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Compactly Supported Orthogonal Symmetric Scaling Functions

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Daubechies (1988, *Comm. Pure Appl. Math.* **41**, 909–996) showed that, except for the Haar function, there exist no compactly supported orthogonal symmetric scaling functions for the dilation $q = 2$. Nevertheless, such scaling functions do exist for dilations $q > 2$ (as evidenced by Chui and Lian's construction (1995, *Appl. Comput. Harmon. Anal.* **2**, 68–84) for $q = 3$); these functions are the main object of this paper. We construct new symmetric scaling functions and introduce the "Batman" family of continuous symmetric scaling functions with very small supports. We establish the exact smoothness of the "Batman" scaling functions using the joint spectral radius technique.

Key Words: wavelets; orthogonal scaling function; symmetric scaling function

1. INTRODUCTION

Compactly supported wavelets are typically constructed from a compactly supported single scaling function that generates a multiresolution analysis [5, 14]. It is important (and nontrivial) to construct scaling functions (and hence wavelets) with desirable properties, such as orthogonality, high regularity, symmetry, and small support.

Recall that a *multiresolution analysis* (MRA) with dilation factor q, where $q \in \mathbb{Z}$ and $q > 1$, is a sequence of nested subspaces of $L^2(\mathbb{R})$

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

such that

$$V_j = \operatorname{span}\{ f(q^j x - k) : k \in \mathbb{Z} \}$$

for some $f(x) \in L^2(\mathbb{R})$, and

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}), \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

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The function $f(x)$ is called the *scaling function* of the MRA. We shall consider only multiresolution analyses generated by a compactly supported scaling function $f(x)$ with orthogonal integer translates. Such an *orthogonal* scaling function generates an orthogonal MRA which leads to an orthonormal wavelet basis for $L^2(\mathbb{R})$ [14].

Let $f(x) \in L^2(\mathbb{R})$ be a compactly supported orthogonal scaling function of a MRA with dilation q. Then $\int_{\mathbb{R}} f(x)\,dx \neq 0$, and $f(x)$ satisfies a *dilation equation*

$$f(x) = \sum_{k\in\mathbb{Z}} c_k f(qx-k), \qquad \sum_{k\in\mathbb{Z}} c_k = q,$$

where c_k are real, and $c_k \neq 0$ for only finitely many $k \in \mathbb{Z}$ [6]. Since the set $\{f(x-k) : k \in \mathbb{Z}\}$ is orthogonal in $L^2(\mathbb{R})$, the coefficients $\{c_k\}$ must satisfy

$$\sum_{k\in\mathbb{Z}} c_k c_{k+qj} = q\delta_{0,j}, \qquad \forall j \in \mathbb{Z}, \tag{1.1}$$

where $\delta_{0,j}$ is the Kronecker symbol. The converse is not true, though; in order for the integer translates of $f(x)$ to be orthogonal, certain conditions (often overlooked in the study of wavelets) in addition to (1.1) must be met [10].

First, Daubechies [5] constructed a family of minimally supported orthogonal scaling functions for dilation $q = 2$ and studied their asymptotically increasing smoothness using Fourier analytic methods. Then, Heller [11], Steffen et al. [15], and Welland and Lundberg [17] constructed compactly supported orthogonal scaling functions for dilations $q > 2$ (we describe Heller's construction briefly in Section 3).

In applications, such as digital imaging, it is often desirable to use scaling functions that are symmetric. Daubechies [5] showed that if $q = 2$, then the only symmetric orthogonal scaling function is the Haar function $\chi_{[0,1)}$. In order to construct symmetric orthogonal scaling functions, one has to consider dilations $q > 2$ (a construction for $q = 3$ is due to Chui and Lian [3]). An alternative approach for $q = 2$ is to give up orthogonality and consider *nearly orthogonal* symmetric scaling functions [1, 12]. Construction of orthogonal symmetric scaling functions for arbitrary dilations $q > 2$ is the main object of this paper.

In Section 2 we introduce definitions and basic results on scaling functions and scaling sequences. In Section 3 we restate Heller's explicit general formula for orthogonal scaling sequences [11]. We use this formula to construct *symmetric* orthogonal scaling functions for an arbitrary dilation $q \geq 3$ in Section 4. We establish necessary and sufficient conditions for scaling functions to be symmetric, based on the modulus of their symbols. Finally, in Section 5 we introduce a new family of symmetric orthogonal scaling functions with short support (the "Batman" family) and compute their smoothness using the joint spectral radius of matrices.

2. PRELIMINARY RESULTS

Fix an integer $q \geq 2$. Let $\mathcal{S}_q(\mathbb{R})$ denote the set of all real sequences $\mathbf{c} = \{c_k : k \in \mathbb{Z}\}$, such that $\sum_{k\in\mathbb{Z}} c_k = q$ and $c_k = 0$ for all but finitely many $k \in \mathbb{Z}$. It is known [7] that for each $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ there exists a unique compactly supported $\Phi_{\mathbf{c}}(x)$ (in the sense of tempered distribution) satisfying

$$\Phi_{\mathbf{c}}(x) = \sum_{k\in\mathbb{Z}} c_k \Phi_{\mathbf{c}}(qx-k) \qquad \text{for almost all } x \in \mathbb{R}, \qquad \text{and} \qquad \hat{\Phi}_{\mathbf{c}}(0) = 1. \tag{2.1}$$

We call $\Phi_{\mathbf{c}}(x)$ the *scaling function corresponding to* $\mathbf{c}$.

DEFINITION 2.1. The *symbol* of $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ is the trigonometric polynomial $M_{\mathbf{c}}(\omega) = (1/q)\sum_{k\in\mathbb{Z}} c_k e^{ik\omega}$. A sequence $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ is q*-orthogonal* if

$$\sum_{k\in\mathbb{Z}} c_k c_{k+qj} = \begin{cases} q & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases} \tag{2.2}$$

We recall some well-known properties of q-orthogonal sequences. A proof of (i) can be found in Gröchenig [10]; Chui and Lian [3] proved (ii).

