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Journal of Computational and Applied Mathematics 220 (2008) 347–354

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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Second-order symmetric duality with cone constraints

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Received 17 March 2007; received in revised form 31 May 2007

Abstract

Wolfe and Mond–Weir type second-order symmetric duals are formulated and appropriate duality theorems are established under η-bonvexity/η-pseudobonvexity assumptions. This formulation removes several omissions in an earlier second-order primal dual pair introduced by Devi [Symmetric duality for nonlinear programming problems involving η-bonvex functions, European J. Oper. Res. 104 (1998) 615–621].

MSC: 26A51; 90C30; 49N15

Keywords: Nonlinear programming; Second-order symmetric duality; η-Bonvexity; Cones

1. Introduction

A pair of primal and dual problems in mathematical programming is called symmetric if the dual of the dual is the primal problem, that is, when the dual is recast in the form of primal, its dual is the primal problem. The duality in linear programming is symmetric. It is not so in nonlinear programming in general. Dorn [6] introduced the concept of symmetric duality in quadratic programming. His results were extended to general nonlinear programs in [4] for convex/concave functions and then in [1] over arbitrary cones.

Mangasarian [9] introduced the concept of second-order duality for nonlinear problems. He has also indicated a possible computational advantage of the second-order dual over the first order dual. This motivated several authors [2,5,8,10,12,13] in this field. Recently, Yang et al. [13] studied second-order multiobjective symmetric dual programs and established the duality relations under F-convexity assumptions.

In this paper, we have studied Wolfe and Mond–Weir type second-order symmetric duality over arbitrary cones and proved duality results under η-bonvexity/η-pseudobonvexity assumptions, respectively. This paper removes several omissions in the definitions, models and proofs (see Remarks 2.1, 3.1 and 3.2) in [5]. It may be noted that as emphasized by Chandra and Kumar [3], the study of second-order symmetric duality under η-pseudobonvexity assumptions has to be on the lines of Mond and Weir [10] and not on the lines of Dantzig et al. [4].

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.08.021

2. Notations and preliminaries

Let C_1 and C_2 be closed convex cones with nonempty interiors in R^n and R^m, respectively. For $i = 1, 2$, $C_i^\star$, called the polar of C_i, is defined as follows:

$$C_i^\star = \{z : x^{\mathrm{T}} z \leqslant 0 \text{ for all } x \in C_i\}.$$

Suppose that $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ are open sets such that $C_1 \times C_2 \subset S_1 \times S_2$.

Definition 2.1 (*Suneja et al. [12]*). A twice differentiable function $k : S_1 \times S_2 \mapsto R$ is said to be η_1-bonvex in the first variable at $u \in S_1$, if there exists a function $\eta_1 : S_1 \times S_1 \mapsto R^n$ such that for $v \in S_2$, $r \in R^n$, $x \in S_1$,

$$k(x, v) - k(u, v) \geqslant \eta_1^{\mathrm{T}}(x, u)[\nabla_x k(u, v) + \nabla_{xx} k(u, v) r] - \tfrac{1}{2} r^{\mathrm{T}} \nabla_{xx} k(u, v) r$$

and $k(x, y)$ is said to be η_2-bonvex in the second variable at $v \in S_2$, if there exists a function $\eta_2 : S_2 \times S_2 \mapsto R^m$ such that for $u \in S_1$, $p \in R^m$, $y \in S_2$,

$$k(u, y) - k(u, v) \geqslant \eta_2^{\mathrm{T}}(y, v)[\nabla_y k(u, v) + \nabla_{yy} k(u, v) p] - \tfrac{1}{2} p^{\mathrm{T}} \nabla_{yy} k(u, v) p.$$

Definition 2.2 (*Suneja et al. [12]*). A twice differentiable function $k : S_1 \times S_2 \mapsto R$ is said to be η_1-pseudobonvex in the first variable at $u \in S_1$, if there exists a function $\eta_1 : S_1 \times S_1 \mapsto R^n$ such that for $v \in S_2$, $r \in R^n$, $x \in S_1$,

$$\eta_1^{\mathrm{T}}(x, u)[\nabla_x k(u, v) + \nabla_{xx} k(u, v) r] \geqslant 0 \Longrightarrow k(x, v) \geqslant k(u, v) - \tfrac{1}{2} r^{\mathrm{T}} \nabla_{xx} k(u, v) r$$

and $k(x, y)$ is said to be η_2-pseudobonvex in the second variable at $v \in S_2$, if there exists a function $\eta_2 : S_2 \times S_2 \mapsto R^m$ such that for $u \in S_1$, $p \in R^m$, $y \in S_2$,

$$\eta_2^{\mathrm{T}}(y, v)[\nabla_y k(u, v) + \nabla_{yy} k(u, v) p] \geqslant 0 \Longrightarrow k(u, y) \geqslant k(u, v) - \tfrac{1}{2} p^{\mathrm{T}} \nabla_{yy} k(u, v) p.$$

It has been revealed in [11] by means of an example that the above class of functions is an extension of bonvex functions. For r and p to be zero vectors, the above definitions reduces to that of η-convex/pseudoconvex functions. The examples of such functions are given in [7].

