AD ⊩ \text{"THE } \kappa_n \text{ ARE JONSSON CARDINALS AND } \kappa_\omega \text{ IS A ROWBOTTOM CARDINAL"}

E.M. KLEINBERG

Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA 02139, USA.

Received 16 September 1976

Dedicated to A. Mesnikoff

1.

The axiom of determinateness (AD) began its rise to a position of almost universal interest among set theorists with the announcement of Solovay’s theorem that AD implies \(\kappa_1\) is a measurable cardinal.\(^1\) There had certainly been earlier work on this axiom—most notably the applications to analysis discovered by Mycielski and his collaborators—but Solovay’s elegant and powerful result indicated a great promise for AD within set theory.

Some time following his original result Solovay proved as a consequence of AD that \(\kappa_2\) is a measurable cardinal, and the prime problem became to show every \(\kappa_n\) measurable. As is fairly well-known this was not to be, for following a good deal of seemingly unrelated work with AD Martin proved that each \(\kappa_n\) for \(2 < n < \omega\) was singular with cofinality \(\kappa_2\). Subsequent work with AD became far more technical and although large cardinal properties were derived for many sets, the cardinals above \(\kappa_2\) and below \(\kappa_\omega\) were dismissed as uninteresting.

As it turns out, AD implies these singular cardinals, the \(\kappa_n\) for \(2 < n < \omega\), and their limit, \(\kappa_\omega\), to all satisfy well-known large cardinal properties—the \(\kappa_n\) are all Jonsson cardinals and \(\kappa_\omega\) is a Rowbottom cardinal. This paper is devoted to a proof of these results.

Before proceeding with the details of the proof, however, it might be well to review some definitions, and to give some background information, concerning Jonsson and Rowbottom cardinals. A cardinal \(\kappa\) is \textit{Jonsson} if every structure of cardinality \(\kappa\) has a proper elementary substructure of cardinality \(\kappa\), and is \textit{Rowbottom}, if for every \(\lambda\) less than \(\kappa\) every two cardinal structure of type \((\kappa, \lambda)\) has a proper elementary substructure of type \((\kappa, \omega)\). Even without the axiom of choice these two model theoretic definitions are equivalent to the well-known combinator-

\(^1\) It is well-known that AD contradicts the axiom of choice. Throughout this paper our base theory will be ZF + DC.
ial ones, $\kappa \rightarrow [\kappa]_2^\omega$ and $\forall \lambda < \kappa (\kappa \rightarrow [\kappa]_2^\omega)$, respectively. In ZFC, here is a very brief description of what is known (most of these results follow in ZF + DC):

1. Every measurable cardinal is Rowbottom (Erdös– Hajnal),
2. under any normal measure almost every cardinal less than a measurable is Rowbottom (Rowbottom),
3. if there exists a Jonsson cardinal, then $\theta^\theta$ exists (Kunen),
4. any singular limit of measurable cardinals is Jonsson (Prikry),
5. $\text{Con}(\text{ZFC} + \exists \text{ Rowbottom cardinal}) \implies \text{Con}(\text{ZFC} + \exists \text{ Rowbottom cardinal} \leq 2^{\aleph_0})$ (Devlin),
6. $\text{Con}(\text{ZFC} + \exists \text{ Rowbottom cardinal})$ if and only if $\text{Con}(\text{ZFC} + \exists \text{ Jonsson cardinal})$ ([2]).
7. In $L[\mu]$ being Jonsson is the same as being Ramsey (Kunen).

It is a prime open question (of Silver) as to whether or not it is consistent with ZFC for $\aleph_\omega$ to be Rowbottom. Silver has shown that if $\aleph_\omega$ is Rowbottom, then there is some inner model in which it is measurable.

2.

Our results are actually somewhat stronger than indicated above. For one thing we do not really use AD itself but rather an appropriate infinite exponent partition relation. In order to be more specific let us recall that for any ordered set $x$ and ordinal $\alpha$, $[x]^\alpha$ denotes the collection of subsets of $x$ of type $\alpha$ and for $\kappa \geq \beta \geq \alpha$ ordinals, $\kappa \rightarrow (\beta)^\alpha$ denotes the partition relation “for any partition $F : [\kappa]^\alpha \rightarrow 2$ there exists a subset $x$ of $\kappa$ of type $\beta$ such that $F$ is constant on $[x]^\alpha$.” Then an infinite exponent partition relation is one of the form $\kappa \rightarrow (\beta)^\alpha$ where $\alpha \geq \omega$.

Such relations (which all happen to contradict the axiom of choice) have been associated with AD for quite some time. Soon after the original result that any $\kappa$ satisfying $\kappa \rightarrow (\kappa)^{\omega^2}$ is measurable ([1]) Martin proved that assuming AD, $\mathbf{N}_1$ satisfies the relation $\mathbf{N}_1 \rightarrow (\mathbf{N}_1)^\omega$. Quite some time later he improved this to $\text{AD} + \mathbf{N}_1 \rightarrow (\mathbf{N}_1)^\omega$ and it is essentially this partition relation, $\mathbf{N}_1 \rightarrow (\mathbf{N}_1)^\omega$, from which we derive our results here. The main theorem is as follows:

**Theorem 2.1.** Assume that $\kappa$ is an uncountable cardinal satisfying $\kappa \rightarrow (\kappa)^\omega$. Then

(a) $\kappa$ is a measurable cardinal,
(b) there exists a measurable cardinal $\kappa_2$ greater than $\kappa$,
(c) there exist countably many Jonsson cardinals $\kappa_n, 2 < n < \omega$, greater than $\kappa_2$ such that for each $n \kappa_n$ is singular with cofinality $\kappa_2$, and
(d) the limit of the $\kappa_n, \kappa_\omega$, is a Rowbottom cardinal.

Furthermore, if for some normal measure $\mu$ on $\kappa$ the ultraproduct $\kappa^+ / \mu$ has type $\kappa^+$ then we can take the $\kappa_n$ to be the first $\omega$-many successors of $\kappa$. (Note: Only parts (c) and (d) of the above theorem are new. Part (a) appears in [1] and part (b) is due
to Martin and Paris. (Our proof of (b) will be different from that of Martin and Paris.)

**Corollary 2.2.** Assuming $\text{AD} \setminus \text{I}$ and $\aleph_1$ are measurable cardinals, the $\aleph_n$ for $2 < n < \omega$ are singular Jonsson cardinals, of cofinality $\aleph_2$, and $\aleph_\omega$ is a Rowbottom cardinal.

**Proof.** It is a theorem of Martin that $\text{AD}$ implies $\text{I} \rightarrow (\aleph_\omega)^{\omega}$: and a theorem of Solovay that $\text{AD}$ implies $\aleph_n^\omega / \mu = \aleph_n$ for any normal measure $\mu$ on $\aleph_1$. □

The remainder of this paper is devoted entirely to a proof of the above theorem. As mentioned earlier the proof will yield somewhat stronger results. While the statements of some of these are technical and best saved for the lemmas, we might mention here that we shall prove that each of the $\kappa_n$ (and hence $\kappa_\omega$) satisfies the partition relation $\gamma \rightarrow (\gamma < \gamma)_{\omega}^\omega$.

**Proof of Theorem 2.1.** The proof of this theorem is extremely long so we begin by outlining it. The proofs of the component lemmas will be given in detail following the outline.

(A) Since $\kappa > \omega$ and $\kappa \rightarrow (\kappa)^{\omega}$, $\kappa \rightarrow (\kappa)^{\omega^2}$ and hence by [1] the filter of $\omega$-closed unbounded sets generates a normal $\kappa$-additive measure on $\kappa$. Henceforth $\mu$ will denote this measure on $\kappa$.

(B) For any ordered set $x$ and ordinal $\alpha \ [x]^\alpha$ denotes the set of subsets of $x$ of type $\alpha$. $[x]^\alpha$ will also be viewed as the set of order-preserving maps from $\alpha$ into $x$.

**Lemma B.1.** For any ordered set $x$ let the relation $<_\mu^\alpha$ on $[x]^\alpha$ be given by $f <^\alpha g$ iff $\mu((\alpha \mid f(\alpha) < g(\alpha))) = 1$. Then if $x$ is a well-ordered set, $<_\mu^\alpha$ is an ordering of $[x]^\alpha$ with no infinite descending chains.

In what follows this ordering $<_\mu^\alpha$ will always be implicitly associated with each set $[x]^\alpha$.

**Lemma B.2.** (First Shuffling Lemma). For any $\alpha < \kappa^+$ there exist maps $bk_\alpha : [\kappa]^\alpha \rightarrow [[\kappa]^\alpha]^\alpha$ and $sf_\alpha : [[\kappa]^\alpha]^\alpha \rightarrow [\kappa]^\alpha$, such that for any $f$ in $[\kappa]^\alpha$ $sf_\alpha (bk_\alpha (f)) = f$ and for any $F \in [[\kappa]^\alpha]^\alpha$, if $F' = bk_\alpha (sf_\alpha (F))$, then for every $\beta < \alpha$, $F(\beta) = F'(\beta)$ a.e. $\mu$.

**Lemma B.3.** (Second Shuffling Lemma). For any $\alpha < \kappa^+$ there exists a map $bk^\alpha : [\kappa]^\alpha \rightarrow [[\kappa]^\alpha]^\alpha$, such that for any $F \in [[\kappa]^\alpha]^\alpha$, if $G = bk^\alpha (sf_\alpha (F))$, then for every $\beta < \alpha$, $F(\beta) = G(\beta)$ a.e. $\mu$.