PROPOSITION 2.2. (i) $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ *is a* q*-orthogonal sequence if and only if*

$$\sum_{k=0}^{q-1} \left| M_{\mathbf{c}}\left(\omega + \frac{2\pi k}{q}\right)\right|^2 = 1, \qquad \textit{for all } \omega \in \mathbb{R}.$$

(ii) *If the sequence* $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ *is* q*-orthogonal, then* $\sum_{j\in\mathbb{Z}} c_{k+qj} = 1$ *for all* $k \in \mathbb{Z}$.

We define two transformations on $\mathcal{S}_q(\mathbb{R})$, the *translation* τ_n for a given $n \in \mathbb{Z}$ and the *reflection* γ, by

$$\tau_n(\{c_k\}) := \{c_{k-n}\} \qquad \text{and} \qquad \gamma(\{c_k\}) := \{c_{-k}\}.$$

The corresponding scaling functions satisfy

$$\Phi_{\gamma(\mathbf{c})}(x) = \Phi_{\mathbf{c}}(-x), \qquad \Phi_{\tau_n(\mathbf{c})}(x) = \Phi_{\mathbf{c}}\left(x + \frac{n}{q-1}\right), \qquad n \in \mathbb{Z}.$$

We also define the *convolution* of $\mathbf{b} = \{b_k\}$ and $\mathbf{c} = \{c_k\}$ in $\mathcal{S}_q(\mathbb{R})$ by

$$\mathbf{b} * \mathbf{c} := \left\{ \frac{1}{q} \sum_i b_i\, c_{k-i} : k \in \mathbb{Z} \right\}.$$

Please note the extra factor $1/q$. It follows that $\mathbf{b} * \mathbf{c} \in \mathcal{S}_q(\mathbb{R})$, and $M_{\mathbf{b}*\mathbf{c}}(\omega) = M_{\mathbf{b}}(\omega) M_{\mathbf{c}}(\omega)$.

DEFINITION 2.3. We say that $\mathbf{b}$ and $\mathbf{c}$ in $\mathcal{S}_q(\mathbb{R})$ are *equivalent* and denote it by $\mathbf{b} \sim \mathbf{c}$, if $\mathbf{c} = \tau_n(\mathbf{b})$, or $\mathbf{c} = \tau_n \circ \gamma(\mathbf{b})$, for some $n \in \mathbb{Z}$.

THEOREM 2.4. *Let* $\mathbf{b}, \mathbf{c} \in \mathcal{S}_q(\mathbb{R})$. *Then* $|M_{\mathbf{b}}(\omega)|^2 = |M_{\mathbf{c}}(\omega)|^2$ *if and only if there exist* $\mathbf{a}, \mathbf{e}, \mathbf{e}' \in \mathcal{S}_q(\mathbb{R})$, *such that*

$$\mathbf{b} = \mathbf{a} * \mathbf{e}, \qquad \mathbf{c} = \mathbf{a} * \mathbf{e}', \qquad \textit{and} \qquad \mathbf{e} \sim \mathbf{e}'. \tag{2.3}$$

Proof. Suppose (2.3) holds. Then $M_{\mathbf{e}'}(\omega) = e^{in\omega} M_{\mathbf{e}}(\omega)$ if $\mathbf{e}' = \tau_n(\mathbf{e})$, and $M_{\mathbf{e}'}(\omega) = e^{in\omega} M_{\mathbf{e}}(-\omega)$ if $\mathbf{e}' = \tau_n \circ \gamma(\mathbf{e})$. In either case, $|M_{\mathbf{b}}(\omega)|^2 = |M_{\mathbf{a}}(\omega)|^2 = |M_{\mathbf{c}}(\omega)|^2$.

Conversely, suppose that $|M_{\mathbf{b}}(\omega)|^2 = |M_{\mathbf{c}}(\omega)|^2$. Without loss of generality we shall assume that $b_0 \neq 0$, and $b_i = 0$ for all $i < 0$, for otherwise we can consider an equivalent (shifted) sequence with this property; we shall assume the same for $\mathbf{c}$. Make the substitution $z = e^{i\omega}$, let $B(z) = M_{\mathbf{b}}(\omega)$ and $C(z) = M_{\mathbf{c}}(\omega)$, and define $\tilde{B}(z) = z^n B(1/z)$, where $n = \deg(B)$. Now the assumption reads $B(z)\tilde{B}(z) = C(z)\tilde{C}(z)$. Note that $B(z)$ and $C(z)$ must have the same degree and hence the same number of zeros (counted with their multiplicity). Let $A(z) = \gcd(B(z), C(z))$. Then $B(z) = A(z)E(z)$ and $C(z) = A(z)E'(z)$ for some $E(z), E'(z) \in \mathbb{R}[z]$. By assumption, $A(z)E(z)\tilde{A}(z)\tilde{E}(z) = A(z)E'(z)\tilde{A}(z)\tilde{E}'(z)$. Since $\gcd(E(z), E'(z)) = 1$, we obtain that $E'(z) = \tilde{E}(z)$. Now (2.3) follows immediately by letting $M_{\mathbf{a}}(\omega) = A(e^{i\omega})$, $M_{\mathbf{e}}(\omega) = E(e^{i\omega})$, $M_{\mathbf{e}'}(\omega) = E'(e^{i\omega})$, and observing that $E'(z) = \tilde{E}(z)$ implies $\mathbf{e}' \sim \mathbf{e}$. ■

Theorem 2.4 suggests that the q-orthogonal sequences can be classified by the square of the modulus of their symbols.

Although Mallat [14] proved the following theorem for $q = 2$, the proof generalizes easily to all $q > 1$.

THEOREM 2.5. *Let $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ be a q-orthogonal sequence. Then $\Phi_{\mathbf{c}}(x) \in L^2(\mathbb{R})$.*

3. SCALING SEQUENCES OF ARBITRARILY HIGH ACCURACY

In this short section we recall Heller's classification [11] of all q-orthogonal sequences for any given accuracy $r \geq 1$ (see also [15, 17]). Let q be a fixed integer and $q \geq 2$. The simplest q-orthogonal sequence is the *Haar sequence* $\mathbf{h} = \{h_k\}$, where $h_k = 1$ for $0 \leq k < q$ and $h_k = 0$ otherwise. We denote its symbol by $H(\omega) := (1/q)\sum_{k=0}^{q-1} e^{ik\omega}$.