Remark 2.1. It may be noted that the η_i involved in the definitions of η_i-bonvex/ pseudobonvex functions in [5] are taken from $C_i \times C_i \mapsto C_i (i = 1, 2)$. Since every convex function is η_1-bonvex with $\eta_1(x, u) = x - u$. Therefore, taking $\eta_1 : C_1 \times C_1 \mapsto C_1$ amounts to assuming that

$$x \in C_1, \quad u \in C_1 \Rightarrow \eta_1(x, u) = x - u \in C_1,$$

which is not true for a closed convex cone C_1. In particular, if $C_1 = R_+^n$, then $x \geqslant 0$, $u \geqslant 0$ does not imply $x - u \geqslant 0$.

3. Wolfe type second-order symmetric duality

We now consider the following pair of Wolfe type second-order symmetric dual nonlinear programming problems over arbitrary cones:

Primal (*WP*):

$$\begin{aligned} &\text{Minimize} && F(x, y, p) = k(x, y) - y^{\mathrm{T}}(\nabla_y k(x, y) + \nabla_{yy} k(x, y) p) - \tfrac{1}{2} p^{\mathrm{T}} \nabla_{yy} k(x, y) p \\ &\text{subject to} && \nabla_y k(x, y) + \nabla_{yy} k(x, y) p \in C_2^\star, && (3.1) \\ & && x \in C_1. && (3.2) \end{aligned}$$

Dual (*WD*):

$$\begin{aligned} &\text{Maximize} && G(u, v, r) = k(u, v) - u^{\mathrm{T}}(\nabla_x k(u, v) + \nabla_{xx} k(u, v) r) - \tfrac{1}{2} r^{\mathrm{T}} \nabla_{xx} k(u, v) r \\ &\text{subject to} && -\nabla_x k(u, v) - \nabla_{xx} k(u, v) r \in C_1^\star, && (3.3) \\ & && v \in C_2. && (3.4) \end{aligned}$$

Remark 3.1. The pair of primal and dual problems studied in [5] are

Primal (*P*):

$$\begin{aligned} &\text{Minimize} && F(x,y) = L(x,y) - y^{\mathrm{T}}\nabla_y k(x,y) \\ &\text{subject to} && (x,y) \in C_1 \times C_2, \\ & && \nabla_y k(x,y), \nabla_{yy} k(x,y)p \in C_2^{\star}. \end{aligned}$$

Dual (*D*):

$$\begin{aligned} &\text{Maximize} && G(u,v) = N(u,v) - u^{\mathrm{T}}\nabla_x k(u,v) \\ &\text{subject to} && (u,v) \in C_1 \times C_2, \\ & && -\nabla_x k(u,v), -\nabla_{xx} k(u,v)r \in C_1^{\star}, \end{aligned}$$

where $L(x,y) = k(x,y) - \frac{1}{2}p^{\mathrm{T}}\nabla_{yy}k(x,y)p$ and $N(u,v) = k(u,v) - \frac{1}{2}r^{\mathrm{T}}\nabla_{xx}k(u,v)r$.

It may be noted that in the above dual pair, the author has taken two constraints

$$\nabla_y k(x,y) \in C_2^{\star} \quad \text{and} \quad \nabla_{yy}k(x,y)p \in C_2^{\star}$$

instead of the single constraint (3.1) in (WP). However, in the proof of the strong duality theorem the above two constraints have been erroneously taken together in $g(z)$ ($g(z) = \nabla_y k(x,y) + \nabla_{yy}k(x,y)p$) and so instead of two only one Lagrangian multiplier is used while writing the Fritz John necessary optimality conditions. Moreover, if the constraints are taken as in Problems (P) and (D) along with two multipliers, then the strong duality theorem does not follow on the lines of the proof in [5] or the one given in this paper.

3.1. The duality results

We now establish the duality results for (WP) and (WD). It may be noted that with the constraints taken as in (WP), we require bonvexity assumptions instead of pseudobonvexity assumptions taken in [5] to prove the weak duality theorem.