The above lemmas revolve about functions which "break" and "shuffle" $\kappa$ sequences from $\kappa$, however, they remain valid and shall later be used in the context of $\kappa$-sequences from any well-ordered set of length at least $\kappa$. The only added clause
in the generalized context is that sequences being "shuffled" must all have the same sup.

(C) In this section we describe the $\kappa_n$. First let us define $\kappa^0$ to be 1 and note that the function $bk_1$ is simply the identity. We are now in a position to define, for each integer $n \geq 2$, $S_n$ to be the range of $bk_{\kappa^{n-2}}$ on $[\kappa]^*$. Suppose $n > 2$ and let $H$ be any member of $[[\kappa]]^{n-2}$. $H$ is a $\kappa^{n-2}$-sequence from $[\kappa]^*$ and we introduce the following associated notation: $H_0$ denotes the $\kappa^{n-3}$-sequence from $[\kappa]^*$ consisting of the first $\kappa^{n-3}$ many members of $H$, $H_i$ denotes the $\kappa^{n-3}$-sequence consisting of the next $\kappa^{n-3}$ many members of $H$, and so forth. In this way we define $H_\alpha$ for every $\alpha < \kappa$. Keep in mind that $H \in [[\kappa]]^{n-2}$ and $H_\alpha \in [[\kappa]]^{n-2}$ for each $\alpha < \kappa$ — the $\kappa$-many $H_\alpha$ are the successive component blocks of $H$. $H_\alpha$ is called the $\alpha$th component of $H$.

We now define by induction on $n \geq 2$ a relation $\sim_n$ on $S_n$ as follows: $\sim_2$ is simply $"=\text{ a.e. (}\mu\text{)}"$ (that is, $f \sim_2 g$ iff $\mu(\{\alpha \mid f(\alpha) = g(\alpha)\}) = 1$) and, if $\sim_k$ has been defined and $H$ and $G$ are two members of $S_{k+1}$, then $H \sim_{k+1} G$ iff $\mu(\{\alpha \mid H_\alpha \sim_k G_\alpha\}) = 1$.

Lemma C.1. For each $n \geq 2$, $\sim_n$ is an equivalence relation on $S_n$.

We next define by induction on $n \geq 2$ a relation $<_{\sim}$ on $S_n/\sim_n$ as follows: if $f$ and $g$ are in $S_2/\sim_2$, then $f <_{\sim_2} g$ iff $\mu(\{\alpha \mid f(\alpha) < g(\alpha)\}) = 1$ and, if $<_{\sim_k}$ has been defined and $H$ and $G$ are members of $S_{k+1}/\sim_{k+1}$, then $H <_{k+1} G$ iff $\mu(\{\alpha \mid H_\alpha <_{k+1} G_\alpha\}) = 1$.

Lemma C.2. For each $n \geq 2$, $<_{\sim}$ is a well-defined well-ordering of $S_n/\sim_n$.

We now simply define, for $n \geq 2$, $\kappa_n$ to be the order-type of $(S_n/\sim_n, <_{\sim})$. In the future the ordinal $\kappa_n$ and $S_n/\sim_n$ will be freely identified, $<_{\sim}$ always being the ordering of $S_n/\sim_n$.

(D) We wish to distinguish among different types of pairs of ordinals from $\kappa_n$. In fact for $n \geq 2$ any pair of ordinals from $\kappa_n$ will fall into precisely one of $n-1$ categories. We do this by induction on $n$ defining for each $n \geq 2$ what we mean for a pair from $\kappa_n$ to be $i$-interlaced for $0 \leq i < n-1$: for $n = 2$, all pairs from $\kappa_2$ are 0-interlaced. Suppose we have completed our induction through $n = k$ and $\{H, G\}$, $\tilde{H} < \tilde{G}$, is a pair from $\kappa_{k+1}$. Then if $\bigcup_{\alpha \in \kappa_n} \tilde{H}_\alpha < \bigcup_{\alpha \in \kappa_n} \tilde{G}_\alpha$ (keep in mind the identification between $S_n/\sim_n$ and $\kappa_n$), we say $\{\tilde{H}, \tilde{G}\}$ is 0-interlaced. If $\bigcup_{\alpha \in \kappa_n} \tilde{H}_\alpha = \bigcup_{\alpha \in \kappa_n} \tilde{G}_\alpha$ then we define $\{\tilde{H}, \tilde{G}\}$ to be $i + 1$-interlaced where $i$ is the unique integer satisfying $\mu(\{\alpha \mid (\tilde{H}_\alpha, \tilde{G}_\alpha) \text{ is } i\text{-interlaced}\}) = 1$.

Lemma D.1. The above definition of "$i$-interlaced" is well-defined as indicated.

(E) For any $C \subseteq \kappa$, $\tilde{C} = \kappa$, let for each $n \geq 2$ $S_n^C$ denote the range of $bk_{\kappa^{n-2}}$ on $[C]^*$.

Lemma E.1. For each $n \geq 2$, if $C \subseteq \kappa$ and $\tilde{C} = \kappa$, then $\langle S_n^C/\sim_n, <_{\sim} \rangle$ and $\langle S_n/\sim_n, <_{\sim} \rangle$ are order-isomorphic.
Lemma E.2. \( \kappa < \kappa_2 \) and for each \( n \geq 2 \), \( \kappa_n < \kappa_{n+1} \).

Lemma E.3. For each \( n \geq 2 \), \( \kappa_n \) is a cardinal.

Lemma E.4. \( \kappa_2 \rightarrow (\kappa_2)^n \) for each countable \( \alpha \) and hence \( \kappa_2 \) is measurable.

Lemma E.5. For each integer \( n \geq 2 \), \( \kappa_n \) has cofinality \( \kappa_2 \).

(F) In this section we prove that \( \kappa \rightarrow (\kappa)^n \) implies \( \kappa \rightarrow (\kappa)^{\omega} \). It is \( \kappa \rightarrow (\kappa)^{\omega} \) which is used later in proving our main theorems.

In the original draft of this paper we did not know that \( \kappa \rightarrow (\kappa)^n \) implied \( \kappa \rightarrow (\kappa)^{\omega} \) and we were forced to carefully analyze those partitions into \( 2^n \) which were needed. We proved there that for those specific partitions, \( \kappa \rightarrow (\kappa)^n \) sufficed to yield homogeneous sets.

The general lemma we are about to prove is due to J.M. Henle. The proof we give, though quite a bit shorter than Henle's, uses many of his ideas.

Lemma F.1. \( \kappa \rightarrow (\kappa)^n \) implies \( \kappa \rightarrow (\kappa)^{\omega} \).

(G) In this section we show the \( \kappa_n \) Jonsson.

Lemma G.1. Given any \( \beta < \kappa^+ \), \( C \in [\kappa]^\beta \), and \( \bar{H} \) in \( \kappa_2 \) there exists a function \( G \) mapping \( \beta \) order-preservingly into \( [C]^\kappa \) such that for every \( \eta < \beta \) \( G(\eta) \) is the \( n^{th} \) member of \( S_{\kappa_2}^{\omega} \) larger than \( \bar{H} \).

Now for \( \gamma > \delta \) cardinals and \( h \) a function from \( \omega \) into \( \omega \), we define the partition relation \( \gamma \rightarrow [\gamma]^{\omega}_{\kappa_3} \) by "for every partition \( F : [\gamma]^{\omega} \rightarrow \delta \) there exists a subset \( C \) of \( \gamma \) of size \( \gamma \) homogeneous for \( F \) in the following sense: for each \( n < \omega \), \( F^{-1}[C] \subseteq h(n) \)."

Lemma G.2. For every \( n \geq 2 \) and \( \gamma < \kappa_3 \),

\[ \kappa_n \rightarrow [\kappa_n]^{\omega}_{\gamma^{\omega}(\gamma-1)} \].

Remark. Lemma G.2 of course implies each \( \kappa_n \) Jonsson.

Remark. The following lemma, although not related to the proof of our main theorem, adds some perspective to the partition relation \( \gamma \rightarrow [\gamma]^{\omega}_{\kappa_3} \):

Lemma G.3. Suppose that \( \gamma > \delta \geq \omega \) are cardinals and that \( h : \omega \rightarrow \omega \) is eventually less than \( \lambda n [r^n] \) for every \( r > 1 \). Then \( \gamma \rightarrow [\gamma]^{\omega}_{\kappa_3} \) implies that \( \gamma \) is a Ramsey cardinal, that is, that \( \gamma \rightarrow (\gamma)^{\omega} \).
Let us define the cardinal \( \kappa_\omega \) to be the limit of the \( \kappa_n \), \( n < \omega \). In this section we show \( \kappa_\omega \) Rowbottom. For each \( n \geq 2 \) consider the binary relation \( \equiv_n \) on \( \kappa_n \) defined by \( \bar{H} \equiv_n \bar{G} \) iff \( \bar{H} = \bar{G} \) or \( \{ \bar{H}, \bar{G} \} \) is \( n - 2 \) interlaced.

**Lemma H.1.** For each \( n \geq 2 \), \( \equiv_n \) is a well-defined equivalence relation on \( \kappa_n \).

**Lemma H.2.** For each \( C \subseteq \kappa, \bar{C} = \kappa, n \geq 2 \), and \( \alpha < \kappa \) there exists a size \( \kappa_{n-1} \) equivalence class under \( \equiv_n \) contained in \( S^\kappa_n / \sim_n - \alpha \).