A sequence $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ is *r-accurate*, or *having accuracy r*, if $M_{\mathbf{c}}(\omega) = H^r(\omega)G(\omega)$ for some trigonometric polynomial $G(\omega)$. Proposition 2.2 states that every q-orthogonal sequence is at least 1-accurate and that a sequence c is q-orthogonal and has accuracy r if and only if there exists a real trigonometric polynomial $G(\omega)$ such that

$$\sum_{k=0}^{q-1} \left| H\left(\omega + \frac{2\pi k}{q}\right)\right|^{2r} \left| G\left(\omega + \frac{2\pi k}{q}\right)\right|^2 = 1. \tag{3.1}$$

There are infinitely many such polynomials G, which becomes evident from the following lemma by Heller [11, Theorem 3.3].

LEMMA 3.1 (Heller).

$$\sum_{k=0}^{q-1} \left| H\left(\omega + \frac{2\pi k}{q}\right)\right|^{2r} \left| G\left(\omega + \frac{2\pi k}{q}\right)\right|^2 = 0$$

if and only if

$$G(\omega) = (1 - \cos\omega)^r \sum_{n \neq qk} c_n e^{in\omega},$$

where $c_n = 0$ for all but finitely many n.

Because of Lemma 3.1, we only need a "particular solution" G_r to (3.1). Heller [11] derived the following formula for such a solution, namely, the minimal degree solution to (3.1);

$$G_r(\omega) := \sum_{n=0}^{r-1} p_n (1-\cos\omega)^n, \tag{3.2}$$

where the coefficients p_n are given by

$$p_n = \frac{q^{2r}}{2^{r(q-1)}} \sum_{k_1+\cdots+k_{q_1}=n} \prod_{j=1}^{q_1} \binom{k_j+2r-1}{2r-1} \left(1-\cos\frac{2\pi k_j}{q}\right)^{-k_j-2r} \tag{3.3}$$

for $q = 2q_1+1$, and

$$p_n = \frac{q^{2r}}{2^{r(q-1)}} \sum_{k_0+k_1+\cdots+k_{q_1}=n} \binom{k_0+r-1}{r-1} \prod_{j=1}^{q_1} \binom{k_j+2r-1}{2r-1} \left(1-\cos\frac{2\pi k_j}{q}\right)^{-k_j-2r} \tag{3.4}$$

for $q = 2q_1+2$. The following theorem summarizes these results and characterizes all q-orthogonal sequences with given accuracy.

THEOREM 3.2 (Heller [11]). *Let $P(\omega)$ be a trigonometric polynomial. Then $P(\omega) = |M_{\mathbf{c}}(\omega)|^2$ for some q-orthogonal sequence $\mathbf{c}$ of accuracy at least r if and only if $P(\omega) \geq 0$ for all $\omega \in \mathbb{R}$ and $P(\omega) = |H(\omega)|^{2r} G(\omega)$ for some $G(\omega)$ of the form*

$$G(\omega) = G_r(\omega) + (1-\cos\omega)^r \sum_{n\neq qk} c_n \cos n\omega,$$

where $G_r(\omega)$ is defined by (3.2)–(3.4).

We shall use Theorem 3.2 to construct symmetric scaling functions for any given accuracy r.

4. SYMMETRIC SCALING FUNCTIONS

A sequence $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ is *symmetric* if $\mathbf{c} = \tau_n \circ \gamma(\mathbf{c})$ for some $n \in \mathbb{Z}$. A function $f(x)$ is *symmetric* if $f(x) = f(a-x)$ for some $a \in \mathbb{R}$.

LEMMA 4.1. *Let $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$. Then $\mathbf{c}$ is symmetric if and only if $\Phi_{\mathbf{c}}(x)$ is.*

Proof. Suppose that $\mathbf{c}$ is symmetric. Then $\mathbf{c} = \tau_n \circ \gamma(\mathbf{c})$ for some $n \in \mathbb{Z}$. Therefore

$$\Phi_{\mathbf{c}}(x) = \Phi_{\tau_n\circ\gamma(\mathbf{c})}(x) = \Phi_{\mathbf{c}}\left(\frac{n}{q-1} - x\right),$$

and so $\Phi_{\mathbf{c}}(x)$ is symmetric.

Conversely, suppose that $\Phi_{\mathbf{c}}(x)$ is symmetric and $\Phi_{\mathbf{c}}(x) = \Phi_{\mathbf{c}}(a-x)$. Without loss of generality, assume that $\mathbf{c} = \{c_k : k \in \mathbb{Z}\}$ so that $c_0 c_n \neq 0$ and $c_k = 0$ for all $k \notin [0, n]$. Then $\Phi_{\mathbf{c}}(x)$ is supported exactly on $[0, n/(q-1)]$, forcing $a = n/(q-1)$. Hence,

$\Phi_{\mathbf{c}}(x) = \Phi_{\tau_n\circ\gamma(\mathbf{c})}(x)$. We claim that $\mathbf{c} = \tau_n \circ \gamma(\mathbf{c})$. Indeed, note that

$$\hat{\Phi}_{\mathbf{c}}(\omega) = M_{\mathbf{c}}\left(\frac{\omega}{q}\right)\hat{\Phi}_{\mathbf{c}}\left(\frac{\omega}{q}\right), \qquad \text{and} \qquad \hat{\Phi}_{\tau_n\circ\gamma(\mathbf{c})}(\omega) = M_{\tau_n\circ\gamma(\mathbf{c})}\left(\frac{\omega}{q}\right)\hat{\Phi}_{\tau_n\circ\gamma(\mathbf{c})}\left(\frac{\omega}{q}\right).$$

Therefore,

$$M_{\mathbf{c}}\left(\frac{\omega}{q}\right) = M_{\tau_n\circ\gamma(\mathbf{c})}\left(\frac{\omega}{q}\right)$$

for all $\omega \in \mathbb{R}$, which yields $\mathbf{c} = \tau_n \circ \gamma(\mathbf{c})$. ■

THEOREM 4.2. (i) *Suppose that* $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ *is symmetric. Let* $|M_{\mathbf{c}}(\omega)|^2 = P(\cos\omega)$, *where* $P(t) \in \mathbb{R}[t]$. *Then*

$$P(t) = G^2(t) \qquad \textit{or} \qquad P(t) = \left(\frac{1+t}{2}\right)G^2(t), \tag{4.1}$$

for some $G(t) \in \mathbb{R}[t]$, $G(1) = 1$.

