length $|x| - k_1$), $t(x') = (\pi', \bar{U}, U')$. Then

$$\mu = 1 \quad \text{and} \quad t(x \cdot 1) = T_B^+(W_x).$$

Let us look at the proof of Lemma 66. As, for every q, $\pi' + |u_q| - 1 < \pi' + k_0 \leqslant \pi' + 2 \cdot k_0 < \pi$, deduction (98) can be replaced by a pure $\mathscr{C}$-deduction:

$$\mathrm{im}(t) \mathrel{\overset{\langle * \rangle}{||\!\!-\!\!\!-}}_{\mathscr{C}} (\pi' + 2 \cdot k_0, \bar{U} \odot u_q, V_q).$$

As deduction (98) was the only one (in the proof of Lemma 66) using rules in $\mathscr{B}_0 - \mathscr{C}$ we conclude that deduction (97) can be replaced by

$$\{t(x'), t(x \cdot 1)\} \cup \mathrm{im}(t) \mathrel{\overset{\langle * \rangle}{||\!\!-\!\!\!-}}_{\mathscr{C}} \tau_{-1}(t(x)). \tag{146}$$

(We recall τ_{-1} consists in replacing the weight of a given weighted equation by its predecessor.) Deduction (146) implies that

$$\exists p' \in \mathbb{N}, \quad (\pi - 1, S, S') \in \mathrm{Cong}_{p'}(\mathrm{im}(t)). \tag{147}$$

As $(\pi, 1, p') < (\pi+1, 1, 0)$, we know from the induction hypothesis that

$$[\pi - 1, S, S', 1] \subseteq \mathrm{Cong}(\mathrm{im}(t)),$$

hence, using (R0),

$$[\pi, S, S', 1] \subseteq \mathrm{Cong}(\mathrm{im}(t)).$$

Subcase 5: T_C applies on x. Then

$$t(x \cdot 1) t(x \cdot 2) \cdots t(x \cdot \mu) = T_C(W_x).$$

Let $W_x = A_1 \cdots A_\ell \cdots A_{|x|+1}$, $\kappa_1 < \cdots < \kappa_i < \kappa_{i+1} < \cdots \kappa_{D+1} = |x|+1$, $\mathscr{S} = (\mathscr{E}_i)_{1 \leqslant i \leqslant D+1}$, where, for every $1 \leqslant i \leqslant D+1$,

$$\mathscr{E}_i = A_{\kappa_i} = \left(\pi_i,\ \sum_{j=1}^{d} \alpha_{i,j}, S_j \sum_{j=1}^{d} \beta_{i,j} S_j \right)$$

and

$$T_C(W_x) = \rho_e(\mathrm{INV}(\mathscr{S})), \mathrm{W}(\mathscr{S}) \neq \bot, \mathrm{D}(\mathscr{S}) = D \leqslant d-1.$$

Let us look at the proof of Lemma 57: the only place where a rule in $\mathscr{B}_0 - \mathscr{C}$ is used, is in deduction (66), when case 2, subcase 1 of the recursive definition of $\mathrm{INV}(\mathscr{S})$ occurs. Let us recall that the word u used in the definition of $\mathscr{E}'_1$ is

$$u = \min(\varphi(\alpha_{1,*}) \triangle \varphi(\beta_{1,*})).$$

Let us notice that $\pi_1 + |u| - 1 < \pi_1 + 2 \cdot |u| < \pi_2 \leqslant \mathrm{W}(\mathscr{S}) + 1 = \pi$. By induction hypothesis

$$\left(\pi_1 + |u|, \left(\sum_{j=1}^{d} \alpha_{i,j} S_j \right) \odot u, \left(\sum_{j=1}^{d} \beta_{i,j} S_j \right) \odot u \right)$$
$$\in \left[\pi_1, \sum_{j=1}^{d} \alpha_{i,j} S_j, \sum_{j=1}^{d} \beta_{i,j} S_j, |u| \right] \subseteq \mathrm{Cong}(\mathrm{im}(t)).$$

Hence deduction (66) can be replaced by

$$\mathscr{E}'_1 \in \mathrm{Cong}(\mathrm{im}(t)). \tag{148}$$

Similarly, for every $i \in [2, D]$, as $\pi_i + 2 \cdot \mathrm{Div}(\alpha_{i,*}^{(i-1)} \beta_{i,*}^{(i-1)}) < \pi_{i+1} \leqslant \mathrm{W}(\mathscr{S}) + 1 = \pi$, and $\mathscr{E}_i^{(i-1)} \in \mathrm{Cong}(\mathrm{im}(t))$,

$$(\mathscr{E}_i^{(i-1)})' \in \mathrm{Cong}(\mathrm{im}(t)). \tag{149}$$

It follows that deduction (65) can be replaced by

$$\mathrm{INV}(\mathscr{S}) \cup \mathrm{im}(t) \overset{\langle * \rangle}{||\!\!-\!\!\!-}_{\mathscr{C}} \tau_{-1}(t(x)). \tag{150}$$

Using the fact that $\rho_e(\mathrm{INV}(\mathscr{S})) \overset{\langle * \rangle}{||\!\!-\!\!\!-}_{\mathscr{C}} \mathrm{INV}(\mathscr{S})$ we may conclude that

$$\{t(x \cdot 1), \ldots, t(x \cdot \mu\} \cup \mathrm{im}(t) \overset{\langle * \rangle}{||\!\!-\!\!\!-}_{\mathscr{C}} \tau_{-1}(t(x)) = (\pi - 1, S, S'). \tag{151}$$

From (151) and the induction hypothesis, we can conclude, as in subcase 4, that

$$[\pi, S, S', 1] \subseteq \mathrm{Cong}(\mathrm{im}(t)).$$

Case 4: $n \geqslant 2, p = 0$. Let us suppose that $(\pi, S, S') \in \mathrm{im}(t)$. Let us consider the decomposition

$$[\pi, S, S', n] = \{(\pi, S, S')\} \cup \left(\bigcup_{x \in X} [\pi + 1, S \odot x, S' \odot x, n - 1] \right). \tag{152}$$

As $\pi + 1 < \pi + n$, by induction hypothesis,

$$\forall x \in X, (\pi + 1, S \odot x, S' \odot x) \in \mathrm{Cong}(\mathrm{im}(t)).$$

Hence there exists $p' \in \mathbb{N}$ such that

$$\bigcup_{x \in X} \{(\pi + 1, S \odot x, S' \odot x)\} \subseteq \mathrm{Cong}_{p'}(\mathrm{im}(t)).$$

As $(\pi + n, n - 1, p') < (\pi + n, n, 0)$, the above inclusion together with the induction hypothesis lead to

$$\bigcup_{x \in X} [\pi + 1, S \odot x, S' \odot x, n - 1] \subseteq \mathrm{Cong}(\mathrm{im}(t)). \tag{153}$$

At last, using (152) and (153) we obtain

$$[\pi, S, S', n] \subseteq \mathrm{Cong}(\mathrm{im}(t)).$$

(End of the induction.) □

Definition 92. Let $P \subseteq \mathscr{A}_1$. P is said *self-generating* iff, for every $(\pi, S, S') \in P$,
(1) $(S \equiv \varepsilon) \Leftrightarrow (S' \equiv \varepsilon)$,
(2) $[\pi, S, S', 1] \subseteq \mathrm{Cong}(P)$.

Remark 93. This notion of "self-generating set (of weighted equations)" is a natural adaptation to our d-space of series of the notion of "self-proving set of pairs" defined in [15, p. 162] for the magma $M(F \cup \Phi, V)$.

Lemma 94. *Every self-generating subset P is a $\mathscr{D}_1$-proof.*

Proof. It suffices to notice that for every $(\pi, S, S') \in \mathscr{A}_1$,

- if $S \equiv S' \equiv \varepsilon$ then $\emptyset \mid\!\!-\!\!-_{\mathscr{D}_1} (\pi, S, S')$.
- if $S \not\equiv \varepsilon$, $S' \not\equiv \varepsilon$, then $[\pi, S, S', 1] \mid\!\!-\!\!-_{\mathscr{D}_1} (\pi, S, S')$. □

Lemma 95. *Let $\pi \in \mathbb{N}, S, S' \in \mathsf{DRB}\langle\langle V \rangle\rangle$. Then, $H_1(\pi, S, S') = \infty$ iff there exists a finite self-generating set P such that $(\pi, S, S') \in P$.*

Proof. Let us consider some true assertion $A_1 = (\pi_1, S_1, S_1') \in \mathscr{A}_1$. Let us define

$$A_0 = (\pi_1, \rho_e(S_1), \rho_e(S_1')), \; t = \mathscr{T}(\mathscr{S}_{ABC}, A_0), \; P = \{A_1\} \cup \mathrm{im}(t).$$

By Lemma 85, im(t) is finite, by Lemma 91 im(t) is consistent and by the hypothesis that A_1 is true, every assertion of P is true. It follows that every $(\pi, S, S') \in \mathrm{im}(t)$ fulfills both conditions of Definition 92. Moreover, owing to meta-rule (R11), $A_1 \in \mathrm{Cong}(\mathrm{im}(t))$. As im($t$) is consistent, it follows that $[\pi_1, S_1, S_1', 1] \subseteq \mathrm{Cong}(\mathrm{im}(t))$. Hence A_1 fulfills also both conditions of Definition 92. Hence P is a finite self-generating set containing A_1. □

Theorem 96. *$\mathscr{D}_1$ is a complete deduction system.*

Proof. Follows from Lemmas 95 and 94. □

10.3. System $\mathscr{D}_2$

We exhibit here a deduction system $\mathscr{D}_2$ which is simpler than $\mathscr{D}_1$ and is still complete.

10.3.1. Rules

Let us *eliminate* the weights in the rules of $\mathscr{D}_1$: we define a new set of assertions, $\mathscr{A}_2$ by

$$\mathscr{A}_2 = \mathsf{DRB}\langle\langle V\rangle\rangle \times \mathsf{DRB}\langle\langle V\rangle\rangle.$$

We define a binary relation $\Vdash\!\!-\; \subseteq \mathscr{P}_f(\mathscr{A}_2) \times \mathscr{A}_2$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

(R21)

$$\{(S,T)\} \Vdash\!\!- (T,S)$$

for $S, T \in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$,

(R22)

$$\{(S,S'),(S',S'')\} \Vdash\!\!- (S,S'')$$

for $S, S', S'' \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

(R23)

$$\emptyset \Vdash\!\!- (S,S)$$

for $S \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

(R′23)

$$\emptyset \Vdash\!\!- (S,T)$$

for $S \in \mathsf{DRB}\langle\langle V\rangle\rangle, T \in \{\emptyset, \varepsilon\}, S \equiv T$,

(R24)

$$\{(S \odot x, T \odot x) | x \in X\} \Vdash\!\!- (S,T)$$

for $S, T \in \mathsf{DRB}\langle\langle V\rangle\rangle, (S \not\equiv \varepsilon \wedge T \not\equiv \varepsilon)$,

(R25)

$$\{(S\cdot T'+S',T')\}\mid\!\!\vdash\!\!-\ (S^{*}\cdot S',T')$$

for $(S,S')\in \mathsf{DRB}_{1,2}\langle\langle V\rangle\rangle, T'\in \mathsf{DRB}\langle\langle V\rangle\rangle, S\not\equiv \varepsilon,$

(R26)

$$\{(S,S'),(T,T')\}\mid\!\!\vdash\!\!-\ (S+T,S'+T')$$

for $(S,T),(S',T')\in \mathsf{DRB}_{1,2}\langle\langle V\rangle\rangle,$

(R27)

$$\{(S,S')\}\mid\!\!\vdash\!\!-\ (S\cdot T,S'\cdot T)$$

for $S,S',T\in \mathsf{DRB}\langle\langle V\rangle\rangle,$

(R28)

$$\{(T,T')\}\mid\!\!\vdash\!\!-\ (S\cdot T,S\cdot T')$$

for $S,T,T'\in \mathsf{DRB}\langle\langle V\rangle\rangle,$

(R29)

$$\{(S,\rho_{\varepsilon}(S))\}$$

for $S\in \mathsf{DRB}\langle\langle V\rangle\rangle,$

(R210)

$$\{(S,\rho_{e}(S))\}$$

for $S\in \mathsf{DRB}\langle\langle V\rangle\rangle.$

We define $\vdash\!\!-_{\mathscr{D}_2}$ by : for every $P\in\mathscr{P}_f(\mathscr{A}_2), A\in\mathscr{A}_2,$

$$P\vdash\!\!- A\Rightarrow P\ \overset{\langle *\rangle}{\mid\!\!\vdash\!\!-}\circ\overset{[1]}{\mid\!\!\vdash\!\!-}_{23,24,29,210}\circ\overset{\langle *\rangle}{\mid\!\!\vdash\!\!-}\{A\},$$

where $\mid\!\!\vdash\!\!-_{23,24,29,210}$ is the relation defined by R23, R′23, R24, R29, R210 only. We define a simpler cost function $H_2:\mathscr{A}_2\to\mathbb{N}\cup\{\infty\}$ by

$$\forall (S,S')\in\mathscr{A}_2,\quad H_2(S,S')=\mathrm{Div}(S,S').$$

We let

$$\mathscr{D}_2=\langle\mathscr{A}_2,H_2,\vdash\!\!-_{\mathscr{D}_2}\rangle.$$

Lemma 97. *$\mathscr{D}_2$ is a deduction system.*

10.3.2. Completeness

Theorem 98. *$\mathscr{D}_2$ is a complete deduction system.*

Proof. Let us consider the map $pr_{2,3}:\mathscr{A}_1\to\mathscr{A}_2$ which erases the weights

$$\forall\pi\in\mathbb{N}, S,\ S'\in\mathsf{DRB}\langle\langle V\rangle\rangle,\quad pr_{2,3}(\pi,S,S')=(S,S').$$

One can check that $pr_{2,3}$ maps any rule of $\mathscr{D}_1$ on an elementary deduction of $\mathscr{D}_2$: if (P,A) is a rule of $\mathscr{D}_1$ then

$$pr_{2,3}(P) \overset{\langle * \rangle}{||\!\!\!-\!\!\!-}_{\mathscr{D}_2} pr_{2,3}(A).$$

Moreover, $pr_{2,3}$ maps the instances of rules (R3), (R′3), (R4) on instances of (R23), (R′23), (R24). Hence, if P is a finite self-generating set, then $pr_{2,3}(P)$ is a finite $\mathscr{D}_2$-proof. As every true assertion in $\mathscr{A}_1$ belongs to some finite self-generating set, every true assertion in $\mathscr{A}_2$ belongs to some finite $\mathscr{D}_2$-proof. □

10.4. Deterministic substitutions

Let $\mathscr{C}_0$ be the formal system consisting of all the instances of the meta-rules R21, R22, R23, R25, R26, R27, R28, R29. One can notice that this system is independant of the automaton $\mathscr{M}$. For every $\delta, \lambda \in \mathbb{N} - \{0\}, S, S' \in \mathsf{DRB}_{\delta,\lambda}\langle\langle V \rangle\rangle$, we shall use the abbreviation

$$[S,S'] = \{(S_{i,j}, S'_{i,j}) \mid 1 \leqslant i \leqslant \delta, 1 \leqslant j \leqslant \lambda\}.$$

Lemma 99. *Let* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{N} - \{0\}$, $S, S' \in \mathsf{DRB}_{\lambda_1,\lambda_2}\langle\langle V \rangle\rangle, T, T' \in \mathsf{DRB}_{\lambda_2,\lambda_3}\langle\langle V \rangle\rangle$. *Then*

$$[S,S'] \cup [T,T'] \overset{\langle * \rangle}{||\!\!\!-\!\!\!-}_{\mathscr{C}_0} [S \cdot T, S' \cdot T'].$$

Proof. It suffices to use meta-rules (R26)–(R28) and the basic meta-rules (R21), (R22). □

From now on, we shall use the deduction of the previous lemma as a derived meta-rule, that will be named "matrix product" (MP).

Lemma 100. *Let* $\delta, \lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{\delta,\delta}\langle\langle V \rangle\rangle, T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle V \rangle\rangle$, *such that* $(S,T) \in \mathsf{DRB}_{\delta,\delta+\lambda}\langle\langle V \rangle\rangle$. *Then* $S^* \cdot T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle V \rangle\rangle$.

Let us recall the well-known formula expressing the entries of S^* as rational expressions in the entries of S. For every $S \in \mathsf{B}_{2,2}\langle\langle V \rangle\rangle$,

$$S^* = \begin{pmatrix} (S_{1,1} + S_{1,2} \cdot S_{2,2}^* \cdot S_{2,1})^* & (S_{1,1} + S_{1,2} \cdot S_{2,2}^* \cdot S_{2,1})^* \cdot S_{1,2} \cdot S_{2,2}^* \\ (S_{2,2} + S_{2,1} \cdot S_{1,1}^* \cdot S_{1,2})^* \cdot S_{2,1} \cdot S_{1,1}^* & (S_{2,2} + S_{2,1} \cdot S_{1,1}^* \cdot S_{1,2})^* \end{pmatrix}. \tag{154}$$

(See [43, Theorem 2.5, p. 618].)

Proof. Let us prove Lemma 100 by induction on δ.

Case 1: $\delta = 1$. By Lemma 29, $(\emptyset, S^* \cdot T) = \Box_1^*(S, T) \in \mathsf{DRB}_{1,1+\lambda}\langle\langle V\rangle\rangle$. It follows that $S^* \cdot T \in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$.

Case 2: $\delta = 2$. By case 1, as $(S_{2,2}, S_{2,1}, \emptyset, T_{2,1}) \in \mathsf{DRB}_{1,3+\lambda}\langle\langle V\rangle\rangle$, $(S_{2,2}^* \cdot S_{2,1}, \emptyset, S_{2,2}^* \cdot T_{2,1}) \in \mathsf{DRB}_{1,2+\lambda}\langle\langle V\rangle\rangle$. It follows that the matrix

$$M = \begin{pmatrix} \mathrm{I}_1 & \emptyset_1^\lambda & \emptyset_1^\lambda \\ S_{2,2}^* \cdot S_{2,1} & \emptyset_1^\lambda & S_{2,2}^* \cdot T_{2,*} \\ \emptyset_\lambda^1 & \mathrm{I}_\lambda & \emptyset_\lambda^\lambda \end{pmatrix},$$

is deterministic. By Lemma 13, it follows that the row-vector

$$(S_{1,1} + S_{1,2} \cdot S_{2,2}^* \cdot S_{2,1}, T_{1,*}, S_{1,2} \cdot S_{2,2}^* \cdot T_{2,*}) = (S_{1,1}, S_{1,2}, T_{1,*}) \cdot M \tag{155}$$

is deterministic. By case 1, the determinism of vector (155) implies that

$$((S_{1,1} + S_{1,2} \cdot S_{2,2}^* \cdot S_{2,1})^* \cdot T_{1,*},\ (S_{1,1} + S_{1,2} \cdot S_{2,2}^* \cdot S_{2,1})^* \cdot S_{1,2} \cdot S_{2,2}^* \cdot T_{2,*}) \tag{156}$$

belongs to $\mathsf{DRB}_{1,2\cdot\lambda}\langle\langle V\rangle\rangle$. It follows that

$$(S_{1,1} + S_{1,2} \cdot S_{2,2}^* \cdot S_{2,1})^* \cdot T_{1,*} + (S_{1,1} + S_{1,2} \cdot S_{2,2}^* \cdot S_{2,1})^* \cdot S_{1,2} \cdot S_{2,2}^* \cdot T_{2,*} \tag{157}$$

which, by formula (154) is the first row of $S^* \cdot T$, belongs to $\mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$. By similar arguments one can show that the second row of $S^* \cdot T$ is rational deterministic too, and finally: $S^* \cdot T \in \mathsf{DRB}_{2,\lambda}\langle\langle V\rangle\rangle$.

Case 3: $\delta > 2$. Let us suppose that the lemma is true for every (δ', λ') such that $\delta' < \delta$. Let us consider block decompositions

$$S = (S_{i,j})_{i,j \in \{1,2\}}, \quad T = (T_{i,k})_{\substack{i \in \{1,2\} \\ k \in [1,\lambda]}},$$

where $\delta_1, \delta_2 \in [1, \delta - 1], \delta_1 + \delta_2 = \delta$, $\forall i, j \in \{1,2\}, \forall k \in [1,\lambda], S_{i,j} \in \mathsf{DRB}_{\delta_i,\delta_j}\langle\langle V\rangle\rangle, T_{i,*} \in \mathsf{DRB}_{\delta_{i},\lambda}\langle\langle V\rangle\rangle$. Let us consider the same formulas as above and let us replace every invocation of case 1 by an invocation of the induction hypothesis. We then have proved that $S^* \cdot T \in \mathsf{DRB}_{\delta,\lambda}\langle\langle V\rangle\rangle$. $\Box$

Definition 101. We call a *deterministic rational* substitution $\mathsf{B}\langle\langle V\rangle\rangle \to \mathsf{B}\langle\langle V\rangle\rangle$ any substitution τ whose componentwise extension as a map $\mathsf{B}_{1,Q}\langle\langle V\rangle\rangle \to \mathsf{B}_{1,Q}\langle\langle V\rangle\rangle$ fulfills

$$\forall q \in Q,\ \forall z \in Z,\ \tau([qz]) \in \mathsf{DRB}_{1,Q}\langle\langle V\rangle\rangle.$$

Lemma 102. *Let* $\lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$. *Let* $\tau : \mathsf{B}\langle\langle V\rangle\rangle \to \mathsf{B}\langle\langle V\rangle\rangle$ *be a deterministic rational substitution. Then* $\tau(S) \in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$.

Let us recall that finite automata can be equivalently seen as matrices (see [1,43]; this way of treating automata goes back to [11]). In the context of *deterministic* rational vectors it can be seen that $S\in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$ iff there exist matrices $A\in \mathsf{DRB}_{1,\delta}\langle\langle V\rangle\rangle$, $B\in \mathsf{DRB}_{\delta,\delta}\langle\langle V\rangle\rangle, C\in \mathsf{DRB}_{\delta,\lambda}\langle\langle V\rangle\rangle$ such that

$$A=\varepsilon_1^{\delta},\ \forall(i,j)\in[1,\delta]\times[1,\delta],\ B_{i,j}\in V,\ \forall(j,k)\in[1,\delta]\times[1,\lambda],\ C_{i,j}\in\{\emptyset,\varepsilon\}$$

and

$$S=A\cdot B^*\cdot C. \tag{158}$$

Proof of Lemma 102. As τ is a substitution, formula (158) implies

$$\tau(S)=\tau(A)\cdot\tau(B)^*\cdot\tau(C). \tag{159}$$

As τ is deterministic, every row-vector of $(\tau(B),\tau(C))$ is deterministic, with finite entries. Hence, by Lemma 100, $\tau(B)^*\cdot\tau(C)$ is rational deterministic too. Moreover $\tau(A)=A$ is rational deterministic. By Lemma 13 $\tau(A)\cdot\tau(B)^*\cdot\tau(C)\in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$, and by formula (159), $\tau(S)\in \mathsf{DRB}_{1,\lambda}\langle\langle V\rangle\rangle$. □

Lemma 103. *Let $\delta,\lambda\in\mathbb{N}-\{0\}$, $S,S'\in \mathsf{DRB}_{\delta,\delta}\langle\langle V\rangle\rangle$, $T,T'\in \mathsf{DRB}_{\delta,\lambda}\langle\langle V\rangle\rangle$, such that $(S,T),(S',T')\in \mathsf{DRB}_{\delta,\delta+\lambda}\langle\langle V\rangle\rangle$. Then*

$$[S,S']\cup[T,T']\ \overset{\langle *\rangle}{|\!\!\vdash\!\!-}_{\mathscr{C}_0}[S^*\cdot T,S'^*\cdot T'].$$

Proof. We prove that the lemma is true for $\delta\in\mathbb{N}-\{0\},\lambda=1$ by induction on δ. We generalize then to arbitrary λ.

Basis: $\delta=\lambda=1$.

Subcase 1: $S\equiv S'\equiv\varepsilon$. Let us first show that under this hypothesis:

$$\rho_\varepsilon(S)=\rho_\varepsilon(S')=\varepsilon \tag{160}$$

and

$$\rho_\varepsilon(S^*\cdot T)=\rho_\varepsilon(T),\ \rho_\varepsilon(S'^*\cdot T')=\rho_\varepsilon(T'). \tag{161}$$

By Lemma 15, $\rho_\varepsilon(S)\equiv\varepsilon$ too. Let us consider the type of the series $S_1=\rho_\varepsilon(S)$. $S_1=\emptyset$ is impossible (because $\emptyset\not\equiv\varepsilon$). Suppose $S_1=[pz]*\Phi$ where $pz\in QZ$ is an ε-free mode. Either $\forall q\in Q,[p,z,q]\cdot\Phi_q\equiv\emptyset$, and then $S_1\equiv\emptyset$, which is impossible, or $\exists q\in Q,\exists u\in X^+,([p,z,q]\cdot\Phi_q)\odot u=\varepsilon$, and then $S_1\not\equiv\varepsilon$ which is impossible too. The only remaining possibility is that $S_1=\varepsilon$. Hence (160) is established. Let us use now some formulas established in the proof of Lemma 15:

$$S=v\cdot\rho_\varepsilon(S)+\sum_{w\in D(v)}w\cdot(S\bullet w), \tag{162}$$

where $\rho_\varepsilon(v)=\varepsilon$ and, for every $w\in D(v)$, $\rho_\varepsilon(w)=\emptyset$. We then have

$$\begin{aligned}\rho_\varepsilon(S^*\cdot T) &= \rho_\varepsilon\left(\left(v+\sum_{w\in D(v)} w\cdot(S\bullet w)\right)^*\cdot T\right)\\ &= \rho_\varepsilon\left(v^*\cdot T + v^*\cdot\left(\sum_{w\in D(v)} w\cdot(S\bullet w)\right)\cdot S^*\cdot T\right)\\ &= \rho_\varepsilon(v^*\cdot T)+\rho_\varepsilon\left(v^*\cdot\left(\sum_{w\in D(v)} w\cdot(S\bullet w)\right)\cdot S^*\cdot T\right).\end{aligned}$$

But $\rho_\varepsilon(v^*\cdot T)=\rho_\varepsilon(T)$ and $\rho_\varepsilon(v^*\cdot(\sum_{w\in D(v)} w\cdot(S\bullet w))\cdot S^*\cdot T)=\sum_{w\in D(v)}\emptyset\cdot\rho_\varepsilon((S\bullet w)\cdot S^*\cdot T)=\emptyset$. Hence

$$\rho_\varepsilon(S^*\cdot T)=\rho_\varepsilon(T),$$

i.e. (161) is established.

Using (R29), formula (161) and (R29) (with the basic rules (R21) and (R22)) we obtain

$$\begin{aligned}[S,S']\cup[T,T']\supseteq\{(T,T')\} &\overset{\langle *\rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (\rho_\varepsilon(T),\rho_\varepsilon(T'))\\ &= (\rho_\varepsilon(S^*\cdot T),\rho_\varepsilon(S'^*\cdot T'))\\ &\overset{\langle *\rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (S^*\cdot T,S'^*\cdot T').\end{aligned}$$

Subcase 2: $S\not\equiv\varepsilon$, $S'\not\equiv\varepsilon$. Let us notice that $S'\cdot(S'^*\cdot T')+T'=S'^*\cdot T'$. Hence, using (R26) and (R27):

$$\begin{aligned}[S,S']\cup[T,T'] &\overset{\langle *\rangle}{\Vdash\!\!-}_{\mathscr{C}_0}(S\cdot(S'^*\cdot T')+T,S'\cdot(S'^*\cdot T')+T')\\ &= S\cdot(S'^*\cdot T')+T,S'^*\cdot T').\end{aligned} \tag{163}$$

Using then (R25) we have

$$(S\cdot(S'^*\cdot T')+T,S'^*\cdot T') \overset{\langle *\rangle}{\Vdash\!\!-}_{\mathscr{C}_0}(S^*\cdot T,S'^*\cdot T'). \tag{164}$$

Combining together (163) and (164) we have

$$[S,S']\cup[T,T'] \overset{\langle *\rangle}{\Vdash\!\!-}_{\mathscr{C}_0}(S^*\cdot T,S'^*\cdot T').$$

First induction step: $\delta=2,\lambda=1$. By the basis case we know that

$$\{(S_{2,2},S'_{2,2}),(S_{2,1},S'_{2,1})\} \overset{\langle *\rangle}{\Vdash\!\!-}_{\mathscr{C}_0}(S^*_{2,2}\cdot S_{2,1},S'^*_{2,2}\cdot S'_{2,1}). \tag{165}$$

Using rule (MP):

$$[S_{1,*}, S'_{1,*}] \cup \{(S^*_{2,2} \cdot S_{2,1}, S'^*_{2,2} \cdot S'_{2,1})\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (S_{1,1} + S_{1,2} \cdot S^*_{2,2} \cdot S_{2,1}, S'_{1,1} + S'_{1,2} \cdot S'^*_{2,2} \cdot S'_{2,1}). \tag{166}$$

By the basis case we know that

$$\{(S_{2,2}, S'_{2,2}), (T_{2,1}, T'_{2,1})\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (S^*_{2,2} \cdot T_{2,1}, S'^*_{2,2} \cdot T'_{2,1}). \tag{167}$$

Using rule (MP) we get

$$\{(S_{1,2}, S'_{1,2}), (S^*_{2,2} \cdot T_{2,1}, S'^*_{2,2} \cdot T'_{2,1})\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (S_{1,2} \cdot S^*_{2,2} \cdot T_{2,1}, S'_{1,2} \cdot S'^*_{2,2} \cdot T'_{2,1}). \tag{168}$$

The vector

$$(S_{1,1} + S_{1,2} \cdot S^*_{2,2} \cdot S_{2,1}, S_{1,2} \cdot S^*_{2,2} \cdot T_{2,1})$$

is a projection of the deterministic vector given in (155), hence is deterministic too. As well, the vector

$$(S'_{1,1} + S'_{1,2} \cdot S'^*_{2,2} \cdot S'_{2,1}, S'_{1,2} \cdot S'^*_{2,2} \cdot T'_{2,1})$$

is deterministic. Using the basis case we have

$$[(S_{1,1} + S_{1,2} \cdot S^*_{2,2} \cdot S_{2,1}, S_{1,2} \cdot S^*_{2,2} \cdot T_{2,1}), (S'_{1,1} + S'_{1,2} \cdot S'^*_{2,2} \cdot S'_{2,1}, S'_{1,2} \cdot S'^*_{2,2} \cdot T'_{2,1})] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (S_{1,1} + S_{1,2} \cdot S^*_{2,2} \cdot S_{2,1})^* \cdot S_{1,2} \cdot S^*_{2,2} \cdot T_{2,1}, (S'_{1,1} + S'_{1,2} \cdot S'^*_{2,2} \cdot S'_{2,1})^* \cdot S'_{1,2} \cdot S'^*_{2,2} \cdot T'_{2,1}). \tag{169}$$

As well

$$[(S_{1,1} + S_{1,2} \cdot S^*_{2,2} \cdot S_{2,1}, T_{1,1}), (S'_{1,1} + S'_{1,2} \cdot S'^*_{2,2} \cdot S'_{2,1}, T'_{1,1})] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} ((S_{1,1} + S_{1,2} \cdot S^*_{2,2} \cdot S_{2,1})^* \cdot T_{1,1}, (S'_{1,1} + S'_{1,2} \cdot S'^*_{2,2} \cdot S'_{2,1})^* \cdot T'_{1,1}) \tag{170}$$

The vector

$$(U_1, U_2) = ((S_{1,1} + S_{1,2} \cdot S^*_{2,2} \cdot S_{2,1})^* \cdot T_{1,1}, (S_{1,1} + S_{1,2} \cdot S^*_{2,2} \cdot S_{2,1})^* \cdot S_{1,2} \cdot S^*_{2,2} \cdot T_{2,1})$$

has been shown deterministic in (156). As well the vector (U'_1, U'_2) obtained by exchanging (S, T) with (S', T') in the definition of (U_1, U_2) is deterministic. By rule (R26) we have

$$[U_1, U'_1] \cup [U_2, U'_2] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (U_1 + U_2, U'_1 + U'_2) = ((S^* \cdot T)_{1,1}, (S'^* \cdot T')_{1,1}). \tag{171}$$

Combining together deductions (165)–(171) we have shown that

$$[S,S'] \cup [T,T'] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} ((S^* \cdot T)_{1,1}, (S'^* \cdot T')_{1,1}).$$

Exchanging the S-indices in the previous arguments leads to

$$[S,S'] \cup [T,T'] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} ((S^* \cdot T)_{2,1}, (S'^* \cdot T')_{2,1}).$$

Hence,

$$[S,S'] \cup [T,T'] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} [S^* \cdot T, S'^* \cdot T'].$$

General induction step: $\delta \geqslant 2, \lambda = 1$. Let us suppose that the lemma is true for every (δ', λ') such that $\delta' < \delta, \lambda' = 1$. Let us consider block decompositions

$$S = (S_{i,j})_{i,j \in \{1,2\}}, \quad T = (T_{i,1})_{i \in \{1,2\}},$$

where $\delta_1, \delta_3 \in [1, \delta - 1], \delta_1 + \delta_3 = \delta$, $\forall i,j \in \{1,2\}, S_{i,j} \in \mathsf{DRB}_{\delta_i,\delta_j}\langle\langle V \rangle\rangle, T_{i,1} \in \mathsf{DRB}_{\delta_i,1}\langle\langle V \rangle\rangle$. Let us consider the same formulas as above and let us replace every invocation of the "basis case" by an invocation of the "induction hypothesis". We then have proved that

$$[S,S'] \cup [T,T'] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} [S^* \cdot T, S'^* \cdot T'].$$

Arbitrary integers: $(\delta, \lambda) \in \mathbb{N} - \{0\} \times \mathbb{N} - \{0\}$. By the above case: $\forall k \in \{1, \lambda\}$,

$$[S,S'] \cup [T_{*,k}, T'_{*,k}] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} ((S^* \cdot T)_{*,k}, (S'^* \cdot T')_{*,k}),$$

hence

$$[S,S'] \cup [T,T'] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} [S^* \cdot T, S'^* \cdot T']. \qquad \square$$

Lemma 104. *Let* $\lambda \in \mathbb{N} - \{0\}, S \in \mathsf{DRB}_{1,\lambda}\langle\langle V \rangle\rangle$ *and let* $\tau : \mathsf{B}\langle\langle V \rangle\rangle \to \mathsf{B}\langle\langle V \rangle\rangle$ *be a deterministic rational substitution and let* $\pi \in \mathbb{N}$. *Then*

$$\{([qzr], \tau([qzr])) \mid q, r \in Q, z \in Z\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (S, \tau(S)).$$

Proof. Let us use the same notation as in the proof of Lemma 102. By Lemma 103,

$$[B, \tau(B)] \cup [C, \tau(C)] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (B^* \cdot C, \tau(B)^* \cdot \tau(C)).$$

As $A = \tau(A)$, the above deduction combined with (MP) gives

$$[B, \tau(B)] \cup [C, \tau(C)] \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (A \cdot B^* \cdot C, \tau(A) \cdot \tau(B)^* \cdot \tau(C)). \tag{172}$$

But the special form of matrices B, C is such that

$$[B,\tau(B)]\cup[C,\tau(C)]\subseteq\{([qzr],\tau([qzr]))\mid q,r\in Q,\ z\in Z\},$$

and the result of deduction (172) is just $(S,\tau(S))$. Hence the conclusion of the lemma is true. □

10.5. System $\mathscr{D}_3$

We prove here that the formal system $\mathscr{D}_3$ obtained by elimination of meta-rule (R210) in $\mathscr{D}_2$ is still complete.