Theorem 3.1. (*Weak duality*). *Let* (x, y, p) *be feasible for the primal problem* (*WP*) *and* (u, v, r) *be feasible for the dual problem* (*WD*). *Let*

(i) $k(., v)$ *be* η_1*-bonvex in the first variable at* u,
(ii) $-k(x, .)$ *be* η_2*-bonvex in the second variable at* y,
(iii) $\eta_1(x,u) + u \in C_1$ *for all* $x \in C_1$,
(iv) $\eta_2(v,y) + y \in C_2$ *for all* $v \in C_2$.

Then $F(x,y,p) \geqslant G(u,v,r)$.

Proof. By η_1-bonvexity of $k(., v)$ and η_2-bonvexity of $-k(x, .)$, we have

$$k(x,v) - k(u,v) \geqslant \eta_1^{\mathrm{T}}(x,u)\{\nabla_x k(u,v) + \nabla_{xx}k(u,v)r\} - \tfrac{1}{2}r^{\mathrm{T}}\nabla_{xx}k(u,v)r, \tag{3.5}$$

$$k(x,y) - k(x,v) \geqslant -\eta_2^{\mathrm{T}}(v,y)\{\nabla_y k(x,y) + \nabla_{yy}k(x,y)p\} + \tfrac{1}{2}p^{\mathrm{T}}\nabla_{yy}k(x,y)p. \tag{3.6}$$

Adding (3.5) and (3.6), we obtain

$$\begin{aligned} k(x,y) - k(u,v) \geqslant{}& \eta_1^{\mathrm{T}}(x,u)\{\nabla_x k(u,v) + \nabla_{xx}k(u,v)r\} - \tfrac{1}{2}r^{\mathrm{T}}\nabla_{xx}k(u,v)r \\ & - \eta_2^{\mathrm{T}}(v,y)\{\nabla_y k(x,y) + \nabla_{yy}k(x,y)p\} + \tfrac{1}{2}p^{\mathrm{T}}\nabla_{yy}k(x,y)p. \end{aligned} \tag{3.7}$$

From (3.3) and hypothesis (iii), we get

$$\{\eta_1(x,u) + u\}^{\mathrm{T}}\{\nabla_x k(u,v) + \nabla_{xx}k(u,v)r\} \geqslant 0$$

or

$$\eta_1^{\mathrm{T}}(x,u)\{\nabla_x k(u,v)+\nabla_{xx}k(u,v)r\}\geqslant -u^{\mathrm{T}}\{\nabla_x k(u,v)+\nabla_{xx}k(u,v)r\}. \tag{3.8}$$

Similarly, (3.1) and hypothesis (iv) yield

$$-\eta_2^{\mathrm{T}}(v,y)\{\nabla_y k(x,y)+\nabla_{yy}k(x,y)p\}\geqslant y^{\mathrm{T}}\{\nabla_y k(x,y)+\nabla_{yy}k(x,y)p\}. \tag{3.9}$$

Inequalities (3.7)–(3.9) give

$$\begin{aligned} k(x,y)-k(u,v)\geqslant & -u^{\mathrm{T}}\{\nabla_x k(u,v)+\nabla_{xx}k(u,v)r\}-\tfrac{1}{2}r^{\mathrm{T}}\nabla_{xx}k(u,v)r \\ & +y^{\mathrm{T}}\{\nabla_y k(x,y)+\nabla_{yy}k(x,y)p\}+\tfrac{1}{2}p^{\mathrm{T}}\nabla_{yy}k(x,y)p, \end{aligned}$$

or

$$F(x,y,p)\geqslant G(u,v,r). \qquad \square$$

Theorem 3.2. (*Strong duality*). *Let $k : R^n \times R^m \mapsto R$ be thrice differentiable and let $(\bar{x},\bar{y},\bar{p})$ be a local optimal solution for (WP). Let*

(i) $\nabla_{yy}k(\bar{x},\bar{y})$ *is nonsingular,*
(ii) *the vector* $\bar{p}^{\mathrm{T}}\nabla_y(\nabla_{yy}k(\bar{x},\bar{y})\bar{p})=0$ *implies* $\bar{p}=0$.

Then $(\bar{x},\bar{y},\bar{r}=0)$ is feasible for (WD) and $F(\bar{x},\bar{y},\bar{p})=G(\bar{x},\bar{y},\bar{r})$.

Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (WP) and (WD), then $(\bar{x},\bar{y},\bar{p}=0)$ and $(\bar{x},\bar{y},\bar{r}=0)$ are global optimal solutions of (WP) and (WD), respectively.