(ii) *Conversely, suppose that* $P(t) = G^2(t)$, *or* $P(t) = ((1+t)/2)G^2(t)$, *where* $G(t) \in \mathbb{R}[t]$ *and* $G(1) = 1$. *Then there exists a unique (up to equivalence) symmetric* $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$, *such that* $|M_{\mathbf{c}}(\omega)|^2 = P(\cos\omega)$.

Proof. (i) If $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ is symmetric, then $\mathbf{c} = \tau_n \circ \gamma(\mathbf{c})$ for some $n \in \mathbb{Z}$, and $M_{\mathbf{c}}(\omega) = e^{in\omega}M_{\mathbf{c}}(-\omega)$. Hence,

$$|M_{\mathbf{c}}(\omega)|^2 = M_{\mathbf{c}}(\omega)\cdot M_{\mathbf{c}}(-\omega) = e^{-im\omega}M_{\mathbf{c}}^2(\omega) = \left(e^{-in\omega/2}M_{\mathbf{c}}(\omega)\right)^2.$$

Since $|M_{\mathbf{c}}(\omega)|^2 \geq 0$ is real, the imaginary part of $e^{-in\omega/2}M_{\mathbf{c}}(\omega)$ must be 0. Now, if $n = 2k$, then

$$e^{-in\omega/2}M_{\mathbf{c}}(\omega) = \mathrm{Re}\left(e^{-ik\omega}M_{\mathbf{c}}(\omega)\right) = G(\cos\omega)$$

for some $G(t) \in \mathbb{R}[t]$. Hence, $|M_{\mathbf{c}}(\omega)|^2 = G^2(\cos\omega)$. If $n = 2k+1$, then

$$e^{-in\omega/2}M_{\mathbf{c}}(\omega) = \mathrm{Re}\left(e^{-i(2k+1)\omega/2}M_{\mathbf{c}}\left(2\cdot\frac{\omega}{2}\right)\right) = \tilde{G}\left(\cos\frac{\omega}{2}\right)$$

for some $\tilde{G}(t) \in \mathbb{R}[t]$. Hence,

$$|M_{\mathbf{c}}(\omega)|^2 = \tilde{G}^2\left(\cos\frac{\omega}{2}\right). \tag{4.2}$$

But $\cos^2(\omega/2) = \frac{1}{2}(1+\cos\omega)$, so

$$\tilde{G}\left(\cos\frac{\omega}{2}\right) = G_1(\cos\omega) + \cos\left(\frac{\omega}{2}\right)\cdot G_2(\cos\omega)$$

for some $G_1(t),\ G_2(t) \in \mathbb{R}[t]$. On the other hand, by (4.2), $|M_{\mathbf{c}}(\omega)|^2 = \tilde{G}^2(\cos(\omega/2)) = P(\cos\omega)$. It follows that either $G_1(t) = 0$ or $G_2(t) = 0$. Since $G_2(t) \neq 0$, it follows that $G_1(t) = 0$. Thus,

$$|M_{\mathbf{c}}(\omega)|^2 = \cos^2\left(\frac{\omega}{2}\right)\cdot G_2^2(\cos\omega) = \frac{1}{2}(1+\cos\omega)\cdot G_2^2(\cos\omega),$$

which proves (i).

(ii) First, we prove the existence. Suppose that $P(t) = G^2(t)$. Then $M_{\mathbf{c}}(\omega) = G(\cos\omega)$ defines a symmetric $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$, because $M_{\mathbf{c}}(-\omega) = M_{\mathbf{c}}(\omega)$. Now suppose that $P(t) = ((1+t)/2)G^2(t)$. Then

$$M_{\mathbf{c}}(\omega) = e^{i\omega/2}\cos\frac{\omega}{2}\cdot G(\cos\omega) = \frac{e^{i\omega}+1}{2\sqrt{2}}\cdot G\left(\frac{e^{i\omega}+e^{-i\omega}}{2}\right) \tag{4.3}$$

defines a symmetric $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$, because $M_{\mathbf{c}}(\omega) = e^{i\omega}M_{\mathbf{c}}(-\omega)$.

Next, we show by contradiction that the symmetric $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ is unique, up to equivalence. Assume that there is another symmetric $\mathbf{c}' \in \mathcal{S}_q(\mathbb{R})$, such that $|M_{\mathbf{c}'}(\omega)|^2 = P(\cos\omega)$. By Theorem 2.4, there exist $\mathbf{a}$, $\mathbf{e}$, $\mathbf{e}' \in \mathcal{S}_q(\mathbb{R})$, such that

$$\mathbf{c} = \mathbf{a} * \mathbf{e}, \qquad \mathbf{c}' = \mathbf{a} * \mathbf{e}', \qquad \text{and} \qquad \mathbf{e} \sim \mathbf{e}'.$$

Therefore there exists some $k \in \mathbb{Z}$, such that

$$M_{\mathbf{c}'}(\omega) = e^{ik\omega}M_{\mathbf{a}}(\omega)M_{\mathbf{e}}(\omega) \qquad \text{or} \qquad M_{\mathbf{c}'}(\omega) = e^{ik\omega}M_{\mathbf{a}}(\omega)M_{\mathbf{e}}(-\omega),$$

depending on the type of equivalence between $\mathbf{e}$ and $\mathbf{e}'$. In the first case, we must have $\mathbf{c}' = \tau_k(\mathbf{c})$, and so $\mathbf{c}$ and $\mathbf{c}'$ are equivalent. In the second case, since both $\mathbf{c}$ and $\mathbf{c}'$ are symmetric, we have

$$M_{\mathbf{a}}(\omega)M_{\mathbf{e}}(\omega) = e^{in_1\omega}M_{\mathbf{a}}(-\omega)M_{\mathbf{e}}(-\omega)$$
$$M_{\mathbf{a}}(\omega)M_{\mathbf{e}}(-\omega) = e^{in_2\omega}M_{\mathbf{a}}(-\omega)M_{\mathbf{e}}(\omega).$$