Let $\mathscr{D}_3=\langle\mathscr{A}_3,H_3,\ \vdash\!\!-_{\mathscr{D}_3}\rangle$ where $\mathscr{A}_3=\mathscr{A}_2, H_3=H_2$ and $\Vdash\!-_{\mathscr{D}_3}$ is defined below.

10.5.1. Rules

We define the elementary deduction relation $\Vdash\!-_{\mathscr{D}_3}$ as the set of all the instances of the the meta-rules (R21)–(R23), (R′23), (R24)–(R29) (i.e. all the meta-rules of $\mathscr{D}_2$ except R210). The deduction relation $\vdash\!\!-_{\mathscr{D}_3}$ is now defined by

$$\vdash\!\!-_{\mathscr{D}_3}\ =\ \overset{\langle *\rangle}{\Vdash\!-}_{\mathscr{D}_3}\circ\overset{[1]}{\Vdash\!-}_{(\mathrm{R23}),(\mathrm{R'23}),(\mathrm{R24}),(\mathrm{R29})}\circ\overset{\langle *\rangle}{\Vdash\!-}_{\mathscr{D}_3}$$

Let us notice that every rule of $\mathscr{C}_0$ is a rule of $\mathscr{D}_3$.

Lemma 105. *$\mathscr{D}_3$ is a deduction system.*

10.5.2. Completeness

Theorem 106. *$\mathscr{D}_3$ is a complete deduction system.*

Proof. It suffices to prove that every instance of R210 is provable in $\mathscr{D}_3$. Let $S\in\mathsf{DRB}\langle\!\langle V\rangle\!\rangle$. As ρ_e is a deterministic substitution, by Lemma 104,

$$\{([qzr],\rho_e([qzr]))\mid q,r\in Q,\ z\in Z\}\ \overset{\langle *\rangle}{\Vdash\!-}_{\mathscr{C}_0}(S,\rho_e(S)).$$

Every pair $([qzr],\rho_e([qzr]))$ is the right-hand side of an instance of (R′23). Hence,

$$\emptyset\ \overset{\langle *\rangle}{\Vdash\!-}_{\mathscr{C}_0}(S,\rho_e(S)).$$

As $\overset{\langle *\rangle}{\Vdash\!-}_{\mathscr{C}_0}\subseteq\overset{\langle *\rangle}{\Vdash\!-}_{\mathscr{D}_3}$, the theorem is proved. □

10.6. System $\mathscr{D}_4$

We exhibit here a deduction system $\mathscr{D}_4$ which is simpler than $\mathscr{D}_3$ and is still complete.

Let us consider

$$\mathscr{D}_4=\langle\mathscr{A}_4,H_4,\ \vdash\!\!-_{\mathscr{D}_4}\rangle,$$

where $\mathscr{A}_4=\mathscr{A}_3, H_4=H_3$ and $\vdash\!\!-_{\mathscr{D}_4}$ is defined below.

10.6.1. Rules

We define the elementary deduction relation $||\!\!-\!\!-_{\mathscr{D}_4}$ as the set of all the instances of meta-rules (R21)–(R29) of $\mathscr{D}_3$ union all the instances of new meta-rule

$$\text{(R}''23\text{)} \quad \emptyset || -- ([qzr], \varepsilon)$$

for $q, r \in Q,\ z \in Z,\ [qzr] \equiv \varepsilon$.

In other words, $\mathscr{D}_4$ is obtained from $\mathscr{D}_3$ by replacing meta-rule (R$'$23) by the weaker meta-rule (R$''$23).

We then define $|\!\!-\!\!-_{\mathscr{D}_4}$ by: for every $P \in \mathscr{P}_f(\mathscr{A}_4),\ A \in \mathscr{A}_4$,

$$P \,|\!\!-\!\!-_{\mathscr{D}_4} A \iff P \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{D}_4} \circ \overset{[1]}{||\!\!-\!\!-}_{\text{(R23),(R}'23\text{),(R24),(R29)}} \circ \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{D}_4} \{A\}.$$

Let us notice that every rule of $\mathscr{C}_0$ is a rule of $\mathscr{D}_4$. As $|\!\!-\!\!-_{\mathscr{D}_4} \subseteq |\!\!-\!\!-_{\mathscr{D}_3}$, $H_4 = H_3$ and the new rule (R$''$23) is recursively enumerable, it is clear that $\mathscr{D}_4$ is a deduction system.

10.6.2. Strategies

Let us define strategies for the system $\mathscr{D}_4$. We shall define new auxiliary strategies $T^0_\emptyset, T_{A,\emptyset}$ and then derive some “compound” strategies from them.

Let us denote by $\mathscr{A}_\emptyset$ the set: $\mathscr{A}_\emptyset = \mathsf{DRB}\langle\langle V \rangle\rangle \times \{\emptyset\}$.

$T^0_\emptyset$:

$T^0_\emptyset(A_1 A_2 \cdots A_n) = B_1 \cdots B_m$ iff

$$A_n = (\emptyset, \emptyset) \quad \text{and} \quad m = 0.$$

$T_{A,\emptyset}$:

$T_{A,\emptyset}(A_1 \cdots A_n) = B_1 \cdots B_m$ iff

(e1) $A_n = ([pzq], \emptyset)$ for some $p, q \in Q, z \in Z$ and $(B_1, B_2, \ldots, B_m)$ is the smallest element of $\mathscr{A}^*_\emptyset$ fulfilling conditions (e2 $\wedge$ e3) below:

(e2) $\forall j \in [1, m],\ B_j = ([p_j, z_j, q_j], \emptyset), H_4(B_j) = \infty$ and

(e3) $\forall x \in X$, for every word $w \in \mathrm{supp}([pzq] \odot x), \exists j \in [1, m], [p_j z_j q_j]$ is a factor of w.

Lemma 107. $T^0_\emptyset, T_{A,\emptyset}$ *are* $\mathscr{D}_4$*-strategies.*

Proof. $T^0_\emptyset : \emptyset \,|\!\!-\!\!- (\emptyset, \emptyset)$ which proves (S1) and $\min\{H_4(B_j) \mid 1 \leqslant j \leqslant m\} = \min\{H_4(\emptyset, \emptyset)\} = \infty$, which proves (S2).

$T_{A,\emptyset}$: By (e3), using (R26)–(R28), $\forall x \in X$,

$$\{B_j, 1 \leqslant j \leqslant m\} \overset{\langle * \rangle}{||\!\!-\!\!-}_{\mathscr{D}_4} ([pzq] \odot x, \emptyset).$$

Using (R24) we obtain that

$$\{B_j, 1 \leqslant j \leqslant m\} \,|\!\!-\!\!-_{\mathscr{D}_4} ([pzq], \emptyset),$$

hence (S1) is fulfilled.

By (e2),

$$\min\{H(B_j) \mid 1 \leqslant j \leqslant m\} = \infty,$$

which establishes (S2). □

Let us consider the following strategy $\mathscr{S}_\emptyset$: for every $W \in \mathscr{A}_4^+$,
(1) if $W \in \mathrm{dom}(T^0_\emptyset)$ then, $\mathscr{S}_\emptyset(W) = T^0_\emptyset(W)$,
(2) elsif $W \in \mathrm{dom}(T_{A,\emptyset})$ then, $\mathscr{S}_\emptyset(W) = T_{A,\emptyset}(W)$,
(3) else $\mathscr{S}_\emptyset(W)$ is undefined.

10.6.3. Completeness

In order to show the completeness of system $\mathscr{D}_4$ it remains to show that every rule in (R′23) is provable in $\mathscr{D}_4$.

Lemma 108. *Let $p, q \in Q, z \in Z$. $[pzq] \equiv \emptyset$ iff there exists a finite $\mathscr{D}_4$-proof of $([pzq], \emptyset)$.*

Proof. Let us suppose $[pzq] \equiv \emptyset$. Let $A = ([pzq], \emptyset)$ and $t = \mathscr{T}(\mathscr{S}_\emptyset, A)$. The definitions of $T^0_\emptyset, T_{A,\emptyset}$ show that the labels of t belong to the finite set $V \times \{\emptyset\}$. Hence every branch of t has a length $\leqslant \mathrm{Card}(V \times \{\emptyset\})$, showing that t is a finite tree.

Let us consider the label of a leaf x of t: $t(x) = ([p'z'q'], \emptyset)$. Let $W_x = A_1 A_2 \cdots A_n$ be the word labelling the branch ending at x. Suppose that $H_4(A) = \infty$. Then, by Lemma 48, $H_4(A_n) = \infty$. It follows that

$$\forall x \in X, \quad [p'z'q'] \odot x \neq \varepsilon,$$

hence, $\forall x \in X$, $\forall w \in \mathrm{supp}([p'z'q'] \odot x)$, $\exists p'', q'' \in Q$, $\exists z'' \in Z$ such that

$$[p''z''q''] \equiv \emptyset \quad \text{and} \quad w \in V^* \cdot [p''z''q''] \cdot V^*. \tag{173}$$

By (173) there exists some $B_1, B_2, \ldots, B_m$ fulfilling conditions (e2), (e3). Hence $W_x \in \mathrm{dom}(T_{A,\emptyset})$. This proves that, every leaf x of t is such that $W_x \in \mathrm{dom}(\mathscr{S}_\emptyset)$. By Lemma 48, P is a finite $\mathscr{D}_4$-proof, containing A. □

Lemma 109. *Let $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$,*
(1) $S \equiv \emptyset$ if and only if there exists some $\mathscr{D}_4$-proof of $(S, \emptyset)$.
(2) $S \equiv \varepsilon$ if and only if there exists some $\mathscr{D}_4$-proof of (S, ε).

Proof. (1) Let us consider the unique substitution $\rho_\emptyset : \mathsf{DRB}\langle\langle V \rangle\rangle \to \mathsf{DRB}\langle\langle V \rangle\rangle$ such that: for every $p, q \in Q$, $z \in Z$,

$$\rho_\emptyset([pzq]) = \emptyset \text{ (if } [pzq] \equiv \emptyset); \quad \rho_\emptyset([pzq]) = [pzq] \text{ (if } [pzq] \not\equiv \emptyset).$$

One can easily check that $\rho_\emptyset$ is a deterministic substitution and that, for every $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

$$S \equiv \emptyset \Leftrightarrow \rho_\emptyset(S) = \emptyset. \tag{174}$$

Let us prove point (1) of the lemma. Let $S \in \mathsf{DRB}\langle\langle V\rangle\rangle$ such that $S \equiv \emptyset$. By (174) $\rho_\emptyset(S) = \emptyset$. By Lemma 104

$$\{([pzq], \rho_\emptyset([pzq])) \mid p, q \in Q, z \in Z\} \mathrel{||\!\!\stackrel{\langle * \rangle}{\vdash\!\!-}}_{\mathscr{C}_0} (S, \emptyset).$$

By Lemma 108, for every $p, q \in Q$, $z \in Z$, there exists a finite $\mathscr{D}_4$-proof, $P_{[pzq]}$ of $([pzq], \rho_\emptyset([pzq]))$. It follows that $(\bigcup_{p,q \in Q, z \in Z} P_{[pzq]}) \cup \{(S, \emptyset)\}$ is a finite $\mathscr{D}_4$-proof of $(S, \emptyset)$.

(2) Let us prove point (2) of the lemma. Let $S \in \mathsf{DRB}\langle\langle V\rangle\rangle$ such that $S \equiv \varepsilon$. We have shown (in the proof of Lemma 103, equation (160)) that, under this hypothesis, $\rho_\varepsilon(S) = \varepsilon$. Using rule (R29), we have: $\emptyset \vdash\!\!-_{\mathscr{D}_4} (S, \rho_\varepsilon(S)) = (S, \varepsilon)$. $\square$

Theorem 110. *$\mathscr{D}_4$ is a complete deduction system.*

Proof. Follows from Theorem 106 and Lemma 109. $\square$

10.7. System $\mathscr{D}_5$

We prove here that the formal system $\mathscr{D}_5$ obtained by elimination of meta-rule (R29) in $\mathscr{D}_4$ is still complete.

Let $\mathscr{D}_5 = \langle \mathscr{A}_5, H_5, \vdash\!\!-_{\mathscr{D}_5} \rangle$ where $\mathscr{A}_5 = \mathscr{A}_4, H_5 = H_4$ and $||\!\!\vdash\!\!-_{\mathscr{D}_5}$, is defined below.

10.7.1. Rules

The rules of $\mathscr{D}_5$ are exactly the rules of $\mathscr{D}_4$, except (R29). Let us recall this set of rules.

(R51)

$$\{(S, T)\} \mathrel{||\!\!\vdash\!\!-} (T, S)$$

for $S, T \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

(R52)

$$\{(S, S'), (S', S'')\} \mathrel{||\!\!\vdash\!\!-} (S, S'')$$

for $S, T \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

(R53)

$$\emptyset \mathrel{||\!\!\vdash\!\!-} (S, S)$$

for $S \in \mathsf{DRB}\langle\langle V\rangle\rangle$,

(R″53)

$$\emptyset \mathrel{||\!\!\vdash\!\!-} ([qzr], \varepsilon)$$

for $q, r \in Q$, $z \in Z$, $[qzr] \equiv \varepsilon$,

(R54)

$$\{(S \odot x, T \odot x) \mid x \in X\} \mathrel{||\!\!-\!\!-} (S, T)$$

for $S, T \in \mathsf{DRB}\langle\langle V \rangle\rangle$, $(S \not\equiv \varepsilon \wedge T \not\equiv \varepsilon)$,

(R55)

$$\{(S \cdot T' + S', T')\} \mathrel{||\!\!-\!\!-} (S^* \cdot S', T')$$

for $(S, S') \in \mathsf{DRB}_{1,2}\langle\langle V \rangle\rangle$, $T' \in \mathsf{DRB}\langle\langle V \rangle\rangle$, $S \not\equiv \varepsilon$,

(R56)

$$\{(S, S'), (T, T')\} \mathrel{||\!\!-\!\!-} (S + T, S' + T')$$

for $(S, T), (S', T') \in \mathsf{DRB}_{1,2}\langle\langle V \rangle\rangle$,

(R57)

$$\{(S, S')\} \mathrel{||\!\!-\!\!-} (S \cdot T, S' \cdot T)$$

for $S, S', T \in \mathsf{DRB}\langle\langle V \rangle\rangle$,

(R58)

$$\{(T, T')\} \mathrel{||\!\!-\!\!-} (S \cdot T, S \cdot T')$$

for $S, T, T' \in \mathsf{DRB}\langle\langle V \rangle\rangle$.

We define $\mathrel{|\!\!-\!\!-}_{\mathscr{D}_5}$ by for every $P \in \mathscr{P}_f(\mathscr{A}_5)$, $A \in \mathscr{A}_5$,

$$P \mathrel{|\!\!-\!\!-}_{\mathscr{D}_5} A \Leftrightarrow P \overset{\langle * \rangle}{\mathrel{||\!\!-\!\!-}}_{\mathscr{D}_5} \circ \overset{[1]}{\mathrel{||\!\!-\!\!-}}_{53,54} \circ \overset{\langle * \rangle}{\mathrel{||\!\!-\!\!-}}_{\mathscr{D}_5} \{A\}.$$

where $\mathrel{||\!\!-\!\!-}_{53,54}$ is the relation defined by (R53), (R$''$53), (R54) only.

10.7.2. Completeness

Lemma 111. *Let* $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$. *Then* $\emptyset \overset{\langle * \rangle}{\mathrel{||\!\!-\!\!-}}_{\mathscr{D}_5} (S, \rho_\varepsilon(S))$.

Proof. Let us use the notation of the proof of Lemma 15. Let $S \in \mathsf{DRB}\langle\langle V \rangle\rangle$.

Case 1: $\rho_\varepsilon(S) = \emptyset$. The definitions of ρ_ε, $\odot$ and $\otimes$ (see Section 2.3.5) are such that

$$\forall x \in X, \quad S \odot x = \rho_\varepsilon(\rho_\varepsilon(S) \otimes x) = \rho_\varepsilon(\emptyset) = \emptyset.$$

Hence, for every $x \in X$,

$$(S \odot x, \rho_\varepsilon(S) \odot x) = (\emptyset, \emptyset).$$

Using rules (R53) and then (R54) we have

$$\emptyset \mathrel{|\!\!-\!\!-}_{\mathscr{D}_5} \{(S \odot x, \rho_\varepsilon(S) \odot x) \mid x \in X\} \mathrel{|\!\!-\!\!-}_{\mathscr{D}_5} \{(S, \rho_\varepsilon(S))\}.$$

Case 2: $\rho_\varepsilon(S) \neq \emptyset$. By Eqs. (26), (25), we know that

$$S = v \cdot \rho_\varepsilon(S) + \sum_{w \in D(v)} w \cdot (S \bullet w). \tag{175}$$

The form of v shows that, using rules (R″53), (R57), (R58):

$$\emptyset \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{D}_5} (v, \varepsilon) \tag{176}$$

Similarly, the form of the elements of $D(v)$ shows that

$$\forall w \in D(v), \quad \{([p_{j+1}, z_{j+1}, q'_{j+1}], \emptyset) \mid 0 \leqslant j \leqslant n-1,\ q'_{j+1} \in Q,\ q'_{j+1} \neq q_{j+1}\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{D}_5} (w, \emptyset). \tag{177}$$

Let $\{w_1, \ldots, w_p\}$ be a bijective enumeration of the elements of $D(v)$. The row-vector $(v, w_1, \ldots, w_p)$ is deterministic. By (175) and (MP) we get

$$\{(v, \varepsilon), (w_1, \emptyset), \ldots, (w_p, \emptyset)\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{C}_0} (S, \rho_\varepsilon(S)).$$

Using (176), (177) and the above deduction we obtain

$$\{([p_{j+1}, z_{j+1}, q'_{j+1}], \emptyset) \mid 0 \leqslant j \leqslant n-1, q'_{j+1} \in Q, q'_{j+1} \neq q_{j+1}\} \overset{\langle * \rangle}{\Vdash\!\!-}_{\mathscr{D}_5} (S, \rho_\varepsilon(S)). \tag{178}$$

By Lemma 108 there exists a finite $\mathscr{D}_4$-proof P_0, such that

$$P_0 \supseteq \{([p_{j+1}, z_{j+1}, q'_{j+1}], \emptyset) \mid 0 \leqslant j \leqslant n-1,\ q'_{j+1} \in Q,\ q'_{j+1} \neq q_{j+1}\}.$$

Moreover, the proof of Lemma 107 does not use rule (R29). Hence P_0 can be chosen so as to be a $\mathscr{D}_5$-proof. By (178), $P_0 \cup \{(S, \rho_\varepsilon(S))\}$ is a $\mathscr{D}_5$-proof. □

Theorem 112. *$\mathscr{D}_5$ is a complete deduction system.*

Proof. Follows from Theorem 110 and Lemma 111. □

11. Coefficients in a group H

We extend here the completeness results to H-pushdown automata, where H is any abelian group.

11.1. Definitions and basic properties

11.1.1. Finite H-automata

Let $(H, \cdot)$ be some group. We call a finite H-automaton over the alphabet W any 5-tuple

$$\mathscr{M} = \langle W, Q, \delta, h_0, q_0, Q' \rangle$$

such that Q is the finite set of states, $\delta \subseteq Q \times H \times W \times Q$ is the finite set of transitions, $h_0 \in H$ is the initial output, $q_0 \in Q$ is the initial state and $Q' \subseteq Q$ is the set of final states. As H is embedded in the semi-ring $K = \mathsf{B}\langle\langle H\rangle\rangle$ such an automaton can be seen as a finite automaton with multiplicities in K and the series recognized by $\mathscr{M}$, $\mathrm{S}(\mathscr{M})$, is defined as usual. It can be defined, for example, as

$$\mathrm{S}(\mathscr{M}) = h_0 \cdot A \cdot B^* \cdot C$$

where $A \in K_{1,Q}\langle\langle W\rangle\rangle, B \in K_{Q,Q}\langle\langle W\rangle\rangle$, and $C \in K_{Q,1}\langle\langle W\rangle\rangle$ are given by

$$A = \varepsilon^{Q}_{q_0}, \quad B_{q,q'} = \sum_{(q,h,v,q') \in \delta} h \cdot v,$$

$$C_{q,1} = \emptyset \text{ (if } q \notin Q'), \quad C_{q,1} = \varepsilon \text{ (if } q \in Q').$$

$\mathscr{M}$ is said *W-deterministic* iff,

$$\forall q \in Q, \ \forall v \in W, \ \mathrm{Card}(\{(h,r) \in H \times Q \mid (q,h,v,r) \in \delta\}) \leqslant 1. \tag{179}$$

11.1.2. Finite m-H-automata

Let $m,n \in \mathbb{N} - \{0\}$ be positive integers. By $\mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle$ we denote the set of matrices of dimension (n,m) with entries in the semi-ring $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. We call a finite *m-H*-automaton over the alphabet W any 5-tuple

$$\mathscr{M} = \langle W, Q, \delta, h_0, q_0, (Q'_j)_{1 \leqslant j \leqslant m}\rangle,$$

such that $\langle W, Q, \delta, h_0, q_0, Q\rangle$ is a finite H-automaton and for every $j \in [1,m]$, $Q'_j \subseteq Q$. For every $j \in [1,m]$ we denote by $\mathscr{M}_j$ the finite H-automaton

$$\mathscr{M}_j = \langle W, Q, \delta, h_0, q_0, Q'_j\rangle.$$

The vector recognized by $\mathscr{M}$, $\mathrm{S}(\mathscr{M})$, is defined by

$$\mathrm{S}(\mathscr{M}) = (S(\mathscr{M}_1), \ldots, S(\mathscr{M}_j), \ldots, S(\mathscr{M}_m)).$$

$\mathscr{M}$ is said *W-deterministic* iff it fulfills the above condition (179).

11.1.3. Pushdown H-automata

We call a *pushdown H-automaton* on the alphabet X any 6-tuple

$$\mathscr{M} = \langle X, Z, Q, \delta, q_0, z_0\rangle,$$

where Z is the finite stack-alphabet, Q is the finite set of states, $q_0 \in Q$ is the initial state, z_0 is the initial stack-symbol and $\delta : QZ \times (X \cup \{\varepsilon\}) \to \mathscr{P}_f(H \times QZ^*)$, is the transition mapping. Let $q,q' \in Q, \omega, \omega' \in Z^*, z \in Z, \ h \in H, \ f \in X^*$ and $a \in X \cup \{\varepsilon\}$; we note $(qz\omega, h, af) \mapsto_{\mathscr{M}} (q'\omega'\omega, h \cdot h', f)$ if $(h', q'\omega') \in \delta(qz, a)$. $\mapsto_* \mathscr{M}$ is the reflexive and transitive closure of $\mapsto_{\mathscr{M}}$. For every $q\omega, q'\omega' \in QZ^*$ and $h \in H, \ f \in X^*$, we note $q\omega \xrightarrow{(h,f)} \mathscr{M} q'\omega'$ iff

$$(q\omega, 1_H, f) \overset{*}{\mapsto}_{\mathscr{M}} (q'\omega', h, \varepsilon).$$

$\mathcal{M}$ is said *deterministic* iff it fulfills conditions (5) and (6) of Section 2.1. A H-dpda $\mathcal{M}$ is said *normalized* iff, for every $qz \in QZ$, $x \in X$:

$$q'\omega' \in \delta_2(qz,x) \Rightarrow |\omega'| \leqslant 2, \quad \text{and} \quad q'\omega' \in \delta_2(qz,\varepsilon) \Rightarrow |\omega'| = 0, \tag{180}$$

where $\delta_2 : QZ \times (X \cup \{\varepsilon\}) \to \mathcal{P}_f(QZ^*)$, is the second component of the map δ. Given some finite set $F \subseteq QZ^*$ of configurations, the *series recognized by* $\mathcal{M}$ *with final configurations* F is defined by

$$\mathrm{S}(\mathcal{M},F) = \sum_{c \in F} \sum_{q_0 z_0 \xrightarrow{hw}_{\mathcal{M}} c} h \cdot w.$$

Intuitively, one can see the coefficient $S_w \in K\langle\langle X\rangle\rangle$ of a word w in the series $\mathrm{S}(\mathcal{M},F)$ either as the "multiplicity" with which the word w is recognized, or as the "output" of the automaton $\mathcal{M}$ on the "input" w. Notice that, from this last point of view, when $\mathcal{M}$ is deterministic and $(H,\cdot) = (\mathbb{Z},+)$, $\mathcal{M}$ can be named a deterministic pushdown *transducer* from *words* to *integers*.

For the same technical reasons as in the boolean case, we suppose that Z contains a special symbol e subject to the property:

$$\forall q \in Q, \quad \delta(qe,\varepsilon) = \{(1_H,q)\} \text{ and } \mathrm{im}(\delta_3) \subseteq \mathcal{P}_f(Q(Z - \{e\})^*). \tag{181}$$

11.1.4. Right-actions

Similarly as in Section 2.3 we fix some H-dpda $\mathcal{M}$ and consider the structured alphabet $(V,\smile)$ associated with $\mathcal{M}$.

11.1.4.1. Action $\bullet$. A σ-right-action of the monoid $H \times W^*$ over $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$ is defined by $\forall S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$, $\forall h \in H$, $\forall w \in W^*$, $T = S \bullet (h,w)$ is the series:

$$\forall v \in W^*, \; T_v = h^{-1} \cdot S_{w \cdot v}.$$

In words, $S \bullet (h,w)$ is the left-quotient of S by the monomial $h \cdot w$. (From now on, we identify the pair $(h,w) \in H \times W^*$ with the monomial $h \cdot w \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$.)