Proof. Since $(\bar{x},\bar{y},\bar{p})$ is a local optimal solution of (WP), by the Fritz John necessary optimality conditions on convex cone domain given in [1], there exist $\lambda_1\in R_+$, $\lambda_2\in C_2$, such that the following conditions are satisfied at $(\bar{x},\bar{y},\bar{p})$:

$$\begin{aligned} &(x-\bar{x})^{\mathrm{T}}\{\lambda_1\nabla_x k(\bar{x},\bar{y})+\nabla_{yx}k(\bar{x},\bar{y})(-\lambda_1\bar{y}+\lambda_2) \\ &\quad +\nabla_x(\nabla_{yy}k(\bar{x},\bar{y})\bar{p})(-\lambda_1\bar{y}+\lambda_2-\tfrac{1}{2}\lambda_1\bar{p})\}\geqslant 0 \quad \text{for all } x\in C_1, \end{aligned} \tag{3.10}$$

$$(\nabla_{yy}k(\bar{x},\bar{y})(-\lambda_1\bar{y}+\lambda_2-\lambda_1\bar{p})+\nabla_y(\nabla_{yy}k(\bar{x},\bar{y})\bar{p})(-\lambda_1\bar{y}+\lambda_2-\tfrac{1}{2}\lambda_1\bar{p})=0, \tag{3.11}$$

$$\nabla_{yy}k(\bar{x},\bar{y})(-\lambda_1\bar{y}+\lambda_2-\lambda_1\bar{p})=0, \tag{3.12}$$

$$\lambda_2^{\mathrm{T}}(\nabla_y k(\bar{x},\bar{y})+\nabla_{yy}k(\bar{x},\bar{y})\bar{p})=0, \tag{3.13}$$

$$(\lambda_1,\lambda_2)\neq 0. \tag{3.14}$$

Since $\nabla_{yy}k(\bar{x},\bar{y})$ is nonsingular, (3.12) gives

$$\lambda_2=\lambda_1(\bar{y}+\bar{p}). \tag{3.15}$$

We claim that $\lambda_1\neq 0$. Indeed if $\lambda_1=0$, then (3.15) implies $\lambda_2=0$, which contradicts (3.14). Hence

$$\lambda_1>0. \tag{3.16}$$

Using (3.15) in (3.11), we get $\nabla_y(\nabla_{yy}k(\bar{x},\bar{y})\bar{p})(\frac{1}{2}\lambda_1\bar{p})=0$, which by hypothesis (ii) and (3.16) yields

$$\bar{p}=0. \tag{3.17}$$

From (3.15) and (3.17), we obtain

$$\lambda_2=\lambda_1\bar{y}. \tag{3.18}$$

Using (3.17) and (3.18) in (3.10), we get

$$(x-\bar{x})^{\mathrm{T}}\nabla_x k(\bar{x},\bar{y})\geqslant 0 \quad \text{for all } x\in C_1. \tag{3.19}$$

Let $x\in C_1$. Then $\bar{x}+x\in C_1$ and so (3.19) implies $x^{\mathrm{T}}\nabla_x k(\bar{x},\bar{y})\geqslant 0$ for all $x\in C_1$. Therefore, $-\nabla_x k(\bar{x},\bar{y})\in C_1^{\star}$. Also, from (3.16), (3.18) and $\lambda_2\in C_2$, we obtain $\bar{y}\in C_2$. Thus, $(\bar{x},\bar{y},\bar{r}=0)$ satisfies the constraints (3.3) and (3.4) and so it is a feasible solution for the dual problem (WD).

Now, letting $x=0$ and $2\bar{x}$ in (3.19), we get $\bar{x}^{\mathrm{T}}\nabla_x k(\bar{x},\bar{y})=0$. Further, from (3.13), (3.16) to (3.18), we obtain $\bar{y}^{\mathrm{T}}\nabla_y k(\bar{x},\bar{y})=0$. Hence,

$$F(\bar{x},\bar{y},\bar{p}=0)=G(\bar{x},\bar{y},\bar{r}=0).$$

Also, it follows from Theorem 3.1 that $(\bar{x},\bar{y},\bar{p}=0)$ and $(\bar{x},\bar{y},\bar{r}=0)$ are global optimal solution for (WP) and (WD), respectively. □

Remark 3.2. Since in symmetric duality, the dual of the dual is the primal problem, the statement and proof of the converse duality theorem go exactly as for the strong duality theorem and hence its proof is not required, e.g., see [8,12,13]. It is not so in [5]. Further, while proving the converse duality theorem, the author obtained:

$$F(x,\bar{y})\geqslant F(\bar{x},\bar{y}) \quad \text{for all feasible } (x,\bar{y}) \text{ for (P)}$$