Hence, $M_{\mathbf{a}}^2(\omega) = e^{i(n_1+n_2)\omega}M_{\mathbf{a}}^2(-\omega)$. This implies that $M_{\mathbf{a}}(\omega) = \pm e^{in\omega}M_{\mathbf{a}}(-\omega)$, where $n = (n_1+n_2)/2$ is clearly an integer. Since $M_{\mathbf{a}}(0) = 1$, we have $M_{\mathbf{a}}(\omega) = e^{in\omega}M_{\mathbf{a}}(-\omega)$. The equivalence of $\mathbf{c}$ and $\mathbf{c}'$ now follows from

$$M_{\mathbf{c}'}(\omega) = e^{ik\omega}M_{\mathbf{a}}(\omega)M_{\mathbf{e}}(-\omega) = e^{i(k+n)\omega}M_{\mathbf{c}}(-\omega). \ \blacksquare$$

Remark. It is possible for a *nonsymmetric* $\mathbf{c} \in \mathcal{S}_q(\mathbb{R})$ to satisfy (4.1). For example, let $c_0 = 4q$, $c_1 = -4q$, $c_2 = q$, and all other $c_k = 0$. This $\mathbf{c}$ is obviously nonsymmetric; nevertheless, $|M_{\mathbf{c}}(\omega)|^2 = (5 - 4\cos\omega)^2$.

EXAMPLE 4.1. For accuracy $r = 1$ and arbitrary $q > 3$, Theorem 3.2 implies that $|M_{\mathbf{c}}(\omega)|^2 = |H(\omega)|^2 G(\omega)$, where

$$G(\omega) = 1 + (1 - \cos\omega)\sum_{n \neq qk} c_n \cos n\omega.$$

Choosing $G(\omega) = 1 + (1-\cos\omega)(c_1\cos\omega + c_2\cos 2\omega)$ and applying (4.1) and (4.3), we obtain two scaling sequences $\mathbf{c}_1$ and $\mathbf{c}_2$, given by

$$M_{\mathbf{c}_1}(\omega) = \frac{1}{2}H(\omega)\left(\alpha + (1-\alpha)e^{i\omega} + (1-\alpha)e^{i2\omega} + \alpha e^{i3\omega}\right)$$
$$M_{\mathbf{c}_2}(\omega) = \frac{1}{2}H(\omega)\left(\beta + (1-\beta)e^{i\omega} + (1-\beta)e^{i2\omega} + \beta e^{i3\omega}\right),$$

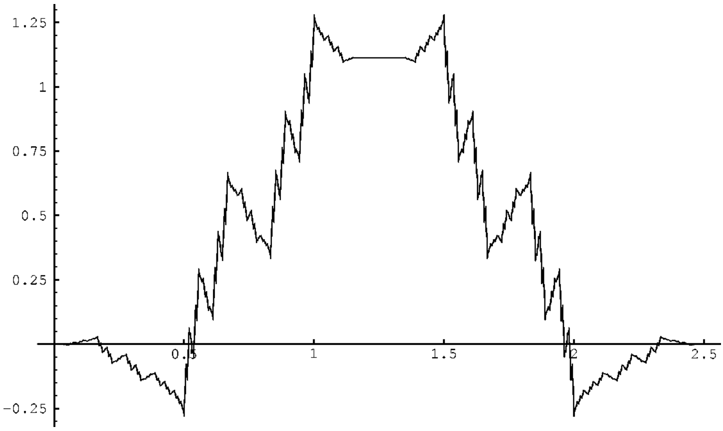


FIG. 1. The "Batman" scaling function $\Phi_{\mathbf{c}_1}$ for $q = 3$ (Example 4.1).

where

$$\alpha = \frac{1}{2} - \frac{\sqrt{6}}{4} \qquad \text{and} \qquad \beta = \frac{1}{2} + \frac{\sqrt{6}}{4}.$$

The scaling sequence $\mathbf{c}_1$ corresponds to the continuous "*Batman*" scaling function (Fig. 1), while $\mathbf{c}_2$ corresponds to a discontinuous scaling function (Fig. 2). For $q = 3$, the corresponding two wavelets (symmetric and antisymmetric) are shown in Fig. 3. In Section 5, we study the "Batman" function in more detail.

EXAMPLE 4.2. Consider q-orthogonal sequences $\mathbf{c}$ for dilation $q = 5$ and accuracy $r = 2$. Choose

$$\begin{aligned} G(\omega) &= 1 + 8(1 - \cos\omega) + (1 - \cos\omega)^2 \big(a_1 \cos\omega + a_2 \cos 2\omega\big) \\ &= 1 + 8t + t^2 + (a_1 + a_2)t^2 - (a_1 + 4a_2)t^3 + 2a_1 t^4, \end{aligned}$$

where $t := 1 - \cos\omega$.

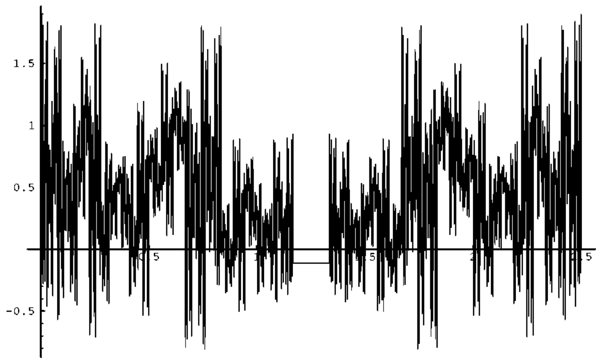
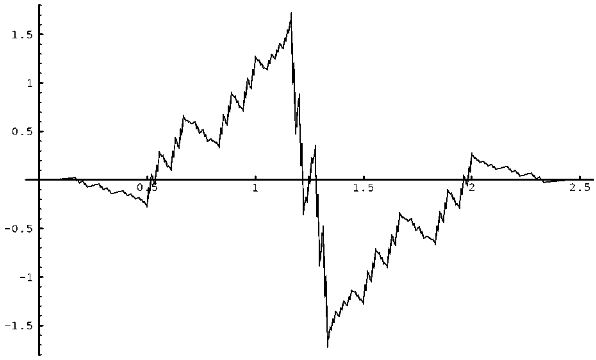


FIG. 2. The (discontinuous) scaling function $\Phi_{\mathbf{c}_2}$ in Example 4.1.