11.1.4.2. Action $\otimes$. Let us consider the set $P_{\mathcal{M}}$ of all the pairs of one of the following forms:

$$([p,z,q], h \cdot x \cdot [p',z_1,p''][p'',z_2,q]), \tag{182}$$

where $p,q,p',p'' \in Q$, $x \in X$, $(h, p'z_1z_2) \in \delta(pz,x)$

$$([p,z,q], h \cdot x \cdot [p',z',q]), \tag{183}$$

where $p,q,p' \in Q$, $x \in X$, $(h, p'z') \in \delta(pz,x)$

$$([p,z,q], h \cdot a), \tag{184}$$

where $p,q, \in Q$, $a \in X \cup \{\varepsilon\}$, $(h,q) \in \delta(pz,a)$. We define a σ-right-action $\otimes$ of the monoid $H \times (X \cup \{e\})^*$ over the semi-ring $(\mathsf{B}\langle\langle H \rangle\rangle)\langle\langle V \rangle\rangle$ by for every $p,q \in Q$, $z \in Z$, $x \in X$, $h \in H$, $k \in \mathsf{B}\langle\langle H \rangle\rangle$,

$$[p,z,q] \otimes x = \sum_{([p,z,q],m) \in P_{\mathscr{M}}} m \bullet (1_H, x), \tag{185}$$

$$[p,z,q] \otimes e = h \quad \text{iff } ([p,z,q],h) \in P_{\mathscr{M}}, \tag{186}$$

$$[p,z,q] \otimes e = \emptyset \quad \text{iff } (\{[p,z,q]\} \times H \cdot V^*) \cap P_{\mathscr{M}} = \emptyset, \tag{187}$$

$$k \otimes x = \emptyset, \quad k \otimes e = \emptyset. \tag{188}$$

The action is extended to all monomials by for every $k \in \mathsf{B}\langle\langle H \rangle\rangle$, $\beta \in V^*$, $y \in X \cup \{e\}$,

$$(k \cdot [p,z,q] \cdot \beta) \otimes y = k \cdot ([p,z,q] \otimes y) \cdot \beta \tag{189}$$

and for every $S \in \mathsf{B}\langle\langle H \rangle\rangle\langle\langle V \rangle\rangle$, $h \in H$,

$$S \otimes h = h^{-1} \cdot S. \tag{190}$$

11.1.4.3. Action $\odot$. We define a map $\rho_\varepsilon : \mathsf{B}\langle\langle H \rangle\rangle\langle\langle V \rangle\rangle \to \mathsf{B}\langle\langle H \rangle\rangle\langle\langle V \rangle\rangle$ as the unique σ-additive map such that

$$\rho_\varepsilon(\emptyset) = \emptyset, \quad \rho_\varepsilon(\varepsilon) = \varepsilon$$

and for every $p \in Q$, $z \in Z$, $q \in Q$, $\beta \in V^*$, $k \in \mathsf{B}\langle\langle H \rangle\rangle$, $S \in \mathsf{B}\langle\langle H \rangle\rangle\langle\langle V \rangle\rangle$,

$$\rho_\varepsilon([p,z,q] \cdot \beta) = \rho_\varepsilon(([p,z,q] \otimes e) \cdot \beta) \text{ if } pz \text{ is } \varepsilon\text{-bound}$$

(the notion of ε-bound mode is defined here as in Section 2.2),

$$\rho_\varepsilon([p,z,q] \cdot \beta) = [p,z,q] \cdot \beta \text{ if } pz \text{ is } \varepsilon\text{-free}$$

and

$$\rho_\varepsilon(k \cdot S) = k \cdot \rho_\varepsilon(S).$$

The right-action $\odot$ of the monoid $H \times X^*$ over the semi-ring $\mathsf{B}\langle\langle H \rangle\rangle\langle\langle V \rangle\rangle$ is then the unique monoid-action fulfilling: for every $S \in \mathsf{B}\langle\langle H \rangle\rangle\langle\langle V \rangle\rangle$, $h \in H$, $x \in X$,

$$S \odot hx = \rho_\varepsilon(\rho_\varepsilon(S) \otimes hx).$$

11.1.4.4. Case where H is abelian. Let us consider the case where H is abelian. Let $\varphi : \mathsf{B}\langle\langle H \rangle\rangle \cup V \to \mathsf{B}\langle\langle H \rangle\rangle\langle\langle X \rangle\rangle$ defined by

$$\forall k \in \mathsf{B}\langle\langle H \rangle\rangle,\ \varphi(k) = k; \quad \forall v \in V,\ \varphi(v) = \sum_{v \odot (h \cdot u) = \varepsilon} h \cdot u.$$

As H is supposed abelian, $\varphi(\mathsf{B}\langle\langle H \rangle\rangle)$ is included in the center of $\mathsf{B}\langle\langle H \rangle\rangle$) and by property (15) there exists a unique σ-additive semi-ring homomorphism $\tilde{\varphi} : \mathsf{B}\langle\langle H \rangle\rangle$

$\langle\langle V\rangle\rangle \to \mathsf{B}\langle\langle H\rangle\rangle\langle\langle X\rangle\rangle$ which extends φ. Let us denote by the same letter the original φ and its extension $\tilde{\varphi}$.

Lemma 113. *For every* $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle V\rangle\rangle$, $h \in H$, $u \in X^*$,
(1) $\varphi(S) = \varphi(\rho_\varepsilon(S))$,
(2) $\varphi(S \odot (h,u)) = \varphi(S) \bullet (h,u)$ (*i.e.* φ *is a morphism of right-actions*).

11.2. Deterministic rational series

11.2.0.5. W-determinism. Let H be a group, let W be an alphabet. Let $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. We define an equivalence relation $\sim$ over $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$ by: for every $S, T \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$,

$$S \sim T \Leftrightarrow \exists h \in H,\ S = h \cdot T.$$

This equivalence is compatible with left-product by elements of H and with right-action $\bullet$: if $S \sim T$ then, for every $h \in H$, $u \in W^*$

$$h \cdot S \sim h \cdot T \quad \text{and} \quad S \bullet (h,u) \sim T \bullet (h,u). \tag{191}$$

Therefore, the left-product by elements of H (resp. the right-action of $H \times W^*$) over $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$ induce a left-product by elements of H (resp. a right-action of $H \times W^*$) over $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle/\sim$. For every $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$, by $\mathsf{Q}(S)$ we denote the set of residuals of S:

$$\mathsf{Q}(S)) = \{S \bullet (h,u) \mid h \in H,\ u \in W^*\}.$$

Let us denote by $(H^0, \cdot, 1_H)$ the submonoid of $(\mathsf{B}\langle\langle H\rangle\rangle, \cdot, 1_H)$ consisting of the empty series and all the singletons $\{h\}$ for $h \in H$. H^0 can be seen as the monoid obtained by "adjoining a zero" to the group H. We sometimes use the symbol 0 for the element $\emptyset \in H^0$ and we identify every $h \in H$ with the corresponding $\{h\} \in H^0$. By $\mathsf{H}^0\langle\langle W\rangle\rangle$ we denote the subset of series in $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$ whose coefficients are all in H^0.

Proposition 114. *Let* $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. *The following properties are equivalent*:
(1) S *is recognized by some* W*-deterministic finite* H*-automaton*
(2) $\forall u \in W^*$, $(S_u \in H^0)$ *and* $\mathsf{Q}(S)/\sim$ *is finite.*

This proposition is established in [74, Proposition 4, p. 93]. Though this author assumed H is a free group, his proof remains valid for any group H. For sake of completeness we restate his arguments.

Proof. (1) $\Rightarrow$ (2): Let us suppose that $\mathscr{M} = \langle W, Q, \delta, h_0, q_0, Q'\rangle$ is a deterministic finite H-automaton such that $S = \mathrm{S}(\mathscr{M})$. One can check that the W-determinism of the automaton implies that every coefficient S_u belongs to H^0. Let us set

$$\forall q \in Q,\ S_q = \varepsilon_q^Q \cdot B^* \cdot C$$

(where B, C are the matrices considered in Section 11.1.1). It should be clear that, $\forall h \in H, u \in W^*$, either $S \bullet (h, u) = \emptyset$ or $\exists q \in Q, S \bullet (h, u) \sim S_q$. Hence

$$\mathrm{Card}(\mathsf{Q}(S)/\sim) \leqslant \mathrm{Card}(Q) + 1.$$

(2) $\Rightarrow$ (1): Let us suppose that $\mathrm{Card}(\mathsf{Q}(S)/\sim) = n < \infty$. Let us denote by Q, Q' the sets

$$Q = \{[S \bullet u]_\sim \mid \exists u' \in W^*,\ S_{u \cdot u'} \neq 0\},$$

$$Q' = \{[S \bullet u]_\sim \mid S_u \neq 0\}.$$

We choose a total ordering over W and consider its short-lex extension to W^*. For every $c \in Q$ we define

$$s(c) = \min\{u \in W^*, c \bullet u \in Q'\}$$

(the letter s stands for "suffix"). One can notice that $\forall c \in \mathsf{Q}(S)/\sim$, $|s(c)| \leqslant n - 1$ and that, if $c \in Q'$ then $s(c) = \varepsilon$.

We define

$$c_0 = [S_\varepsilon]_\sim,$$

$$h_0 = S_{s(c_0)} \text{ (if } c_0 \in Q); \quad h_0 = 1_H \text{ (if } c_0 \notin Q).$$

We let δ be the set of all the 4-tuples $(c, h, v, c') \in Q \times H \times W \times Q$ such that

$$c = [S \bullet u]_\sim, \quad c' = [S \bullet uv]_\sim, \quad h = (S_{u \cdot s(c)})^{-1} \cdot S_{uv \cdot s(c')}$$

for some $u \in W^*$, $v \in W$.

Let us remark that, if $S \bullet u = g \cdot (S \bullet u')$ (for some $u, u' \in W^*$, $g \in H$), then, by (191),

$$c' = [S_{u' \cdot v}]_\sim, \quad S_{u \cdot s(c)} = g \cdot S_{u' \cdot s(c)}, \quad S_{uv \cdot s(c')} = g \cdot S_{u'v \cdot s(c')},$$

so that

$$h = (S_{u \cdot s(c)})^{-1} \cdot S_{uv \cdot s(c')} = (S_{u' \cdot s(c)})^{-1} \cdot (S_{u'v \cdot s(c')}).$$

Hence condition (179) is fulfilled by δ. Let us consider the deterministic finite H-automaton $\mathscr{M} = \langle W, Q, \delta, h_0, q_0, Q' \rangle$ associated with the above values of $W, Q, \delta, h_0, q_0, Q'$. We prove by induction on the integer p that for every path

$$c_0, (h_1, v_1), c_1, \ldots, (h_i, v_i), c_i, \ldots, (h_p, v_p), c_p \tag{192}$$

in the automaton $\mathscr{M}$, the "labels" of the path

$$h = h_0 \cdot h_1 \cdots h_i \cdots h_p \in H, \quad u = v_1 \cdots v_i \cdots v_p \in W^*$$

fulfill the relation

$$h = S_{u \cdot s(c_p)}. \tag{193}$$

If $p = 0$: $h = h_0$, $u = \varepsilon$. As (192) is a path in $\mathscr{M}$, c_0 is assumed to belong to Q, hence, by definition of h_0, $h_0 = S_{s(c_0)} = S_{\varepsilon \cdot s(c_0)}$.

If $p=m+1$: $h=h'\cdot h_p, u=u'\cdot v_p$, where, by induction hypothesis,

$$h' = S_{u'\cdot s(c_m)}. \tag{194}$$

By the definition of δ,

$$h_p = (S_{u'\cdot s(c_m)})^{-1}\cdot S_{u\cdot s(c_p)}. \tag{195}$$

Multiplying relations (194), (195), we obtain, as required:

$$h = h'\cdot h_p = S_{u'\cdot s(c_m)}\cdot (S_{u'\cdot s(c_m)})^{-1}\cdot S_{u\cdot s(c_p)} = S_{u\cdot s(c_p)}.$$

Applying invariant (193) to the case where $c_p\in Q'$ we have: if (h,u) labels any path in $\mathscr{M}$, ending in a state $c\in Q'$, then $h=S_u$. Moreover, one can check that the projection of $\mathscr{M}$ on W^* (i.e. the boolean automaton obtained by sending every coefficient in H to the boolean constant 1) recognizes exactly $\mathrm{supp}(S)$. It follows that

$$S = \mathrm{S}(\mathscr{M}). \qquad \square$$

Definition 115. Let $S\in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. S is said W-deterministic rational iff it fulfills one of points (1), (2) of Proposition 114.

11.2.0.6. Length and norm. Let us suppose now that H admits a presentation over a finite alphabet $\hat{Y}$: $\varphi_H:\hat{Y}^*\to H$ is a surjective monoid-homomorphism. We suppose the presentation φ_H is "symmetric" in the following sense:

- $\hat{Y}=Y\cup\bar{Y}$; $Y\cap\bar{Y}=\emptyset$,
- a map $y\mapsto\bar{y}$, from $\hat{Y}$ to $\hat{Y}$ is given; this map is an involution (i.e. $\bar{\bar{y}}=y$) , which fixes no letter of $\hat{Y}$, and which sends Y on $\bar{Y}$ (hence $\bar{Y}$ on Y),
- $\forall y\in\hat{Y}, \varphi_H(y\cdot\bar{y})=1_H$.

For every $h\in H$, the *length* of h, relative to the presentation φ_H, is defined by

$$\ell(h) = \min\{|u|\mid u\in\hat{Y}^*, \varphi_H(u)=h\}.$$

One can notice that the map $(h,h')\mapsto\ell(h^{-1}\cdot h')$ is a distance over H. Let us denote by $\mathsf{F}(W)$ the free-group over the alphabet W. It has a standard presentation over the symetric alphabet $\hat{W}=W\cup\bar{W}$:

$$\mathsf{F}(W)\approx \hat{W}^*/\underset{T}{\overset{*}{\leftrightarrow}},$$

where T is the set of relations

$$T = \{(w\cdot\bar{w},\varepsilon)\mid w\in W\}\cup\{(\bar{w}\cdot w,\varepsilon)\mid w\in W\}.$$

(The notion of length over $\mathsf{F}(W)$ relative to this standard presentation is defined as above for H,φ_H.)

Let us notice that the distance $(u,v)\mapsto\ell(u^{-1}\cdot v)=\mathrm{d}(u,v)$ restricted to $W^*\subseteq\mathsf{F}(W)$ can be equivalently defined by

$$\mathrm{d}(u,v) = |u|+|v|-2\cdot|gcp(u,v)|,$$

where $gcp(u,v)$ is the greatest common prefix of u,v. Let us consider a W-deterministic, finite, H-automaton $\mathscr{M} = \langle W, Q, \delta, h_0, q_0, Q' \rangle$. We define the length of $\mathscr{M}$, $\bar{k}(\mathscr{M})$, the initial length of $\mathscr{M}$, $k_0(\mathscr{M})$ and the norm of $\mathscr{M}$, $\|\mathscr{M}\|$ as

$$\bar{k}(\mathscr{M}) = \sup\{\ell(h) \mid \exists q \in Q,\ v \in W,\ r \in Q,\ (q,h,v,r) \in \delta\}; \quad k_0(\mathscr{M}) = \ell(h_0),$$

$$\|\mathscr{M}\| = \mathrm{Card}(Q).$$

(The sup is taken in $\mathbb{N} \cup \{\infty\}$. Notice that, when $\delta = \emptyset$, $\bar{k}(\mathscr{M}) = 0$.) Similarly, we define the length of a pushdown H-automaton $\mathscr{M} = \langle X, Z, Q, \delta, q_0, z_0 \rangle$, by

$$\bar{k}(\mathscr{M}) = \sup\{\ell(h) \mid \exists q \in Q, z \in Z, a \in X \cup \{\varepsilon\}, r \in Q, \omega \in Z^*, (h, r\omega) \in \delta(qz, a)\}.$$

Let us consider now a series $S \in \mathsf{H}^0\langle\langle W \rangle\rangle$. We define the length of S, $\bar{\ell}(S)$, the initial length of S, $\ell_0(S)$, and the norm of S, $\|S\|$ by

$$\bar{\ell}(S) = \inf\{\mu \in \mathsf{R}_+ \mid \forall u, v \in W^*, S_u \neq 0 \Rightarrow \ell((S_u)^{-1} \cdot S_v)) \leqslant \mu \cdot \ell(u^{-1} \cdot v)\},$$

$$\ell_0(S) = \ell(S_{u_0}),\ \text{where } u_0 \text{ is the minimum word of } \mathrm{supp}(S),$$

$$\|S\| = \mathrm{Card}(\mathsf{Q}(S)/\sim).$$

(One can check that $\bar{\ell}(\emptyset) = 0$ and we define $\ell_0(\emptyset) = 0$.) In general $\bar{\ell}(S)$, $\|S\|$ belong to $\mathbb{N} \cup \{\infty\}$ and $\ell_0(S)$ belongs to $\mathbb{N}$.

Lemma 116. *Let $S, T \in \mathsf{H}^0\langle\langle W \rangle\rangle$. If $S \sim T$ then $\bar{\ell}(S) = \bar{\ell}(T)$ and $\|S\| = \|T\|$.*

Lemma 117. *Let $\mathscr{M}$ be some W-dfa and let $\mathsf{S}(\mathscr{M}) = S \in \mathsf{H}^0\langle\langle W \rangle\rangle$. Then $\bar{\ell}(S) \leqslant \bar{k}(\mathscr{M}), \ell_0(S) \leqslant k_0(\mathscr{M}) + \bar{k}(\mathscr{M}) \cdot \|\mathscr{M}\|$, $\|S\| \leqslant \|\mathscr{M}\| + 1$.*

Lemma 118. *For every W-deterministic rational series $S \in \mathsf{B}\langle\langle H \rangle\rangle\langle\langle W \rangle\rangle$, there exists some W-dfa $\mathscr{M}$ such that $\mathsf{S}(\mathscr{M}) = S$ and: $\bar{k}(\mathscr{M}) \leqslant 2 \cdot \bar{\ell}(S) \cdot \|S\|$, $k_0(\mathscr{M}) \leqslant \ell_0(S)$, $\|\mathscr{M}\| \leqslant \|S\|$.*

Proof. Let us consider the W-dfa $\mathscr{M}$ constructed in the proof of Lemma 114 and let $(c, h, v, c') \in \delta$. By definition

$$h = (S_{u \cdot s(c)})^{-1} \cdot S_{uv \cdot s(c')},$$

where $u \in W^*, v \in W$. Hence,

$$\ell(h) \leqslant \bar{\ell}(S)(|s(c)| + |v| + |s(c')|) \leqslant \bar{\ell}(S)(\|S\| - 1 + 1 + \|S\| - 1)$$
$$\leqslant 2 \cdot \bar{\ell}(S) \cdot \|S\|. \qquad \square$$

11.2.0.7. $\smile$-Determinism. We use here the relation $\sim$ defined at the beginning of Section 11.2. Let us notice that $\sim$ induces an equivalence relation $\sim$ on H^0 with only two classes H and $\{0\}$.

Definition 119. Let $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. S is said *left-$\smile$-deterministic* iff either
(1) $S \sim \emptyset$ or
(2) $S \sim \varepsilon$ or
(3) $\exists w_0 \in W^*$, $S_{w_0} \neq 0$ and $\forall w, w' \in W^*$,

$$S_w \sim S_{w'} \sim 1_H$$
$$\Rightarrow [\exists A, A' \in W, w_1, w_1' \in W^*, A \smile A', w = A \cdot w_1 \text{ and } w' = A' \cdot w_1'].$$

A left-$\smile$-deterministic series S is said to have the type $\emptyset$ (resp. ε, $[A]_\smile$) if case (1) (resp. (2), (3)) occurs.

Definition 120. Let $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. S is said *$\smile$-deterministic* iff, for every $u \in W^*$, $S \bullet u$ is left-$\smile$-deterministic.

Let us notice that, if S is $\smile$-deterministic, then every coefficient S_u belongs to H^0 and supp(S) is deterministic in the sense of Definition 2. We denote by $\mathsf{DH}^0\langle\langle W\rangle\rangle$ the set of $\smile$-deterministic series in $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$.

A finite H-automaton $\mathscr{M} = \langle W, Q, \delta, h_0, q_0, Q'\rangle$ will be said *$\smile$-deterministic* if and only if, for every $q \in Q$, $A, A' \in W$, $h, h' \in H$, $r, r' \in Q$:

$$((q, h, A, r) \in \delta \quad \text{and} \quad (q, h', A', r') \in \delta) \Rightarrow A \smile A'. \tag{196}$$

11.2.0.8. Full determinism.

Proposition 121. *Let $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. The following properties are equivalent*:
(1) *S is both W-deterministic rational and $\smile$-deterministic.*
(2) *$\mathsf{Q}(S)/\sim$ is finite and S is $\smile$-deterministic.*
(3) *S is W-deterministic rational and* supp(S) *is deterministic.*
(4) *S is recognized by some finite H-automaton which is both W-deterministic and $\smile$-deterministic.*

Definition 122. Let $S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. S is said fully deterministic rational (deterministic rational, for short) iff it fulfills one of points (1)–(4) of Proposition 121.

As point (2) of Proposition 121 is very close to the definition used in the boolean case (Definition 2), we shall mostly use point (2) as the main definition of *deterministic rational series* in the sequel. We denote by $\mathsf{DRH}^0\langle\langle W\rangle\rangle$ the set of Deterministic Rational series with coefficients in H^0 and undeterminates in W.

11.3. Vectors, matrices

We recall that, for every $n,m \in \mathbb{N} - \{0\}$, $\mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle$ denotes the set of matrices of dimension (n,m) with entries in $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. The external product $k \in \mathsf{B}\langle\langle H\rangle\rangle$, $S \in \mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle \mapsto k \cdot S \in \mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle$ is defined, as usual by

$$\forall i \in [1,n],\ \forall j \in [1,m],\ (k \cdot S)_{i,j} = k \cdot S_{i,j}.$$

11.3.0.9. W-deterministic rational matrices. The equivalence relation $\sim$ is adapted to $\mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$ by

$$S \sim T \Leftrightarrow \exists h \in H, \quad S = h \cdot T.$$

It is then extended to $\mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle$ by

$$S \sim T \Leftrightarrow \forall i \in [1,n], \quad S_{i,*} \sim T_{i,*}.$$

The right-action $\bullet$ is extended compentwise to $\mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle$ by for every $S \in \mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle$, $h \in H$, $u \in W^*$,

$$(S \bullet (h,u))_{i,j} = S_{i,j} \bullet (h,u).$$

For every $S \in \mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle$ we define the set of residuals of S, $\mathsf{Q}(S)$ and the set of row-residuals of S, $\mathsf{Q}_r(S)$, by

$$\mathsf{Q}(S) = \{S \bullet (h,u) \mid h \in H, u \in W^*\}, \quad \mathsf{Q}_r(S) = \bigcup_{1 \leqslant i \leqslant n} \mathsf{Q}(S_{i,*}).$$

Proposition 123. *Let* $m \geqslant 1$, $S \in \mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$. *The following properties are equivalent*:
(1) *S is recognized by some W-deterministic finite m-H-automaton*
(2) $\forall j \in [1,m]$, $\forall u \in W^*$, $((S_j)_u \in H^0)$ *and* $\mathsf{Q}(S)/\!\sim$ *is finite*

Definition 124. Let $S \in \mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$. S is said W-deterministic rational iff it fulfills one of points (1) and (2) of Proposition 123.

11.3.0.10. Length and norm. Let us consider a W-deterministic, finite, m-H-automaton $\mathscr{M} = \langle W, Q, \delta, h_0, q_0, (Q'_j)_{1 \leqslant j \leqslant m}\rangle$. We define the length of $\mathscr{M}$, $\bar{k}(\mathscr{M})$, the initial length of $\mathscr{M}$, $k_0(\mathscr{M})$ and the norm of $\mathscr{M}$, $\|\mathscr{M}\|$ as

$$\bar{k}(\mathscr{M}) = \max\{\ell(h) \mid \exists q \in Q,\ v \in W,\ r \in Q,\ (q,h,v,r) \in \delta\}, \quad k_0(\mathscr{M}) = \ell(h_0),$$

$$\|\mathscr{M}\| = \mathrm{Card}(Q).$$

Let us consider now a vector $S \in \mathsf{H}^0_{1,m}\langle\langle W\rangle\rangle$. We define the length of S, $\bar{\ell}(S)$, the initial length of S, $\ell_0(S)$, and the norm of S, $\|S\|$ by

$$\bar{\ell}(S) = \inf\{\mu \in \mathsf{R}_+ \mid \forall i,j \in [1,m],\ \forall u,v \in W^*,\ S_{i,u} \neq 0 \Rightarrow \ell((S_{i,u})^{-1} \cdot S_{j,v})) \leqslant \mu \cdot \ell(u^{-1} \cdot v)\},$$

$$\ell_0(S) = \ell(S_{u_0}),$$

where $S_{j,u}$ denotes the coefficient of S_j on the word u and u_0 is the minimum word of $\cup_{j=1}^{m}\operatorname{supp}(S_j)$. We define $\ell_0(\emptyset^m)=0$ and

$$\|S\| = \mathrm{Card}(\mathsf{Q}(S)/\sim).$$

The three following lemmas can be proved in a similar way as Lemmas 116–118.

Lemma 125. *Let $S,T \in \mathsf{H}^0_{1,m}\langle\langle W\rangle\rangle$. If $S\sim T$ then $\bar{\ell}(S)=\bar{\ell}(T)$ and $\|S\|=\|T\|$.*

Lemma 126. *Let $\mathscr{M}$ be some m-W-dfa and let $\mathrm{S}(\mathscr{M})=S\in \mathsf{H}^0_{1,m}\langle\langle W\rangle\rangle$. Then*

$$\bar{\ell}(S)\leqslant \bar{k}(\mathscr{M}),\quad \ell_0(S)\leqslant k_0(\mathscr{M})+\bar{k}(\mathscr{M})\cdot\|\mathscr{M}\|,\quad \|S\|\leqslant\|\mathscr{M}\|+1.$$

Lemma 127. *For every W-deterministic rational vector $S\in \mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$, there exists some m-W-dfa $\mathscr{M}$ such that $\mathrm{S}(\mathscr{M})=S$ and: $\bar{k}(\mathscr{M})\leqslant 2\cdot\bar{\ell}(S)\cdot\|S\|$, $k_0(\mathscr{M})\leqslant\ell_0(S)$, $\|\mathscr{M}\|\leqslant\|S\|$.*

Let us consider now a matrix $S\in \mathsf{H}^0_{n,m}\langle\langle W\rangle\rangle$. We define the length of S, $\bar{\ell}(S)$, the initial length of S, $\ell_0(S)$, and the norm of S, $\|S\|$ by

$$\bar{\ell}(S)=\max\{\bar{\ell}(S_{i,*}),1\leqslant i\leqslant n\},\quad \ell_0(S)=\max\{\ell_0(S_{i,*}),1\leqslant i\leqslant n\}$$

and

$$\|S\| = \mathrm{Card}(\mathsf{Q}_r(S)/\sim).$$

In general, $\bar{\ell}(S)$, $\|S\|$ belong to $\mathbb{N}\cup\{\infty\}$ and $\ell_0(S)$ belongs to $\mathbb{N}$.

11.3.0.11. $\smile$-deterministic matrices.

Definition 128. Let $m\geqslant 1$, $S\in \mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$. S is said *left-$\smile$-deterministic* iff either
(1) $\forall j\in[1,m]$, $S_j\sim\emptyset$ or
(2) $\exists j_0\in[1,m]$, $S_{j_0}\sim\varepsilon$ and $\forall j\neq j_0, S_j\sim\emptyset$ or
(3) $\exists j_0\in[1,m]$, $S_{j_0}\not\sim\emptyset$ and $\forall w,w'\in W^*$, $\forall i,j\in[1,m]$, $(S_i)_w\sim(S_j)_{w'}\sim 1_H\Rightarrow [\exists A,A'\in W,\ w_1,w_1'\in V^*,\ A\smile A',\ w=A\cdot w_1$ and $w'=A'\cdot w_1']$.

A left-$\smile$-deterministic series S is said to have the type $\emptyset$ (resp. ε, $[A]_\smile$) if case (1) (resp. (2), (3)) occurs.

Definition 129. Let $m\geqslant 1, S\in \mathsf{B}_{1,m}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$. S is said *$\smile$-deterministic* iff, for every $u\in W^*$, $S\bullet u$ is left-$\smile$-deterministic.

Let us notice that, if S is $\smile$-deterministic, then every coefficient $S_{j,u}$ belongs to H^0 and supp(S) is deterministic in the sense of Definition 5. We denote by $\mathsf{DH}^0_{1,m}\langle\langle W\rangle\rangle$ the set of $\smile$-deterministic vectors in $\mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$.

A finite m-H-automaton $\mathscr{M} = \langle W, Q, \delta, h_0, q_0, (Q'_j)_{1 \leqslant j \leqslant m}\rangle$ will be said $\smile$-*deterministic* if and only if it fulfills condition (196).

11.3.0.12. Deterministic rational matrices.

Proposition 130. *Let* $m \geqslant 1$, $S \in \mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$. *The following properties are equivalent*:
(1) *S is both W-deterministic rational and $\smile$-deterministic.*
(2) *$\mathsf{Q}(S)/\!\sim$ is finite and S is $\smile$-deterministic.*
(3) *S is W-deterministic rational and* $\mathrm{supp}(S)$ *is deterministic.*
(4) *S is recognized by some finite m-H-automaton which is both W-deterministic and $\smile$-deterministic.*

Definition 131. Let $m \geqslant 1$, $S \in \mathsf{B}\langle\langle H\rangle\rangle_{1,m}\langle\langle W\rangle\rangle$. The vector S is said fully deterministic rational (deterministic rational, for short) iff it fulfills one of points (1)–(4) of Proposition 130.

Definition 132. Let $n, m \geqslant 1$, $S \in \mathsf{B}\langle\langle H\rangle\rangle_{n,m}\langle\langle W\rangle\rangle$. The matrix S is said fully deterministic rational (deterministic rational, for short) iff every row-vector $S_{i,*}$, for $1 \leqslant i \leqslant n$, is fully deterministic rational.

We denote by $\mathsf{DRH}^0_{n,m}\langle\langle W\rangle\rangle$ the set of Deterministic Rational matrices of dimension (n, m), with coefficients in $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$.

11.3.1. Ordering

We define a partial ordering on $\mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$ by: for every $S, T \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$,

$$S \sqsubseteq T \Leftrightarrow (\forall u \in W^*, S_u = 0 \text{ or } S_u = T_u).$$

Given $S, T \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$ such that $S \sqsubseteq T$ we define $T - S \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$ by

$$\forall u \in W^*,\ (T - S)_u = T_u \text{ (if } S_u = 0); \quad (T - S)_u = 0 \text{ (if } S_u = T_u).$$

One can easily check the following

Fact 133. *Let* $S, T \in \mathsf{B}\langle\langle H\rangle\rangle\langle\langle W\rangle\rangle$ *such that* $S \sqsubseteq T$.
(1) *If T is $\smile$-deterministic, then S is $\smile$-deterministic.*
(2) *If T is $\smile$-deterministic, then $(S, T - S)$ is a $\smile$-deterministic vector.*

11.4. Algebraic properties

Let us fix now some *abelian* group $(H, \cdot)$. We adapt here the main results concerning $\mathsf{B}_{n,m}\langle\langle W\rangle\rangle$ obtained in Section 3 to the matrices in $\mathsf{H}^0_{n,m}\langle\langle W\rangle\rangle$. Most of the proofs are so close to the proofs given in Section 3 that we just mention the corresponding lemma of Section 3 and leave to the reader the necessary adaptations. Some *new* statements concerning the functions $\bar{\ell}, \ell_0$ are introduced.

11.4.1. Residuals

Lemma 134. *Let* $S \in \mathsf{DH}^0_{1,m}\langle\langle W\rangle\rangle$, $T \in \mathsf{B}\langle\langle H\rangle\rangle_{m,s}\langle\langle W\rangle\rangle$, $u \in W^*$ *and* $U = S \cdot T$. *Exactly one of the following cases is true*:
(1) $\exists j, S_j \bullet u \notin H^0$
in this case $U \bullet u = (S \bullet u) \cdot T$.
(2) $\exists j_0, \exists u', u'', u = u' \cdot u'', S_{j_0} \bullet u' = h \in H$;
in this case $U \bullet u = h \cdot T_{j_0,*} \bullet u''$.
(3) $\forall j, \forall u' \leqslant u,\ S_j \bullet u = \emptyset,\ S_j \bullet u' \notin H$;
in this case $U \bullet u = \emptyset^s = (S \bullet u) \cdot T$.

(See Lemma 11.)

Lemma 135. *Let* $S \in \mathsf{DRH}^0_{1,m}\langle\langle V\rangle\rangle$, *such that, for every* $j \in [1,m]$, $S_j \neq \emptyset$. *Let* $T, T' \in \mathsf{B}\langle\langle H\rangle\rangle_{m,s}\langle\langle W\rangle\rangle$. *If* $S \cdot T \sim S \cdot T'$ *then* $T \sim T'$.