**Lemma H.3.** \( \kappa_\omega \) is a Rowbottom cardinal.

*Note.* Each \( \kappa_n \) (and hence \( \kappa_\omega \)) satisfying the relation \( \gamma \rightarrow (\kappa^\alpha) \) for each \( \alpha < \aleph_1 \), is a corollary of the proof of Lemma H.3.

Now the first part of our theorem follows routinely from (A), Lemma E.4, Lemma E.5, Lemma G.2, and Lemma H.3. To establish the second part we first note that for each \( n \geq 1 \) \( \kappa_{n+1} \) is a subset of the ultraproduct \( \kappa_n''/\mu \). (Think of \( \kappa_1 \) as \( \kappa \)) Now if \( \kappa''/\mu = \kappa'' \), then since \( \kappa < \kappa_2 \subseteq \kappa''/\mu \) we must have \( \kappa_2 = \kappa_1'' = \kappa_1''/\mu \). Continuing inductively we can show that for each \( n \), \( \kappa_{n+1} = \kappa_n'' = \kappa_n''/\mu \). For suppose \( \kappa_{k+1} = \kappa_k'' = \kappa_k''/\mu \) and we wish to show \( \kappa_{k+2} = \kappa_{k+1}'' = \kappa_{k+1}''/\mu \). Well since the cofinality of \( \kappa_{k+1} \) is \( \kappa_2 \) and \( \kappa_2 > \kappa_1 \), it must follow that the constant functions in \( \kappa_k''/\mu \) are cofinal. But then any initial segment of \( \kappa_k''/\mu \) must have cardinality \( \kappa_k''/\mu \) and so \( \kappa_{k+1}''/\mu \leq \kappa_{k+1}''/\mu \). But \( \kappa_{k+1} < \kappa_{k+2} \subseteq \kappa_k''/\mu \) and so \( \kappa_{k+2} = \kappa_{k+1}'' = \kappa_{k+1}''/\mu \).

**Proofs of Lemmas** (Carried out in ZF+DC+"\( \kappa > \omega \) and \( \kappa \rightarrow (\kappa^\alpha) \)).

(A) See [1].

(B) **Proof of Lemma B.1.** Given an infinite descending chain in \( [x]^\kappa \), the additivity of \( \mu \) gives us an infinite descending chain in \( x \) — contradiction.

**Proof of Lemma B.2.** These next two lemmas (which will be most important for our later work) are basically consequences of the normality of \( \mu \). Before launching into their proofs note that the normality of \( \mu \) implies the following: suppose \( f : \kappa \rightarrow [\kappa]^\kappa \) is such that \( \bigcup f(\alpha) < \alpha \) for all \( \alpha \). Then \( f \) is constant almost everywhere. For if \( f_1 \) and \( f_2 \) are the maps which satisfy \( f(\alpha) = \{ f_1(\alpha), f_2(\alpha) \} \) for all \( \alpha \), then the normality of \( \mu \) yields measure 1 sets \( A_1 \) and \( A_2 \) on which \( f_1 \) and \( f_2 \), resp., are constant. \( f \) is now constant on \( A_1 \cap A_2 \). Now to prove Lemma B.2.

There are basically two cases to consider, \( \alpha \geq \kappa \), and \( \alpha < \kappa \).

\( \alpha \geq \kappa \): Let \( h \) be a 1-1 map of \( \kappa \) onto \( \alpha \). When we refer to the "\( h \)-ordering on \( \kappa \)" we will mean the linear ordering of \( \kappa \), determined by \( h : \eta_1 < h, \eta_2 \iff h(\eta_1) \in h(\eta_2) \). Now given \( f \in [\kappa]^\kappa \) we must describe \( bk_\alpha(f) \). We will do this as follows: imagine a square of size \( \kappa \) by \( \kappa \). Given \( f \) we will proceed to fill the lattice points in this square.
using, in order, ordinals from the range of $f$ (plus some filler). We will fill the array inductively a row at a time and when we are done the $\kappa$-many columns will essentially constitute $bk_\kappa(f)$—indeed if for $\eta < \kappa$ $f_\eta$ denotes the $\eta^{th}$ column, then $\beta \rightarrow f_\eta^{-1}(\beta)$ will be the map $bk_\kappa(f)$. Here is how we distribute the ordinals in the range of $f$ into such an array: the $0^{th}$ row of the array, all $\kappa$ entries in it, are 0's—this is just filler. Proceeding inductively suppose we have filled all rows below the $\nu^{th}$ and in doing have used up some initial segment of $f$. Then we fill the $\nu^{th}$ row as follows: the first $\nu$-many entries of the $\nu^{th}$ row are filled with the first $\nu^*$-many (some $\nu^* < \kappa$) ordinals in the range of $f$, which we haven't yet used. These ordinals are placed in the first $\nu$-many spots of the $\nu^{th}$ row so as to preserve the $h$ ordering of $\kappa$, (this $h$ ordering determines $\nu^*$). From entry $\nu$ on in the $\nu^{th}$ row we fill every spot with the ordinal $\nu$. This stuff is just filler. Proceeding in this way we build our array as described and it is routine to verify that if the map $f_\kappa \in [\kappa]^{\kappa}$ is given by $f_\kappa(\beta) = \nu$ "the $\langle \beta, \nu \rangle^{th}$ entry in array", then $\eta \rightarrow f_\kappa^{-1}(\eta)$ is a member of [[\kappa]^{\kappa}]^\kappa$. This member of [[\kappa]^{\kappa}]^\kappa is $bk_\kappa(f)$ and since $f$ was arbitrary, our description of $bk_\kappa(f)$ is complete.

Now to describe $sf_\kappa$. $sf_\kappa$ is basically the reverse of $bk_\kappa$. Given $H \in [[\kappa]^{\kappa}]^\kappa$ we can arrange $H$ so that its $\alpha$-many $\kappa$-sequences are the columns in a $\kappa$ by $\kappa$ array. We do this by using our map $h : \kappa \rightarrow \alpha$. Indeed consider the $\kappa$ by $\kappa$ array, where the $\langle \eta, \delta \rangle^{th}$ entry is $H(h(\delta))(\eta)$. We are going to build a member of $[\kappa]^{\kappa}$ by listing its elements in order and these elements we are going to choose from among the entries in our $\kappa$ by $\kappa$ array determined by $H$. The member of $[\kappa]^{\kappa}$ we so build will be $sf_\kappa(H)$. Our construction of $sf_\kappa(H)$ will take place inductively by stages—at the $\beta^{th}$ stage for $\beta < \kappa$ we will say how to add a $\beta^*$-sequence (some $\beta^* < \kappa$) of ordinals to the amount of $sf_\kappa(H)$ constructed to that point. This $\beta^*$-sequence of ordinals itself will be described inductively. Here is the construction: the $0^{th}$ stage of our construction involves nothing. Now suppose we have completed the $\eta^{th}$ stage of our construction for every $\eta < \beta$ and we wish to describe the $\beta^{th}$ stage. For this $\beta^{th}$ stage we will inductively pick a $\beta^*$-sequence (some $\beta^* < \kappa$) of ordinals from our $\kappa$ by $\kappa$ array each of which is larger than any ordinal we have put into $sf_\kappa(H)$ so far. Here is the first ordinal in the $\beta^*$-sequence: look at the first $\beta$ columns of our array and consider their ordering in the $<_\kappa$-ordering. (\beta^* is the order-type of this ordering.) The ordinal we want is the least ordinal larger than everything yet used in building $sf_\kappa(H)$ which lies in the $<_\kappa$-least column among the first $\beta$. Continuing, inductively on $\eta < \beta^*$, the $\eta^{th}$ ordinal we add to $sf_\kappa(H)$ at stage $\beta$ is the least ordinal larger than everything yet used which lies in the $\eta^{th}$ $<_\kappa$-least column among the first $\beta$ columns. This describes the inductions and $sf_\kappa(H)$ is the $\kappa$ sequence from $\kappa$ we so build. As $H$ was arbitrary we have defined $sf_\kappa$.

Now it is routine to verify that for any $f \in [\kappa]^{\kappa}$, $sf_\kappa(bk_\kappa(f)) = f$. What we must show is that for any $H$ in $[[\kappa]^{\kappa}]^\kappa$, if $H' = bk_\kappa(sf_\kappa(H))$, then for all $\beta < \alpha$ $H'(\beta) = H(\beta)$ a.e.$(\mu)$. We do this as follows: let $g_\beta : \kappa \rightarrow \alpha$ as follows: $g_\beta(\beta) = \nu$ "the least $\eta$ such that during the $\eta^{th}$ stage of the definition of $sf_\kappa(H)$ ordinals appearing in row $\beta$ or higher of our $\kappa$ by $\kappa$ array associated with $H$ are used". It is easy to see that $g_\beta(\beta) < \beta$ for each $\beta$ and if $g_\beta(\beta) < \beta$ a.e.$(\mu)$, then by
normality this would mean that for some $\beta_0 < \kappa \, g_1(\beta) = \beta_0 \text{ a.e.}(\mu)$. It is clear from our construction that this cannot be and so we must have an $A_1$, such that $\mu(A_1) = 1$ and on $A_1$, $g_1(\beta) = \beta$. Next consider $g_2 : \kappa \rightarrow [\kappa]^2$ given as follows: $g_2(\beta) = \alpha$ if the least pair $\langle \beta_1, \beta_2 \rangle \prec$ such that on row $\beta$ of the $\kappa$ by $\kappa$ array for $H$ the ordinal in column $\beta_1$ and the ordinal in column $\beta_2$ violate the $h$ ordering or $\{0, \beta\}$, whichever pair has smaller sup. Now if $\bigcup g_2(\beta) < \beta \text{ a.e.}(\mu)$, then by the extension of normality observed earlier there would be a pair $\langle \beta_0^1, \beta_0^2 \rangle \in [\kappa]^2$ such that $g_3(\beta) = \{\beta_0^1, \beta_0^2\} \text{ a.e.}(\mu)$. But this would mean that the $\beta_0^1$ and $\beta_0^2$ columns of our $\kappa$ by $\kappa$ array for $H$ violates the $h$ ordering on full columns (i.e. $f < h g$ iff $\mu(\{\alpha \mid f(\alpha) < h g(\alpha)\}) = 1$) and this contradicts our very construction of the array. So let $A_2$ be such that $\mu(A_2) = 1$ and on $A_2$ $g_2(\beta) = \{0, \beta\}$, i.e., for any $\beta$ in $A_2$ the first $\beta$ many ordinals in the $\beta$'th row preserve the $h$ ordering.