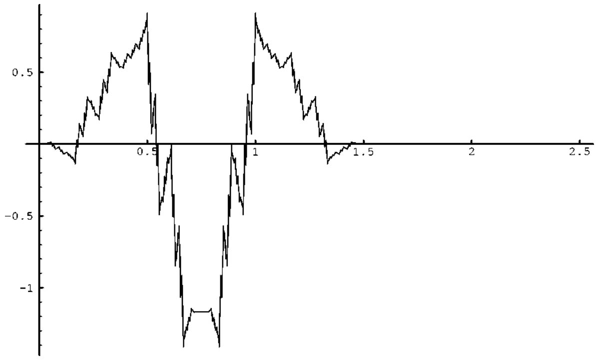


FIG. 3. Two "Batman" wavelets (long and short) for dilation $q = 3$.

By Theorem 3.2, the symbol $M_{\mathbf{c}}(\omega)$ of $\mathbf{c}$ satisfies $|M_{\mathbf{c}}(\omega)|^2 = |H(\omega)|^4 G(\omega)$. Solving for a_1, a_2 to complete the square for $G(\omega)$, we obtain two solutions:

$$G^{(1)}(\omega) = \left(1 + 4t - 4t^2\right)^2 \qquad \text{and} \qquad G^{(2)}(\omega) = \left(1 + 4t - \tfrac{8}{3}t^2\right)^2.$$

The two symmetric scaling sequences are:

$$\mathbf{c}_1 = \frac{1}{5}\{-1, 0, 0, 2, 3, 6, 5, 6, 3, 2, 0, 0, -1\} \tag{4.4}$$

$$\mathbf{c}_2 = \frac{1}{15}\{-2, -2, 1, 6, 9, 16, 19, 16, 9, 6, 1, -2, -2\}. \tag{4.5}$$

Figures 4 and 5 depict the corresponding continuous scaling functions. As we shall see in Section 5, only $\Phi_{\mathbf{c}_2}(x)$ is differentiable.

EXAMPLE 4.3. As r grows, it becomes increasingly harder to find symmetric scaling sequences by hand. Fortunately, Theorem 4.2 can be aided by standard software tools, such as `Mathematica`. Figure 6 shows two minimal support symmetric scaling functions in the case $q = 4$ and $r = 3$; the polynomial $P(t)$, defined in Theorem 4.2, has the form $P(t) = \frac{1}{2}(1 + t)G^2(t)$.

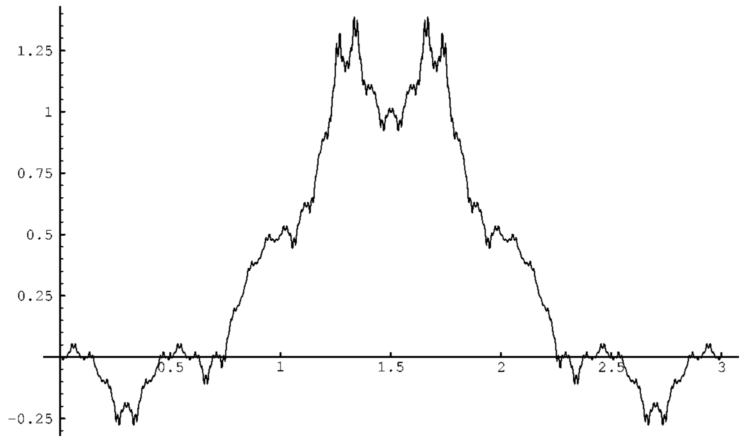


FIG. 4. Continuous scaling function $\Phi_{\mathbf{c}_1}$, $q = 5$, $r = 2$ (Example 4.2).

5. THE "BATMAN" SCALING FUNCTION

Let $q \geq 3$. In Example 4.1 (Fig. 1) we introduced the "Batman" scaling function, which corresponds to the q-orthogonal sequence

$$\mathbf{c} = \left\{\alpha, \frac{1}{2}, 1-\alpha, \underbrace{1, \ldots, 1}_{q-3}, 1-\alpha, \frac{1}{2}, \alpha\right\}, \qquad \text{where } \alpha = \frac{1}{2} - \frac{\sqrt{6}}{4}. \tag{5.1}$$

The refinement equation has the form

$$\begin{aligned} f(x) = \alpha f(qx) + \frac{1}{2} f(qx-1) + (1-\alpha) f(qx-2) + f(qx-3) + \cdots \\ + f(qx-q+1) + (1-\alpha) f(qx-q) + \frac{1}{2} f(qx-q-1) + \alpha f(qx-q-2). \end{aligned}$$

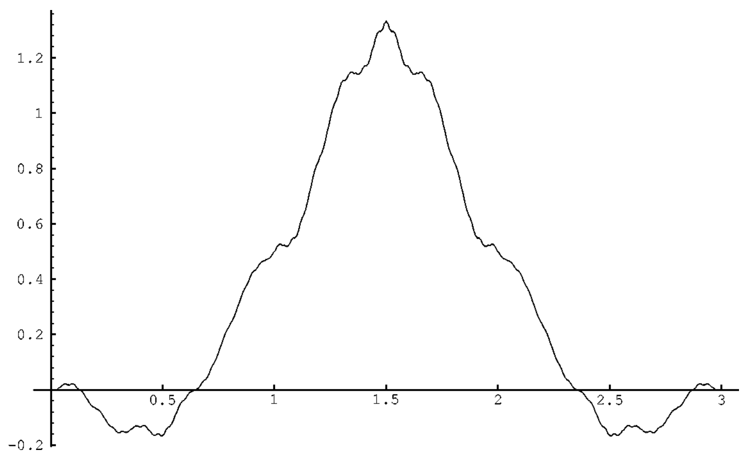


FIG. 5. Smooth scaling function $\Phi_{\mathbf{c}_2}$ for $q = 5$, $r = 2$ (Example 4.2).