Proof. Suppose that $S \cdot T = h \cdot S \cdot T'$ (where S, T, T' fulfill the above hypotheses). Let $u_j \in \mathrm{supp}(S_j)$ (for every $j \in [1,m]$). For every $j \in [1,m]$,

$$(S \cdot T) \bullet u_j = (h \cdot S \cdot T') \bullet u_j$$

which, by Lemma 134, case 2, can be rewritten as

$$h_j \cdot T_{j,*} = h \cdot h_j \cdot T'_{j,*}$$

(where $S_j \bullet u_j = h_j \in H$). Multiplying by h_j^{-1} the above equality, we obtain

$$\forall j \in [1,m],\ T_{j,*} = h \cdot T'_{j,*},$$

hence $T \sim T'$. $\square$

Lemma 136. *Let* $n, m \in \mathbb{N} - \{0\}$, $S \in \mathsf{H}^0\langle\langle W\rangle\rangle$, $u \in W^*$,
(1) $\bar{\ell}(S \bullet u) \leqslant \bar{\ell}(S)$.
(2) $\ell_0(S \bullet u) \leqslant \ell_0(S) + \bar{\ell}(S) \cdot (|u| + 2 \cdot \|S\|)$.
(3) $\|S \bullet u\| \leqslant \|S\|$.

Proof. Points (1) and (3) are obvious. Let us prove point (2). If $S \bullet u = \emptyset$, the inequality is clearly true. Let us suppose now that $S \bullet u \neq \emptyset$, $\min(\mathrm{supp}(S)) = u_0$, $\min(\mathrm{supp}(S \bullet u)) = u'_0$.

$$(S \bullet u)_{u'_0} = S_{u_0} \cdot (S_{u_0}^{-1} \cdot S_{u \cdot u'_0})$$

and

$$\ell(u_0^{-1} \cdot u \cdot u'_0) \leqslant \ell(u) + \ell(u_0) + \ell(u'_0) \leqslant \ell(u) + 2 \cdot \|S\|.$$

It follows that

$$\ell((S \bullet u)_{u'_0}) \leqslant \ell_0(S) + \bar{\ell}(S)(\ell(u) + 2\|S\|). \qquad \square$$

11.4.2. Product

Lemma 137. *For every* $S \in \mathsf{DH}^0_{n,m}\langle\langle W\rangle\rangle$, $T \in \mathsf{DH}^0_{m,s}\langle\langle W\rangle\rangle$, $S \cdot T \in \mathsf{DH}^0_{n,s}\langle\langle W\rangle\rangle$.

(See Lemma 13.)

Lemma 138. *Let* $S \in \mathsf{DH}^0_{n,m}\langle\langle W\rangle\rangle$, $T \in \mathsf{H}^0_{m,s}\langle\langle W\rangle\rangle$. *Then* $\|S \cdot T\| \leqslant \|S\| + \|T\|$.

(See Lemma 14.)

Lemma 139. *Let* $S \in \mathsf{DH}^0\langle\langle W\rangle\rangle$, $T \in \mathsf{H}^0\langle\langle W\rangle\rangle$. *Then*
(1) $\bar{\ell}(S \cdot T) \leqslant \max\{\bar{\ell}(S), \bar{\ell}(T)\}$,
(2) $\ell_0(S \cdot T) \leqslant \ell_0(S) + \ell_0(T)$.

Proof. In order to prove the first inequality we consider $h = (S \cdot T)^{-1}_{u\cdot v} \cdot (S \cdot T)_{u\cdot w}$ where $u, v, w \in W^*$, $gcp(v, w) = \varepsilon$ and $(S \cdot T)_{u\cdot v} \neq 0, (S \cdot T)_{u\cdot w} \neq 0$. As supp$(S)$ is deterministic, one of the following two cases must occur:

Case 1: $u \cdot v_1 \in \text{supp}(S)$, $u \cdot w_1 \in \text{supp}(S)$, $v = v_1 \cdot v_2$, $w = w_1 \cdot w_2$. Using the commutativity of H we have

$$\begin{aligned}\ell(h) &= \ell((S_{uv_1} T_{v_2})^{-1} \cdot (S_{uw_1} T_{w_2})) = \ell((S^{-1}_{uv_1} \cdot S_{uw_1}) \cdot (T^{-1}_{v_2} \cdot T_{w_2}))\\ &\leqslant \bar{\ell}(S)(|v_1| + |w_1|) + \bar{\ell}(T)(|v_2| + |w_2|)\\ &\leqslant \max\{\bar{\ell}(S), \bar{\ell}(T)\} \cdot (|v| + |w|) = \max\{\bar{\ell}(S), \bar{\ell}(T)\} \cdot \ell((uv)^{-1} \cdot (uw)).\end{aligned}$$

Case 2: $u_1 \in \text{supp}(S), u_2 \cdot v \in \text{supp}(T), u_2 \cdot w \in \text{supp}(T), u = u_1 \cdot u_2$.

$$\ell(h) = \ell(T^{-1}_{u_2 v} \cdot T_{u_2 w}) \leqslant \bar{\ell}(T)(|v| + |w|) \leqslant \max\{\bar{\ell}(S), \bar{\ell}(T)\} \cdot \ell((uv)^{-1} \cdot (uw)).$$

This ends the proof of the first inequality. The second inequality is straightforward. □

Lemma 140. *Let* $n, m, s \in \mathbb{N} - \{0\}$, $S \in \mathsf{DH}^0_{n,m}\langle\langle W\rangle\rangle$, $T \in \mathsf{DH}^0_{m,s}\langle\langle W\rangle\rangle$. *Then*
(1) $\bar{\ell}(S \cdot T) \leqslant \max\{\bar{\ell}(S), \bar{\ell}(T)\} + 2 \cdot \ell_0(T) + 2 \cdot \bar{\ell}(T) \cdot \|T\|$.
(2) $\ell_0(S \cdot T) \leqslant \ell_0(S) + \ell_0(T)$.

Proof. Let us prove point (1). We treat first the

Case 1: $n = 1, s = 1$. Let us consider $u, v, w \in W^*$, $h \in H$ such that

$$h = (S \cdot T)^{-1}_{u\cdot v} \cdot (S \cdot T)_{u\cdot w}$$

and $gcp(v, w) = \varepsilon$, $(S \cdot T)_{u\cdot v} \neq 0$, $(S \cdot T)_{u\cdot w} \neq 0$.
As supp(S) is deterministic, one of the following two cases must occur:

Subcase 1.1: $i, j \in [1, m]$, $u \cdot v_1 \in \text{supp}(S_i)$, $u \cdot w_1 \in \text{supp}(S_j)$, $v = v_1 \cdot v_2$, $w = w_1 \cdot w_2$. Using the commutativity of H we have

$$\begin{aligned}\ell(h) &= \ell((S_{i,uv_1} T_{i,v_2})^{-1} \cdot (S_{j,uw_1} T_{j,w_2})) = \ell((S^{-1}_{i,uv_1} \cdot S_{j,uw_1}) \cdot (T^{-1}_{i,v_2} \cdot T_{j,w_2}))\\ &\leqslant \bar{\ell}(S)(|v_1| + |w_1|) + \ell(T_{i,v_2}) + \ell(T_{j,w_2}).\end{aligned} \quad (197)$$

Let us consider a general series $U \in \mathsf{H}^0\langle\langle W\rangle\rangle$ and a word $w' \in W^*$. Let $u_0 = \min(\mathrm{supp}(U))$ and $w' = u' \cdot u''$ with $|u'| = |u_0|$. By definition of the length of a series we then have

$$\begin{aligned}\ell(U_{w'}) &\leqslant \ell(u_0) + \bar{\ell}(U) \cdot \ell(u_0^{-1} \cdot w') \\ &\leqslant \ell_0(U) + \bar{\ell}(U)[\ell(w') + \ell(u_0)],\end{aligned}$$

and, as every finite automaton recognizes at least one word of length smaller or equal to its number of states $\ell(u_0) \leqslant \|U\|$, hence

$$\ell(U_{w'}) \leqslant \bar{\ell}(U)\ell(w') + \ell_0(U) + \bar{\ell}(U) \cdot \|U\|. \tag{198}$$

Applying inequality (198) to the series T_i, T_j in inequality (197) we get

$$\begin{aligned}\ell(h) &\leqslant \bar{\ell}(S)(|v_1| + |w_1|) + \bar{\ell}(T_i)\ell(v_2) + \ell_0(T_i) + \bar{\ell}(T_i) \cdot \|T_i\| \\ &\quad + \bar{\ell}(T_j)\ell(w_2) + \ell_0(T_j) + \bar{\ell}(T_j) \cdot \|T_j\| \\ &\leqslant \max\{\bar{\ell}(S), \bar{\ell}(T)\} \cdot (|v| + |w|) + 2 \cdot \ell_0(T) + 2 \cdot \bar{\ell}(T) \cdot \|T\|.\end{aligned}$$

Subcase 1.2: $i \in [1, m]$, $u_1 \in \mathrm{supp}(S_i)$, $u_2 \cdot v \in \mathrm{supp}(T_i)$, $u_2 \cdot w \in \mathrm{supp}(T_i)$, $u = u_1 \cdot u_2$. Using the commutativity of H we have

$$\ell(h) = \ell(T_{i,u_2v}^{-1} \cdot T_{i,u_2w}) \leqslant \bar{\ell}(T)(|v| + |w|) \leqslant \max\{\bar{\ell}(S), \bar{\ell}(T)\} \cdot (|v| + |w|).$$

Let us now consider the

Case 2: $n = 1, s \geqslant 1$. For every $T \in \mathsf{DH}^0_{m,s}\langle\langle W\rangle\rangle$ we define $\bar{T} \in \mathsf{DH}^0_{m,1}\langle\langle W\rangle\rangle$ by

$$\forall j \in [1, m],\ \bar{T}_j = \sum_{k=1}^{s} T_{j,k}.$$

By the above case 1,

$$\bar{\ell}(S \cdot \bar{T}) \leqslant \max\{\bar{\ell}(S), \bar{\ell}(\bar{T})\} + 2 \cdot \ell_0(\bar{T}) + 2 \cdot \bar{\ell}(\bar{T}) \cdot \|\bar{T}\|. \tag{199}$$

But one can easily check the following relations:

$$\bar{\ell}(S \cdot T) = \bar{\ell}(S \cdot \bar{T}),\ \bar{\ell}(T) = \bar{\ell}(\bar{T}), \quad \ell_0(\bar{T}) = \ell_0(T), \quad \|\bar{T}\| \leqslant \|T\|.$$

By (199) and the above relations:

$$\begin{aligned}\bar{\ell}(S \cdot T) &\leqslant \bar{\ell}(S \cdot \bar{T}) \leqslant \max\{\bar{\ell}(S), \bar{\ell}(\bar{T})\} + 2 \cdot \ell_0(\bar{T}) + 2 \cdot \bar{\ell}(\bar{T}) \cdot \|\bar{T}\| \\ &\leqslant \max\{\bar{\ell}(S), \bar{\ell}(T)\} + 2 \cdot \ell_0(T) + 2 \cdot \bar{\ell}(T) \cdot \|T\|.\end{aligned}$$

Case 3: $n \geqslant 1, s \geqslant 1$. For every $i \in [1, n]$, by case 2, we have

$$\bar{\ell}(S_{i,*} \cdot T) \leqslant \max\{\bar{\ell}(S_{i,*}), \bar{\ell}(T)\} + 2 \cdot \ell_0(T) + 2 \cdot \bar{\ell}(T) \cdot \|T\|,$$

hence,

$$\bar{\ell}(S_{i,*} \cdot T) \leqslant \max\{\bar{\ell}(S), \bar{\ell}(T)\} + 2 \cdot \ell_0(T) + 2 \cdot \bar{\ell}(T) \cdot \|T\|.$$

It follows that, the maximum of the numbers $\bar{\ell}(S_{i,*} \cdot T)$ (for $1 \leqslant i \leqslant n$) is smaller than $\max\{\bar{\ell}(S), \bar{\ell}(T)\} + 2 \cdot \ell_0(T) + 2 \cdot \bar{\ell}(T) \cdot \|T\|$. $\square$

Let $S \in \mathsf{DH}^0_{1,m}\langle\langle W \rangle\rangle$. S is said *totally unitary* iff, for every $j \in [1,m]$, $u \in W^*$, $(S_j)_u \in \{0, 1_H\}$. S is said *special totally unitary* iff it is totally unitary and, for every $1 \leqslant i < j \leqslant m$, there exists $u \in W^*$, $v, v' \in W$, $S_{i,u\cdot v} = S_{j,u\cdot v'} = 1_H$.

Lemma 141. *Let $S, U \in \mathsf{DH}^0_{1,m}\langle\langle W \rangle\rangle$, $T \in \mathsf{H}^0_{m,s}\langle\langle W \rangle\rangle$, such that S is special totally unitary and U is totally unitary. Then $\bar{\ell}(U \cdot T) \leqslant \bar{\ell}(S \cdot T)$.*

Proof. Let us treat first the

Case 1: $s = 1$. Let us consider $u, v, w \in W^*$, $h \in H$ such that

$$h = (U \cdot T)^{-1}_{u\cdot v} \cdot (U \cdot T)_{u\cdot w}$$

and $gcp(v, w) = \varepsilon$, $(U \cdot T)_{u\cdot v} \neq 0$, $(U \cdot T)_{u\cdot w} \neq 0$.

As in the proof of Lemma 140, we consider two subcases.

Subcase 1.1: $i, j \in [1,m]$, $u \cdot v_1 \in \mathrm{supp}(U_i)$, $u \cdot w_1 \in \mathrm{supp}(U_j)$, $v = v_1 \cdot v_2$, $|v_1| \geqslant 1$, $w = w_1 \cdot w_2, |w_1| \geqslant 1$. Then, as U is totally unitary:

$$\ell(h) = \ell((U_{i,uv_1} T_{i,v_2})^{-1} \cdot (U_{j,uw_1} T_{j,w_2})) = \ell(T^{-1}_{i,v_2} \cdot T_{j,w_2}).$$

As S is special totally unitary, there exists $u' \in W^*$, $\alpha_i, \alpha_j \in W$,

$$\begin{aligned}\ell(T^{-1}_{i,v_2} \cdot T_{j,w_2}) &= \ell((S_{i,u'\alpha_i} T_{i,v_2})^{-1} \cdot (S_{j,u'\alpha_j} T_{j,w_2})) \\ &= \ell((S \cdot T)^{-1}_{i,u'\alpha_i\cdot v_2} \cdot (S \cdot T)_{j,u'\alpha_j\cdot w_2}) \leqslant \bar{\ell}(S \cdot T) \cdot (1 + |v_2| + 1 + |w_2|) \\ &\leqslant \bar{\ell}(S \cdot T) \cdot (|v| + |w|).\end{aligned}$$

Subcase 1.2: $i \in [1,m]$, $u_1 \in \mathrm{supp}(S_i)$, $u_2 \cdot v \in \mathrm{supp}(T_i)$, $u_2 \cdot w \in \mathrm{supp}(T_i)$, $u = u_1 \cdot u_2$. Then

$$\ell(h) = \ell(T^{-1}_{i,u_2 v} \cdot T_{i,u_2 w}).$$

Let us consider some $u' \in \mathrm{supp}(S_i)$ (such a word does exist, by definition of "totally unitary"):

$$\ell(h) = \ell((S_i T_i)^{-1}_{u' u_2 v} \cdot (S_i T_i)_{u' u_2 w}) \leqslant \bar{\ell}(S \cdot T)(|v| + |w|).$$

In both subcases we have checked that

$$\bar{\ell}(U \cdot T) \leqslant \bar{\ell}(S \cdot T).$$

Case 2: $s \geqslant 1$. As in the proof of Lemma 140, case 2, considering $\bar{T} \in \mathsf{DH}^0_{m,1}\langle\langle W \rangle\rangle$ we see that

$$\bar{\ell}(U \cdot T) = \bar{\ell}(U \cdot \bar{T}) \leqslant \bar{\ell}(S \cdot \bar{T}) = \bar{\ell}(S \cdot T). \qquad \square$$

11.4.3. $W = V$

Let $(W, \smile)$ be the structured alphabet $(V, \smile)$ associated with a given H-dpda $\mathcal{M}$. As the monoid $(\mathsf{B}, \cdot, 1)$ is embedded in $(H^0, \cdot, 1_H)$, all the particular series, vectors and matrices ($[p\omega q], [p\omega], [\omega], \varepsilon_j^m, \ldots$) introduced in Section 3.1.4 embed in the corresponding set of series, vectors, matrices with coefficients in H^0. The notions of Q-form, Q-λ-form, Q-product are defined analogously.

Lemma 142. *Let* $\lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle$.
(1) *there exists* $v \in V^*$ *such that* $\rho_\varepsilon(S) \sim S \bullet v$, $\rho_\varepsilon(S) = S \otimes e^{|v|}$ *and* $|v| \leqslant \|S\| - 1$.
(2) $\rho_\varepsilon(S) \equiv S$.

(See Lemma 15.)

Corollary 143. (1) $\forall \lambda \in \mathbb{N} - \{0\}, \forall S \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle, \|\rho_\varepsilon(S)\| \leqslant \|S\|$.
(2) $\forall \lambda \in \mathbb{N} - \{0\}$, $\forall S \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle$, $\rho_\varepsilon(S) \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle$.
(3) $\forall \lambda \in \mathbb{N} - \{0\}$, $\forall S \in \mathsf{DRH}^0_{1,\lambda}\langle\langle V \rangle\rangle$, $\rho_\varepsilon(S) \in \mathsf{DRH}^0_{1,\lambda}\langle\langle V \rangle\rangle$.

Lemma 144. *Let* $\lambda \in \mathbb{N} - \{0\}$, $S \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle, u \in X^+$. *One of the three following cases must occur*:
(1) $S \odot u \sim \emptyset^\lambda$,
(2) $S \odot u \sim \varepsilon_j^\lambda$ *for some* $j \in [1, \lambda]$,
(3) $\exists u_1, u_2 \in X^*$, $v_1 \in V^*$, $q \in Q$, $z \in Z$, $h_1 \in H, \Phi$ *Q-λ-form such that*

$$u = u_1 \cdot u_2, \quad \rho_\varepsilon(S) \odot u_1 = S \bullet (h_1, v_1) = [qz] * \Phi \quad \textit{and} \quad S \odot u = ([qz] \odot u_2) * \Phi.$$

(See Lemma 19.)

Corollary 145. (1) $\forall S \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle$, $u \in X^*$, $S \odot u \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle$.
(2) $\forall S \in \mathsf{DRH}^0_{1,\lambda}\langle\langle V \rangle\rangle$, $u \in X^*$, $S \odot u \in \mathsf{DRH}^0_{1,\lambda}\langle\langle V \rangle\rangle$.

Lemma 146. *Let* $S \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle$, $u \in X^*$.
(1) $\bar{\ell}(S \odot u) \leqslant \bar{\ell}(S)$.
(2) *if S is ε-free then*
$\ell_0(S \odot u) \leqslant \ell_0(S) + \|S\| \cdot \bar{k}(\mathcal{M}) \cdot |u| + K_0 \cdot \bar{k}(\mathcal{M}) \cdot |u|^2$.
(3) $\|S \odot u\| \leqslant \|S\| + K_0 \cdot |u|$.

Proof. Let us prove point (1). We consider the 3 cases distinguished in Lemma 144. If case 1 or 2 occurs, then clearly $\bar{\ell}(S \odot u) = 0 \leqslant \bar{\ell}(S)$.

Let us suppose that case 3 occurs. One can notice that $[qz], [qz] \odot u_2$ are special totally unitary vectors. By Lemma 141, we have

$$\bar{\ell}(S \odot u) = \bar{\ell}(([qz] \odot u_2) * \Phi) \leqslant \bar{\ell}([qz] * \Phi),$$

and by Lemma 136, point (1):

$$\bar{\ell}([qz] * \Phi) = \bar{\ell}(S \bullet v_1) \leqslant \bar{\ell}(S).$$

The two above inequalities prove point (1) of the lemma.

Let us prove point (2). Suppose that S is ε-free and $|u| = 1$, i.e. $u = x \in X$. Then

$$\ell_0(S \otimes x) \leqslant \ell_0(S) + \bar{k}(\mathscr{M}); \quad \|S \otimes x\| \leqslant \|S\| + K_0. \tag{200}$$

By Lemma 142, there exists $v \in V^*$, $|v| \leqslant \|S \otimes x\| - 1$ such that

$$\rho_\varepsilon(S \otimes x) = (S \otimes x) \otimes e^{|v|}.$$

Hence,

$$\begin{aligned}\ell_0(\rho_\varepsilon(S \otimes x)) &= \ell_0(S \otimes xe^{|v|}) \\ &\leqslant \ell_0(S) + |xe^{|v|}| \cdot \bar{k}(\mathscr{M}) \\ &\leqslant \ell_0(S) + \|S \otimes x\| \cdot \bar{k}(\mathscr{M}) \\ &\leqslant \ell_0(S) + (\|S\| + K_0) \cdot \bar{k}(\mathscr{M}).\end{aligned} \tag{201}$$

Suppose now that $|u| \geqslant 1$. Applying $|u|$ times inequality (201), we get

$$\begin{aligned}\ell_0(S \odot u) &\leqslant \ell_0(S) + \sum_{i=1}^{|u|} (\|S\| + i \cdot K_0) \cdot \bar{k}(\mathscr{M}) \\ &= \ell_0(S) + \|S\|\bar{k}(\mathscr{M}) \cdot |u| + K_0\bar{k}(\mathscr{M}) \cdot |u|(|u| + 1)/2 \\ &\leqslant \ell_0(S) + \|S\|\bar{k}(\mathscr{M}) \cdot |u| + K_0\bar{k}(\mathscr{M}) \cdot |u|^2.\end{aligned}$$

This proves point (2) of the lemma.

Let us prove point (3). Applying Corollary 143 point (1) and $|u|$ times the second inequality of (200), we obtain point (3). □

Remark 147. In fact, inequality (2) can be strengthened into the following:

$$\ell_0(S \odot u) \leqslant \ell_0(S) + (\|S\| + K_0 \cdot |u| + |u|) \cdot \bar{k}(\mathscr{M}).$$

But the proof would be more delicate while the result is not needed for our purposes.

Lemma 148. *Let $S \in \mathsf{DH}^0_{1,m}\langle\langle W \rangle\rangle$, $T \in \mathsf{H}^0_{m,s}\langle\langle W \rangle\rangle$, $u \in X^+$ and $U = S \cdot T$. Exactly one of the following cases is true*:

(1) $S \odot u \notin \{\emptyset^m\} \cup \{h \cdot \varepsilon^m_j \mid h \in H, 1 \leqslant j \leqslant m\}$
in this case $U \odot u = (S \odot u) \cdot T$.

(2) $\exists j_0, \exists u', u'',\ u = u' \cdot u'',\ h \in H,\ \rho_\varepsilon(S \odot u') = h \cdot \varepsilon^m_{j_0}$;
in this case $U \odot u = h \cdot \rho_\varepsilon(T_{j_0,*} \odot u'')$.

(3) $\forall j, \forall u' \leqslant u,\ S \odot u = \emptyset^m$ *and* $\rho_\varepsilon(S \odot u') \not\sim \varepsilon_j^m$;
in this case $U \odot u = \emptyset^s = (S \odot u) \cdot T$.

(See Lemma 22.)

Lemma 149. *For every* $S \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle$,
(1) $\rho_e(S) \in \mathsf{DH}^0_{1,\lambda}\langle\langle V \rangle\rangle$,
(2) $\bar{\ell}(\rho_e(S)) \geqslant \bar{\ell}(S)$.
(3) $\|\rho_e(S)\| \leqslant \|S\|$,
(4) $S \equiv \rho_e(S)$.

Proof. Points (1),(3),(4) can be proved as in Lemma 23. Let us prove point (2). Let $u, v \in V^*$, $S_u \neq 0$. Let us note $T = \rho_e(S)$,

$$\begin{aligned}\ell(S_u^{-1} \cdot S_v) &= \ell(T_{\rho_e(u)}^{-1} \cdot T_{\rho_e(v)}) \leqslant \bar{\ell}(T) \cdot \ell(\rho_e(u)^{-1} \cdot \rho_e(v)) \\ &\leqslant \bar{\ell}(T) \cdot \ell(u^{-1} \cdot v) = \bar{\ell}(\rho_e(S)) \cdot \ell(u^{-1} \cdot v)\end{aligned}$$

(we use the fact that $w \mapsto \rho_e(w)$ is contracting). □

11.4.4. Equivalence on row-vectors

Lemma 150. *Let* $\lambda \in \mathbb{N} - \{0\}$, S, $S' \in \mathsf{DB}_{1,\lambda}\langle\langle V \rangle\rangle$. *Then* $S \equiv S'$ *if and only if*, $\forall h \in H$, $\forall u \in X^*$, $\forall j \in [1, \lambda]$, $\rho_\varepsilon(S \odot (h, u)) = \varepsilon_j^\lambda \Leftrightarrow \rho_\varepsilon(S' \odot (h, u)) = \varepsilon_j^\lambda$.

(See Corollary 26.)

Definition 151. For every $\lambda \in \mathbb{N} - \{0\}$, S, $S' \in \mathsf{B}_{1,\lambda}\langle\langle V \rangle\rangle$ we define $\mathrm{Div}(S, S') = \inf\{|u|, u \in X^*, \exists j \in [1, \lambda], \exists h \in H\ (\rho_\varepsilon(S \odot (h, u)) = \varepsilon_j^\lambda) \Leftrightarrow (\rho_\varepsilon(S' \odot (h, u)) \neq \varepsilon_j^\lambda)\}$.

(See the alternative definition (38) in the boolean case.)

11.5. Operations on row-vectors

Given $A, B \in \mathsf{H}^0_{1,m}\langle\langle W \rangle\rangle$ and $1 \leqslant j_0 \leqslant m$ we define the vector $C = A \,\square_{j_0}\, B$ as follows: if $A = (a_1, \ldots, a_j, \ldots, a_m)$, $B = (b_1, \ldots, b_j, \ldots, b_m)$ then $C = (c_1, \ldots, c_j, \ldots, c_m)$, where

$$c_j = a_j + a_{j_0} \cdot b_j \text{ if } j \neq j_0, \quad c_j = \emptyset \text{ if } j = j_0.$$

Let us notice that $A \,\square_{j_0}\, B \in \mathsf{B}\langle\langle H \rangle\rangle_{1,m}\langle\langle W \rangle\rangle$ but need not belong to $\mathsf{H}^0_{1,m}\langle\langle W \rangle\rangle$ in general.

Lemma 152. *Let* $A, B \in \mathsf{H}^0_{1,m}\langle\langle W \rangle\rangle$ *and* $1 \leqslant j_0 \leqslant m$:
(1) *if* A, B *are left-deterministic, then* $A \,\square_{j_0}\, B$ *is left-deterministic.*
(2) *if* A, B *are deterministic, then* $A \,\square_{j_0}\, B$ *is deterministic.*
(3) *if* A, B *are deterministic, then* $\|A \,\square_{j_0}\, B\| \leqslant \|A\| + \|B\|$.

(See Lemma 28.)

Lemma 153. *Let* $A \in \mathsf{DH}^0_{1,m}\langle\langle W \rangle\rangle$ *and* $1 \leqslant j_0 \leqslant m$. *Then* $\square^*_{j_0}(A) \in \mathsf{DH}^0_{1,m}\langle\langle W \rangle\rangle$ *and* $\|\square^*_{j_0}(A)\| \leqslant \|A\|$.

(See Lemma 29.)

11.6. Deterministic spaces

The notions of d-space, linear combination, generating set are defined as in Section 3.2 but where B is replaced by H^0 everywhere.

Lemma 154. *Let* $S_1, \ldots, S_j, \ldots, S_m \in \mathsf{DRH}^0\langle\langle V \rangle\rangle$. *The following are equivalent*:
(1) $\exists \vec{\alpha}, \vec{\beta} \in \mathsf{DRH}^0_{1,m}\langle\langle V \rangle\rangle, \vec{\alpha} \not\equiv \vec{\beta}$, *such that* $\sum_{1 \leqslant j \leqslant m} \alpha_j \cdot S_j \equiv \sum_{1 \leqslant j \leqslant m} \beta_j \cdot S_j$,
(2) $\exists j_0 \in [1, m], \exists \vec{\gamma} \in \mathsf{DRH}^0_{1,m}\langle\langle V \rangle\rangle, \forall h \in H, \vec{\gamma} \not\equiv h \cdot \varepsilon^m_{j_0}$, *such that* $S_{j_0} \equiv \sum_{1 \leqslant j \leqslant m} \gamma_j \cdot S_j$,
(3) $\exists j_0 \in [1, m], \exists \vec{\gamma}' \in \mathsf{DRH}^0_{1,m}\langle\langle V \rangle\rangle, \gamma'_{j_0} \equiv \emptyset$, *such that* $S_{j_0} \equiv \sum_{1 \leqslant j \leqslant m} \gamma'_j \cdot S_j$,
(4) $\exists j_0 \in [1, m]$, *such that* $\mathsf{V}((S_j)_{1 \leqslant j \leqslant m}) \equiv \mathsf{V}((S_j)_{1 \leqslant j \leqslant m, j \neq j_0})$.

(See Lemma 30.)

11.7. Height, defect and linearity

Here also, the definitions of height and defect of a deterministic rational series (or Q-series) are those of Section 3.3 where B is replaced by H^0.

Lemma 155. *Let* $S \in \mathsf{DRH}^0\langle\langle V \rangle\rangle$, $x \in X$, $d, d' \in \mathbb{N}$.
(1) $\mathrm{rd}(S \odot x) \leqslant \mathrm{rd}(S)$
(2) S *is* (d, d')*-linear* $\Rightarrow$ $S \odot x$ *is* $(d+1, d')$*-linear*

(See Lemma 32.)

Lemma 156. *Let* $B, A \in Z$, $\Phi \in \mathsf{DH}^0_{Q,1}\langle\langle V \rangle\rangle$. *If* $\|[A] * \Phi\| > \|\Phi\|$ *then*, $\forall q \in Q, [[qBA] * \Phi]_\sim \notin \mathsf{Q}_r([A] * \Phi)/\sim$.

The proof is analogous to the proof of Lemma 33. We use Lemma 135 to conclude in case 1 that $[A] * \Phi \sim \Phi$.

Lemma 157. *Let* $\omega \in Z^+$, $A', A \in Z$, $p \in Q$, $\Phi \in \mathsf{DH}^0_{Q,1}\langle\langle V \rangle\rangle$. *If* $\|[A] * \Phi\| > \|\Phi\|$, *then*
(1) $\|[\omega A] * \Phi\| = |Q| \cdot |\omega| + \|[A] * \Phi\|$
(2) $\|[pA'\omega A] * \Phi\| = 1 + |Q| \cdot |\omega| + \|[A] * \Phi\|$.

(See Lemma 34.)

Lemma 158. *Let* $U = [p\omega] * \Phi, U' = U \odot u$ *where* $p \in Q$, $\omega \in Z^*$, $|\omega| \geqslant 1, \Phi$ *is a* Q*- form*, $|\Phi| \geqslant 1$, $u \in X^*$, $|u| \leqslant k$. *Let us suppose that* $\|U'\| \geqslant 1 + k|Q| + \|\Phi\|$. *Then* $U' = ([p\omega] \odot u) * \Phi$ *where* $[p\omega] \odot u = [q\omega']$ *for some* $q \in Q$, $|\omega'| \geqslant k$.