Now notice what happens during stage $\beta$ of $s_{k_0}(H)$ for any $\beta$ in $A_1 \cap A_2$: the $\beta$'-sequence added to the $\kappa$-sequence from $\kappa$ we are building would be precisely the ordinals (under correct $h$ ordering) appearing in the first $\beta$ columns of the $\beta$'th row of our $\kappa$ by $\kappa$ array. Now let's look at the process of $b_{k_0}$. We define $g_3 : \kappa \rightarrow \kappa$ as follows: $g_3(\beta) = \alpha$ if the row of the $\kappa$ by $\kappa$ array into which $b_{k_0}(f)$ inserts the $\beta$'th ordinal of $f$'. Clearly $g_3(\beta) \leq \beta \text{ a.e.}(\mu)$ and if $g_3(\beta) < \beta \text{ a.e.}(\mu)$, then we would have a $\beta_0 < \kappa$, such that $g_3(\beta) = \beta_0 \text{ a.e.}(\mu)$. But since any $b_{k_0}(f)$ fills each row using fewer than $\kappa$-many members of $f$ this would be a contradiction. Thus let $A_3$ be such that $\mu(A_3) = 1$ and $g_3(\beta) = \beta$ on $A_3$. Finally let $g_4 : \kappa \rightarrow \kappa$ as follows: $g_4(\beta) = \alpha$ if the stage which contributed the $\beta$'th ordinal to $s_{k_0}(H)$'. Clearly $g_4(\beta) \leq \beta$ but if $g_4(\beta) < \beta \text{ a.e.}(\mu)$, then we would have $g_4(\beta) = \beta_0 \text{ a.e.}(\mu)$ and this would be impossible as $\beta^- < \kappa$. Thus let $A_4$ be such that $\mu(A_4) = 1$ and on $A_4$, $g_4(\beta) = \beta$.

So suppose $\beta \in A_1 \cap A_2 \cap A_3 \cap A_4$. Then what are the entries of the first $\beta$ columns of the $\beta$'th row of the array associated with $b_{k_0}(s_{k_0}(H))$? We claim they are the same as the entries in the first $\beta$ columns of the $\beta$'th row of $H$. For what happens during stage $\beta$ of $s_{k_0}(H)$? Well as $\beta \in A_1 \cap A_2$ stage $\beta$ throws precisely those ordinals appearing in the first $\beta$ columns of row $\beta$ of the $H$ array into our $\kappa$-sequence from $\kappa$ being built. Furthermore, it throws them in as the $\beta$'th to $\beta + \beta$ entries in our $\kappa$ sequences as $\beta \in A_4$. Now during $b_{k_0}$ of $s_{k_0}(H)$ these same $\beta$'-many ordinals are returned in the same order into row $\beta$ of the array being built. This is simply because $\beta \in A_3$. So if $H' = b_{k_0}(s_{k_0}(H))$ and $\eta < \alpha$ do we have $H'(\eta) = H(\eta)$ a.e.$(\mu)$? Certainly, for if $\beta_0 > h^{-1}(\eta)$, then for any $\beta$ in $A_1 \cap A_2 \cap A_3 \cap A_4 - \beta_0$, $H'(\eta)(\beta) = H(\eta)(\beta)$.

The second case of $\alpha < \kappa$ is really a subcase of the case just completed. Its proof is identical with the above proof except here we deal with arrays with $\kappa$-many rows and only $\alpha$-type-many columns (under $\prec$), and as $\alpha < \kappa$, we can break and shuffle entire rows without having to worry about staying above the diagonal.

The proof of the first shuffling lemma is thus complete. 

**Proof of Lemma B.3.** We must first define our map $b_{k_0} : [\kappa]^2 \rightarrow [[\kappa]^2]^2$. We do this almost exactly as we did $b_{k_0}$ but here we double everything. We will take our
starting \( f \in [\kappa]^\kappa \) and build from it two \( \kappa \) by \( \kappa \) arrays. We again proceed by filling in ordinal entries inductively a row at a time but this time we use the appropriate number to build row \( \alpha \) of the first array and then use the same appropriate number of very next (after these) ordinals in \( f \)'s range to build row \( \alpha \) of the second array. In this way we build two arrays and it is clear from our construction and previous discussion that these two arrays together determine a member of \( ([\kappa]^\kappa)^{\aleph_2} \). This is our \( bk^\kappa (f) \) and as \( f \) was arbitrary we have described \( bk^\kappa \). Now suppose \( H \in ([\kappa]^\kappa)^\kappa \) and \( G \) denotes \( bk^\kappa (sf_\mu (H)) \). We must show that for every \( \beta < \alpha \), \( G(\beta) = H(\beta) \) a.e.(\( \mu \)). The proof of this is almost identical with our proof of \( H'(\beta) = H(\beta) \) a.e.(\( \mu \)) in the first shuffling lemma but with one extra twist: we revise our definition of \( g_\beta (\beta) \) to be \( g_\beta (\beta) = \text{the row of either } \kappa \text{ by } \kappa \text{ array into which } bk_\mu (f) \text{ inserts the } \beta^\text{th} \text{ ordinal of } f \). As before we have \( A_\beta \), such that \( \mu (A_\beta) = 1 \) and \( g_\beta (\beta) = \beta \) on \( A_\beta \). Also as before, if \( \beta \in A_\beta \), then for any \( f \), \( bk^\kappa (f) \) must for the first time be starting on row \( \beta \) of the first array when placing the \( \beta^\text{th} \) ordinal of \( f \). Since row \( \beta \) of the first array is completed before row \( \beta \) of the second array is started, we must have by an argument similar to our previous one that the entries in the first \( \beta \) columns of row \( \beta \) of the first array of \( bk^\kappa (sf_\mu (H)) \) must be the same as the entries in the first \( \beta \) columns of row \( \beta \) of the array for \( H \). By our previous argument then, for all \( \eta < \alpha \), \( G(\eta) = H(\eta) \) a.e.(\( \mu \)).

This completes the proof of the second shuffling lemma. \( \square \)

(C) **Proof of Lemma C.1.** By induction on \( n \). Routine using the additivity of \( \mu \).


(D) **Proof of Lemma D.1.** By induction on \( n \). Routine.

(E) **Proof of Lemma E.1.** \( S_\kappa \sim \kappa \) is a subset of \( S_n \sim \kappa \) and so we need only map \( S_n \sim \kappa \) order-preservingly into \( S_\kappa \sim \kappa \). Here's how: let \( t \) be an order-preserving map of \( \kappa \) into \( C \). Then we can verify that the map \( \varphi : bk_{\kappa \sim \kappa}(f) \mapsto bk_{\kappa \sim \kappa}(t \circ f) \) is well-defined and order-preserving from \( S_n \sim \kappa \) into \( S_\kappa \sim \kappa \). For suppose \( f \) and \( g \) are two members of \( [\kappa]^\kappa \). Then for any \( \alpha \) less than \( \kappa \) the relationship between \( f(\alpha) \) and \( g(\alpha) \) is precisely that between \( t \circ f(\alpha) \) and \( t \circ g(\alpha) \). This observation shows that \( \varphi \) is our desired map.

Proof of Lemm E.2. We first note that given any well-ordered set of length at least \( \kappa \) and any nondecreasing map \( f : \kappa \rightarrow x \), \( f \) is either constant almost everywhere or is equal to a strictly increasing map almost everywhere. This is a routine consequence of the normality of \( \mu \) and so for any such \( f : \kappa \rightarrow x \), which is not almost everywhere constant let \( f^* \) be the natural canonical increasing map which equals \( f \) almost everywhere.

We now define by induction on \( n \geq 1 \) maps
and
\[ 2 : [\kappa]^\kappa \to [\kappa]^\kappa \]
as follows: (by convention \( \kappa^\kappa = 1 \) and \( [\kappa]^\kappa = \kappa \)). In the base step of \( n = 1 \), \( f + \alpha = (\lambda \beta [f(\beta) + \alpha])^* \) and \( 2\alpha = (\lambda \alpha \cdot 2) \). Assuming we have the definition for \( n = k \), and \( H \in [\kappa]^\kappa \) and \( h \in [\kappa]^\kappa \), we define \( H + h (2h) \) to be the member of \( [\kappa]^\kappa \) whose \( \beta \)-th component is \( H_\beta + h_\beta (2h_\beta) \).