(where the problem (P) is defined in Remark 3.1) and concluded that $(\bar{x},\bar{y})$ is the optimal solution for the primal problem (P). This is not correct since in order to show optimality of $(\bar{x},\bar{y})$ one needs to prove

$$F(x,y)\geqslant F(\bar{x},\bar{y}) \quad \text{for all feasible } (x,y) \text{ for (P)}.$$

Theorem 3.3. (*Converse duality*). *Let $k: R^n\times R^m\mapsto R$ be thrice differentiable and let $(\bar{u},\bar{v},\bar{r})$ be a local optimal solution for* (*WD*). *If*

(i) $\nabla_{xx}k(\bar{u},\bar{v})$ *is nonsingular*,
(ii) *the vector* $\bar{r}^{\mathrm{T}}\nabla_x(\nabla_{xx}k(\bar{u},\bar{v})\bar{r})=0$ *implies* $\bar{r}=0$,

then $(\bar{u},\bar{v},\bar{p}=0)$ *is feasible for* (*WP*) *and* $F(\bar{u},\bar{v},\bar{p})=G(\bar{u},\bar{v},\bar{r})$.

Also, if the hypotheses of Theorem 3.1 are satisfied for all feasible solutions of (WP) and (WD), then $(\bar{u},\bar{v},\bar{p}=0)$ and $(\bar{u},\bar{v},\bar{r}=0)$ are global optimal solutions of (WP) and (WD), respectively.

Proof. Follows on the lines of Theorem 3.2. □

4. Mond–Weir type second-order symmetric duality

In this section we establish duality theorems for the following pair of Mond–Weir type second-order symmetric dual nonlinear programming problems over arbitrary cones:

Primal (*MP*):

$$\begin{aligned}
&\text{Minimize} && L(x,y,p)=k(x,y)-\tfrac{1}{2}p^{\mathrm{T}}\nabla_{yy}k(x,y)p\\
&\text{subject to} && \nabla_y k(x,y)+\nabla_{yy}k(x,y)p\in C_2^{\star}, && (4.1)\\
& && y^{\mathrm{T}}(\nabla_y k(x,y)+\nabla_{yy}k(x,y)p)\geqslant 0, && (4.2)\\
& && x\in C_1. && (4.3)
\end{aligned}$$

Dual (*MD*):

$$\begin{aligned}
&\text{Maximize} && M(u,v,r)=k(u,v)-\tfrac{1}{2}r^{\mathrm{T}}\nabla_{xx}k(u,v)r\\
&\text{subject to} && -\nabla_x k(u,v)-\nabla_{xx}k(u,v)r\in C_1^{\star}, && (4.4)\\
& && u^{\mathrm{T}}(\nabla_x k(u,v)+\nabla_{xx}k(u,v)r)\leqslant 0, && (4.5)\\
& && v\in C_2. && (4.6)
\end{aligned}$$

Theorem 4.1. (*Weak duality*). *Let* (x, y, p) *be feasible for the primal problem* (*MP*) *and* (u, v, r) *be feasible for the dual problem* (*MD*). *Let*

(i) $k(., v)$ *be* η_1-*pseudobonvex in the first variable at* u,
(ii) $-k(x, .)$ *be* η_2- *pseudobonvex in the second variable at* y,
(iii) $\eta_1(x, u) + u \in C_1$ *for all* $x \in C_1$,
(iv) $\eta_2(v, y) + y \in C_2$ *for all* $v \in C_2$.

Then

$$L(x, y, p) \geqslant M(u, v, r).$$

Proof. Since (x, y, p) is feasible for (MP) and (u, v, r) is feasible for (MD), so by the dual constraint (4.4) and hypothesis (iii), we have

$$(\eta_1(x, u) + u)^{\mathrm{T}}[\nabla_x k(u, v) + \nabla_{xx} k(u, v) r] \geqslant 0.$$

This together with (4.5) implies

$$\eta_1^{\mathrm{T}}(x, u)[\nabla_x k(u, v) + \nabla_{xx} k(u, v) r] \geqslant 0.$$

Now, by the η_1-pseudobonvexity of $k(., v)$ at u, we have

$$k(x, v) \geqslant k(u, v) - \tfrac{1}{2} r^{\mathrm{T}} \nabla_{xx} k(u, v) r. \tag{4.7}$$

Similarly, (4.1), hypothesis (iv) and (4.2) give

$$-\eta_2^{\mathrm{T}}(v, y)[\nabla_y k(x, y) + \nabla_{yy} k(x, y) p] \geqslant 0.$$