(See Lemma 37.)

Lemma 159. *Let* $D \geqslant 0$. *Let* $\Phi = (\Phi_q)_{q\in Q}$ *be a Q-form and let* $S \in \mathsf{V}((\Phi_q)_{q\in Q})$ *such that*
(1) $\|\Phi\| \geqslant D + |Q|, |\Phi| \geqslant 2$,
(2) $\mathrm{rd}(S) \leqslant D$.
Then, $\exists\, \omega \in Z^*, \exists p \in Q,\ S = [p\omega] * \Phi$.

The proof of Lemma 38 can be adapted in the following way. One proves first that

$$\exists q \in Q,\ \exists u \in \mathrm{supp}(\alpha_q),\ \exists u', u'' \in V^*,\ u = u' \cdot u'' \quad \text{and} \quad S \bullet u' \sim \Psi_q. \tag{202}$$

is impossible. Eq. (48) is then established in the same way. All the remaining of the proof is still valid (provided "$u \in \alpha_q$" is replaced by "$u \in \mathrm{supp}(\alpha_q)$", everywhere).

11.7.1. Derivations

The notions of derivations and sub-derivations are adapted in a straightforward way to the case of series in $\mathsf{DRH}^0\langle\langle V\rangle\rangle$.

For every $u \in X^*$ we define the binary relation $\uparrow(u)$ over $\mathsf{DH}^0\langle\langle V\rangle\rangle$ by for every $S, S' \in \mathsf{DH}^0\langle\langle V\rangle\rangle$, $S \uparrow (u) S' \Leftrightarrow \exists z \in Z,\ \omega \in Z^+,\ p, q \in Q,\ h \in H,\ \Psi \in \mathsf{DH}^0_{Q,1}\langle\langle V\rangle\rangle$ such that

$$S = [pz] * \Psi, \quad [pz] \odot u = h \cdot [q\omega], \quad S' = h \cdot [q\omega] * \Psi.$$

A derivation $S_0, S_1, \ldots, S_n$ is said to be *stacking* iff it is the derivation associated to a pair (S, u) such that $S = S_0$ and $S_0 \uparrow (u) S_n$.

Definition 160. A vector $S \in \mathsf{DRH}^0_{1,\lambda}\langle\langle V\rangle\rangle$ is said *loop-free* if and only if for every $v \in V^+$, $S \bullet v \not\sim S$.

Lemma 161. *Let* $S \in \mathsf{DRH}^0_{1,\lambda}\langle\langle V\rangle\rangle$, $u \in X^*$, *such that* $\|S \odot u\| > \|S\|$. *Then* $S \odot u$ *is loop-free.*

(See Lemma 41.)

Lemma 162. *Let* $S \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$, $w \in X^*$, *such that*
(1) *S is ε-free and loop-free,*
(2) $\forall v \preccurlyeq w, \|S \odot v\| \geqslant \|S\|$. *Then the derivation* $S \xrightarrow{w} S \odot w$ *is stacking.*

(See Lemma 42.)

Lemma 163. *Let* $S, S' \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$, $w \in X^*$, $k \in \mathbb{N}$, *such that* $S \odot w = S'$ *and* $\|S'\| \geqslant \|S\| + k \cdot K_0 + 1$. *Then the derivation* $S \xrightarrow{w} S'$ *contains some stacking sub-derivation of length k.*

(See Lemma 43.)

Lemma 164. *Let* $S, S' \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$, $w \in X^*$, $k, d, d' \in \mathbb{N}$, *such that* S *is* ε*-free*, (d, d')*-linear and*
(1) *the derivation* $S \xrightarrow{w} S'$ *contains no stacking sub-derivation of length* k.
(2) $|w| \geqslant d \cdot k$.
Then S' *is* $(0, d')$*-linear.*

(See Lemma 44.)

11.8. Formal system $\mathscr{H}_0$

We define here a particular deduction system $\mathscr{H}_0$ "Taylored for the equivalence problem for H-dpda's".

Given a fixed H-dpda $\mathscr{M}$ over the terminal alphabet X, we consider the variable alphabet V associated to $\mathscr{M}$ (see Section 11.1.4) and the set $\mathsf{DRH}^0\langle\langle V\rangle\rangle$ (the set of Deterministic Rational series over V^*, with coefficients in H^0). The set of assertions is defined by

$$\mathscr{A} = \mathbb{N} \times \mathsf{DRH}^0\langle\langle V\rangle\rangle \times \mathsf{DRH}^0\langle\langle V\rangle\rangle,$$

i.e. an assertion is here a *weighted equation* over $\mathsf{DRH}^0\langle\langle V\rangle\rangle$.

The "cost-function" $J : \mathscr{A} \to \mathbb{N} \cup \{\infty\}$ is defined by

$$J(n, S, S') = n + 2 \cdot \mathrm{Div}(S, S').$$

(We recall $\mathrm{Div}(S, S')$ is introduced in Definition 151.) Here also

$$\chi(n, S, S') = 1 \Leftrightarrow S \equiv S'.$$

We define a binary relation $||\!\!-\!\!- \subset \mathscr{P}_f(\mathscr{A}) \times \mathscr{A}$, the *elementary deduction relation*, as the set of all the pairs having one of the following forms:

(H0)

$$\{(p, S, T)\} \;||\!\!-\!\!-\; (p + 1, S, T)$$

for $p \in \mathbb{N}$, $S, T \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$,

(H1)

$$\{(p, S, T)\} \;||\!\!-\!\!-\; (p, T, S)$$

for $p \in \mathbb{N}$, $S, T \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$,

(H2)

$$\{(p, S, S'), (p, S', S'')\} \;||\!\!-\!\!-\; (p, S, S'')$$

for $p \in \mathbb{N}$, $S, S', S'' \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$,

(H3)

$$\emptyset \;||\!\!-\!\!-\; (0, S, S)$$

for $S \in \mathsf{DRH}^0\langle\langle V\rangle\rangle$,

(H′3)

$$\emptyset \mid\!\vdash\!- (0, S, T)$$

for $S \in \mathsf{DRH}^0\langle\langle V\rangle\rangle,\ T \in \{\emptyset, \varepsilon\},\ S \equiv T,$

(H4)

$$\{(p+1, S \odot x, T \odot x) \mid x \in X\} \mid\!\vdash\!- (p, S, T)$$

for $p \in \mathbb{N},\ S,\ T \in \mathsf{DRH}^0\langle\langle V\rangle\rangle,\ (\forall h \in H,\ S \not\equiv h, \wedge T \not\equiv h),$

(H5)

$$\{(p, S, S')\} \mid\!\vdash\!- (p+2, S \odot x, S' \odot x)$$

for $p \in \mathbb{N},\ S, T \in \mathsf{DRH}^0\langle\langle V\rangle\rangle, x \in X,$

(H6)

$$\{(p, S \cdot T' + S', T')\} \mid\!\vdash\!- (p, S^* \cdot S', T')$$

for $p \in \mathbb{N},\ (S, S') \in \mathsf{DRH}^0_{1,2}\langle\langle V\rangle\rangle,\ T' \in \mathsf{DRH}^0\langle\langle V\rangle\rangle,\ (\forall h \in H,\ S \not\equiv h),$

(H7)

$$\{(p, S, S'), (p, T, T')\} \mid\!\vdash\!- (p, S+T, S'+T')$$

for $p \in \mathbb{N},\ (S, T), (S', T') \in \mathsf{DRH}^0_{1,2}\langle\langle V\rangle\rangle,$

(H8)

$$\{(p, S, S')\} \mid\!\vdash\!- (p, S \cdot T, S' \cdot T)$$

for $p \in \mathbb{N},\ S, S', T \in \mathsf{DRH}^0\langle\langle V\rangle\rangle,$

(H9)

$$\{(p, T, T')\} \mid\!\vdash\!- (p, S \cdot T, S \cdot T')$$

for $p \in \mathbb{N},\ S, T, T' \in \mathsf{DRH}^0\langle\langle V\rangle\rangle,$

(H10)

$$\emptyset \mid\!\vdash\!- (0, S, \rho_\varepsilon(S))$$

for $S \in \mathsf{DRH}^0\langle\langle V\rangle\rangle,$

(H11)

$$\emptyset \mid\!\vdash\!- (0, S, \rho_e(S))$$

for $S \in \mathsf{DRH}^0\langle\langle V\rangle\rangle.$

Though we did not prove this result formally, it should be clear that the operations $+, \cdot$ and $\square_1^*$ over $\mathsf{DRH}^0_m\langle\langle V\rangle\rangle$ correspond to some computable functions on deterministic finite m-H-automata and that the equality in $\mathsf{DRH}^0_m\langle\langle V\rangle\rangle$ corresponds to some computable predicate on pairs of deterministic finite m-H-automata (i.e. the equivalence problem for deterministic finite m-H-automata is decidable). Hence, modulo an encoding of $\cup_{m\geqslant 1}\mathsf{DRH}^0_m\langle\langle V\rangle\rangle$ into integers, based on deterministic finite m-H-automata, the above set of rules is recursively enumerable. Let us define $\models\!\!-$ by: for every $P\in\mathscr{P}_f(\mathscr{A})$, $A\in\mathscr{A}$,

$$P \Vdash\!\!- A \Leftrightarrow P \overset{\langle * \rangle}{\Vdash\!\!-} \circ \Vdash\!\!-^{[1]}_{0,3,4,10,11} \circ \overset{\langle * \rangle}{\Vdash\!\!-} \{A\}.$$

where $\Vdash\!\!-_{0,3,4,10,11}$ is the relation defined by (H0), (H3), (H′3), (H4), (H10), (H11) only. We let

$$\mathscr{H}_0 = \langle \mathscr{A}, J, \Vdash\!\!- \rangle.$$

Lemma 165. *$\mathscr{H}_0$ is a deduction system.*

11.9. Triangulations

Let $S_1, S_2, \ldots, S_d$ be a family of deterministic rational series over the structured alphabet V, with coefficients in H^0 (i.e. $S_i\in\mathsf{DRH}^0\langle\langle V\rangle\rangle$). We recall V is the alphabet associated with some dpda $\mathscr{M}$ as defined in Section 11.1.4. Let us consider a sequence $\mathscr{S}$ of n "weighted" linear equations:

$$(\mathscr{E}_i):\quad p_i, \sum_{j=1}^{d}\alpha_{i,j}S_j,\ \sum_{j=1}^{d}\beta_{i,j}S_j, \tag{203}$$

where $p_i\in\mathbb{N}-\{0\}$, and $A=(\alpha_{i,j})$, $B=(\beta_{i,j})$ are deterministic rational matrices of dimension (n,d), with indices $m\leqslant i\leqslant m+n-1, 1\leqslant j\leqslant d$.

For any weighted equation, $\mathscr{E}=(p,S,S')$, we recall the "cost" of this equation is $J(\mathscr{E})=p+2\mathrm{Div}(S,S')$.

Let us adapt the construction of the system INV($\mathscr{S}$) to the case of series with coefficients in H^0. We assume a total ordering $\leqslant$, is given on X and we denote also by $\leqslant$ its short-lex extension to X^*. We denote by $\leqslant_H$ some well-ordering on H.

11.9.1. Restricted systems

We assume here that

$$\forall j\in[1,d],\ S_j\not\equiv\emptyset, \tag{204}$$

$$\forall i\in[m,m+n-1],\ \forall j\in[1,d],\ \alpha_{i,j},\beta_{i,j} \text{ are } \varepsilon\text{-free} \tag{205}$$

and

$$\forall i\in[m,m+n-1],\ \alpha_{i,*} \text{ is unitary.} \tag{206}$$

(We recall it means that $\ell_0(\alpha_{i,*})=0$.) A system $\mathscr{S}$ fulfilling the three hypotheses (204)–(206) will be called a *restricted* system of weighted linear equations.

Let us define INV($\mathscr{S}$), W($\mathscr{S}$) $\in \mathbb{N} \cup \{\perp\}$, D($\mathscr{S}$) $\in \mathbb{N}$, by induction on n. W($\mathscr{S}$) is the *weight* of $\mathscr{S}$. D($\mathscr{S}$) is the *weak codimension* of $\mathscr{S}$.

Case 1: $\alpha_{m,*} \equiv \beta_{m,*}$

$$\mathrm{INV}(\mathscr{S}) = ((\mathrm{W}(\mathscr{S}), \alpha_{m,j}, \beta_{m,j}))_{1 \leqslant j \leqslant d}, \quad \mathrm{W}(\mathscr{S}) = p_m - 1, \quad \mathrm{D}(\mathscr{S}) = 0.$$

Case 2: $\alpha_{m,*} \not\equiv \beta_{m,*}$, $n \geqslant 2$, $p_{m+1} - p_m \geqslant 2 \cdot \mathrm{Div}(\alpha_{m,*}, \beta_{m,*}) + 1$. Let us consider

$$(h,u) = \min\{(k,v) \in H \times X^* \mid \exists j \in [1,d], (\alpha_{m,*} \odot (k,v) = \varepsilon_j^d) \Leftrightarrow (\beta_{m,*} \odot (k,v) \neq \varepsilon_j^d)\}. \tag{207}$$

(Lemma 150 and the ε-freeness assumption (205) ensure the existence of such a pair (h,u)). Let $j_0 \in [1,n]$ such that $(\alpha_{m,*} \odot u = \varepsilon_{j_0}^d) \Leftrightarrow (\beta_{m,*} \odot u \neq \varepsilon_{j_0}^d.)$

Subcase 1: $\alpha_{m,j_0} \odot (h,u) = \varepsilon$, $\beta_{m,j_0} \odot (h,u) \neq \varepsilon$. Let us consider the equation

$$(\mathscr{E}'_m): \quad p_m + 2 \cdot |u|, S_{j_0}, \sum_{\substack{j=1 \\ j \neq j_0}}^{d} (\beta_{m,j_0} \odot (h,u))^* (\beta_{m,j} \odot (h,u)) S_j$$

and define a new system of weighted equations $\mathscr{S}' = (\mathscr{E}'_i)_{m+1 \leqslant i \leqslant m+n-1}$ by

$$(\mathscr{E}'_i): \quad p_i, \sum_{j \neq j_0} [\alpha_{i,j} + \alpha_{i,j_0} (\beta_{m,j_0} \odot (h,u))^* (\beta_{m,j} \odot (h,u))] \cdot S_j,$$
$$\sum_{j \neq j_0} [\beta_{i,j} + \beta_{i,j_0} (\beta_{m,j_0} \odot (h,u))^* (\beta_{m,j} \odot (h,u))] \cdot .$$

(The above equation is seen as an equation between two linear combinations of the S_i's, $1 \leqslant i \leqslant d$, where the j_0th coefficient is $\emptyset$ on both sides.) We then define

$$\mathrm{INV}(\mathscr{S}) = \mathrm{INV}(\mathscr{S}'), \mathrm{W}(\mathscr{S}) = \mathrm{W}(\mathscr{S}'), \quad \mathrm{D}(\mathscr{S}) = \mathrm{D}(\mathscr{S}') + 1.$$

Subcase 2: $\alpha_{m,j_0} \odot (h,u) \neq \varepsilon$, $\beta_{m,j_0} \odot (h,u) = \varepsilon$. (analogous to subcase 1).

Case 3: $\alpha_{m,*} \not\equiv \beta_{m,*}$, $n = 1$. We then define

$$\mathrm{INV}(\mathscr{S}) = \perp, \quad \mathrm{W}(\mathscr{S}) = \perp, \quad \mathrm{D}(\mathscr{S}) = 0,$$

where $\perp$ is a special symbol which can be understood as meaning "undefined".

Case 4: $\alpha_{m,*} \not\equiv \beta_{m,*}$, $n \geqslant 2$, $p_{m+1} - p_m \leqslant 2 \cdot \mathrm{Div}(\alpha_{m,*}, \beta_{m,*})$. We then define

$$\mathrm{INV}(\mathscr{S}) = \perp, \quad \mathrm{W}(\mathscr{S}) = \perp, \quad \mathrm{D}(\mathscr{S}) = 0.$$

Let us consider the function F defined by

$$F(d,n) = \max\{\mathrm{Div}(A,B) \mid A,B \in \mathsf{DRB}_{1,d}\langle\langle V \rangle\rangle, \|A\| \leqslant n, \|B\| \leqslant n, A \not\equiv B\}.$$

For every integer parameters $K_0, K_1, K_2, \bar{K}_3, \bar{K}_4, K_3^0, K_4^0, K_3, K_4 \in \mathbb{N} - \{0\}$, we define integer sequences $(\delta_i, \ell_i, L_i, \bar{s}_i, s_i^0, s_i, S_i, \Sigma_i)_{m \leqslant i \leqslant m+n-1}$ by

$$\delta_m = 0, \quad \ell_m = 0, \quad L_m = K_2, \tag{208}$$

$$\bar{s}_m = \bar{K}_3 \cdot K_2 + \bar{K}_4, \quad s_m^0 = K_3^0 \cdot K_2 + K_4^0, \tag{209}$$

$$s_m = K_3 \cdot K_2 + K_4, \quad S_m = 0, \quad \Sigma_m = 0, \tag{210}$$

$$\begin{aligned}
\delta_{i+1} &= 2 \cdot F(d, s_i + \Sigma_i) + 1,\\
\ell_{i+1} &= 2 \cdot \delta_{i+1} + 3,\\
L_{i+1} &= K_1 \cdot (L_i + \ell_{i+1}) + K_2,\\
\bar{s}_{i+1} &= \bar{K}_3 \cdot L_{i+1} + \bar{K}_4,\\
s_{i+1}^0 &= K_3^0 \cdot L_{i+1} + K_4^0,\\
s_{i+1} &= K_3 \cdot L_{i+1} + K_4,\\
S_{i+1} &= s_i + \Sigma_i + K_0 F(d, s_i + \Sigma_i),\\
\Sigma_{i+1} &= \Sigma_i + S_{i+1}
\end{aligned} \tag{211}$$

for $m \leqslant i \leqslant m + n - 2$.

These sequences are intended to have the following meanings when $K_0, K_1, K_2, \bar{K}_3, \bar{K}_4, K_3^0, K_4^0, K_3, K_4$ are chosen to be the constants defined in Section 11.10 and Eqs. ($\mathscr{E}_i$) are labelling nodes of a N-stacking sequence (see Section 11.12.1):

$\delta_{i+1} \leqslant$ increase of weight between $\mathscr{E}_i, \mathscr{E}_{i+1}$,
$\ell_{i+1} \geqslant$ increase of depth between $\mathscr{E}_i, \mathscr{E}_{i+1}$,
$L_{i+1} \geqslant$ increase of depth between $\mathscr{E}_m, \mathscr{E}_{i+1}$,
$\bar{s}_{i+1} \geqslant$ length of the coefficients of $\mathscr{E}_{i+1}$,
$s_{i+1}^0 \geqslant$ initial length of the coefficients of $\mathscr{E}_{i+1}$,
$s_{i+1} \geqslant$ norm of the coefficients of $\mathscr{E}_{i+1}$,
$S_{i+1} \geqslant$ norm of the coefficients of $\mathscr{E}_{i+1}^{(i+1-m)}$ (these systems were introduced in the proof of Lemma 59),
$\Sigma_{i+1} \geqslant$ increase of the norm of the coefficients between $\mathscr{E}_k^{(i-m)}, \mathscr{E}_k^{(i+1-m)}$(for $k \geqslant$
$i+1$).

For every linear equation $\mathscr{E} = (p, \sum_{j=1}^{d} \alpha_j S_j, \sum_{j=1}^{d} \beta_j S_j)$, we define

$$\rho_1(\mathscr{E}) = \left(p, h_0^{-1} \cdot \sum_{j=1}^{d} \alpha_j S_j, h_0^{-1} \cdot \sum_{j=1}^{d} \beta_j S_j \right),$$

where h_0 is the coefficient of the smallest word u_0 of $\cup_{j=1}^{d} \mathrm{supp}(\alpha_j)$

$$\bar{\ell}(\mathscr{E}) = \max\{\bar{\ell}(\alpha_1, \ldots, \alpha_d), \bar{\ell}(\beta_1, \ldots, \beta_d)\},$$

$$\ell^0(\mathscr{E}) = \ell_0(h_0^{-1} \cdot (\beta_1, \ldots, \beta_d)),$$

$$|||\mathscr{E}||| = \max\{\|(\alpha_1, \ldots, \alpha_d)\|, \|(\beta_1, \ldots, \beta_d)\|\}.$$

(Notice that $\rho_1(\mathscr{E})$ is left-unitary and that for every system $\mathscr{S}$, $\mathrm{INV}(\mathscr{S}) = \mathrm{INV}(\rho_1(\mathscr{S}))$.) We define the constant

$$\bar{K}_2 = \max\{\bar{\ell}(\rho_e(\mathrm{INV}(\mathscr{S}))) \mid \mathscr{S} \text{ system of } d_0 \text{ equations such that, } \bar{\ell}(\mathscr{S}) \leqslant \bar{s}_{d_0}, \ell^0(\mathscr{S}) \leqslant s^0_{d_0}, |||\mathscr{S}||| \leqslant s_{d_0}\}. \quad (212)$$

Let us check that the integer $\bar{K}_2$ is well defined.

For given integers $M^0, \bar{M}, M$, the set

$$\{S \in \mathsf{DRH}^0_{1,d}\langle\langle V\rangle\rangle \mid \ell_0(S) \leqslant M^0, \bar{\ell}(S) \leqslant \bar{M}, \|S\| \leqslant M\}$$

is finite (by Lemma 127). It follows that the set of pairs

$$\{(\alpha, \beta) \in \mathsf{DRH}^0_{1,d}\langle\langle V\rangle\rangle \mid \ell_0(\alpha) = 0, \ell_0(\beta) \leqslant M^0, \bar{\ell}(\alpha) \leqslant \bar{M}, \bar{\ell}(\beta) \leqslant \bar{M}, \|\alpha\| \leqslant M, \|\beta\| \leqslant M\}$$

is finite. Hence, the set of left-unitary equations $\mathscr{E} = (p, \sum_{j=1}^d \alpha_j S_j, \sum_{j=1}^d \beta_j S_j)$ such that

$$\ell^0(\mathscr{E}) \leqslant M^0, \quad \bar{\ell}(\mathscr{E}) \leqslant \bar{M}, \quad |||\mathscr{E}||| \leqslant M$$

is finite. But $\mathrm{INV}(\mathscr{S}) = \mathrm{INV}(\rho_1(\mathscr{S}))$ and the map $\mathscr{E} \mapsto \rho_1(\mathscr{E})$ preserves the three maps $\ell^0, \bar{\ell}, |||*|||$.We can conclude that the set in the right-hand side of (212) is finite. This shows that $\bar{K}_2$ is a well-defined integer.

Lemma 166. *Let $\mathscr{S} = (\mathscr{E}_i)_{m \leqslant i \leqslant m+d-1}$ be a restricted system of d linear equations such that $J(\mathscr{E}_i) = \infty$ (for every i) and*
(1) $\forall i \in [m, m+d-1]$, $\bar{\ell}(\mathscr{E}_i) \leqslant \bar{s}_i$,
(2) $\forall i \in [m, m+d-1]$, $\ell^0(\mathscr{E}_i) \leqslant s^0_i$,
(3) $\forall i \in [m, m+d-1]$, $|||\mathscr{E}_i||| \leqslant s_i$,
(4) $\forall i \in [m, m+d-2], \mathrm{W}(\mathscr{E}_{i+1}) - \mathrm{W}(\mathscr{E}_i) \geqslant \delta_{i+1}$.
Then $\mathrm{INV}(\mathscr{S}) \neq \perp, \mathrm{D}(\mathscr{S}) \leqslant d-1$, *and for every* $\mathscr{E} \in \mathrm{INV}(\mathscr{S})$,
(5) $\bar{\ell}(\mathscr{E}) \leqslant \bar{\ell}(\rho_e(\mathscr{E})) \leqslant \bar{K}_2$,
(6) $|||\mathscr{E}||| \leqslant \Sigma_{m+\mathrm{D}(\mathscr{S})} + s_{m+\mathrm{D}(\mathscr{S})}$.

Sketch of proof. The proof of Lemma 59 can be adapted in the following way. The word u_i introduced in (78) must be now defined by

$$(h_i, u_i) = \min\{(k, v) \in H \times X^* \mid \exists j \in [1, d], (\alpha^{(i-m)}_{i,*} \odot (h, v) = \varepsilon^d_j) \Leftrightarrow (\beta^{(i-m)}_{i,*} \odot (h, v) \neq \varepsilon^d_j)\}. \quad (213)$$

It follows that, for example

$$\alpha^{(i-m)}_{i,*} \odot (h_i, u_i) = \varepsilon^d_j \quad (214)$$

while

$$\beta^{(i-m)}_{i,*} \odot (h_i, u_i) \neq \varepsilon^d_j. \quad (215)$$

But, if $\beta^{(i-m)}_{i,*} \odot (h_i, u_i) = h \cdot \varepsilon^d_j$, for some $h \in H$, then $S_j \equiv h \cdot S_j$ which, by hypothesis (204), implies that $h = 1_H$. Hence, hypotheses (214) and (215) imply that

$$\mathrm{supp}(\alpha^{(i-m)}_{i,*}) \odot u_i = \varepsilon^d_j \quad \text{and} \quad \mathrm{supp}(\beta^{(i-m)}_{i,*}) \odot u_i \neq \varepsilon^d_j$$

and finally

$$|u_i| \leqslant F(d, \|\|\mathrm{supp}(\mathscr{E}^{(i-m)}_i)\|\|) \leqslant F(d, \|\|\mathscr{E}^{(i-m)}_i\|\|) \leqslant F(d, s_i + \Sigma_i).$$

(the case where α, β are exchanged in (214), (215) leads to the same upper-bound on $|u_i|$).
As well, the word u introduced in (80) is now defined by

$$(h,u) = \min\{(k,v) \in H \times X^* \mid \exists j \in [1,d], (\alpha^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S}),*} \odot (h,v) = \varepsilon^d_j) \\ \Leftrightarrow (\beta^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S}),*} \odot (h,v) \neq \varepsilon^d_j)\} \qquad (216)$$

and by the same trick as above about the supports we obtain

$$|u| \leqslant F(d, \|\|\mathscr{E}^{(\mathrm{D}(\mathscr{S}))}_{m+\mathrm{D}(\mathscr{S})}\|\|) \leqslant F(d, s_{m+\mathrm{D}(\mathscr{S})} + \Sigma_{m+\mathrm{D}(\mathscr{S})}).$$

The remaining of the proof is unchanged. □

11.9.2. General systems

We consider now the general case where assumptions (204)–(206) are removed. We only suppose that

$$\exists d_1 \in [1,d], \quad S_{d_1} \not\equiv \emptyset. \qquad (217)$$

Under the same assumption (82) we construct similarly a system $\hat{\mathscr{S}}$ of n linear equations:

$$(\hat{\mathscr{E}}_i): \quad \rho_1\left(p_i, \sum_{j=1}^{\hat{d}} \rho_\varepsilon(\alpha_{i,j}) \cdot S_j, \quad \sum_{j=1}^{\hat{d}} \rho_\varepsilon(\beta_{i,j}) \cdot S_j\right)$$

where $m \leqslant i \leqslant m+n-1$.

We then define

$$\mathrm{INV}(\mathscr{S}) = \mathrm{INV}(\hat{\mathscr{S}}), \quad \mathrm{W}(\mathscr{S}) = W(\hat{\mathscr{S}}), \quad \mathrm{D}(\mathscr{S}) = \mathrm{D}(\hat{\mathscr{S}}).$$

Lemma 167. *Let $\mathscr{S} = (\mathscr{E}_i)_{m \leqslant i \leqslant m+d-1}$ be a system of d linear equations such that $J(\mathscr{E}_i) = \infty$ (for every i) and*
(0) $\exists j \in [1,d],\ S_j \not\equiv \emptyset$,
(1) $\forall i \in [m, m+d-1],\ \bar{\ell}(\mathscr{E}_i) \leqslant \bar{s}_i$,
(2) $\forall i \in [m, m+d-1],\ \ell^0(\mathscr{E}_i) \leqslant s^0_i$,
(3) $\forall i \in [m, m+d-1],\ \|\|\mathscr{E}_i\|\| \leqslant s_i$,
(4) $\forall i \in [m, m+d-2],\ \mathrm{W}(\mathscr{E}_{i+1}) - \mathrm{W}(\mathscr{E}_i) \geqslant \delta_{i+1}$.

Then $\mathrm{INV}(\mathscr{S}) \neq \perp, \mathrm{D}(\mathscr{S}) \leqslant d-1$, *and for every* $\mathscr{E} \in \mathrm{INV}(\mathscr{S})$,
(5) $\bar{\ell}(\mathscr{E}) \leqslant \bar{K}_2$,
(6) $|||\mathscr{E}||| \leqslant \Sigma_{m+\mathrm{D}(\mathscr{S})} + s_{m+\mathrm{D}(\mathscr{S})}$.