**Claim.** The maps \( "+" \) and \( "2" \) are well-defined as indicated.

(Pf: we need only check (by induction) that the range sets of the maps are as indicated. If \( n = 1 \) this is clear and so suppose we have the case \( n = k \). The only thing to verify in the \( n = k + 1 \) case is that components remain distinct, i.e., that any \( \kappa \)-sequence in a given component of the form \( H_\beta + h_\beta (2h_\beta) \) never is \( \geq \text{a.e.}(\mu) \) any \( \kappa \)-sequence from a component \( H_\alpha + h_\alpha (2h_\alpha) \) for \( \beta < \alpha \). The case \( n = 2 \) is special here and follows simply because \( "\alpha_1 < "\alpha_2 \) and \( \beta_1 < \beta_2 \) imply \( \eta_1 + \beta_1 < \eta_2 + \beta_2 \). The cases for \( n > 2 \) quickly reduce to the following problem: if \( f_1 \) and \( f_2 \) are two members of \( [\kappa]^\kappa \) and \( \kappa \)-many distinct \( g \) satisfy \( f_1 < \text{a.e.} g < \text{a.e.} f_2 \), then for any \( \eta \) and \( \delta, f_1 + \eta < \text{a.e.} f_2 + \delta \). Now how do we prove this? Well suppose with \( f_1 \) and \( f_2 \) as above \( f_1 + \eta \geq \text{a.e.} f_2 + \delta \) for some \( \eta \) and \( \delta \); for simplicity \( \delta = 0 \). Then there are at most \( \eta \)-many \( g \) such that \( f_i < \text{a.e.} g < \text{a.e.} f_j \) for if \( f_i < \text{a.e.} g < \text{a.e.} f_j \), let, for each \( \beta < \kappa \), \( l(\beta) \) be the ordinal such that \( f_i(\beta) + l(\beta) = g(\beta) \). Then almost everywhere \( l(\beta) \) is between \( 0 \) and \( \eta \) and hence the additivity of \( \mu \) tells us that \( l \) is constant almost everywhere, i.e., \( g = f_i + \tau \) some \( \tau \) between \( 0 \) and \( \eta \). This contradicts that there exist \( \kappa \)-many \( g \)'s between \( f_1 \) and \( f_2 \). (The case for the components \( 2h_\alpha \) is immediate by induction.)

Now that we have the maps \( "+" \) and \( "2" \) here's what we do with them: we define for \( n \geq 1 \) maps
\[ + : \kappa_{n+1} \times \kappa_n \to \kappa_{n+1} \]
and
\[ 2 : \kappa_n \to \kappa_n \]
as follows (recall that \( \kappa_1 = \kappa \)): \( \vec{H} + \vec{h} = (\lambda \vec{H} + h) \) and \( 2\vec{h} = (\lambda 2\vec{h}) \).

**Claim.** The maps \( + \) and \( 2 \) are well-defined as indicated.

(Pf: The previous claim and the first shuffling lemma tell us that the range sets of \( + \) and \( 2 \) are as indicated. That \( + \) and \( 2 \) are well-defined is immediate by induction. \( \square \))

**Claim.** For each \( n \geq 1 \), given any \( \vec{h} \in \kappa_n \), \( \vec{H} + \vec{h} < \text{a.e.} \vec{2H} \) for every \( \vec{H} \in \kappa_{n+1} \).

(Pf: This is routine for \( n = 1 \) and generalizes immediately by induction to all \( n \). \( \square \))
Theorem. For any \( n \geq 1 \) and \( \vec{H} \in \kappa_{n+1} \) the map \( \vec{h} \mapsto \vec{h} + \vec{h} \) from \( \kappa_n \) into \( \kappa_{n+1} \) is order-preserving.

(Pf: By induction on \( n \), \( n = 1 \) being immediate and the inductive step routine. \( \square \))

These last two claims show that for each \( n \geq 1 \) \( \kappa_n \) can be mapped order-preservingly into a proper initial segment of \( \kappa_{n+1} \) and hence that for each \( n \geq 1 \) \( \kappa_n \prec \kappa_{n+1} \). \( \square \)

**Proof of Lemma E.3.** Motivation for this proof might be the standard proof of "\( \gamma \rightarrow (\gamma)^2 \rightarrow \gamma \) is a cardinal": if

\[
\lambda : \gamma \rightarrow \gamma
\]

let \( G : [\gamma]^2 \rightarrow 2 \) by \( G(\{\alpha, \beta\}) = 0 \) iff \( f(\alpha) < f(\beta) \). Then if \( C \) is any type \( \gamma \)-set homogeneous for \( G \) the fact that \( \gamma \) is infinite tells us that \( G''[C]^2 = \{0\} \) and hence we know that \( f \) is order-preserving on \( C \). Since \( C \) has type \( \gamma \) there is an order-preserving map of \( \gamma \) 1-1 onto \( \vec{\gamma} \) and hence \( \gamma = \vec{\gamma} \).

Our proof here is similar except we generally don't have \( \kappa_n \rightarrow (\kappa_n)^2 \). Our trick is to use \( \kappa \rightarrow (\kappa)^* \). For \( n = 1 \) this lemma is covered in (A). So assume \( n \geq 2 \). We first need some notation:

**Notation.** Suppose \( H \in S_n \). We wish to define for any \( m < \omega \) and \( i, 0 < i < n - 1 \), the "canonical" \( i \)-interlaced \( m \)-tuple \( \{H^{m,i,1}, H^{m,i,2}, \ldots, H^{m,i,m}\} \) associated with \( H \).

We do this by induction on \( n > 2 \) as follows: suppose \( H \in S_3 \) and \( m < \omega \). Then for each \( j \) between 1 and \( m \) \( H^{m,j} \) has as its \( \alpha^m \) component \( H_\alpha \) where \( H_\alpha \) is the unique component of \( H \) satisfying \( \vec{H}_\alpha = ((b_k^m(\lambda[\vec{H}_\lambda]))_{k})(\alpha) \). In general, if we have our induction through \( n = k \) and \( H \in S_{k+1} \) and \( m < \omega \), then, by induction on \( i \), \( 0 < i < k \), \( H^{m,i,j} \) has as its \( \alpha^m \) component \( H_\alpha \) where \( H_\alpha \) is the unique component of \( H \) satisfying \( \vec{H}_\alpha = ((b_k^m(\lambda[\vec{H}_\lambda]))_{k})(\alpha) \) and \( H^{m,i+1,j} \) has as its \( \alpha^m \) component \( (H_\alpha)^{m,i+1,j} \). This completes our notational definition. It is routinely verified that for each \( H \in S_n \), \( m \), and \( i \),

\[
\{H^{m,i,1}, H^{m,i,2}, \ldots, H^{m,i,m}\}
\]

is an \( i \)-interlaced \( m \)-element subset of \( \kappa_n \).

Given this notation we are now ready to run our argument. (Note that for each integer \( l > 0 \), \( \kappa \rightarrow (\kappa)^{l} \) implies \( \kappa \rightarrow (\kappa)^{l} \) — this follows by a routine induction on \( l \).)

Suppose \( g \) is a given map of \( \kappa_n \) 1-1 onto \( \vec{\kappa}_n \). We wish to produce a subset \( D \) of \( \kappa_n \) of order-type \( \kappa_n \) on which \( g \) is order-preserving so let \( G \) be a partition of \( [\kappa ]^{\kappa} \) into \( 2^{n-1} \) defined as follows:

given \( f \in [\kappa]^n \) let \( H_{0,f} \) and \( H_{1,f} \) be the two successive blocks of \( \kappa^{n-2} \)-many \( \kappa \)-sequences from \( \kappa \) which make up \( bk^{n-2}(f) \). Our partition \( G \) now sends \( f \) to the \( n-1 \) sequence of 0's and 1's defined as follows:

\[
G(f)_0 = 0, \quad \text{iff} \quad g(\vec{H}_{0,f}) < g(\vec{H}_{1,f})
\]
and for $0 < i < n - 1$,
\[ G(f)_i = 0, \text{ iff } g(\tilde{H}^{i+1}_{0,i}) < g(\tilde{H}^{i+2}_{0,i}). \]

Now suppose $C$ is a size $\kappa$ subset of $\kappa$ homogeneous for $G$.

**Claim.** $G''[C]^{\kappa} = \{0\}$.

(Pf: The idea here is to use the second shuffling lemma and the fact that there exist no infinite descending chains of ordinals.

We proceed formally as follows: suppose the claim is false and that $f \in [C]^{\kappa}$ and $0 \leq i < n - 1$ are such that $G(f)_i = 1$.

**Case 1.** $i = 0$: In this case $g(\tilde{H}_{0,f}) > g(\tilde{H}_{1,f})$. Let $f_1 = s f_{i-2}(\tilde{H}_{1,f})$. Then by the second shuffling lemma (Lemma B.3), $\tilde{H}_{0,f} = \tilde{H}_{1,f}$ and as we clearly have $f_1 \in [C]^\kappa$, the homogeneity of $C$ tells us that $g(\tilde{H}_{1,f}) = g(\tilde{H}_{0,f}) > g(\tilde{H}_{1,f})$. Continuing in this way we get an infinite descending chain of ordinals
\[ g(\tilde{H}_{0,f}) > g(\tilde{H}_{1,f}) > g(\tilde{H}_{0,f}) > g(\tilde{H}_{1,f}), \ldots \]
a contradiction.