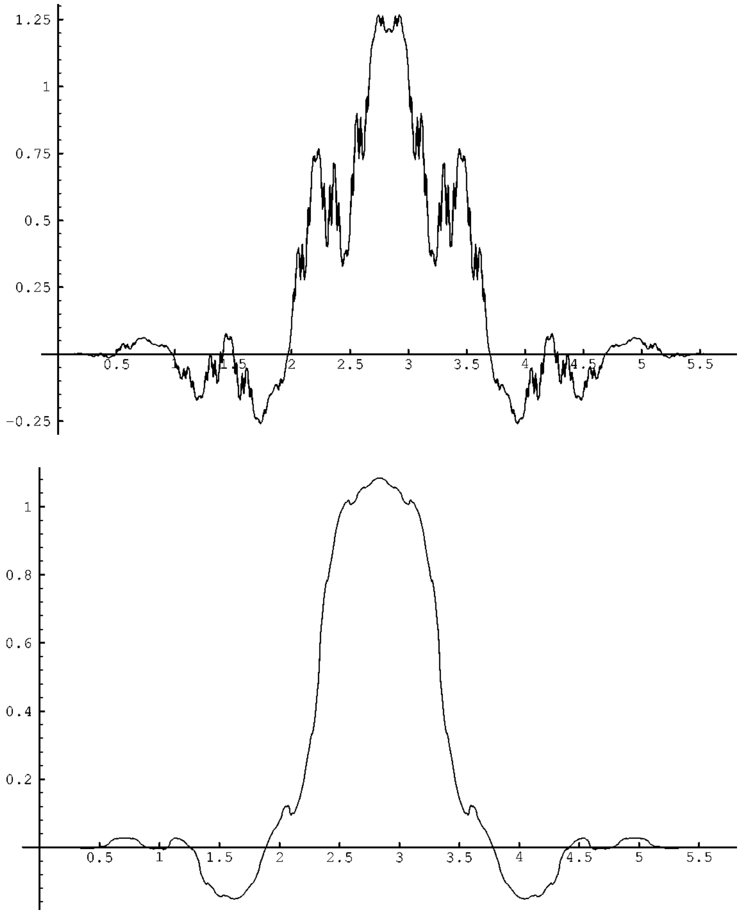


FIG. 6. Two scaling functions for $q = 4$, $r = 3$ (Example 4.3).

Let $\phi_q(x)$ denote the "Batman" scaling function corresponding to the scaling sequence given by (5.1). The support of $\phi_q(x)$ is precisely $[0, (q+2)/(q-1)]$, which yields $[0, 2.5]$ for $q = 3$, and $[0, 2]$ for $q = 4$.

In what follows, we show that $\phi_q(x)$ is continuous for every $q \geq 3$. We use the joint spectral radius to compute the Hölder exponent of $\phi_q(x)$. The joint spectral radius technique is presented in more detail by Daubechies and Lagarias [9], Berger and Wang [2], and Lagarias and Wang [13].

Consider the general two-scale dilation equation

$$f(x) = \sum_{n=0}^{N} c_n f(qx - n), \tag{5.2}$$

where $c_0 c_N \neq 0$ and $q \geq 2$. If $f(x)$ is a compactly supported solution to (5.2), then $\operatorname{supp} f = [0, N/(q-1)]$. Let $L = [N/(q-1)]$. Define the L-dimensional vector $\mathbf{v}(x)$ by

$$\mathbf{v}(x) = [f(x), f(x+1), \ldots, f(x+L-1)]^T, \qquad 0 \leq x \leq 1.$$

Denote the space of real $L \times L$ matrices by $\mathcal{M}_L(\mathbb{R})$, and define k matrices $P_k \in \mathcal{M}_L(\mathbb{R})$, $0 \leq k \leq q-1$, by

$$(P_k)_{ij} = c_{q(i-1)-(j-1)+k}. \tag{5.3}$$

Now the dilation equation (5.2) can be rewritten in the form

$$\mathbf{v}(x) = P_{d_1}\mathbf{v}(\sigma x), \tag{5.4}$$

where $x \in [0, 1]$ has the base q expansion

$$x = 0.d_1 d_2 d_3 \ldots, \qquad 0 \leq d_j \leq q-1, \;\; j = 1, 2, \ldots,$$

and σx is the fractional part of qx,

$$\sigma x = 0.d_2 d_3 d_4 \ldots .$$

Iterating (5.4), we obtain

$$\mathbf{v}(x) = P_{d_1} P_{d_2} \cdots P_{d_n} \mathbf{v}(\sigma^n x).$$

By Proposition 2.2, the matrices P_k are *column stochastic*; i.e., the entries in each column add up to one. Hence, the vector $[1, 1, \ldots, 1]$ is a common left 1-eigenvector of all P_k. Therefore, all P_k can be simultaneously block-triangularized by a real nonsingular $L \times L$ matrix Q whose first row is $[1, 1, \ldots, 1]$:

$$QP_kQ^{-1} = \begin{bmatrix} 1 & 0 \\ * & A_k \end{bmatrix}, \qquad 0 \leq k \leq q-1.$$

The following statement by Collela and Heil [4, Theorem 3] and Wang [16, Theorem 2.5] relates the Hölder exponent of the scaling function f to the spectral properties of the matrices A_k; we omit the proof.

PROPOSITION 5.1. *Denote the joint spectral radius* $\hat{\rho}(A_0, A_1, \ldots, A_{q-1})$ *by* $\hat{\rho}$, *and let* $s = \log_q(1/\hat{\rho})$. *Suppose that* $\hat{\rho} < 1$. *Then*

(i) *the scaling function* $f(x)$ *satisfying (5.2) is continuous;*
(ii) $f(x) \in C^{s-\epsilon}$, *but* $f(x) \notin C^{s+\epsilon}$, *for any* $0 < \epsilon < s$;
(iii) $f(x) \in C^s$ *if and only if the semigroup of matrices generated by* $\{A_k/\hat{\rho} : 0 \leq k \leq q-1\}$ *is bounded.*

Now we can compute the exact smoothness of the Batman scaling function as a direct application of Proposition 5.1.

THEOREM 5.2. *The "Batman" scaling function* $\phi_q(x)$ *is continuous for each* $q \geq 3$, *and its Hölder exponent is* $\log_q(4/\sqrt{6})$.

Proof. We consider the case $q \geq 4$ first. In this case, $L = [(q+2)/(q-1)] = 2$, and the q matrices $P_0, \ldots, P_{q-1}$ are 2×2 matrices. Let

$$Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$QP_kQ^{-1} = \begin{bmatrix} 1 & 0 \\ * & A_k \end{bmatrix},$$

where all A_k are scalars. By the definition (5.3),

$$P_k = \begin{bmatrix} c_k & c_{k-1} \\ c_{q+k} & c_{q+k-1} \end{bmatrix},$$

and a straightforward computation yields

$$\hat{\rho}(A_0, A_1, \ldots, A_{q-1}) = \max(|A_0|, |A_1|, \ldots, |A_{q-1}|) = \frac{\sqrt{6}}{4}.$$

By Proposition 5.1, the theorem is proved for $q \geq 4$.