Sketch of proof. Applying Lemma 166 on the restricted system $\hat{\mathscr{S}}$ we obtain Lemma 167. □

11.10. New constants

Let us *fix* a normalized H-dpda $\mathscr{M}$ and an initial equation

$$A_0 = (\Pi_0, S_0^-, S_0^+) \in \mathbb{N} \times \mathsf{DRH}^0\langle\langle V \rangle\rangle \times \mathsf{DRH}^0\langle\langle V \rangle\rangle.$$

The constants $k_0, k_1, D_1, k_2, K_0, K_1, K_2, K_3, K_4, d_0$ are still defined by the formulas (85)–(89) of Section 6. In addition we introduce

$$\bar{K}_3 = 4 \cdot K_0 \cdot k_0 \cdot k_1 \cdot \bar{k}(\mathscr{M}), \quad \bar{K}_4 = (4 \cdot K_0 \cdot k_1 \cdot k_2 + 4 \cdot K_0 \cdot k_0^2 + 6 \cdot k_0) \cdot \bar{k}(\mathscr{M}). \tag{218}$$

$$K_3^0 = \bar{K}_3, \quad K_4^0 = (4 \cdot K_0 \cdot k_1 \cdot k_2 + K_0 \cdot k_1^2 + 4 \cdot K_0 \cdot k_0^2 + 3 \cdot k_1 + 6 \cdot k_0) \cdot \bar{k}(\mathscr{M}). \tag{219}$$

We still consider the same function F as in Section 6 (see the trick in the proof of Lemma 166). We recall it is defined by

$F(d,n) = \max\{\mathrm{Div}(A,B) \mid A,B \in \mathsf{DRB}_{1,d}\langle\langle V \rangle\rangle, \|A\| \leqslant n, \|B\| \leqslant n, A \not\equiv B\}$. We consider now the integer sequences $(\delta_i, \ell_i, L_i, \bar{s}_i, s_i^0, s_i, S_i, \Sigma_i)_{m \leqslant i \leqslant m+n-1}$ defined by relations (211) of Section 11.9 where the parameters $K_1, \ldots, K_4$ are chosen to be the above constants and $m=1$, $n=d=d_0$. Equivalently, they are defined by

$$\delta_1 = 0, \ \ell_1 = 0, \ L_1 = K_2, \tag{220}$$

$$\bar{s}_1 = \bar{K}_3 \cdot K_2 + \bar{K}_4, \quad s_1^0 = K_3^0 \cdot K_2 + K_4^0, \tag{221}$$

$$s_1 = K_3 \cdot K_2 + K_4, \quad S_1 = 0, \quad \Sigma_1 = 0. \tag{222}$$

$$\begin{aligned}
\delta_{i+1} &= 2 \cdot F(d_0, s_i + \Sigma_i) + 1, \\
\ell_{i+1} &= 2 \cdot \delta_{i+1} + 3, \\
L_{i+1} &= K_1 \cdot (L_i + \ell_{i+1}) + K_2, \\
\bar{s}_{i+1} &= \bar{K}_3 \cdot L_{i+1} + \bar{K}_4, \\
s_{i+1}^0 &= K_3^0 \cdot L_{i+1} + K_4^0, \\
s_{i+1} &= K_3 \cdot L_{i+1} + K_4, \\
S_{i+1} &= s_i + \Sigma_i + K_0 F(d_0, s_i + \Sigma_i), \\
\Sigma_{i+1} &= \Sigma_i + S_{i+1}
\end{aligned} \tag{223}$$

for $1 \leqslant i \leqslant d_0 - 1$. The constants D_2, N_0 are still defined by formulas (93), (94) of Section 6. We recall the two following constants introduced in Section 11.9:

$$\bar{K}_2 = \max\{\bar{\ell}(\rho_e(\mathrm{INV}(\mathscr{S}))) \mid \mathscr{S} \text{ system of } d_0 \text{ equations such that } \bar{\ell}(\mathscr{S})) \leqslant \bar{s}_{d_0}, \ell^0(\mathscr{S}) \leqslant s^0_{d_0}, |||\mathscr{S}||| \leqslant s_{d_0}\}, \tag{224}$$

$$\bar{L}_2 = \max\{\bar{\ell}(S_0^-), \bar{\ell}(S_0^+), \bar{K}_2\}. \tag{225}$$

Let $\Psi : \mathsf{DRH}^0\langle\langle V\rangle\rangle \to \mathsf{DRH}^0\langle\langle V\rangle\rangle/\!\sim$ be the canonical projection. For every integers $D, N, L \in \mathbb{N}$, we consider the set

$$\mathsf{C}(D,N,L) = \Psi\{S \in \mathsf{DRH}^0\langle\langle V\rangle\rangle \mid \bar{\ell}(S) \leqslant L, \|S\| \leqslant N\} \cup (Q \times Z^{\leqslant D}) \times \Psi\{S \in \mathsf{DRH}^0_{Q,1}\langle\langle V\rangle\rangle \mid \bar{\ell}(S) \leqslant L, \|S\| \leqslant N\}. \tag{226}$$

We introduce the new constants:

$$K_8 = 5 \cdot (\mathrm{Card}(\mathsf{C}(D_1, N_0, \bar{L}_2)))^2; \quad K_7 = (K_8 + 1) \cdot k_0 \cdot K_0. \tag{227}$$

11.11. Strategies for $\mathscr{H}_0$

By some slight adaptations of the strategies devised for the system $\mathscr{D}_0$ (see Section 7), we obtain strategies for the particular system $\mathscr{H}_0$.

T_{cut}: $T_{\mathrm{cut}}(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $\exists i \in [1, n-1], \exists S_i, S'_i, S_n, S'_n \in \mathsf{DRH}^0\langle\langle V\rangle\rangle, h \in H$

$$O_i \sqsubseteq S_i, \quad O'_i \sqsubseteq S'_i, \quad O_n \sqsubseteq S_n, \quad O'_n \sqsubseteq S'_n,$$

$$O_i \equiv O'_i \equiv O_n \equiv O'_n \equiv \emptyset,$$

$$A_i = (p_i, S_i, S'_i), \quad A_n = (p_n, S_n, S'_n), \quad p_i < p_n,$$

$$S_i - O_i = h \cdot (S_n - O_n), \quad S'_i - O'_i = h \cdot (S'_n - O'_n), \quad \text{and} \quad m = 0$$

$T_\emptyset$: $T_\emptyset(A_1A_2 \cdots A_n) = B_1 \cdots B_m$ iff $\exists S, T,\ A_n = (p, S, T), p \geqslant 0, S \equiv T \equiv \emptyset$ and $m = 0$

T_H: $T_H(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $A_n = (p, S, T), p \geqslant 0, \exists h \in H, S \equiv T \equiv h$ and $m = 0$

T_A: $T_A(A_1 \cdots A_n) = B_1 \cdots B_m$ iff

$$A_n = (p, S, T), \quad m = |X|, \quad B_1 = (p+1, S \odot x_1, T \odot x_1), \ldots,$$

$$B_m = (p+1, S \odot x_m, T \odot x_m),$$

where $\forall h \in H,\ S \not\equiv h,\ T \not\equiv h$

T_B^+: $T_B^+(A_1 \cdots A_n) = B_1 \cdots B_m$ iff $n \geqslant k_1 + 1$, $A_{n-k_1} = (\pi, \bar{U}, U')$, (where $\bar{U}$ is unmarked)

$$U' = \sum_{q \in Q} [\bar{p}zq] \cdot V_q \quad (\text{for some } \bar{p} \in Q,\ z \in Z,\ V_q \in \mathsf{BH}^0\langle\langle V\rangle\rangle),$$

$A_i = (\pi + k_1 + i - n, U_i, U'_i)$ for $n - k_1 \leqslant i \leqslant n$, $(U_i)_{n-k_1 \leqslant i \leqslant n}$ is a derivation, $(U'_i)_{n-k_1 \leqslant i \leqslant n}$ is a "stacking derivation" (see definitions in Section 3.4),

$$U'_n = \sum_{q \in Q} h \cdot [p\tau q] \cdot V_q, \quad \text{for some } h \in H,\ p \in Q,\ \tau \in Z^+,$$

$m = 1, B_1 = (\pi + k_1 - 1, V, V')$, $V = U_n$, $V' = \sum_{q \in Q'} h \cdot [p\tau q] \cdot [qeq] \cdot (\bar{U} \odot (h_q, u_q))$, where $Q' = \{q \in Q \mid [\bar{p}zq] \not\equiv \emptyset\}$, $\forall q \in Q', (h_q, u_q) = \min(\varphi([\bar{p}zq]))$.

T_B^-: T_B^- is defined in the same way as T_B^- by exchanging the left series (S^-) and right series (S^+) in every assertion (p, S^-, S^+).

T_C: $T_C(A_1 \cdots A_n) = B_1 \cdots B_m$ iff there exists $d \in [1, d_0]$, $D \in [0, d-1]$, $S_1, S_2, \ldots, S_d \in \mathsf{DRH}^0\langle\langle V \rangle\rangle, 1 \leqslant \kappa_1 < \kappa_2 < \cdots < \kappa_{D+1} = n$, such that,

(C1) every equation $(\mathscr{E}_i) = (p_{\kappa_i}, S^-_{p_{\kappa_i}}, S^+_{p_{\kappa_i}})$, for $1 \leqslant i \leqslant D+1$, is a weighted equation over $S_1, S_2, \ldots, S_d$,

(C2) $\mathscr{S} = (\mathscr{E}_i)_{1 \leqslant i \leqslant D+1}$ is such that, $\mathrm{INV}(\mathscr{S}) \neq \perp$, $\mathrm{D}(\mathscr{S}) = D$ and $\bar{\ell}(\mathscr{S}) \leqslant \bar{s}_{d_0}$, $\ell^0(\mathscr{S}) \leqslant s^0_{d_0}$, $|||\mathscr{S}||| \leqslant s_{d_0}$,

(C3) $(\kappa_1, \kappa_2, \ldots, \kappa_{D+1}, S_1, \ldots, S_d) \in \mathbb{N}^* \times (\mathsf{DRH}^0\langle\langle V \rangle\rangle)^*$ is the minimal vector satisfying conditions (C1,C2) for the given sequence $(A_1 \cdots A_n)$ and

(C4) $B_1 \cdots B_m = \rho_e(\mathrm{INV}(\mathscr{S}))$.

The strategies $\mathscr{S}_{AB}, \mathscr{S}_{ABC}$ are then defined from the above elementary strategies as in Section 7.

Lemma 168. *$T_{\mathrm{cut}}, T_\emptyset, T_H, T_A, T_B^+, T_B^-, T_C$ are $\mathscr{H}_0$-strategies. Moreover, $\mathscr{S}_{AB}, \mathscr{S}_{ABC}$ are closed $\mathscr{H}_0$-strategies.*

(See all the lemmas of Section 7.)

11.12. Tree analysis

We adapt here the statements of Section 8. We fix throughout the remaining of this subsection a tree

$$\tau = \mathscr{T}(\mathscr{S}_{AB}, (\pi_0, U_0^-, U_0^+))$$

(i.e. τ is the proof tree associated to the assertion (π_0, U_0^-, U_0^+) by the strategy $\mathscr{S}_{AB}$). We suppose that, for every $\alpha \in \{-, +\}$

$$\bar{\ell}(U_0^\alpha) \leqslant \bar{L}_2, \quad \mathrm{rd}(U_0^\alpha) \leqslant D_2 \tag{228}$$

and

$$U_0^-, U_0^+ \text{ are both unmarked.} \tag{229}$$

Lemma 169. *For every label (π, U^-, U^+) of τ,*

(1) *$\exists \alpha \in \{-, +\}, U^\alpha$ is unmarked.*

(2) *If U^α is unmarked, then $\bar{\ell}(U^\alpha) \leqslant \bar{L}_2$ and $\mathrm{rd}(U^\alpha) \leqslant D_2$.*

(3) *If U^α is marked, then $U^\alpha = \sum_{q \in Q} [p\omega q][qeq] V_q$ for some $p \in Q$, $\omega \in Z^+$, V_q $\in \mathsf{DRH}^0\langle\langle V \rangle\rangle$, with $|\omega| \leqslant D_1$, $\bar{\ell}(V_q) \leqslant \bar{L}_2$, $\mathrm{rd}(V_q) \leqslant D_2$.*

Sketch of proof. Analogous to Lemma 72. We use the fact that, for every $S \in \mathsf{DRH}^0\langle\langle V \rangle\rangle$, $u \in X^*$, $\bar{\ell}(S \odot u) \leqslant \bar{\ell}(S)$. $\square$

11.12.1. N-stacking sequences

The maps $U \mapsto |||U|||$, $x \mapsto \mathrm{N}(x)$ are defined as in Section 8.3. Let $\sigma = (x_i)_{i\in I}$ be a path in τ, where $I \subseteq \mathbb{N}$ is a non-empty interval and $i_0 = \min(I)$. As in Section 8.3, σ is called an N-*stacking sequence* iff

$$\forall i \in I,\ \mathrm{N}(x_i) \geqslant \mathrm{N}(x_{i_0}) \quad \text{and} \quad \mathrm{N}(x_{i_0}) \geqslant N_0. \tag{230}$$

From now on and until Lemma 174, we fix an N-stacking sequence $\sigma = (x_i)_{i\in I}$. We call $\mathrm{Card}(I) - 1$ the *length* of σ (denoted $|\sigma|$). We use the simplified notation $\mathrm{N}(i)$ for $\mathrm{N}(x_i)$ and we note $\tau_s(x_i) = (U_i^-, U_i^+)$. All the definitions and properties (114)–(122), all the Lemmas 73–81 and definition (127) remain unchanged. Let us fix a total ordering on $\mathscr{G}_1$:

$$\mathscr{G}_1 = \{\theta_1, \theta_2, \dots, \theta_d\}, \quad \text{where } d = \mathrm{Card}(\mathscr{G}_1).$$

Let us remark that $d \leqslant 2 \cdot |Q| \cdot (\mathrm{Card}(X^{\leqslant k_1}) + 1) = d_0$.

Lemma 170. *Let $L \geqslant 0$ such that $(i_0 + K_1 L + K_2) + k_1 \in I$. There exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2]$ and, for every $\alpha \in \{-,+\}$, there exists a deterministic rational family $(\beta^\alpha_{i,j})_{1\leqslant j\leqslant d}$ fulfilling*

(1) $U_i^\alpha = \sum_{j=1}^d \beta^\alpha_{i,j} \cdot \theta_j$ (*for every* $\alpha \in \{-,+\}$),
(2) $\bar{\ell}(\beta^\alpha_{i,*}) \leqslant \bar{K}_3 \cdot (i - i_0) + \bar{K}_4$ (*for every* $\alpha \in \{-,+\}$),
(3) $\ell^0(\beta^-_{i,*}, \beta^+_{i,*}) \leqslant K^0_3 \cdot (i - i_0) + K^0_4$,
(4) $\|\beta^\alpha_{i,*}\| \leqslant K_3 \cdot (i - i_0) + K_4$ (*for every* $\alpha \in \{-,+\}$).

Proof. We follow the lines of the proof of Lemma 82 but the new upper bounds (2), (3) require new arguments. We know that there exists $i \in [i_0 + L, i_0 + K_1 \cdot L + K_2], \alpha \in \{-,+\}$, such that T^α_B occurs at i. Up to a left-translation of both sides by h_0^{-1} (where h_0 is the coefficient of $\min(\mathrm{supp}(U^\alpha_{i-k_1-1}))$ in $U^{-\alpha}_{i-k_1-1}$), we can suppose that $U^{-\alpha}_{i-k_1-1}$ is unitary. Hence,

$$\bar{U} = U^{-\alpha}_{i-k_1-1} = [r\omega] * \Phi^{\alpha'} \quad \text{for some } r \in Q,\ \omega \in Z^*,\ \alpha' \in \{-,+\}, \tag{231}$$

$$U^{+\alpha}_{i-k_1-1} = \sum_{q\in Q} [\bar{p}zq] \cdot V_q \quad \text{for some } \bar{p} \in Q, z \in Z, \alpha' \in \{-,+\}, V_q \in \mathsf{DRH}^0\langle\langle V \rangle\rangle, \tag{232}$$

$$U_i^{-\alpha} = \bar{U} \odot u \quad \text{for some } u \in X^{k_1}, \tag{233}$$

$$U_i^{+\alpha} = \sum_{q\in Q'} h_i \cdot [p\tau q][qeq](\bar{U} \odot (h_q, u_q))$$
$$\text{for some } Q' \subseteq Q,\ h_i \in H,\ \tau \in Z^*,\ h_q \in H,\ u_q \in X^{\langle 1,k\rangle}, \tag{234}$$

where

$$h_i \cdot [p\tau q] = [\bar{p}zq] \odot u \quad \text{and} \quad (h_q, u_q) = \min\{\varphi([\bar{p}zq])\}.$$

Let us analyze the coefficients of $U_i^{-\alpha}, U_i^{+\alpha}$ expressed as a linear combination of the set $\{\Phi_q^{\alpha'} \odot w \mid 0 \leqslant |w| \leqslant k_1\} \cup \{\rho_\varepsilon(\Phi_q^{\alpha'}) \mid q \in Q\}$.

(C1) **Coefficients of** $U_i^{-\alpha}$

1.1. Suppose that $U_i^{-\alpha} = ([r\omega] \odot u) * \Phi^{\alpha'}$, with $r \in Q,\ \omega \in Z^*$.

Using Lemma 146, point (1) we obtain

$$\bar{\ell}([r\omega] \odot u) \leqslant \bar{\ell}([r\omega]) = 0$$

and by Lemma 146, point (2) we have

$$\ell_0([r\omega] \odot u) \leqslant |Q||\omega|\bar{k}(\mathscr{M})k_1 + K_0\bar{k}(\mathscr{M})k_1^2.$$

1.2. Suppose that $U_i^{-\alpha} = h \cdot \rho_\varepsilon(\Phi_q^{\alpha'} \odot u'')$ with $q \in Q,\ u = u' \cdot u'',\ u', u'' \in X^*$. We then have

$$\bar{\ell}(\emptyset, \ldots, h, \ldots, \emptyset) = 0,$$

and by Lemma 146 point (2)

$$\ell_0(\emptyset, \ldots, h, \ldots, \emptyset) = \ell(h) = \ell_0([r\omega] \odot u') \leqslant |Q||\omega|\bar{k}(\mathscr{M})k_1 + K_0\bar{k}(\mathscr{M})k_1^2.$$

In any case, we have proved that

$$\bar{\ell}(\beta_{i,*}^{-\alpha}) = 0, \tag{235}$$

$$\ell_0(\beta_{i,*}^{-\alpha}) \leqslant K_0 k_1^2 \bar{k}(\mathscr{M}) + K_0 k_1 \bar{k}(\mathscr{M}) \cdot |\omega|. \tag{236}$$

(C2) **Coefficients of** $U_i^{+\alpha}$

In order to deal with matrices we fix some total orderings of the sets Q and $\mathscr{G}_1^{\alpha'}$:

$$Q = \{q_1, q_2, \ldots, q_n\}, \quad \text{where } n = \mathrm{Card}(Q),$$
$$\mathscr{G}_1^{\alpha'} = \{\xi_1, \xi_2, \ldots, \xi_m\}, \quad \text{where } m = \mathrm{Card}(\mathscr{G}_1^{\alpha'}).$$

Let us consider the following matrices $A \in \mathsf{DRH}_{1,n}^0\langle\langle V\rangle\rangle, B \in \mathsf{DRH}_{n,m}^0\langle\langle V\rangle\rangle$ where

$$a_{1,j} = h_i \cdot [p\tau q_j] \cdot [q_j e q_j]$$
$$b_{j,l} = [r\omega q_k] \odot (h_{q_j}, u_{q_j}), \quad \text{if } q_j \in Q', [r\omega] \odot u_{q_j} \notin \{\emptyset^n\}$$
$$\cup \{h' \cdot \varepsilon_{k'}^n \mid h' \in H,\ 1 \leqslant k' \leqslant n\}, \xi_l = \Phi_{q_k}^{\alpha'},$$
$$b_{j,l} = h \quad \text{if } q_j \in Q', u_{q_j} = u_j' \cdot u_j'', [r\omega] \odot (h_{q_j}, u_j') = h \cdot \varepsilon_k^n \text{ and } \xi_l = \rho_\varepsilon(\Phi_{q_k}^{\alpha'} \odot u_j'').$$
$$b_{j,l} = \emptyset \quad \text{if } (j, l) \text{ does not fulfill any of the two above conditions.}$$

Then

$$U_i^{-\alpha} = A \cdot B \cdot \bar{\xi}$$

where $\bar{\xi} \in \mathsf{DRH}_{m,1}^0\langle\langle V\rangle\rangle$ is the column-vector defined by

$$\bar{\xi}_{\ell,1} = \xi_\ell \quad (\text{for } 1 \leqslant \ell \leqslant m).$$

2.1. **Upper-bounds for** A

As $A \sim \sum_{j=1}^{n} [p\tau q_j][q_j e q_j]$, which is unitary,

$$\bar{\ell}(A) = 0. \tag{237}$$

Using Lemma 146 point (2) we obtain

$$\ell_0(A) = \ell_0([\bar{p}z] \odot u) \leqslant 0 + 3\bar{k}(\mathscr{M})k_1 + K_0\bar{k}(\mathscr{M})k_1^2 = (K_0 \cdot k_1^2 + 3k_1) \cdot \bar{k}(\mathscr{M}). \tag{238}$$

2.2. **Upper-bounds for** B

We analyze B row by row. We distinguish three types of rows $B_{j,*}$.

2.2.1. $[r\omega] \odot (h_{q_j}, u_{q_j}) = h'_j \cdot [s\omega'_j]$ for some $h'_j \in H$, $s \in Q$, $\omega'_j \in Z^*$. In this case

$$\begin{aligned}\ell_0([r\omega] \odot (h_{q_j}, u_{q_j})) &\leqslant \ell(h_{q_j}) + \ell_0([r\omega] \odot u_{q_j})\\ &= \ell_0([\bar{p}zq_j] \odot u_{q_j}) + \ell_0([r\omega] \odot u_{q_j})\\ &\leqslant (K_0 \cdot k_0^2 + 3k_0) \cdot \bar{k}(\mathscr{M}) + K_0k_0\bar{k}(\mathscr{M}) \cdot |\omega| + K_0\bar{k}(\mathscr{M})k_0^2\\ &\leqslant (2K_0 \cdot k_0^2 + 3k_0) \cdot \bar{k}(\mathscr{M}) + (K_0k_1\bar{k}(\mathscr{M})) \cdot |\omega|\end{aligned}$$

Hence, for every row-index j fulfilling case 2.2.1, we have

$$\bar{\ell}(B_{j,*}) = 0; \quad \ell_0(B_{j,*}) \leqslant (2K_0 \cdot k_0^2 + 3k_0) \cdot \bar{k}(\mathscr{M}) + (K_0k_1\bar{k}(\mathscr{M})) \cdot |\omega|. \tag{239}$$

2.2.2. $[r\omega] \odot (h_{q_j}, u'_j) = h \cdot \varepsilon_k^n$ for some $u'_j \leqslant u_{q_j}, h \in H, k \in [1, n]$. By the same calculations as in the above subcase:

$$\bar{\ell}(B_{j,*}) = 0; \quad \ell_0(B_{j,*}) \leqslant (2K_0 \cdot k_0^2 + 3k_0) \cdot \bar{k}(\mathscr{M}) + (K_0k_1\bar{k}(\mathscr{M})) \cdot |\omega|.$$

2.2.3. $B_{j,*} = \emptyset^n$

In this last case, we clearly have

$$\bar{\ell}((B_{j,*}) = 0; \quad \ell_0(B_{j,*}) = 0.$$

(C3) **Upper-bounds for** $(\beta_{i,*}^{-}, \beta_{i,*}^{*})$

Using the definition of $(\beta_{i,*}^{+\alpha})$, Lemma 140 and the fact that $\bar{\ell}(A) = \bar{\ell}(B) = 0$, we obtain:

$$\begin{aligned}\bar{\ell}(\beta_{i,*}^{+\alpha}) = \bar{\ell}(A \cdot B) &\leqslant \max\{\bar{\ell}(A), \bar{\ell}(B)\} + 2 \cdot \ell_0(B) + 2 \cdot \bar{\ell}(B) \cdot \|B\|\\ &= 2 \cdot \ell_0(B).\end{aligned}$$

By Lemmas 76 and 80

$$|\omega| \leqslant 2k_0(i - i_0 - k_1 - 1) + 2(k_2 + k_1k_0 + 2). \tag{240}$$

Combining the upper bounds (239) and (240) we get

$$\begin{aligned}\bar{\ell}(\beta_{i,*}^{+\alpha}) &\leqslant (4K_0\cdot k_0^2+6k_0)\cdot\bar{k}(\mathcal{M})+(2K_0k_1\bar{k}(\mathcal{M}))\cdot|\omega|\\ &\leqslant [(4K_0\cdot k_0^2+6k_0)\cdot\bar{k}(\mathcal{M})+(4K_0k_1\bar{k}(\mathcal{M}))\cdot k_2]\\ &\quad +(4K_0k_0k_1\bar{k}(\mathcal{M}))\cdot(i-i_0)\\ &= \bar{K}_3(i-i_0)+\bar{K}_4. \end{aligned}\tag{241}$$

Let us give now an upper bound for $\ell_0(\beta_{i,*}^{+\alpha})$. By Lemma 140

$$\begin{aligned}\ell_0(\beta_{i,*}^{+\alpha}) &\leqslant \ell_0(A)+\ell_0(B)\\ &\leqslant (K_0\cdot k_1^2+3k_1)\cdot\bar{k}(\mathcal{M})+(2K_0\cdot k_0^2+3k_0)\cdot\bar{k}(\mathcal{M})+(K_0k_1\bar{k}(\mathcal{M}))\cdot|\omega|\\ &\leqslant \bar{K}_3(i-i_0)+\bar{K}_4+(K_0\cdot k_1^2+3k_1)\cdot\bar{k}(\mathcal{M})\\ &= K_3^0(i-i_0)+K_4^0. \end{aligned}\tag{242}$$

Inequations (235) and (241) establish point (2) of the lemma.

As the right-hand side of (236) is smaller than the second line of (242), (242) is sufficient to establish point (3) of the lemma.

Point (4) can be established as in Lemma 82. □

Lemma 171. *Let us suppose that* $|\sigma|\geqslant L_d+k_1$. *Then, there exists* $i_0\leqslant\kappa_1<\kappa_2<\cdots<\kappa_d$ *and deterministic rational vectors* $(\beta_{i,j}^{\alpha})_{1\leqslant j\leqslant d}$ *(for every* $i\in[1,d]$*) such that*
(0) $W(\kappa_1)\geqslant 1$,
(1) $\forall i,\forall\alpha, U_{\kappa_i}^{\alpha}=\sum_{j=1}^{d}\beta_{i,j}^{\alpha}\theta_j\in\overline{V_1}$,
(2) $\forall i,\forall\alpha,\bar{\ell}(\beta_{i,*}^{\alpha})\leqslant\bar{s}_i$,
(3) $\forall i,\ell^0(\beta_{i,*}^{-},\beta_{i,*}^{+})\leqslant s_i^0$,
(4) $\forall i,\forall\alpha,\|\beta_{i,*}^{\alpha}\|\leqslant s_i$,
(5) $\forall i, W(\kappa_{i+1})-W(\kappa_i)\geqslant\delta_{i+1}$.

Sketch of proof. Points (0),(1),(4),(5) can be proved as for Lemma 83. Points (2),(3) are obtained by replacing every invocation of Lemma 82 by an invocation of Lemma 170. □

The adaptation of Lemma 84 turns out to be more technical. Let us prove two auxiliary lemmas.

Lemma 172. *Let* $m\geqslant 1, S\in\mathsf{DRH}^0\langle\langle V\rangle\rangle,\alpha\in\mathsf{DRH}^0_{1,m}\langle\langle V\rangle\rangle,\ T,T'\in\mathsf{DRH}^0_{m,1}\langle\langle V\rangle\rangle$ *such that* $S\equiv\sum_{i=1}^m\alpha_i\cdot T_i,\ S\equiv\sum_{i=1}^m\alpha_i\cdot T_i'$ *and* $\forall i\in[1,m],\ T_i\sim T_i',\ \alpha_i\not\equiv\emptyset,\ T_i'\not\equiv\emptyset$. *Then,* $T=T'$.

Proof. Suppose that

$$S\equiv\sum_{i=1}^{m}\alpha_i\cdot T_i,\quad T_i=h_i\cdot T_i'\quad(\forall i\in[1,m]).$$

As $\alpha_i \not\equiv \emptyset$, there exists $u_i \in X^*$ such that $\alpha_i \odot u_i \in H$. For every $i \in [1, m]$:

$$S \odot u_i \equiv (\alpha_i \odot u_i) \cdot T_i \equiv (\alpha_i \odot u_i) \cdot T_i'.$$

Hence $T_i \equiv T_i'$ i.e. $h_i \cdot T_i' \equiv T_i'$. As $T_i' \not\equiv \emptyset$, this implies $h_i = 1_H$, hence $T = T'$. □

Let us consider the map

$$\Upsilon : \mathsf{DRH}^0\langle\langle V \rangle\rangle \to \mathsf{DRH}^0\langle\langle V \rangle\rangle / \sim \cup (Q \times Z^* \times \mathsf{DRH}^0_{Q,1}\langle\langle V \rangle\rangle / \sim,$$

defined by

$$\Upsilon(S) = (p, \omega, [\Phi]_\sim), \text{ if } S \text{ is marked and } S = [p\omega e] * \Phi \text{ where } \Phi \text{ is unmarked,}$$
$$\Upsilon(S) = [S]_\sim, \text{ otherwise.}$$

(Notice that, in particular, when S is unmarked, $\Upsilon(S) = [S]_\sim$.)

Lemma 173. *Let $(x_i)_{i \in I}$ be a path in τ (we suppose $I \subseteq \mathbb{N}$ is a non-empty interval). Suppose that $i, j \in I$, $i < j < \max(I)$ and $\Upsilon(U_i^-) = \Upsilon(U_j^-)$, $\Upsilon(U_i^+) = \Upsilon(U_j^+)$. Then,*

$$\exists h \in H,\ \exists O_k^\alpha \sqsubseteq U_k^\alpha \quad (for\ all\ \alpha \in \{-,+\}, k \in \{i,j\})$$

such that

$$O_i^\alpha \equiv O_j^\alpha \equiv \emptyset \quad and \quad U_i^\alpha - O_i^\alpha = h \cdot (U_j^\alpha - O_j^\alpha) \quad (for\ all\ \alpha \in \{-,+\}).$$

Proof. *Case* 1: $U_i^-, U_j^-, U_i^+, U_j^+$ are unmarked. As $\Upsilon(U_i^\alpha) = \Upsilon(U_j^\alpha)$, for all $\alpha \in \{-,+\}$, there exists $h^+, h^- \in H$ such that

$$U_i^- = h^- \cdot U_j^-, \quad U_i^+ = h^+ \cdot U_j^+.$$

As $U_i^- \equiv U_i^+$ we have $h^- \cdot U_j^- \equiv h^+ \cdot U_j^+$. Hence

$$(h^+)^{-1} \cdot h^- \cdot U_j^- \equiv U_j^+. \tag{243}$$

As $j < \max(I)$, we know that

$$U_j^- \not\equiv \emptyset, \quad U_j^+ \not\equiv \emptyset \tag{244}$$

otherwise $T_\emptyset$ would apply on x_j and x_j would be a leaf of τ, contradicting the hypothesis "$j < \max(I)$". Assertions (243), (244) imply that $(h^+)^{-1} \cdot h^- = 1_H$. Taking $h = h^+ = h^-$ and $O_k^\alpha = \emptyset$ (for all α, k), the required property is true.

Case 2: U_i^-, U_j^- are unmarked while U_i^+, U_j^+ are marked. Owing to Lemma 169 and to the definition of Υ, this means that

$$U_i^- = h^- \cdot U_j^- \tag{245}$$

and

$$U_i^+ = [p\omega] * \Phi, \quad U_j^+ = [p\omega] * \Phi', \quad \Phi \sim \Phi'$$

for some $p \in Q$, $\omega \in Z^*$, $\Phi, \Phi' \in \mathsf{DRH}^0_{Q,1}\langle\langle V \rangle\rangle$. Let us define the subsets of states

$$Q' = \{q \in Q \mid [p\omega q] \not\equiv \emptyset \text{ and } \Phi_q \not\equiv \emptyset\}, \quad Q'' = Q - Q'$$

and the series

$$O_i^- = O_j^- = \emptyset, \quad O_i^+ = \sum_{q \in Q''} [p\omega q]\Phi_q, \quad O_j^+ = \sum_{q \in Q''} [p\omega q]\Phi'_q.$$

Let us notice that, by (245),

$$U_i^- \equiv \sum_{q \in Q'} [p\omega q]\Phi_q \quad \text{and} \quad U_i^- \equiv \sum_{q \in Q'} [p\omega q](h^- \cdot \Phi'_q)$$

where, for every $q \in Q'$,

$$[p\omega q] \not\equiv \emptyset, \quad \Phi_q \not\equiv \emptyset, \quad \Phi'_q \not\equiv \emptyset, \quad \Phi_q \sim h^- \cdot \Phi'_q.$$

By Lemma 172 we get

$$\forall q \in Q', \ \Phi_q = h^- \cdot \Phi'_q. \tag{246}$$

By (245) (resp. (246)) we have

$$U_i^- - O_i^- = h^- \cdot (U_j^- - O_j^-) \quad (\text{resp.} U_i^+ - O_i^+ = h^- \cdot (U_j^+ - O_j^+)).$$

Taking $h = h^-$, the required property is true.