Therefore, by the η_2-pseudobonvexity of $-k(x, .)$ at y, we obtain

$$k(x, v) \leqslant k(x, y) - \tfrac{1}{2} p^{\mathrm{T}} \nabla_{yy} k(x, y) p. \tag{4.8}$$

Finally, inequalities (4.7) and (4.8) yield

$$k(x, y) - \tfrac{1}{2} p^{\mathrm{T}} \nabla_{yy} k(x, y) p \geqslant k(u, v) - \tfrac{1}{2} r^{\mathrm{T}} \nabla_{xx} k(u, v) r$$

or

$$L(x, y, p) \geqslant M(u, v, r). \qquad \square$$

Theorem 4.2. (*Strong duality*). *Let* $k : R^n \times R^m \mapsto R$ *be thrice differentiable and let* $(\bar{x}, \bar{y}, \bar{p})$ *be a local optimal solution for* (*MP*). *If*

(i) *either* $\nabla_{yy} k(\bar{x}, \bar{y})$ *is positive definite and* $\bar{p}^T \nabla_y k(\bar{x}, \bar{y}) \geqslant 0$ *or* $\nabla_{yy} k(\bar{x}, \bar{y})$ *is negative definite and* $\bar{p}^{\mathrm{T}} \nabla_y k(\bar{x}, \bar{y}) \leqslant 0$,
(ii) $\nabla_y k(\bar{x}, \bar{y}) + \nabla_{yy} k(\bar{x}, \bar{y}) \bar{p} \neq 0$,

then $\bar{p} = 0$, $(\bar{x}, \bar{y}, \bar{r} = 0)$ *is feasible for* (*MD*) *and* $L(\bar{x}, \bar{y}, \bar{p}) = M(\bar{x}, \bar{y}, \bar{r})$.

Also, if the hypotheses of Theorem 4.1 are satisfied for all feasible solutions (MP) and (MD), then $(\bar{x}, \bar{y}, \bar{p} = 0)$ and $(\bar{x}, \bar{y}, \bar{r} = 0)$ are global optimal solutions of (MP) and (MD), respectively.

Proof. Since $(\bar{x}, \bar{y}, \bar{p})$ is a local optimal solution for (MP), by the Fritz John necessary optimality conditions [1] there exists $\lambda_1 \in R_+$, $\lambda_2 \in C_2$, $\lambda_3 \in R_+$ such that the following conditions are satisfied at $(\bar{x}, \bar{y}, \bar{p})$:

$$\begin{aligned}(x - \bar{x})^{\mathrm{T}}\{&\lambda_1 \nabla_x k(\bar{x}, \bar{y}) + \nabla_{xy} k(\bar{x}, \bar{y})(\lambda_2 - \lambda_3 \bar{y}) \\ &+ \nabla_x(\nabla_{yy} k(\bar{x}, \bar{y}) \bar{p})(\lambda_2 - \tfrac{1}{2}\lambda_1 \bar{p} - \lambda_3 \bar{y})\} \geqslant 0 \quad \text{for all } x \in C_1,\end{aligned} \tag{4.9}$$

$$\begin{aligned}&(\lambda_1 - \lambda_3) \nabla_y k(\bar{x}, \bar{y}) + \nabla_{yy} k(\bar{x}, \bar{y})(\lambda_2 - \lambda_3 \bar{y} - \lambda_3 \bar{p}) \\ &\quad + \nabla_x(\nabla_{yy} k(\bar{x}, \bar{y}) \bar{p})(\lambda_2 - \tfrac{1}{2}\lambda_1 \bar{p} - \lambda_3 \bar{y}) = 0,\end{aligned} \tag{4.10}$$

$$\nabla_{yy}k(\bar{x},\bar{y})(\lambda_2-\lambda_1\bar{p}-\lambda_3\bar{y})=0, \tag{4.11}$$

$$\lambda_2^{\mathrm{T}}[\nabla_y k(\bar{x},\bar{y})+\nabla_{yy}k(\bar{x},\bar{y})\bar{p}]=0, \tag{4.12}$$

$$\lambda_3\bar{y}^{\mathrm{T}}[\nabla_y k(\bar{x},\bar{y})+\nabla_{yy}k(\bar{x},\bar{y})\bar{p}]=0, \tag{4.13}$$

$$(\lambda_1,\lambda_2,\lambda_3)\neq 0. \tag{4.14}$$

By hypothesis (i), (4.11) yields

$$\lambda_2=\lambda_1\bar{p}+\lambda_3\bar{y}. \tag{4.15}$$

Now, we claim that $\lambda_1\neq 0$. Indeed, if $\lambda_1=0$, then (4.15) gives

$$\lambda_2=\lambda_3\bar{y}$$

which together with (4.10) yields

$$\lambda_3[\nabla_y k(\bar{x},\bar{y})+\nabla_{yy}k(\bar{x},\bar{y})\bar{p})]=0.$$