**Case 2.** $i > 0$: In this case
\[ g(\tilde{H}^{i+1}_{0,i}) > g(\tilde{H}^{i+2}_{0,i}). \]

Let $f_1 = s f_{i-2}(\tilde{H}^{i+2}_{1,f})$. Then it is routine to verify by induction on $i$ (using Lemma B.3 and the fact that $s f_i$ is the identity) that $\tilde{H}_{0,f} = \tilde{H}^{i+1}_{0,i}$. Hence by Lemma B.3,
\[ \tilde{H}^{i+1}_{1,f} = \tilde{H}^{i+1}_{0,i} \]
and so by the homogeneity of $C$,
\[ g(\tilde{H}^{i+1}_{0,i}) > g(\tilde{H}^{i+2}_{0,i}) > g(\tilde{H}^{i+3}_{0,i}). \]
Continuing inductively in this way we produce an infinite descending chain of ordinals as we did in Case 1. This contradiction yields the claim. \qed

From this claim we can now show that $g$ is order-preserving on $S^{\kappa}_{\kappa}/\sim$. For if $\tilde{H} < \tilde{K}$ is a given pair from $S^{\kappa}_{\kappa}/\sim$, $i$-interlaced say, we can use the first shuffling lemma to produce a member $f$ of $[\kappa]^\kappa$, such that $\tilde{H}_{0,f} = \tilde{H}$ and $\tilde{H}_{1,f} = \tilde{K}$ (if $i = 0$) or $\tilde{H}^{i+1}_{0,f} = \tilde{H}$ and $\tilde{H}^{i+2}_{0,f} = \tilde{K}$ (if $i > 0$). In either case, the fact that $G(f)_i = 0$ yields $g(\tilde{H}) < g(\tilde{K})$. As $\tilde{H} < \tilde{K}$ was arbitrary we must have by that $g$ is order-preserving on $S^{\kappa}_{\kappa}/\sim$. By Lemma E.1 $S^{\kappa}_{\kappa}/\sim$ has type $\kappa$ and so g mapping $S^{\kappa}_{\kappa}/\sim$ order-preservingly into $\kappa$, must imply $\kappa = \tilde{\kappa}$. \qed

**Proof of Lemma E.4.** The proof of this is fairly easy. Suppose $F : [\kappa]^\alpha \rightarrow 2$ for $\alpha < \aleph_1$. Define $G : [\kappa]^{\kappa} \rightarrow 2$ as follows: for any $f \in [\kappa]^{\kappa}$, $G(f) = F(\alpha^\beta[(\beta k_{i,f}(f))]_\beta)$ and suppose $C \subseteq \kappa$ is such that $\tilde{C} = \kappa$ and $C$ is homogeneous for $G$. Let $D = \alpha S^{\kappa}_{\kappa}/\sim$. Then $D$ has type $\kappa_2$ and we claim that $D$ is homogeneous for $F$. Indeed it is easy to see that $G''[C]^{\kappa} = F''[D]^{\kappa}$ for suppose $l \in [D]^{\kappa}$. Using
countable choice let $H$ be a member of $[[C]^*]$ such that for each $\beta < \alpha \\overline{H(\beta)} = 1(\beta)$. Then clearly $F(i) = G[s_f(H)]$ and since $s_f(H) \in [C]^*$, we have our proof. □

**Proof of Lemma E.5.** Given $\beta$ fixed, the proof of Lemma G.1 is entirely uniform. Hence, for example, there exists a map $M : [\kappa]^* \times \kappa^{-2} \rightarrow [\kappa]^*$, such that for any $f$ in $[\kappa]^*$ and $\beta < \kappa^{-2}$, $M(f, \beta)$ is the $\beta^{th}$ member of $\kappa_2$ greater than $\vec{f}$. Thus for any $f$ in $[\kappa]^*$, $\lambda \beta[M(f_n, \beta)]$ is a member of $S_n$ and it is routine to verify that $\alpha \rightarrow \lambda \beta[M(f_n, \beta)]$ (where $f_n$ is any member of $[\kappa]^*$ such that $f_n = \alpha$) is a well-defined map of $\kappa_2$ into $\kappa_n$.

Now suppose $\vec{H}$ is a given member of $\kappa_n$. Then $\beta \rightarrow \vec{H}(\beta)$ is a well-defined map of $\kappa^{-2}$ into $\kappa_2$ and as $\kappa_2$ is regular let $\alpha$ exceed the sup of the range of this map. Then $\lambda \beta[M(f_n, \beta)]$ is easily checked to exceed $\vec{H}$ as a member of $\kappa_n$ and hence we have that our above map of $\kappa_2$ into $\kappa_n$ is cofinal. But it is clear by its definition that this map is nondecreasing and so as $\kappa_2$ is regular we must in fact have $cf(\kappa_n) = \kappa_2$. □

**(F) Proof of Lemma F.1.** Given a number $p$ of $[\kappa]^*$, let $\omega p$, a member of $[\kappa]^*$, be defined by

$$\omega p(\alpha) = \bigcup_{n < \alpha} p(\omega \cdot \alpha + n).$$

It is clear that if $p$ and $q$ are two members of $[\kappa]^*$ which interlock (i.e. are such that for all $\alpha p(\alpha) < q(\alpha) < p(\alpha + 1)$, then $\omega p = \omega q$. For this reason, it is immediate that

$$Q = \{ A \subseteq [\kappa]^* \mid \text{for some } p \in [\kappa]^*, \omega q \in A \text{ for all } q \in [p]^* \}$$

is a filter on $[\kappa]^*$ and, using countable choice, that it is a countably additive filter.

**Claim.** Given any partition $F : [\kappa]^* \rightarrow 2$, $\kappa \rightarrow (\kappa)^*$ implies that the collection of sets homogeneous for $F$ is a member of $Q$.

**(Pf:** Given $F : [\kappa]^* \rightarrow 2$, let $G : [\kappa]^* \rightarrow 2$ be given by $G(p) = F(\omega p)$. Then since $r \in [\omega q]^*$ implies $r = \omega s$ for some $s \in [q]^*$, it is clear that if $q$ is homogeneous for $G$, then $\omega s$ is homogeneous for $F$ for every $s \in [q]^*$ □.)

Now given $F : [\kappa]^* \rightarrow 2^\omega$, let, for each $n < \omega$, $F_n : [\kappa]^* \rightarrow 2$ be given by $F_n(p) = (F(p))(n)$. By the claim, the collection of sets homogeneous for $F_n$ is in $Q$ for each $n$ and so, as $Q$ is countably additive, let $C$ be a single set homogeneous for each $F_n$. By the definition of the $F_n$, $C$ is homogeneous for $F$. □

**(G) Proof of Lemma G.1.** Let $g$ be a map of $\kappa$ 1–1 onto $\beta$. We will use $g$ to define by induction on $\eta \leq \beta$ maps $G_{\eta} : \eta \rightarrow [C]^*$, such that for every $\nu < \eta$, $G_{\eta}(\nu)$ is the $\nu^{th}$ member of $[C]^*$ larger than $\vec{H}$ and such that $\eta_1 < \eta_2 < \beta$ implies $G_{\eta_2}$ is an extension of $G_{\eta_1}$. Our desired map $G$ will then be taken to be $G_{\beta}$. 
Now before actually proceeding with the definition of the $G_\alpha$ let us note the following: given any ordinal $\tau < \beta$, which is a limit ordinal we have, using $g$, a "canonical" at-most-$\kappa$-sequence with $\sup \tau$ given as follows: since $g$ maps $\kappa$ 1-1 onto $\beta$ and $\tau < \beta$ let $Q$ be $g^{-1}\tau$. We now take our "canonical" sequence to be that whose first point is $g$ of the least member of $Q$ and, inductively, whose $\sigma^{th}$ point is $g$ of the least ordinal in $Q \geq \sigma$, which $g$ sends above every point in the sequence picked so far. It is routine to verify that this "canonical" sequence is indeed cofinal in $\tau$.

Now for the $G_\alpha$: for $x \subseteq \alpha < \kappa$ let us denote by $\alpha + x$ 1 the least member of $C$ greater than $\alpha$ and by $\bigcup x$ the least member of $C$ greater than or equal to $\bigcup x$. Given this notation let $G_\alpha(0) = \sigma H$; if $G_\alpha$ has been defined and $\tau$ is a successor ordinal $\sigma + 1$, then

$$G_{\tau+1} = \sigma \nu [(G_\tau(\sigma))(\nu) + 1]$$

and $G_{\tau+1} \tau = G_\tau$; if $G_\alpha$ has been defined and $\tau$ is a limit ordinal of cofinality less than $\kappa$ and $\alpha_0, \alpha_1, \ldots, \alpha_\xi < \sigma < \kappa$ is the canonical cofinal sequence for $\tau$, then we set

$$G_{\tau+1}(\tau) = \sigma \nu \left[ \bigcup_{\xi < \sigma} (G_\tau(\alpha_\xi))(\nu) \right]$$

and $G_{\tau+1} \tau = \sigma G_\tau$; if $G_\alpha$ has been defined and $\tau$ is a limit ordinal of cofinality $\kappa$ with $\alpha_0, \alpha_1, \ldots, \alpha_\xi < \kappa$ the canonical cofinal sequence for $\tau$ we set

$$G_{\tau+1}(\tau) = \sigma \nu \left[ \bigcup_{\xi < \sigma} (G_\tau(\alpha_\xi))(\nu) \right]$$

and $G_{\tau+1} \tau = \sigma G_\tau$; if $G_\alpha$ has been defined for $\sigma < \tau$ where $\tau$ is a limit ordinal then we set $G_\tau = \sigma \bigcup _{\xi < \tau} G_\alpha$. This completes our inductive definition of the $G_\alpha$ and hence of $G$ which we take to be $G_\#$. It is routine to see that $G$ is as desired.