Case 3: U_i^-, U_j^- are marked while U_i^+, U_j^+ are unmarked. Same proof as for case 2. □

Lemma 174. *Let* $(x_i)_{i \in I}$ *be a path in* τ (*we suppose* $I \subseteq \mathbb{N}$ *is a non-empty interval*). *Let* $L > 0$. *One of the following cases is true*:
(0) $\mathrm{N}(i_0) \geqslant N_0$, *where* $i_0 = \min(I)$,
(1) $|I| \leqslant K_7 \cdot L + K_8$,
(2) $(x_i)_{i \in I}$ *contains a* N-*stacking sequence of length* $\geqslant L$.

Proof. Suppose that neither (0) nor (2) is realized. By Lemma 169, the set $\{\Upsilon(\tau_s(x_i)) \mid i \in I\}$ is included in the set $C(D_1, N_0, \bar{L}_2)$ (the sets $C(D, N, L)$ were defined in Section 11.10 by Eq. (226)). Hence,

$$\mathrm{Card}\{(\Upsilon(U_i^-), \Upsilon(U_i^+)) \mid i \in I, \ \mathrm{N}(i) < N_0\} \leqslant K_8/5. \tag{247}$$

By Lemma 71, if $i_0 \leqslant i < j \leqslant \max(I)$ and $j - i \geqslant 4$, then $\pi_j - \pi_i \geqslant 1$. It follows that, if $i_0 \leqslant i < j < \max(I)$, $j - i \geqslant 4$ and $\Upsilon(U_i^\alpha) = \Upsilon(U_j^\alpha)$ (for $\alpha \in \{-, +\}$), by Lemma 173,

$$\tau(x_0)\tau(x_1) \cdots \tau(x_i) \cdots \tau(x_j) \in \mathrm{dom}(T_{\mathrm{cut}}),$$

which is impossible because x_j is not a leaf (this is implied by "$j < \max(I)$"). Hence, for every $i \in I$,

$$\mathrm{Card}\{j \in I \mid j \geqslant i, \Upsilon(U_i^-) = \Upsilon(U_j^-), \Upsilon(U_i^+) = \Upsilon(U_j^+)\} \leqslant 5. \tag{248}$$

Upper bounds (247) and (248) together show that

$$\mathrm{Card}\{i \in I \mid \mathrm{N}(i) < \mathrm{N}(i_0)\} \leqslant 5 \cdot K_8/5 = K_8.$$

As in the proof of Lemma 84, we conclude that

$$|I| \leqslant \mathrm{Card}(I) \leqslant K_8 + (K_8+1) \cdot L \cdot k_0 \cdot K_0 = K_7 \cdot L + K_8$$

i.e. property (1) is realized. □

11.13. Completeness of $\mathscr{H}_0$

By the same arguments (mutatis mutandis) as in Section 9, one can prove successively the three next statements.

Lemma 175. *Let A_0 be some true assertion which is supposed unmarked. Then the tree $\mathscr{T}(\mathscr{S}_{ABC}, A_0)$ is finite.*

Theorem 176. *The system $\mathscr{H}_0$ is complete.*

Theorem 177. *The equivalence problem for deterministic pushdown H-automata is decidable.*

12. Examples

In order to make pratically feasible the computation of proofs (in $\mathscr{D}_0$, and, after erasure of the weights, in $\mathscr{D}_5$), we introduce some variants of the strategies defined in Section 7 and used for the completeness proof:

- We apply T_{cut} on $A_i = (p_i, S, T), A_n = (p_n, S', T')$, with $p_i < p_n$, provided that $\rho_e(S) = \rho_e(S'),\ \rho_e(T) = \rho_e(T')$.
- We introduce a new strategy T_{eq} defined by
  $T_{eq}(A_1A_2 \cdots A_n) = B_1 \cdots B_m$ iff $\exists S, T,\ A_n = (p, S, T),\ p \geqslant 0,\ \rho_e(S) = \rho_e(T)$ and $m = 0$.
- We allow T_B^+, applied on $A_1 \cdots A_n$, to give the result described in Section 7 but where the fixed integer k_1 is replaced by any integer k_1' provided that

  $$k_1' \geqslant 1 + 2 \cdot \max\{|u_q|, q \in Q'\}.$$

  (Hence T_B^α, for $\alpha \in \{-,+\}$, become now binary relations, which need not be functional in general.)
- We remove the *minimality* condition in point (C3) of the definition of T_C. (Hence T_C becomes a binary relation too.)
- In case 1 of the definition of INV, we do not require any more that $\alpha_{m,*} \equiv \beta_{m,*}$. Hence INV, W, D are also binary relations. The drawback of this modification is that point (2) of Lemma 57 is not valid any more but point (1) remains valid since our proof of point (1) does not use the hypothesis that, in case 1, $\alpha_{m,*}$ must be

equivalent to $\beta_{m,*}$. It follows that, when one uses such a modified T_C, *if the result obtained is a finite closed tree* t, i.e. a tree where every leaf x is such that the word W labelling its branch has an image ε by $T_{\text{cut}} \cup T_{\emptyset} \cup T_{\varepsilon} \cup T_{eq}$, then the set of labels of t is a proof. [12]

- We define a generalized version of T_C, that we name T'_C where, in case 2 of the definition of INV, one can choose two (or more) words u, u' such that

$$\exists j \in [1,d], \quad (\alpha_{m,*} \odot u = \varepsilon_j^d) \Leftrightarrow (\beta_{m,*} \odot u \neq \varepsilon_j^d),$$

such that u (resp. u') correspond to different values j_0 (resp. j'_0) of the index j. One can then consider the two equations

$$(\mathscr{E}'_m): \quad p_m + 2 \cdot |u|, S_{j_0}, \sum_{\substack{j=1 \\ j \neq j_0}}^{d} (\beta_{m,j_0} \odot u)^* (\beta_{m,j} \odot u) S_j$$

$$(\mathscr{E}''_m): \quad p_m + 2 \cdot |u'|, S_{j'_0}, \sum_{\substack{j=1 \\ j \neq j_0}}^{d} (\beta_{m,j_0} \odot u')^* (\beta_{m,j} \odot u') S_j$$

and then eliminate both series $S_{j_0}, S_{j'_0}$ in the other equations.

We also allow to *stop* the development of a branch at a node x, with label (p, S, T) when there exists another node y in the tree with label (p', S, T) where $p' < p$. (As y needs not be an ancestor of x, ε needs not belong to $T_{\text{cut}}(W_x)$ in general.)

12.1. Example 1

12.1.1. The automaton

Let $\mathscr{M} = \langle X, Z, Q, \delta, q_0, \Omega \rangle$ with $X = \{x, a, b, c, t, \bar{t}\}$, $Z = \{\Omega, A, B, D, T\}$, $Q = \{q_0, q_1, q_2, \bar{q}\}$ and δ consists of the transitions:

$$\begin{array}{lll} q_0\Omega \xrightarrow{x} q_0 A\Omega, & q_0 A \xrightarrow{a} q_1, & q_0 A \xrightarrow{c} q_2; \\ q_1\Omega \xrightarrow{b} q_1\Omega, & q_1\Omega \xrightarrow{c} q_1 D, & q_1 D \xrightarrow{d} \bar{q}; \\ q_2\Omega \xrightarrow{d} \bar{q}, & q_0 A \xrightarrow{t} q_0 TA, & q_0 T \xrightarrow{\bar{t}} q_0; \\ q_0 T \xrightarrow{t} q_0 TT. & & \end{array} \quad \begin{array}{lll} q_1\Omega \xrightarrow{x} q_1 A\Omega, & q_1 A \xrightarrow{a} q_1 B, & q_1 B \xrightarrow{b} q_1; \\ q_1 A \xrightarrow{c} q_2, & q_1 B \xrightarrow{x} q_1 A, & q_1 B \xrightarrow{c} q_2; \\ q_1 A \xrightarrow{t} q_1 TA, & q_1 T \xrightarrow{\bar{t}} q_1, & q_1 T \xrightarrow{t} q_1 TT. \end{array}$$

12.1.2. The equivalence proof

A finite proof of the assertion $[q_0\Omega\bar{q}] \equiv [q_1\Omega\bar{q}]$ is exhibited in Figs. 6–8. It can be considered as a proof in the deduction system $\mathscr{D}_0$, where the weight of the root-assertion is 0 and all the other weights can be deduced (just add 1 at each T_A-node, substract 1 at each T_B or T_C node). By the results of Section 10, as it is represented,

[12] We chosed to treat with full rigor only the simpler *functional* strategies used in our completeness proof. The adaptations made here are done just for the practical purpose of giving examples, which was not the main goal of this work. Any real implementation of our proof-system should include such non-functional strategies and will require the corresponding rigorous proofs.

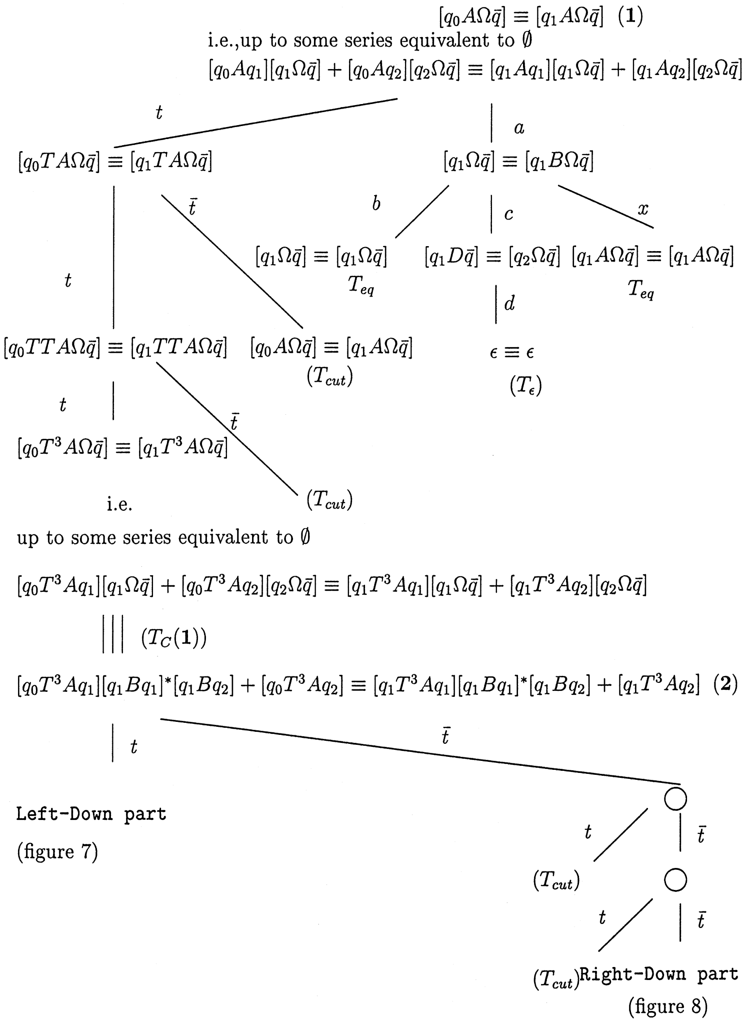

Fig. 6. Proof of Example 1: the top part.

it is a $\mathscr{D}_5$-proof. The boldface numbers are just labels used for distinguishing some important nodes (they are *not* part of the proof in the technical sense of Section 4.3 or Section 10.7). Let us compute explicitly the steps $T_C(\mathbf{1})$ and $T_B^+(\mathbf{2})$ appearing in Fig. 6.

Computation of $T_C(\mathbf{1})$. Let us stick to the notation of Sections 7 and 5 (concerning the computation of INV). Here $n=5, d=|Q|=4, S_1=[q_1\Omega\bar{q}], S_2=[q_2\Omega\bar{q}], S_3=[q_0\Omega\bar{q}]$, $S_4=[\bar{q}\Omega\bar{q}]$, $D=1$, $\kappa_1=2$, $\kappa_2=5=n$, and

$$\mathscr{E}_1=(1,[q_0A\Omega\bar{q}],[q_1A\Omega\bar{q}]),$$
$$\mathscr{E}_2=(4,[q_0T^3A\Omega\bar{q}],[q_1T^3A\Omega\bar{q}]).$$

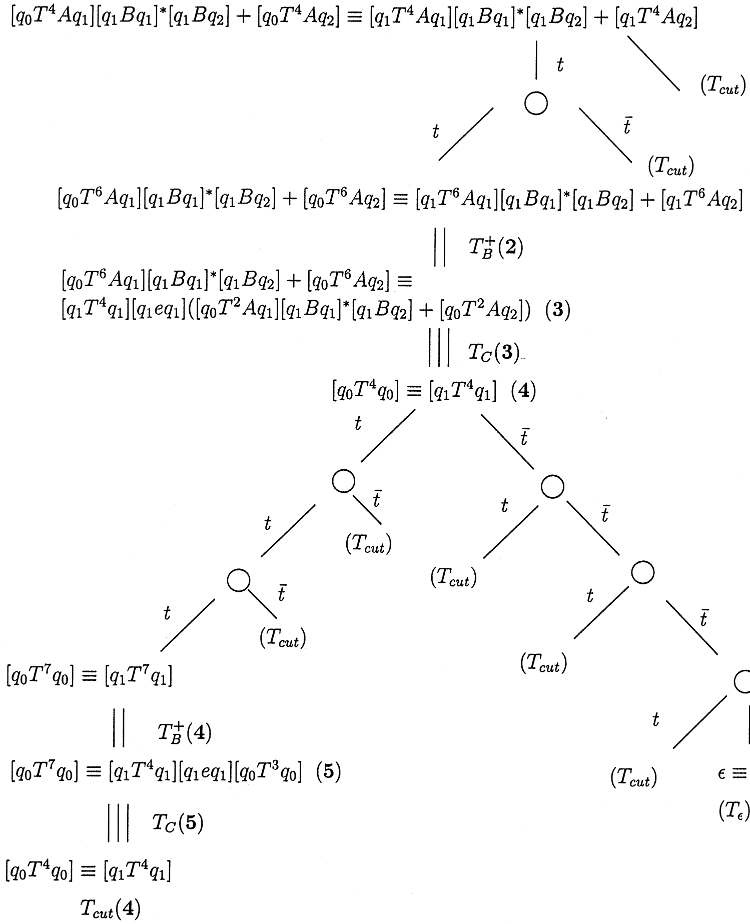

Fig. 7. Proof of Example 1: the left-down part.

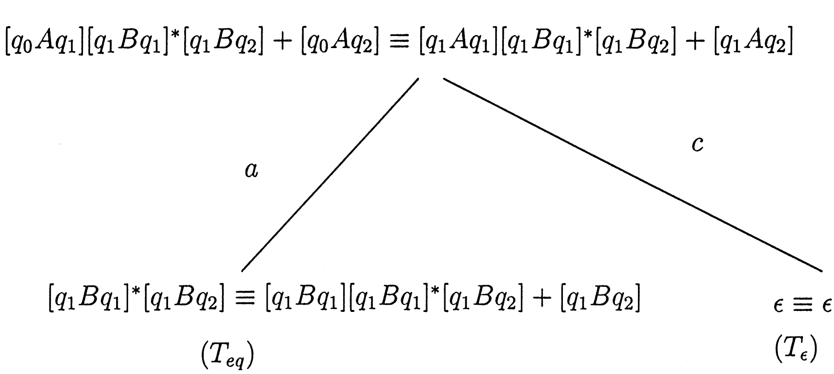

Fig. 8. Proof of Example 1: the right-down part.

which can be rewritten as

$$\mathscr{E}_1 = \left(1,\ \sum_{q\in Q}[q_0 A q]\cdot[q\Omega\bar{q}],\ \sum_{q\in Q}[q_1 A q]\cdot[q\Omega\bar{q}]\right),$$

$$\mathscr{E}_2 = \left(4,\ \sum_{q\in Q}[q_0 T^3 A q]\cdot[q\Omega\bar{q}],\ \sum_{q\in Q}[q_1 T^3 A q]\cdot[q\Omega\bar{q}]\right).$$

One can check that, for $q\in Q-\{q_1,q_2\}$, $[q\Omega\bar{q}]\equiv\emptyset$. Hence the new system $\hat{\mathscr{S}}$ (defined in Section 5.2) consists of the equations:

$$\hat{\mathscr{E}}_1 = (1, [q_0Aq_1]\cdot S_1 + [q_0Aq_2]\cdot S_2, [q_1Aq_1]\cdot S_1 + [q_1Aq_2]\cdot S_2),$$
$$\hat{\mathscr{E}}_2 = (4, [q_0T^3Aq_1]\cdot S_1 + [q_0T^3Aq_2]\cdot S_2, [q_1T^3Aq_1]\cdot S_1 + [q_1T^3Aq_2]\cdot S_2).$$

Let $u=a$. The right-action of a on equation $\hat{\mathscr{E}}_1$ gives the equation

$$(3, S_1, [q_1Bq_1]\cdot S_1 + [q_1Bq_2]\cdot S_2),$$

which, as $[q_1Bq_1]\not\equiv\varepsilon$, leads to

$$\hat{\mathscr{E}}_1' = (3, S_1, [q_1Bq_1]^*[q_1Bq_2]\cdot S_2).$$

"Plugging" $\hat{\mathscr{E}}_1'$ into $\hat{\mathscr{E}}_2$ we obtain:

$$\hat{\mathscr{E}}_2' = (4, ([q_0T^3Aq_1][q_1Bq_1]^*[q_1Bq_2] + [q_0T^3Aq_2])\cdot S_2,$$
$$([q_1T^3Aq_1][q_1Bq_1]^*[q_1Bq_2] + [q_1T^3Aq_2])\cdot S_2).$$

Let us choose case 1 of the definition of INV (see the adaptation defined above; intuitively, this means that we *guess* that the coefficients on both sides of $\hat{\mathscr{E}}_2'$ are equivalent). Hence,

$$\mathrm{INV}(\mathscr{S}) = (3, [q_0T^3Aq_1][q_1Bq_1]^*[q_1Bq_2] + [q_0T^3Aq_2],$$
$$[q_1T^3Aq_1][q_1Bq_1]^*[q_1Bq_2] + [q_1T^3Aq_2]),$$
$$\mathrm{W}(\mathscr{S}) = 3,\ \mathrm{D}(\mathscr{S}) = 1.$$

Computation of $T_B^+(2)$. Let us stick to the notation of Section 7. Here $n=8$, $k_1'=3$,

$$\begin{aligned}
\bar{U} &= [q_0T^3Aq_1][q_1Bq_1]^*[q_1Bq_2] + [q_0T^3Aq_2],\\
U' &= [q_1T^3Aq_1][q_1Bq_1]^*[q_1Bq_2] + [q_1T^3Aq_2]\\
&= \sum_{q\in Q}[q_1Tq]\cdot([qT^2Aq_1][q_1Bq_1]^*[q_1Bq_2] + [qT^2Aq_2]),\\
U_8 &= [q_0T^6Aq_1][q_1Bq_1]^*[q_1Bq_2] + [q_0T^6Aq_2],\\
U_8' &= \sum_{q\in Q}[q_1T^4q]\cdot([qT^2Aq_1][q_1Bq_1]^*[q_1Bq_2] + [qT^2Aq_2]).
\end{aligned}$$

The assertions A_5, A_6, A_7, A_8 consist of the four equations:

$$[q_0 T^i A q_1][q_1 B q_1]^*[q_1 B q_2] + [q_0 T^i A q_2]$$
$$\equiv [q_1 T^i A q_1][q_1 B q_1]^*[q_1 B q_2] + [q_1 T^i A q_2] \text{ for } i \in [3,6].$$

One can check that (U_5', U_6', U_7', U_8') is a stacking derivation. We also have:

$$Q' = \{q_1\}, \quad u_{q_1} = \bar{t}.$$

$$\bar{U} \odot u_{q_1} = [q_0 T^2 A q_1][q_1 B q_1]^*[q_1 B q_2] + [q_0 T^2 A q_2],$$

hence, the result of T_B^+ is

$$V = [q_0 T^6 A q_1][q_1 B q_1]^*[q_1 B q_2] + [q_0 T^6 A q_2],$$
$$V' = [q_1 T^4 q_1][q_1 e q_1] \cdot ([q_0 T^2 A q_1][q_1 B q_1]^*[q_1 B q_2] + [q_0 T^2 A q_2]).$$

12.2. Example 2

This example is more advanced in the sense that the automaton considered here is not real time any more and there is an occurrence of application of T_C' which transforms a system of two equations over *four* non-null series into two new equations.

12.2.1. The automaton

Let $\mathscr{M} = \langle X, Z, Q, \delta, q_1, A\rangle$ with $X = \{x, a, b\}$, $Z = \{\Omega, A, B\}$, $Q = \{q_1, q_2, q_3, q_3', \bar{q}_3, q_4, q_5, \bar{q}\}$ and δ consists of the transitions:

$$q_1 A \xrightarrow{a} q_3, \quad q_1 A \xrightarrow{b} q_5, \quad q_1 A \xrightarrow{x} q_1 AA; \; q_2 A \xrightarrow{a} q_4 AA, \; q_2 A \xrightarrow{b} \bar{q}, \quad \bar{q} A \xrightarrow{\varepsilon} \bar{q};$$
$$q_3 A \xrightarrow{a} q_3, \quad q_5 A \xrightarrow{\varepsilon} q_5, \quad q_5 \Omega \xrightarrow{\varepsilon} q_3; \quad \bar{q}\Omega \xrightarrow{\varepsilon} q_3, \quad q_2 A \xrightarrow{x} q_2 AA, \; q_4 A \xrightarrow{a} q_4;$$
$$q_3 \Omega \xrightarrow{a} q_3'\Omega, \; q_3'\Omega \xrightarrow{a} \bar{q}_3 \Omega; \; \bar{q}_3 \Omega \xrightarrow{a} q_3, \quad q_4 \Omega \xrightarrow{a} q_3.$$

12.2.2. The equivalence proof

A finite proof of the assertion $[q_1 A\Omega.] \equiv [q_2 A\Omega.]$ is exhibited in Figs. 9 and 10. The expression $[q\omega.]$ (for every $q \in Q, \omega \in Z^*$), denotes the polynomial $\sum_{q' \in Q} [q\omega q']$. Let us compute explicitly the steps $T_C'(\mathbf{1}), T_C(\mathbf{3})$ appearing in Fig. 9 and $T_C(\mathbf{2})$ appearing in Fig. 10.

Computation of $T_C'(1)$. Here

$$n = 4, \quad d = |Q| = 8, \quad \kappa_1 = 1, \quad \kappa_2 = 4 = n, \quad D = 1,$$

$$S_1 = [q_3\Omega.], \quad S_2 = [q_5\Omega.], \quad S_3 = [q_4\Omega.], \quad S_4 = [\bar{q}\Omega.],$$

$$S_5 = [q_1\Omega.], \quad S_6 = [q_2\Omega.], \quad S_7 = [q_3'\Omega.], \quad S_8 = [\bar{q}_3\Omega.]$$

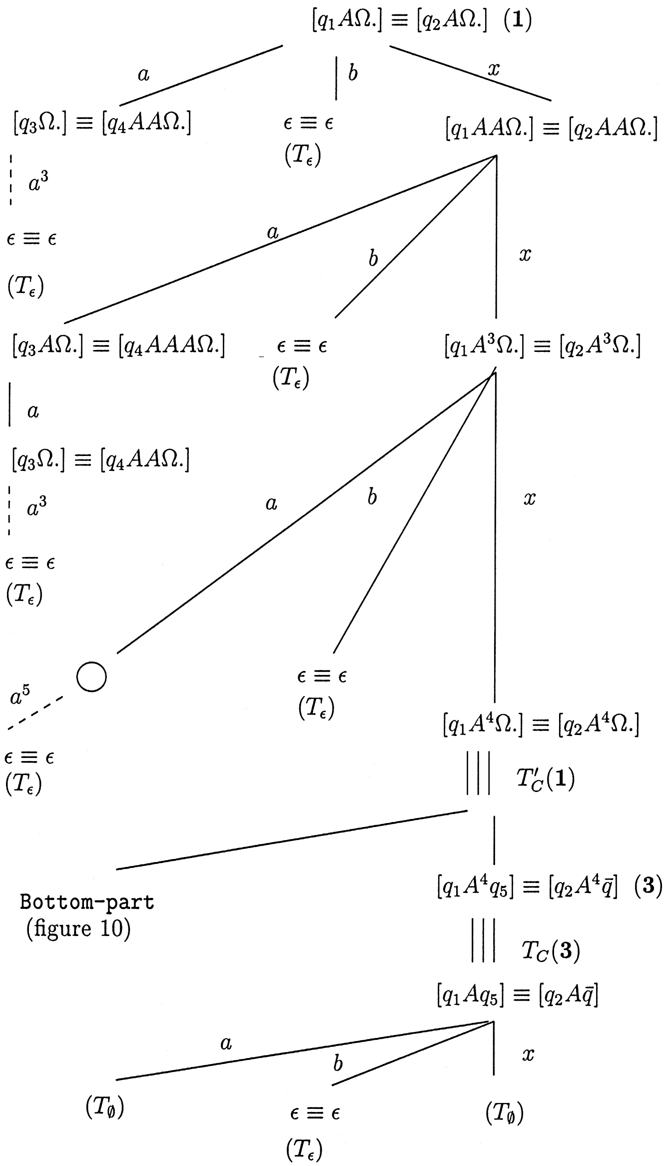

Fig. 9. Proof of Example 2: the top part.

and the system of equations $\mathcal{S}$ consists of the two equations:

$$\mathcal{E}_1 = (0, [q_1 A\Omega.], [q_2 A\Omega.]),$$

$$\mathcal{E}_2 = (3, [q_1 A^4\Omega.], [q_2 A^4\Omega.]).$$

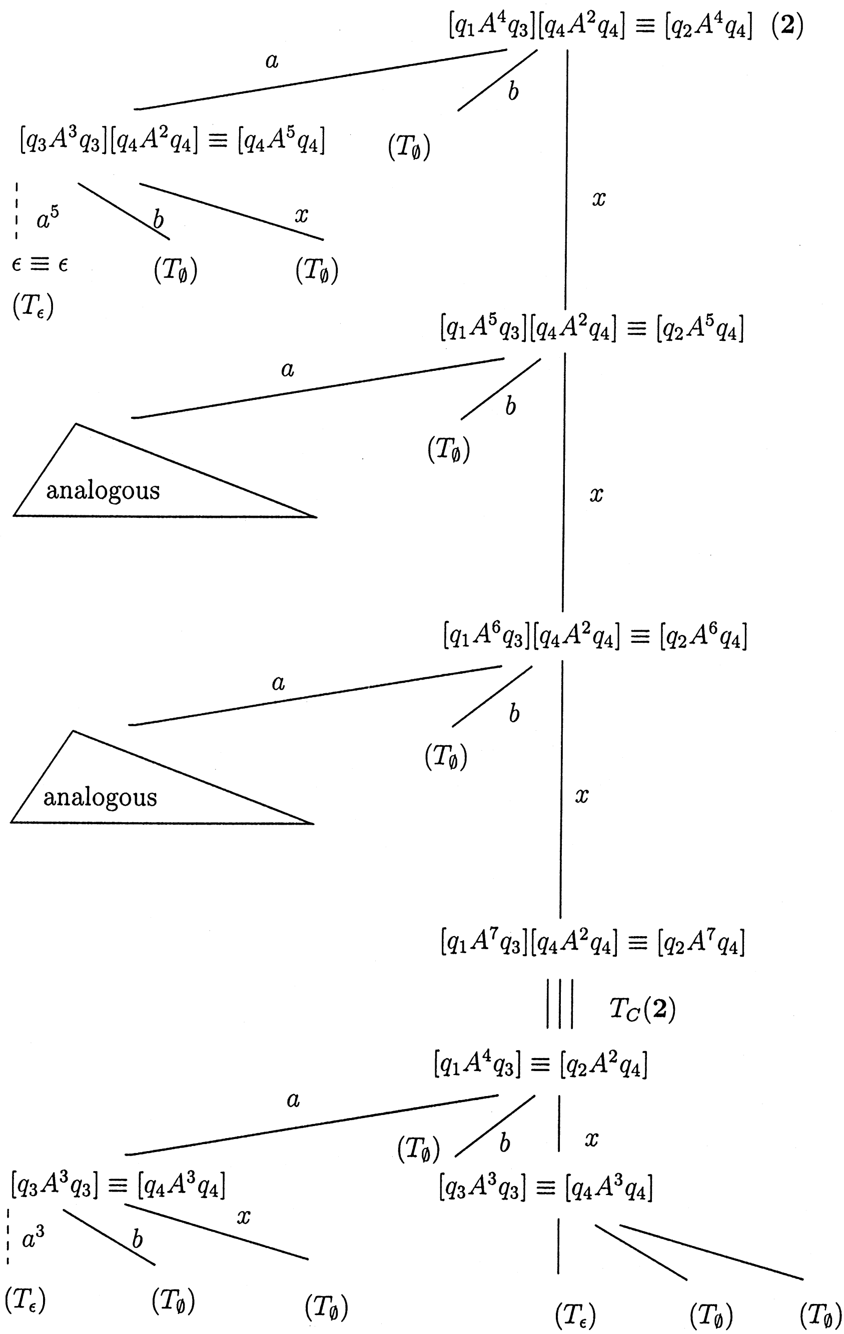

Fig. 10. Proof of Example 2: the bottom part.

Owing to the equivalences:

$$S_5 = [q_1\Omega] \equiv \emptyset, \quad S_6 = [q_2\Omega] \equiv \emptyset \quad \text{and} \quad [q\omega q_3'] \equiv \emptyset,$$
$$[q\omega\bar{q}_3] \equiv \emptyset \text{ (for every } q \in Q, \ \omega \in A^+)$$

one can simplify [13] the equations $\hat{\mathscr{E}}_1, \hat{\mathscr{E}}_2$ into

$$\hat{\mathscr{E}}_1 = (0, [q_1Aq_3] \cdot [q_3\Omega.] + [q_1Aq_5] \cdot [q_5\Omega.], \ [q_2Aq_4] \cdot [q_4\Omega.] + [q_2A\bar{q}] \cdot [\bar{q}\Omega.]),$$
$$\hat{\mathscr{E}}_2 = (3, [q_1A^4q_3] \cdot [q_3\Omega.] + [q_1A^4q_5] \cdot [q_5\Omega.], \ [q_2A^4q_4] \cdot [q_4\Omega.] + [q_2A^4\bar{q}] \cdot [\bar{q}\Omega.]).$$

Let $u = a$, $u' = b$. The right-action of u (resp. u') on equation $\hat{\mathscr{E}}_1$ gives the equations:

$$\hat{\mathscr{E}}_1' = (2, [q_3\Omega.], [q_4AAq_4] \cdot [q_4\Omega.]),$$
$$\hat{\mathscr{E}}_1'' = (2, [q_5\Omega.], [\bar{q}\Omega.]).$$

"Plugging" $\hat{\mathscr{E}}_1'$, $\hat{\mathscr{E}}_1''$ into $\hat{\mathscr{E}}_2$ we obtain

$$\hat{\mathscr{E}}_2' = (3, ([q_1A^4q_3][q_4AAq_4]) \cdot [q_4\Omega.] + [q_1A^4q_5] \cdot [\bar{q}\Omega.], [q_2A^4q_4] \cdot [q_4\Omega.]$$
$$+ [q_2A^4\bar{q}] \cdot [\bar{q}\Omega.]).$$

Let us choose case 1 of the definition of INV:

$$\mathrm{INV}(\mathscr{S}) = \{(2, [q_1A^4q_3][q_4AAq_4], [q_2A^4q_4]), (2, [q_1A^4q_5], [q_2A^4\bar{q}])\}.$$

Computation of $T_C(3)$. Here

$$n = 5, \quad d = 16, \quad D = 0, \quad \kappa_1 = 5 = n,$$

in principle there are 16 series $S_1, \dots, S_{16}$ corresponding to the set $\{[qA^3q_5], q \in Q\} \cup \{[qA^3\bar{q}], q \in Q\}$. Let

$$S_1 = [q_5A^3q_5], \quad S_2 = [\bar{q}A^3\bar{q}].$$

The system of equations $\mathscr{S}$ consists of one equation:

$$\mathscr{E}_1 = (2, [q_1A^4q_5], [q_2A^4\bar{q}]).$$

After simplification we obtain

$$\hat{\mathscr{E}}_1 = (2, [q_1Aq_5] \cdot [q_5A^3q_5], [q_2A\bar{q}] \cdot [\bar{q}A^3\bar{q}]).$$

But one can easily check that: $\rho_\varepsilon(S_1) = \rho_\varepsilon(S_2) = \varepsilon$. The following equation is then provable from $\emptyset$ within the system $\mathscr{D}_0$:

$$\mathscr{E}_0' = (0, S_1, S_2).$$

[13] Here again, we use a small simplification-trick which does not fully correspond to the simplification explained in Section 5.2. We claim that, owing to rule (R′3), Lemma 60 remains valid with this slightly more powerful simplification.