By hypothesis (ii), $\nabla_y k(\bar{x},\bar{y})+\nabla_{yy}k(\bar{x},\bar{y})\bar{p}\neq 0$. Therefore, $\lambda_3=0$ and hence $\lambda_2=0$, which contradicts (4.14). Thus we obtain

$$\lambda_1>0. \tag{4.16}$$

Subtracting (4.13) from (4.12) and using (4.15), (4.16), we get

$$\bar{p}^{\mathrm{T}}[\nabla_y k(\bar{x},\bar{y})+\nabla_{yy}k(\bar{x},\bar{y})\bar{p}]=0. \tag{4.17}$$

Now suppose $\bar{p}\neq 0$. Then hypothesis (i) implies that $\bar{p}^{\mathrm{T}}[\nabla_y k(\bar{x},\bar{y})+\nabla_{yy}k(\bar{x},\bar{y})\bar{p}]\neq 0$, contradicting (4.17). Hence,

$$\bar{p}=0. \tag{4.18}$$

Therefore, (4.15) implies

$$\lambda_2=\lambda_3\bar{y}. \tag{4.19}$$

This together with (4.18), (4.10) and hypothesis (ii) yields $\lambda_3=\lambda_1>0$. Therefore, from (4.19)

$$\bar{y}=\frac{\lambda_2}{\lambda_3}\in C_2.$$

Further, using (4.18) and (4.19) in (4.9), we get

$$(x-\bar{x})^{\mathrm{T}}\nabla_x k(\bar{x},\bar{y})\geqslant 0 \quad \text{for all } x\in C_1. \tag{4.20}$$

Let $x\in C_1$. Then $\bar{x}+x\in C_1$ and so (4.20) implies $x^{\mathrm{T}}\nabla_x k(\bar{x},\bar{y})\geqslant 0$ for all $x\in C_1$. Therefore, $-\nabla_x k(\bar{x},\bar{y})\in C_1^*$. Also by letting $x=0$ and $2\bar{x}$ in (4.20), we get $\bar{x}^T\nabla_x k(\bar{x},\bar{y})=0$ and hence $(\bar{x},\bar{y},\bar{r}=0)$ satisfies the constraints (4.4)–(4.6), that is, it is a feasible solution for the dual problem (MD). Moreover,

$$L(\bar{x},\bar{y},\bar{p}=0)=M(\bar{x},\bar{y},\bar{r}=0).$$

Now, by Theorem 4.1, $(\bar{x},\bar{y},\bar{p}=0)$ and $(\bar{x},\bar{y},\bar{r}=0)$ are global optimal solutions for (MP) and (MD), respectively. □

Theorem 4.3. (*Converse duality*). *Let $k: R^n\times R^m\mapsto R$ be thrice differentiable and let $(\bar{u},\bar{v},\bar{r})$ be a local optimal solution for* (*MD*). *If*

(i) *either $\nabla_{xx}k(\bar{u},\bar{v})$ is positive definite and $\bar{r}^{\mathrm{T}}\nabla_x k(\bar{u},\bar{v})\geqslant 0$ or $\nabla_{xx}k(\bar{u},\bar{v})$ is negative definite and $\bar{r}^{\mathrm{T}}\nabla_x k(\bar{u},\bar{v})\leqslant 0$,*
(ii) *$\nabla_x k(\bar{u},\bar{v})+\nabla_{xx}k(\bar{u},\bar{v})\bar{r}\neq 0$,*

then $\bar{r}=0$, $(\bar{u},\bar{v},\bar{p}=0)$ is feasible for (*MP*) *and $L(\bar{u},\bar{v},\bar{p})=M(\bar{u},\bar{v},\bar{r})$.*

Also, if the hypotheses of Theorem 4.1 are satisfied for all feasible solutions (MP) and (MD), then $(\bar{u}, \bar{v}, \bar{p} = 0)$ and $(\bar{u}, \bar{v}, \bar{r} = 0)$ are global optimal solutions of (MP) and (MD), respectively.

Proof. Follows on the lines of Theorem 4.2. □

5. Example

Let $n = m = 2$, $C_1 = \{(x, y) : x - y = 0, x, y \geqslant 0\}$ and $C_2 = \{(x, y) : x + y = 0, x \leqslant 0, y \geqslant 0\}$. Then $C_1^\star = \{(x, y) : x + y \leqslant 0\}$ and $C_2^\star = \{(x, y) : x - y \geqslant 0\}$.