**Proof of Lemma G.2.** Motivation here might be the following proof of the fact that $\tau \rightarrow (\tau)^{<\omega}$ implies "for any $\sigma < \tau$, $\tau \rightarrow (\tau)^{<\omega}$": if $F : [\tau]^{<\omega} \rightarrow \sigma$, let $G : [\tau]^{<\omega} \rightarrow 2$ by $G(\{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n}\}) = 0$, iff

$$F(\{\alpha_1, \ldots, \alpha_n\}) = F(\{\alpha_{n+1}, \ldots, \alpha_{2n}\}).$$

Now suppose $C$ is homogeneous for $G$ and of cardinality $\tau$. There are two parts to the completion of the proof. Part 1 is showing $G''[C]^{2n} = \{0\}$ for each $n$ and part 2 is showing that this implies $C$ homogeneous for $F$:

(Part 1). For any $n$ there are $\tau$-many nonoverlapping $n$-element subsets of $C$ and since $\sigma < \tau$ we must have that $F$ is equal on some two of them, say

$$F(\{\alpha_1, \ldots, \alpha_n\}) = F(\{\beta_1, \ldots, \beta_n\}).$$

Since $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ are nonoverlapping $G(\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}) = 0$ and so for each $n$ $G''[C]^{2n} = \{0\}$. 
From Part 1 it immediately follows that $C$ is homogeneous for $F$ for if

$\{\alpha_1, \ldots, \alpha_n\} \text{ and } \{\beta_1, \ldots, \beta_n\}$ are any two $n$-element subsets of $C$, let $\{\gamma_1, \ldots, \gamma_n\}$ be an $n$-element subset of $C$ which overlaps neither $\{\alpha_1, \ldots, \alpha_n\}$ nor $\{\beta_1, \ldots, \beta_n\}$. Then since $G(\{\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n\}) = 0$ and $G(\{\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n\}) = 0$, we have

$$F(\{\alpha_1, \ldots, \alpha_n\}) = F(\{\gamma_1, \ldots, \gamma_n\}) = F(\{\beta_1, \ldots, \beta_n\}).$$

The idea in our context of the $\kappa_\eta$ is to use the above trick but always falling back on the partition relation $\kappa \to (\kappa)^{\eta} \rightarrow$ rather than $\kappa_\eta \to (\kappa_\eta)^{\eta} \rightarrow$ (which is, in fact, usually false).

Suppose $\gamma < \kappa_2$ and $F : [\kappa_\eta]^{\eta} \rightarrow \gamma$ is a given partition. For simplicity let us first look at $F \upharpoonright [\kappa_\eta]^2$. Using the notation, introduced in the Proof of Lemma E.3, let us define $G : [\kappa]^* \rightarrow 2$ as follows: given $f \in [\kappa]^*$, let $U_f, X_f, Y_f$ and $Z_f$ be the four successive blocks of length $\kappa_\eta - 2$ which make up $b_{\kappa_\eta - 2}(f)$. Then $G(f) = 0$, iff

$$F(\{U_f, X_f\}) = F(\{Y_f, Z_f\}) = F(\{Y_f, X_f\}) = F(\{U_f, Z_f\}),$$

and

$$F(\{U_f^{2, \kappa_\eta - 2}, X_f^{2, \kappa_\eta - 2}\}) = F(\{Y_f^{2, \kappa_\eta - 2}, Z_f^{2, \kappa_\eta - 2}\}).$$

Now let $C$ be homogeneous for $G$, $\bar{C} = \kappa$. Then following our motivational example we will show that $S^C_{\kappa_\eta} / \sim_n$ is homogeneous for $F$ in the sense that $F''[S^C_{\kappa_\eta} / \sim_n]^2 \leq n - 1$.

**Part I.** $G''[C]^* = \{0\}$.

**Proof.** Mark off $S^C_{\kappa_\eta} / \sim_2$ into successive blocks each of length $\kappa_\eta - 2$. There are $\kappa_\eta$-many such blocks since $\kappa_2$ is a cardinal — let $K_\alpha$ be the $\alpha$-th. Now by Lemmas G.1 and B.2, for each $\alpha < \kappa$ there is an $n - 1$ tuple of ordinals less than $\gamma$ associated with $K_\alpha$, namely

$$t_\alpha = (F(\{U, X\}), F(\{U, U^{2,1,1}, U^{2,1,2}\}), \ldots, F(\{U^{2, \kappa_\eta - 2,1}, U^{2, \kappa_\eta - 2,2}\})),$$

where $U$ and $X$ are members of $S^C_{\kappa_\eta}$, such that for each $\beta < \kappa_\eta$, $U(\beta) = K_\alpha(\beta)$ and $X(\beta) = K_\alpha(\kappa_\eta - 2 + \beta)$. (It is routine to see by induction that this $n - 1$-tuple of ordinals is independent of which $U$ and $X$ we choose.) We thus have a map $\alpha \mapsto t_\alpha$ from $\kappa_\eta$ to $\gamma^{n - 1}$ and since $\gamma < \kappa_2$ and $\kappa_2$ is a cardinal, there must exist $\alpha < \beta < \kappa$ such that $t_\alpha = t_\beta$. But then by Lemmas G.1 and B.2 we can put together an $f$ in $[C]^*$, such that $G(f) = 0$. Hence $G''[C]^* = \{0\}$. 

**Part II.** $S^C_{\kappa_\eta} / \sim_n$ is homogeneous for $F$ in that $F''[S^C_{\kappa_\eta} / \sim_n]^2 \leq n - 1$.

**Proof.** We show that for each $i$, $0 \leq i < n - 1$, $F$ is equal on all $i$-interlaced pairs from $S^C_{\kappa_\eta} / \sim_n$. For suppose $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ are two $i$-interlaced pairs from
Using the first shuffling lemma, the regularity of $\kappa$ and Lemma G.1 we can routinely put together two members $f$ and $g$ of $[\mathcal{C}]^\kappa$, such that if $i = 0$,

$$U_1 \times \tilde{X} = \{\alpha_1, \alpha_2\} \quad U_2 \times \tilde{Y} = \{\beta_1, \beta_2\} \quad \tilde{Y}_3 \times \tilde{Z} = \{\tilde{Y}_3, \tilde{Z}_3\}$$

and if $i > 0$,

$$U_{i+1} \times \tilde{X}_{i+1} = \{\alpha_1, \alpha_2\} \quad U_{i+1} \times \tilde{Y}_{i+1} = \{\beta_1, \beta_2\} \quad \tilde{Y}_{i+1} \times \tilde{Z}_{i+1} = \{\tilde{Y}_{i+1}, \tilde{Z}_{i+1}\}.$$

Since $G(f) = G(g)$, $F(\{\alpha_1, \alpha_2\}) = F(\{\beta_1, \beta_2\})$. \qed

Now what about $F \upharpoonright [\kappa_\gamma]^\kappa$? By an entirely similar argument we produce a partition $\mathcal{G} : [\kappa]^{\kappa} \to 2$ such that if $C'$ is any set homogeneous for $\mathcal{G}$, $C' = \kappa$, then $S_{\kappa}^n / \sim_n$ is homogeneous for $F$ in the sense that $F^n(S_{\kappa}^n / \sim_n)^3 \leq (n - 1)^3$. The main difference here is that there are $(n - 1)^3$ different types of triples from $\kappa_n$ depending upon the interlacings of the first two ordinals in the triple and of the last two ordinals in the triple. Similarly the above argument generalizes to $F \upharpoonright [\kappa_\gamma]^\kappa$ for each $m$. Now what about getting one subset of $\kappa_\gamma$ of size $\kappa_\gamma$ homogeneous simultaneously for $F \upharpoonright [\kappa_\gamma]^\kappa$ for all $m$, i.e. homogeneous for $F$. Well, we would need one subset of $\kappa_\gamma$ homogeneous simultaneously for the countably many associated partitions of $[\kappa]^\kappa$ into 2. Countably many partitions $G_m : [\kappa]^\kappa \to 2$ can be coded into a single partition $\mathcal{G} : [\kappa]^\kappa \to 2$ given by $(\mathcal{G}(f))(m) = G_m(f)$ and it is clear that any set homogeneous for $\mathcal{G}$ is homogeneous for each $G_m$. Thus by Lemma F.1, $(\kappa)^\omega_\omega$, our proof is complete.

$$\kappa_\gamma \upharpoonright \kappa_\gamma = \kappa_\gamma / \mathcal{A}_\kappa / \kappa_\gamma^{(n - 1)^3}$$

**Proof of Lemma G.3.** Suppose $F : [\gamma]^\omega \to 2$. We wish to find a subset $C$ of $\gamma$ of cardinality $\gamma$ such that for each $n$ $F[C]^n = 1$, and so let us define $G : [\gamma]^\omega \to \omega$ as follows:

$$G(\{\alpha_1, \ldots, \alpha_2k\}) = \pi^2(x_1) \cdot 3^{p(\alpha_2)} \cdot \ldots \cdot p^{p(\alpha_2)} \cdot \ldots \cdot p^{p(\alpha_2)} \cdot \ldots \cdot p^{p(\alpha_2)}$$

where $k$ is a prime number exceeding 2, where for each $j$, $p_j$ is the $j^{th}$ prime number, and where $x_1$ is the $i$-tuple consisting of the least $i$-many members of $\{\alpha_1, \ldots, \alpha_2k\}$, $x_2$ is the next $i$-many members, etc. Now suppose $D$ is homogeneous for $G$ in the sense that for each $n$ $G[D]^n \leq h(n)$.