In the case $q = 3$, the support of $\phi_3(x)$ is $[0, 2.5]$, so $L = 3$ and the matrices P_k are 3×3. We have

$$QP_kQ^{-1} = \begin{bmatrix} 1 & 0 \\ * & A_k \end{bmatrix}, \qquad \text{by taking} \qquad Q = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here,

$$A_0 = \begin{bmatrix} \alpha & 0 \\ \alpha & \frac{1}{2} - \alpha \end{bmatrix}, \qquad A_1 = \begin{bmatrix} \frac{1}{2} - \alpha & \alpha \\ 0 & \alpha \end{bmatrix}, \qquad A_2 = \begin{bmatrix} \frac{1}{2} - \alpha & \frac{1}{2} - \alpha \\ 0 & 0 \end{bmatrix},$$

and $\alpha = 1/2 - c\sqrt{6}/4$. Note that

$$\|A\|_1 := \max\left\{|a_{11}| + |a_{21}|,\ |a_{12}| + |a_{22}|\right\}$$

defines a matrix norm on $\mathcal{M}_2(\mathbb{R})$ (actually, the operator norm induced by the l^1-norm in $\mathbb{R}^2$). Since

$$2|\alpha| = \frac{\sqrt{6}}{2} - 1 < \frac{1}{2} - \alpha = \frac{\sqrt{6}}{4},$$

it follows that $\|A_k\|_1 = \frac{1}{2} - \alpha = \sqrt{6}/4$, for $k = 0, 1, 2$. As a result, $\hat{\rho}(A_0, A_1, A_2) \leq \sqrt{6}/4$, and the semigroup generated by $\{4/\sqrt{6}A_k : k = 0, 1, 2\}$ is bounded [2, Lemma II]. On the other hand, $\hat{\rho}(A_0, A_1, A_2) \geq \sqrt{6}/4$, because $\sqrt{6}/4$ is an eigenvalue of A_0. Thus,

$$\hat{\rho} = \hat{\rho}(A_0, A_1, A_2) = \frac{\sqrt{6}}{4},$$

which proves the theorem for $q = 3$. ■

Using the same technique, we can show that the "smooth hat" scaling function (Fig. 5), which corresponds to the 5-orthogonal scaling sequence defined in (4.5), is differentiable.

THEOREM 5.3. *Let* $q = 5$, *and* $\mathbf{c} = \frac{1}{15}\{-2, -2, 1, 6, 9, 16, 19, 16, 9, 6, 1, -2, -2\}$. *The symmetric scaling function* $\Phi_{\mathbf{c}}(x)$ *is differentiable.*

Sketch of a proof. To prove the theorem, we assemble the five matrices P_k, $0 \leq k \leq 4$. A straightforward computation reveals that they have a common left $\frac{1}{5}$-eigenvector $[1, -2, 3]$, in addition to the common left 1-eigenvector $[1, 1, 1]$. The rest of the proof involves simultaneously triangularizing P_k and applying results by Daubechies and Lagarias [8]; we omit the details. ■

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REFERENCES

1. E. Adelson, E. Simoncelli, and R. Hingorani, Orthogonal pyramid transforms for image coding, *in* "Proc. SPIE Visual Comm. and Image Processing II," Cambridge, MA, 1987.
2. M. A. Berger and Y. Wang, Bounded semigroup of matrices, *Linear Algebra Appl.* **166** (1992), 21–27.
3. C. Chui and J. A. Lian, Construction of compactly supported symmetric and antisymmetric orthonormal wavelets with scale $= 3$, *Appl. Comput. Harmon. Anal.* **2** (1995), 68–84.
4. D. Collela and C. Heil, Characterizations of scaling functions: Continuous solutions, *SIAM J. Matrix Anal. Appl.* **15** (1994), 496–518.
5. I. Daubechies, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 909–996.
6. I. Daubechies, "Ten Lectures on Wavelets," CBMS-NSF Regional Conference Series in Applied Mathematic, Vol. 61, SIAM, Philadelphia, 1992.
7. I. Daubechies and J. C. Lagarias, Two-scale difference equations I: existence and global regularity of solutions, *SIAM J. Appl. Math.* **22** (1991), 1388–1410.
8. I. Daubechies and J. C. Lagarias, Two-scale difference equations II: local regularity, infinite products of matrices and fractals, *SIAM J. Appl. Math.* **23** (1992), 1031–1079.
9. I. Daubechies and J. C. Lagarias, Sets of matrices all infinite products of which converge, *Linear Algebra Appl.* **162** (1992), 227–263.
10. K. Gröchenig, Orthogonality criteria for compactly supported scaling functions, *Appl. Comput. Harmon. Anal.* **1** (1994), 242–245.
11. P. Heller, Rank M wavelets with N vanishing moments, *SIAM J. Matrix Anal. Appl.* **16** (1995), 502–519.
12. P. Heller, J. Shapiro, and R. O. Wells, Jr., Optimally smooth symmetric quadrature mirror filters for image coding, *in* "Proc. SPIE 2491, Wavelet Applications for Dual Use," Orlando, FL, April 1995.
13. J. C. Lagarias and Y. Wang, The finiteness conjecture for the generalized spectral radius of a set of matrices, *Linear Algebra Appl.* **214** (1995), 17–42.
14. S. Mallat, Multiresolution analysis and wavelets, *Trans. Amer. Math. Soc.* **315** (1989), 69–88.
15. P. Steffen, P. Heller, R. Gopinath, and C. Burrus, Theory of regular M-band wavelet bases, *IEEE Trans. Signal Processing* **41** (1993), 3497–3511.
16. Y. Wang, Two-scale dilation equations and the mean spectral radius, *Random Comput. Dynam.* **4** (1996), 49–72.
17. G. Welland and M. Lundberg, Construction of compact p-wavelets, *Constr. Approx.* **9** (1993), 347–370.