Let $k : R^2 \times R^2 \mapsto R$, $k(x, y) = x_1^2 + x_2^2 - e^{y_1} - e^{y_2}$. Then, our problems (WP) and (WD) reduce to

Primal (*EWP*):

$$\begin{aligned} &\text{Minimize} \quad F(x, y, p) = x_1^2 + x_2^2 + \left(-1 + y_1(1 + p_1) + \frac{p_1^2}{2}\right) e^{y_1} + \left(-1 + y_2(1 + p_2) + \frac{p_2^2}{2}\right) e^{y_2} \\ &\text{subject to} \quad e^{y_1}(1 + p_1) \leqslant e^{y_2}(1 + p_2), \quad x_1 - x_2 = 0, \quad x_1, x_2 \geqslant 0. \end{aligned}$$

Dual (*EWD*):

$$\begin{aligned} &\text{Maximize} \quad G(u, v, r) = -(u_1 + r_1)^2 - (u_2 + r_2)^2 - e^{v_1} - e^{v_2} \\ &\text{subject to} \quad (u_1 + r_1) + (u_2 + r_2) \geqslant 0, \quad v_1 + v_2 = 0, \quad v_1 \leqslant 0, \quad v_2 \geqslant 0. \end{aligned}$$

The function k is η_1-bonvex at $u \in R^2$ with $\eta_1 = x - u$ and η_2-boncave at $y \in R^2$ with $\eta_2 = v - y$. Therefore, the assumptions of weak duality theorem are satisfied. Further, for the above function $k(x, y)$,

(i) $\nabla_{yy} k(x, y)$ is nonsingular, and
(ii) the vector $p^{\mathrm{T}} \nabla_y (\nabla_{yy} k(x, y) p) = 0$ implies $p = 0$.

That is, the assumptions of strong duality theorem (Theorem 3.2) are also satisfied. Therefore, the duality results studied in Section 3 hold for the problems (EWP) and (EWD). This can be easily seen since the optimal value of the primal and the dual problems is -2.

Acknowledgments

The authors are thankful to a reviewer for his valuable suggestions. The second author was a Research Scholar at Indian Institute of Technology, Roorkee (India) during this work and is thankful to UGC, New Delhi for providing financial support.

References

[1] M.S. Bazaraa, J.J. Goode, On symmetric duality in nonlinear programming, Oper. Res. 21 (1973) 1–9.
[2] C.R. Bector, S. Chandra, Generalized bonvex function and second-order duality in mathematical programming, Research Report 85-2. The University of Manitoba, Winnipeg, Manitoba, January 1985.
[3] S. Chandra, V. Kumar, A note on pseudoinvexity and symmetric duality, European J. Oper. Res. 105 (1998) 626–629.
[4] G.B. Dantzig, E. Eisenberg, R.W. Cottle, Symmetric dual nonlinear programming, Pacific J. Math. 15 (1965) 809–812.
[5] G. Devi, Symmetric duality for nonlinear programming problems involving η-bonvex functions, European J. Oper. Res. 104 (1998) 615–621.
[6] W.S. Dorn, A symmetric dual theorem for quadratic programming, J. Oper. Res. Soc. Japan 2 (1960) 93–97.
[7] R.N. Kaul, S. Kaur, Optimality criteria in nonlinear programming involving nonconvex functions, J. Math. Anal. Appl. 105 (1985) 104–112.
[8] S. Khurana, Symmetric duality in multiobjective programming involving generalized cone-invex functions, European J. Oper. Res. 165 (2005) 592–597.
[9] O.L. Mangasarian, Second and higher-order duality in nonlinear programming, J. Math. Anal. Appl. 51 (1975) 607–620.
[10] B. Mond, T. Weir, Generalized concavity and duality, in: S. Schaible, W.T. Ziemba (Eds.), Generalized Concavity in Optimization and Economics, Academic Press, New York, 1981, pp. 263–279.
[11] S. Pandey, Duality for multiobjective fractional programming involving generalized η-bonvex functions, Opsearch 28 (1991) 36–43.
[12] S.K. Suneja, C.S. Lalitha, S. Khurana, Second order symmetric duality in multiobjective programming, European J. Oper. Res. 144 (2003) 492–500.
[13] X.M. Yang, X.Q. Yang, K.L. Teo, S.H. Hou, Multiobjective second-order symmetric duality with F-convexity, European J. Oper. Res. 165 (2005) 585–591.