"Plugging" $\hat{\mathscr{E}}_0'$ into $\hat{\mathscr{E}}_1$ we obtain

$$\hat{\mathscr{E}}_1 = (2, [q_1Aq_5] \cdot S_1, [q_2A\bar{q}] \cdot S_1).$$

Chosing case 1 of the definition of INV:

$$\mathrm{INV}(\mathscr{S}) = (1, [q_1Aq_5], [q_2A\bar{q}]).$$

Computation of T_C(2). Here

$$n = 8, \quad d = 16, \quad D = 1, \quad \kappa_1 = 5, \quad \kappa_2 = 8 = n,$$

the system of equations $\mathscr{S}$ consists of two equations:

$$\mathscr{E}_1 = (2, [q_1A^4q_3][q_4AAq_4], [q_2A^4q_4]),$$

$$\mathscr{E}_2 = (5, [q_1A^7q_3][q_4AAq_4], [q_2A^7q_4]).$$

After simplification we obtain

$$\hat{\mathscr{E}}_1 = (2, [q_1Aq_3] \cdot ([q_3A^3q_3][q_4AAq_4]) + [q_1Aq_5] \cdot ([q_5A^3q_3][q_4AAq_4]), [q_2A^4q_4]),$$

$$\hat{\mathscr{E}}_2 = (5, [q_1A^4q_3] \cdot ([q_3A^3q_3][q_4AAq_4]) + [q_1A^4q_5] \cdot ([q_5A^3q_3][q_4AAq_4]),$$
$$[q_2A^2q_4] \cdot [q_4A^5q_4] + [q_2A^2\bar{q}] \cdot [\bar{q}A^5q_4]).$$

Let us note

$$S_1 = [q_3A^3q_3][q_4AAq_4], \quad S_2 = [q_5A^3q_3][q_4AAq_4], \quad S_3 = [q_4A^5q_4],$$
$$S_4 = [\bar{q}A^5q_4].$$

With these notations,

$$\hat{\mathscr{E}}_2 = (5, [q_1A^4q_3] \cdot S_1 + [q_1A^4q_5] \cdot S_2,\ [q_2A^2q_4] \cdot S_3 + [q_2A^2\bar{q}] \cdot S_4).$$

As $S_2 \equiv S_4 \equiv \emptyset$, $\hat{\mathscr{E}}_2$ can be simplified as

$$(5, [q_1A^4q_3] \cdot S_1, [q_2A^2q_4] \cdot S_3). \tag{249}$$

Let $u = a$. The right-action of u on equation $\hat{\mathscr{E}}_1$ gives

$$\hat{\mathscr{E}}_1' = (4, S_1, S_3).$$

"Plugging" $\hat{\mathscr{E}}_1'$ into Eq. (49) we obtain

$$\hat{\mathscr{E}}_2' = (5, [q_1A^4q_3] \cdot S_3, [q_2A^2q_4] \cdot S_3).$$

Let us choose case 1 of the definition of INV:

$$\mathrm{INV}(\mathscr{S}) = \{(4, [q_1A^4q_3], [q_2A^2q_4])\}.$$

12.3. Example 3

Let us consider the subgroup H of $(\mathbb{Q} - \{0\}, \cdot)$ defined by

$$H = \{2^n \mid n \in \mathbb{Z}\}.$$

(Of course, up to isomorphism, H is just the additive group of integers. We choose this definition of H in order to use the multiplicative notation, as we did throughout Section 11.)

12.3.1. The automaton

Let $\mathscr{M} = \langle X, Z, Q, \delta, q_0, \Omega\rangle$ with $X = \{a, b, c\}$, $Z = \{\Omega, A, B\}$, $Q = \{q_0, q_1, q_a, q_b, \bar{q}_0, \bar{q}_1, \bar{q}_a, \bar{q}_b\}$ and δ consists of the transitions:

$$\begin{array}{lll}
q_0\Omega \xrightarrow{a} 2\cdot q_0A\Omega, & q_0\Omega \xrightarrow{b} 2^{-1}\cdot q_0B\Omega; & \\
q_0A \xrightarrow{a} 2\cdot q_0AA, & q_0A \xrightarrow{b} 2^{-1}\cdot q_0; & \\
q_0B \xrightarrow{a} 2\cdot q_0, & q_0B \xrightarrow{b} 2^{-1}\cdot q_0BB; & \\
q_0\Omega \xrightarrow{c} q_1, & q_0A \xrightarrow{c} q_a, & q_0B \xrightarrow{c} 2\cdot q_b; \\
q_aA \xrightarrow{\varepsilon} q_a, & q_a\Omega \xrightarrow{\varepsilon} q_1; & \\
q_bB \xrightarrow{\varepsilon} 2\cdot q_b, & q_b\Omega \xrightarrow{\varepsilon} q_1; &
\end{array}$$

$$\begin{array}{lll}
\bar{q}_0\Omega \xrightarrow{a} \bar{q}_0A\Omega, & \bar{q}_0\Omega \xrightarrow{b} \bar{q}_0B\Omega; & \\
\bar{q}_0A \xrightarrow{a} \bar{q}_0AA, & \bar{q}_0A \xrightarrow{b} \bar{q}_0; & \\
\bar{q}_0B \xrightarrow{a} \bar{q}_0, & \bar{q}_0B \xrightarrow{b} \bar{q}_0BB; & \\
\bar{q}_0\Omega \xrightarrow{c} \bar{q}_1, & \bar{q}_0A \xrightarrow{c} 2\cdot\bar{q}_a, & \bar{q}_0B \xrightarrow{c} \bar{q}_b; \\
\bar{q}_aA \xrightarrow{\varepsilon} 2\cdot\bar{q}_a, & \bar{q}_a\Omega \xrightarrow{\varepsilon} \bar{q}_1; & \\
\bar{q}_bB \xrightarrow{\varepsilon} \bar{q}_b, & \bar{q}_b\Omega \xrightarrow{\varepsilon} \bar{q}_1. &
\end{array}$$

12.3.2. The equivalence proof

A finite proof of the assertion $[q_0\Omega q_1] \equiv [\bar{q}_0\Omega\bar{q}_1]$ is exhibited in Figs. 11–13. One can also check directly that $\varphi([q_0\Omega q_1]) = \varphi([\bar{q}_0\Omega\bar{q}_1]) = S$ where

$$S = \sum_{\substack{u\in\{a,b\}^* \\ |u|_a \geqslant |u|_b}} 2^{|u|_a - |u|_b}\cdot u\cdot c + \sum_{\substack{u\in\{a,b\}^* \\ |u|_a < |u|_b}} u\cdot c.$$

Computation of $T_B^+(1)$. Let us stick to the notation of Section 7. Here $n = 5$, $k_1' = 3$,

$$\bar{U} = 2\cdot[q_0A\Omega q_1], \quad U' = [\bar{q}_0A\Omega\bar{q}_1] = \sum_{q\in Q}[\bar{q}_0Aq][q\Omega\bar{q}_1],$$

$$U_5 = 2^4\cdot[q_0A^4\Omega q_1], \quad U_5' = [\bar{q}_0A^4\Omega\bar{q}_1],$$

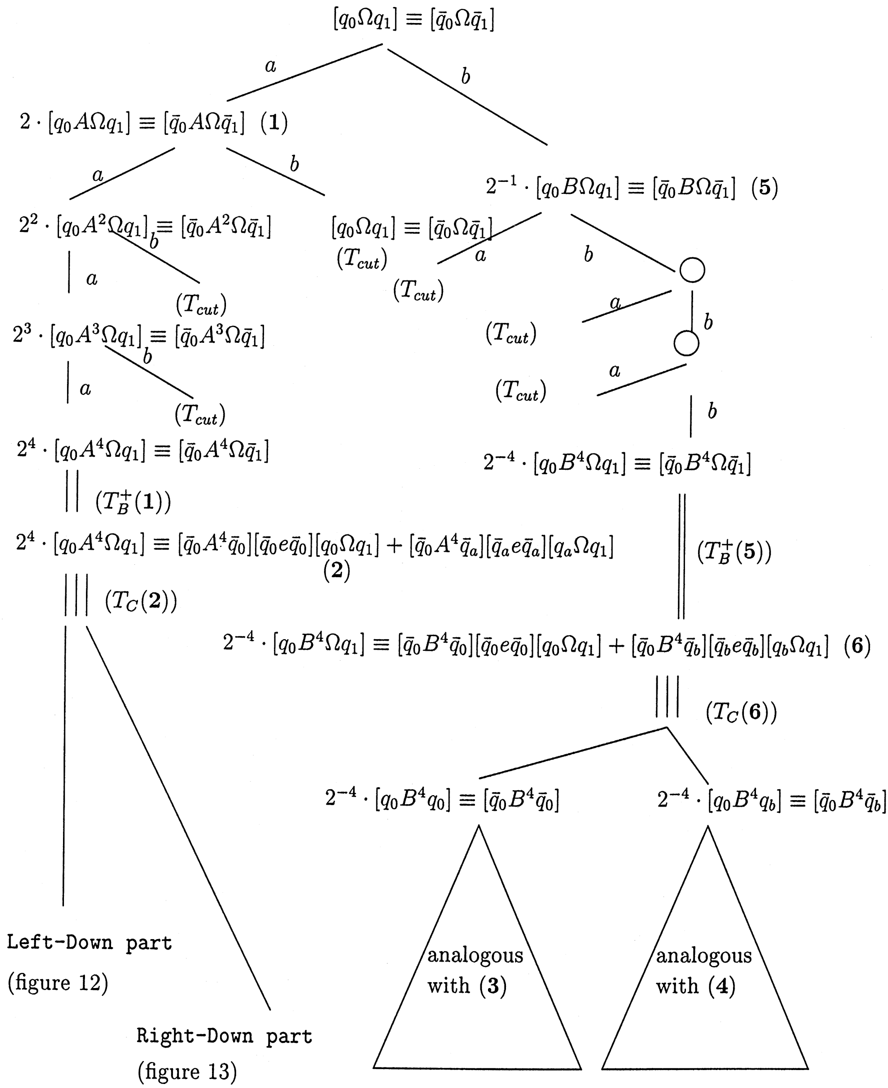

Fig. 11. Proof of Example 3: the top part.

$$Q' = \{\bar{q}_0, \bar{q}_a\}.\ (h_{\bar{q}_0}, u_{\bar{q}_0}) = (1,b),\ \ (h_{\bar{q}_a}, u_{\bar{q}_a}) = (2,c)$$

$$\bar{U} \odot (1,b) = 2\cdot[q_0A\Omega q_1] \odot b = [q_0\Omega q_1],$$

$$\bar{U} \odot (2,c) = 2\cdot[q_0A\Omega q_1] \odot (2,c) = [q_a\Omega q_1],$$

hence, the result of T_B^+ is

$$V = 2^4\cdot[q_0A^4\Omega q_1],\quad V' = [\bar{q}_0A^4\bar{q}_0][\bar{q}_0e\bar{q}_0][q_0\Omega q_1] + [\bar{q}_0A^4\bar{q}_a][\bar{q}_ae\bar{q}_a][q_a\Omega q_1].$$

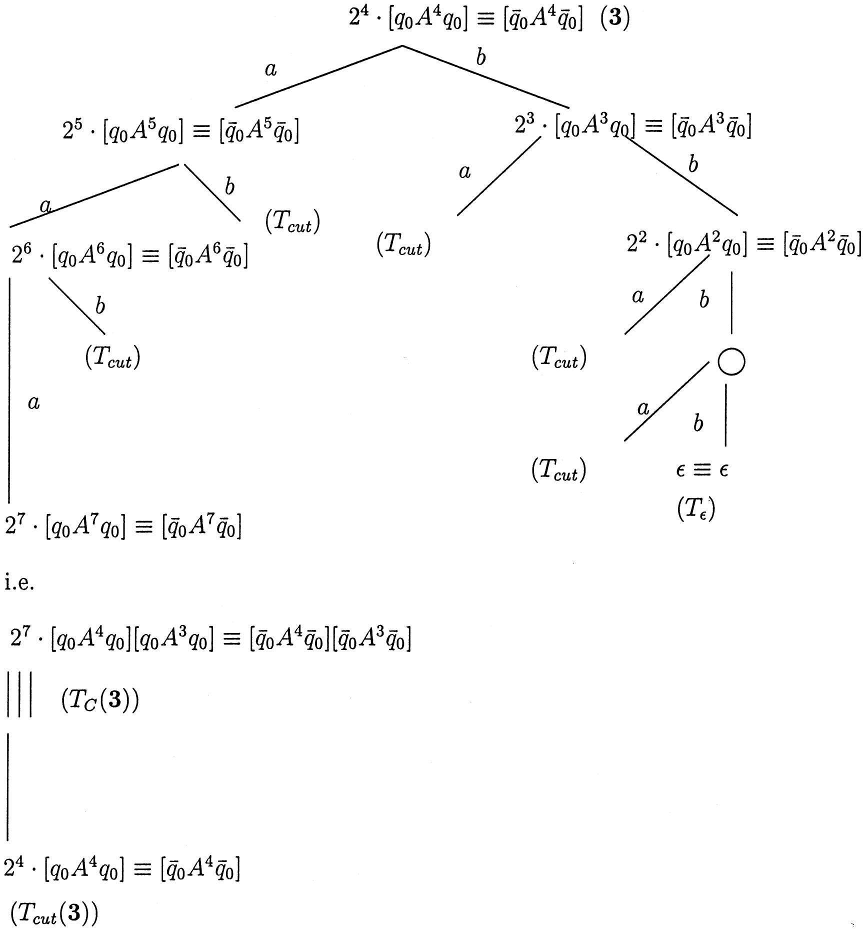

Fig. 12. Proof of Example 3: the left-down part.

Computation of T_C(2). Easy ($D=0$), hence left to the reader.

Computation of T_C(3). Here $n=10$, $d=2\cdot|Q|=16$, $S_1=[q_0A^3q_0]$, $S_2=[\bar{q}_0A^3\bar{q}_0],\ldots,$ $D=1$, $\kappa_1=7$, $\kappa_2=10=n$, and

$$\mathscr{E}_1=(2,2^4\cdot[q_0A^4q_0],[\bar{q}_0A^4\bar{q}_0]),$$

$$\mathscr{E}_2=(5,2^7\cdot[q_0A^7q_0],[\bar{q}_0A^7\bar{q}_0]).$$

One can check that, for $q\in Q-\{q_0\}$, $[qA^3q_0]\equiv\emptyset$ and for $q\in Q-\{\bar{q}_0\}$, $[qA^3\bar{q}_0]\equiv\emptyset$. Hence the new system $\hat{\mathscr{S}}$ (defined in Section 5.2) consists of the equations:

$$\hat{\mathscr{E}}_1=(2,2^4\cdot[q_0Aq_0]\cdot S_1,[\bar{q}_0A\bar{q}_0]\cdot S_2),$$

$$\hat{\mathscr{E}}_2=(5,2^7\cdot[q_0A^4q_0]\cdot S_1,[\bar{q}_0A^4\bar{q}_0]\cdot S_2).$$

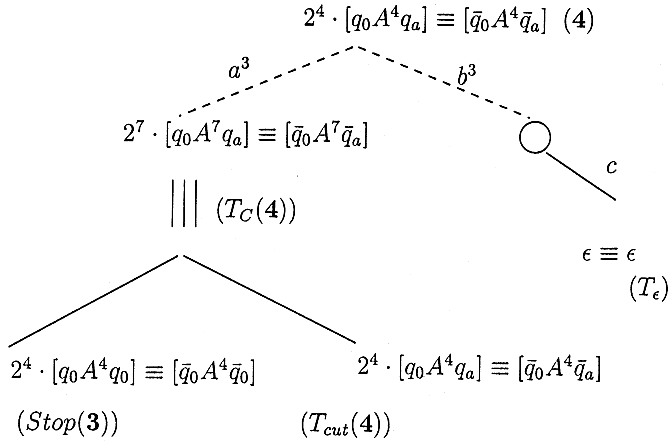

Fig. 13. Proof of Example 3: the right-down part.

Let $h=2^3, u=b$. The right-action of $(2^3, b)$ on equation $\hat{\mathscr{E}}_1$ gives the equation:

$$\hat{\mathscr{E}}_1' = (4, S_1, 2^{-3} \cdot S_2),$$

"Plugging" $\hat{\mathscr{E}}_1'$ into $\hat{\mathscr{E}}_2$ we obtain

$$\hat{\mathscr{E}}_2' = (5, 2^4 \cdot [q_0 A^4 q_0] \cdot S_2, [\bar{q}_0 A^4 \bar{q}_0] \cdot S_2).$$

Let us choose case 1 of the definition of INV. We obtain

$$\mathrm{INV}(\mathscr{S}) = (4, 2^4 \cdot [q_0 A^4 q_0], [\bar{q}_0 A^4 \bar{q}_0])$$

$$\mathrm{W}(\mathscr{S}) = 4, \quad \mathrm{D}(\mathscr{S}) = 1.$$

Computation of T_C(4). Here $n=10$, $d=2 \cdot |Q|=16$, $S_1=[q_0 A^3 q_a]$, $S_2=[\bar{q}_0 A^3 \bar{q}_a]$, $S_3=[q_a A^3 q_a]$, $S_4=[\bar{q}_a A^3 \bar{q}_a], \ldots, D=1$, $\kappa_1=7$, $\kappa_2=10=n$, and

$$\mathscr{E}_1 = (2, 2^4 \cdot [q_0 A^4 q_a], [\bar{q}_0 A^4 \bar{q}_a]),$$

$$\mathscr{E}_2 = (5, 2^7 \cdot [q_0 A^7 q_a], [\bar{q}_0 A^7 \bar{q}_a]).$$

One can check that, for $q \in Q - \{q_0, q_a\}$, $[qA^3 q_a] \equiv \emptyset$ and for $q \in Q - \{\bar{q}_0, \bar{q}_a\}$, $[qA^3 \bar{q}_a] \equiv \emptyset$. Hence the new system $\hat{\mathscr{S}}$ (defined in Section 5.2) consists of the equations

$$\hat{\mathscr{E}}_1 = (2, 2^4 \cdot [q_0 A q_0] \cdot S_1 + 2^4 \cdot [q_0 A q_a] \cdot S_3, [\bar{q}_0 A \bar{q}_0] \cdot S_2 + [\bar{q}_0 A \bar{q}_a] \cdot S_4),$$

$$\hat{\mathscr{E}}_2 = (5, 2^7 \cdot [q_0 A^4 q_0] \cdot S_1 + 2^7 \cdot [q_0 A^4 q_a] \cdot S_3, [\bar{q}_0 A^4 \bar{q}_0] \cdot S_2 + [\bar{q}_0 A^4 \bar{q}_a] \cdot S_4).$$

Let $h=2^3$, $u=b$, $h'=2^4$, $u'=c$. The right-action of $(2^3, b)$ on equation $\hat{\mathscr{E}}_1$ gives the equation:

$$\hat{\mathscr{E}}_1' = (4, S_1, 2^{-3} \cdot S_2).$$

The right-action of $(2^4, c)$ on equation $\hat{\mathscr{E}}_1$ gives the equation:

$$\hat{\mathscr{E}}_1'' = (4, S_3, 2^{-3} \cdot S_4).$$

"Plugging" $\hat{\mathscr{E}}_1'$, $\hat{\mathscr{E}}_1''$ into $\hat{\mathscr{E}}_2$ we obtain

$$\hat{\mathscr{E}}_2' = (5, 2^4 \cdot [q_0 A^4 q_0] \cdot S_2 + 2^4 \cdot [q_0 A^4 q_a] \cdot S_4, [\bar{q}_0 A^4 \bar{q}_0] \cdot S_2 + [\bar{q}_0 A^4 \bar{q}_a] \cdot S_4).$$

Let us choose case 1 of the definition of INV. We obtain

$$\mathrm{INV}(\mathscr{S}) = \{(4, 2^4 \cdot [q_0 A^4 q_0], [\bar{q}_0 A^4 \bar{q}_0]), (4, 2^4 \cdot [q_0 A^4 q_a], [\bar{q}_0 A^4 \bar{q}_a])\}$$

$$\mathrm{W}(\mathscr{S}) = 4, \quad \mathrm{D}(\mathscr{S}) = 1.$$

The remaining computation of $T_C(7)$ is analogous with that of $T_C(2)$, the computation of $T_B^+(5)$ is analogous with that of $T_B^+(1)$.

13. Applications and perspectives

We describe here some immediate applications of our main result (Theorem 87).[14]

13.1. Applications

13.1.1. Formal languages: words

Corollary 178. *The equivalence problem is decidable for LR-regular grammars.*

This follows from Theorem 87 and the reduction given in [51]. This result *extends* Theorem 87 because the class of LR-regular languages strictly contains the class of deterministic context-free languages. The class of LR-regular languages is in turn a subclass of the class of non-ambiguous context-free languages; the equivalence-problem for this last class remains open.

13.1.2. Formal languages: trees

Corollary 179. *The equivalence problem is decidable for simple deterministic tree grammars.*

This follows from Theorem 87 and the reduction given in [13, Theorem 4.17].

13.1.3. Program schemes

Corollary 180. *The equivalence problem for monadic recursion schemes* (*with interpreted* `if-then-else`), *is decidable.*

This follows from Theorem 87 and the reduction given in [28].

[14] They are *immediate* in the sense that they follow from reductions constructed in *previous* works; but of course, most of these reductions are by no means "immediate".

Corollary 181. *The equivalence problem for recursive polyadic program schemes* (*with completely uninterpreted function symbols*) *is decidable.*

This follows from Theorem 87 and the reduction given in [12, Theorem 3.25] or the reduction given in [26, Corollary 4.4].

13.1.4. Equational graphs

Corollary 182. *The bisimulation problem for rooted deterministic equational graphs is decidable.*

This follows from Theorem 87 and the reduction given in [6, Proposition 5.9]. This kind of reduction was initiated in [2]. The extension of Corollary 182 to rooted equational graphs of finite out-degree (which may be *non-deterministic*) is established in [68].

13.1.5. Term rewriting

Corollary 183. *The bisimulation problem for prefix transition graphs of term deterministic context-free grammars is decidable.*

This follows from Theorem 87 and the reduction given in [6, Corollary 5.7]. Corollary 183 is interesting because the class of *graphs* involved is strictly more general than the class of rooted deterministic equational graphs (the transition graphs of term deterministic context-free grammars may have infinite tree-width, hence they need not be equational, see [6, p. 15]), though the associated languages remain exactly the deterministic context-free languages.

13.1.6. Thue systems

We recall a *semi-Thue* system over an alphabet X is a subset of $X^* \times X^*$. We denote by $\overset{*}{\underset{S}{\longleftrightarrow}}$ the smallest congruence of the monoid $(X^*, \cdot)$ which contains S. For every subset $K \subseteq X^*$, $[K]_{\overset{*}{\underset{S}{\longleftrightarrow}}}$ denotes the smallest subset of X^* which is saturated by $\overset{*}{\underset{S}{\longleftrightarrow}}$ and contains K:

$$[K]_{\overset{*}{\underset{S}{\longleftrightarrow}}} = \{u \in X^*, \exists k \in K, k \overset{*}{\underset{S}{\longleftrightarrow}} u\}.$$

We denote by $\mathrm{IRR}(S) \subseteq X^*$, the set of all irreducible words $(\bmod S)$. (See [5,67].)

Corollary 184. *Let X be some finite terminal alphabet. Given a dpda A over X, a finite semi-Thue system S, which is assumed confluent and noetherian, and a rational subset $K \subseteq \mathrm{IRR}(S)$, one can decide whether $\mathrm{L}(A) = [K]_{\overset{*}{\underset{S}{\longleftrightarrow}}}$ or not.*

This follows from Theorem 87 and the reduction given in [65, Theorem III.3].

Corollary 185. *Let X be some finite alphabet. Given a finite semi-Thue system S, which is assumed left-basic, confluent, strictly length-reducing and a word $w \in X^*$, one can decide whether S is confluent over w.*

This follows from Theorem 87 and the reduction given in [66, Theorem 5.17]. Let us notice that the same decision-problem becomes *undecidable* if we remove the hypothesis "left-basic" in Corollary 185 ([53] or [66, Proposition 5.32]) and becomes solvable in P-time if we strengthen the hypothesis "left-basic" into "basic" [73, Theorem 3.7].

13.2. Perspectives

13.2.1. Other applications

Some other applications of Theorem 87 seem plausible and interesting:

1. It is known that two graphs Γ, Γ' are bisimilar iff they have a common quotient: $\Gamma \rightarrow \Gamma'' \leftarrow \Gamma'$. In view of Corollary 182 it is natural to ask whether the "quotient-problem" for two rooted deterministic equational graphs (i.e. $\Gamma \rightarrow \Gamma'$?) is decidable. We think it is decidable (work in preparation); the generalisation to *non-deterministic* rooted equational graphs of finite out-degree is open.
2. Corollary 181 might be seen as a result over any algebraic structure, provided that this structure is isomorphic to the magma of infinite trees (the free-interpretation introduced in [52]). In particular, it is possible to find a nice free-interpretation whose domain is $F_p[[X_1, \dots, X_n]]$, the ring of formal power series with n commutative undeterminates and coefficients in the finite field F_p (for a prime p), and with polynomial operators [46].

13.2.2. Extensions

We hope to extend the main ideas of this work to other equivalence problems.

3. Let us recall that the complexity of the equivalence problem is unknown even for the subclass of strict-real-time dpda's (i.e. the dpda's without ε-transition and recognizing by empty stack only). It is not known if the equivalence problem for this subclass is *primitive recursive* ([54, comment p. 11] or [59, last line of first paragraph, p. 689] or [75, conclusion]). Concerning our proof, nothing is said about the function $F(d,n)$ introduced in Section 5 by Eq. (74). As the constants D_2, N_0, C_2, K_5, K_6 are depending on F (see Section 6), our proof just shows *decidability* of the equivalence problem.

   It would be interesting to explore more closely the *complexity* of this problem, by experimental means (Section 12 shows the possibility of computing examples of reasonable size) and by theoretical means too.

   In contrast, let us mention that the equivalence problem for dpda's without ε-transition and with one state only (the so-called "simple" dpda's) has been finally shown to be decidable in *polynomial* time [35] and the equivalence problem for 2-tape deterministic finite automata is also decidable in *polynomial* time [25].

4. Let K be a commutative field, $\mathscr{M}$ be a K-dpda (i.e. a dpda with outputs in the multiplicative group $K-\{0\}$) and V the associated alphabet. It seems plausible that the equivalence between two rational series $S, S' \in \mathsf{K}\langle\langle V \rangle\rangle$ which are $\smile$-deterministic but *not necessarily* V-deterministic, remains decidable. Notice that, in this case, the supports of $\varphi(S)$, $\varphi(S')$ need not be context-free languages.
5. One could investigate which groups (or even monoids) H have the property that the equivalence problem for deterministic pushdown transducers $X^* \to H$, remains decidable. The case where H is a free group of rank $\geqslant 2$ is particularly interesting. A positive result for free groups (work in preparation) will of course imply the decidability of the equivalence problem for deterministic pushdown transducers $X^* \to Y^*$. Partial results in this last direction were proved in [20,37,77].
6. The extension of Corollary 182 to rooted equational graphs of finite out-degree is done in [68]. The general case of rooted equational graphs (without restriction on the out-degree) is open (this problem is raised in [6,75]).
7. One can think of generalizing our results to automata with a more general kind of "storage type". For example, various notions of pushdowns of pushdowns are defined in [22,23,45,81,82]) and might be studied from this point of view.
8. One can think of generalizing our results on polyadic recursion schemes (Corollary 181) to higher-level recursion-schemes (such general schemes are defined for example in [21,27]; from this point of view, recursive schemes appear to be just the level 1 recursion-schemes). A link with perspective 7 above can be expected (such a link is explicitely conjectured by J. Gallier in [27, p. 773]).
9. Let us recall that the isomorphism problem for *equational* graphs has been solved in [16,18] while our Corollary 181 amounts to solve the isomorphism problem for *algebraic* ordered trees. It is tempting to try to unify both results into a decidability result for *algebraic* graphs. This class of graphs has to be defined properly; it might be the infinite graphs which are the values of some "infinite algebraic term" in the magma of graphs (see [3,17] for a definition of this magma). A link with perspectives 7 and 8 is expected too.
10. We feel that the proof of Theorem 87, its generalization to coefficients in H^0 (Theorem 177) and the main result of [68] demonstrate the usefulness of the notion of *deterministic space*. The author introduced this notion as a systematization of previous ideas from [34,47,48] (the central idea of [75] is not far either).

    In turn, it might be fruitful for further generalizations, to develop a systematic, *general study of deterministic spaces*. The paragraph "dimension" of Section 3.3 of [68] goes this way. A comparison with the classical notion of left vector space over a division-ring (which is central in the elegant work of [32]) would perhaps shed new light on deterministic spaces.

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Note added in proof. Since the manuscript was submitted, some progress has been made on the subject:

- a short exposition of the result and methods of [68] has been given in [72],
- the equivalence problem for deterministic pushdown transducers from a free monoid into a free group has been solved positively in [74] (this is a partial solution of our perspective 5 of Section 13.2),
- C. Stirling has found [76] some nice simplifications in the proof of Theorem 87:
  - Instead of the generating set $\mathscr{G}_1$ that we build in Section 8, he constructs a generating set $\mathscr{G}$ of the form $\{S_0^\alpha \odot u \mid 0 \leqslant |u| \leqslant K\}$, for some suitable $\alpha \in \{-,+\}$, $K \in \mathbb{N}$. As this definition is not "geometric" anymore (it does not use the notions of "height" or "defect"), it is possible to make the initial deterministic grammar *proper* and *reduced*. Consequently, all the technicalities concerning ρ_ε, in particuliar the distinction between $\otimes$ and $\odot$, can be avoided.
  - Instead of manipulating directly rational series, C. Stirling prefers to introduce another alphabet of "second-level undeterminates" representing such series. In this way, he essentially avoids the consideration of the *norm* of series.

Let us mention that, after minor adaptation, the improvements of [76] can be applied to [72,74] as well.

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