**Claim.** For some $\alpha < \gamma$ $D - \alpha$ is homogeneous for $F \upharpoonright [\gamma]^\omega$ in the sense that $F[D - \alpha]^3 = 1$. \(\) **Proof.** Since $h$ is eventually less than $\lambda n[(\sqrt{\gamma})^\gamma]$, let $k$ be a prime $> 2$ so large that $h(4k) < (\sqrt{\gamma})^{4k}$, that is, such that $h(4k) < 2^k$. Now if the claim is false let $x^0, x^1, x^2, x^3, \ldots x^l, x^0, x^1, \ldots x^l$, $x^l$ be nonoverlapping 2-tuples in $D$ such that for each $l$ between 1 and $k$

$$0 = F(x^l) \neq F(x^i) = 1.$$
Then it is routine to see that by forming $2^k$-tuples from $D$ using appropriate combinations of the $x^i$ and $x^j$ as the first $2^k$ many ordinals, we can get $2^k$ many distinct things in the range of $G$ on $[D]^{2^k}$. But this contradicts $G(\mathcal{D})^{2^k} \leq h(4k) < 2^k$. The claim is thus proved. \)

Now by the claim we first observe that we at least have $\gamma \to (\gamma)^2$ and hence $\gamma$ is regular. But in addition, it is routine to check that the claim remains valid in a general context:

“For each $n > 1$ there exists an ordinal $\alpha < \gamma$, such that $F^{\alpha}[D-\alpha] = 1$.” So by putting these two observations together we can now define for each $n \geq 1$ $\beta_n$ to be the least ordinal $\alpha$ satisfying $F^{\alpha}[D-\alpha] = 1$ and take $\beta < \gamma$ to be the sup of the $\beta_n$. Then $C = D - \beta$ is a subset of $\gamma$ of size $\gamma$ such that for each $n$, $F^{\alpha}[C] = 1$. This completes our proof. \)

(H) Proof of Lemma H.1. The only point to verify is transitivity and this is routine by induction. \)

Proof of Lemma H.2. We define by induction on $n > 2$ a map $I_n$ from $\kappa_n$ into $\kappa_{n-1}$ with the property that for any $H$ and $G$ in $\kappa_n$, $I_n(H) = I_n(G)$ implies $H = G$. This will yield the lemma for as $\kappa_n$ is a cardinal greater than $\kappa_{n-1}$, for any subset of $\kappa_n$ of size $\kappa_n$ there must be a subset of it of size $\kappa_{n-1}$ on which $I_n$ is constant.

So what is $I_n$? Given $\bar{H} \in \kappa_n$, $I_n(\bar{H}) = \sup \{H | \alpha < \kappa_n\}$. Since $cf(\kappa_n) = \kappa_n > \kappa$, and since $\bar{H}_\alpha < \kappa_n$ for each $\alpha < \kappa_n$, $I_n(\bar{H})$ is easily seen to be well-defined. Furthermore, it is easy to see that $I_n(\bar{H}) = I_n(\bar{G})$ implies $\bar{H} = \bar{G}$. The definition of $I_n$ in general is similar to this but first we need some notation and machinery. First note that there exists a map $K$ from $S_3$ into $S_2$ such that for any $G$ in $S_3$, $K(G)$ is equal to $\alpha$ for all $\alpha < \kappa$. Let us call this map $K$. We simply let $K(G)(\beta) = (G(\beta))$. We do have to check that $K_n(G) \in S_{n-1}$ but this is routine by induction and Lemma B.2. It is now routine to verify that $I_n(\bar{H}) = \sup \{H | \alpha < \kappa_n\}$ is well-defined and as desired. \)

Proof of Lemma H.3. Suppose $\gamma < \kappa_n$ and $F : [\kappa_n]^{<\omega} \to \gamma$ is a given partition. We wish a subset $D$ of $\kappa_n$ of cardinality $\kappa_n$ such that $F^{\omega}(D) = \omega$.

The idea of the proof is to use the methods of above to define a partition $G : [\gamma]^{<\omega} \to 2$, such that given a homogeneous set $C$ for $G$ we can convert $C$ to a desired homogeneous set for $F$. Now if we were just interested in showing $\kappa_n$ Jonsson (or if we had $\gamma < \kappa_3$), then we could do this by an appropriate generalization of the arguments used in proving Lemma G.2. In this case if $C$ were the subset of $\kappa$ homogeneous for $G$, then $D = \bigcup_{\alpha \kappa_n \cap \kappa_n = \kappa_n$ for each $n$, a
situation which provably cannot happen for certain partitions $F: [\kappa_n]^{<\omega} \rightarrow \gamma$, if $\gamma \geq \kappa_n$. In the argument we are about to do our homogeneous set $D$ will have the property that from some $n_0$ on $D \cap \kappa_n = \kappa_{n-1}$.

So $\gamma < \kappa_n$ and $F: [\kappa_n]^{<\omega} \rightarrow \gamma$ is a given partition for which we wish a homogeneous set. We first describe a countable collection $\mathcal{G}$ of associated partitions of $[\kappa]^\omega$ into 2. First of all we will want to throw into $\mathcal{G}$ basically those partitions of $[\kappa]^\omega$ which arose in the proof of Lemma G.2. The only modification here is that in associating partitions of $[\kappa]^\omega$ with say $F[I[K]]^\omega$ we will only want to consider the action of $F$ on fully $n-2$ interlaced $m$-element subsets of $\kappa_n$. So, for example, instead of throwing in the partition $G: [\kappa]^\omega \rightarrow 2$ as given in the proof of G.2, we would throw in the partition which sends $f$ in $[\kappa]^\omega$ to 0 iff

$$F(\{U_{j}\}_{j<n}^{n-2}) = F(\{U_{j}\}_{j<k}^{n-2}).$$

Now throw into $\mathcal{G}$ all such partitions of $[\kappa]^\omega$ associated with the restriction of $F$ to sets of the form $[\kappa_n]^\omega$ where $n$ is larger than $n_0$, $n_0$ being the least integer satisfying $\kappa_n > \gamma$. We throw at most countably many partitions into $\mathcal{G}$ during this step.

What else do we throw into $\mathcal{G}$? Well we need "mixture partitions", that is partitions which worry about the action of $F$ on $m$-element subsets of $\kappa_n$ whose members come from different $S_m$. For example, we would want a partition in $\mathcal{G}$ which considered the action of $F$ on triples from $\kappa_n$ the first largest element of which is a member of $S_l/\sim_k$, the second largest element a member of $S_k/\sim_k$ $k-2$-interlaced with the first largest element, and the third largest element a member of $S_l/\sim_l$ ($l > k$) which does not interlace with the first two at all. Such triples we call $S_k < S_l$-triples and our associated partition $G$ to be thrown into $\mathcal{G}$ is simply as follows: given $f$ in $[\kappa]^\omega$, let $H$ denote $bk_{k-1}(f)$ and let $K$ denote the least $K_{k-1}$-many $n$-sequences in $H$. Then we simply have our partition $G$ send $f$ to 0 iff

$$F(\{U_{j}\}_{j<n}^{k-1}H_{j<n}^{k-1}) = F(\{U_{j}\}_{j<k}^{k-1}H_{j<n}^{k-1}).$$

(Keep in mind that, for example, $K_{k-1}$ is the $\sim_k$ equivalence class of $K^{k-1}$ whereas $H_{k-1}$ is a $\sim$ equivalence class.)

At any rate, we complete our definition of $\mathcal{G}$ by throwing in all such "mixture partitions". Just keep in mind that we only throw in mixture partitions for those $S_k < S_l < \cdots < S_m$-tuples, such that $i_1 \leq i_2 \leq \cdots \leq i_m$ and $i_k = i_{k+1}$ the $i_k$th largest and $i_{k+1}$th largest members of the tuple are $i_k - 2$-interlaced and $i_k < i_{k+1}$ the $i_k$th largest and $i_{k+1}$th largest members of the tuple are not at all "interlaced", that is, they are equivalence classes of some $K$ and $H$ in $S_k$ and $S_{k+1}$ respectively where $K$ is an initial segment of $H$.

Given this definition of $\mathcal{G}$ it is clear that $\mathcal{G}$ is at most countable and hence by the same argument used in the proof of Lemma G.2, the argument using Lemma F.1, we can find a single size $\kappa$ subset $C$ of $\kappa$ which is homogeneous for each member of $\mathcal{G}$.
Claim. For any $G$ in $\mathcal{G}$, $G''[C]^* = \{0\}$.

(Proof. The argument here is different from that for “Part I” appearing in the Proof of Lemma G.2. We proceed as follows: suppose $G \in \mathcal{G}$, suppose for example $G$ is the partition given above. By Lemma H.2 let $D_1$ be a size $\kappa_{k-1}$ equivalence class under $\equiv_k$ contained in $S_{\xi}^{\sigma}/\sim_{\xi}$, let $\alpha_0$ be a member of $D_1$, let $\alpha$ be a member of $\kappa$ such that for some $H$ in $S_\kappa$ such that $\tilde{H} = \alpha$, there is an initial segment $K$ of $H$, such that $\tilde{K} = \alpha_0$, and let (again by Lemma H.2) $D_2$ be a size $l - 1$ equivalence class under $\equiv_l$ contained in $S_{\xi}^{\sigma}/\sim_{\xi} - \alpha$. Now associated with any triple $t = \{\alpha_1 < \alpha_2 < \alpha_3\}$, $\alpha_1, \alpha_2 \in D_1$, $\alpha_3 \in D_2$, we have an ordinal $\alpha_i < \gamma$ namely $\alpha_i = F(\{\alpha_1, \alpha_2, \alpha_3\})$. Consider the following $\kappa_{k-1}$-many triples of this sort: the first, $t_0$, has as its first two ordinals the first two members of $D_1$ and as its third ordinal the first element of $D_2$, the second, $t_1$, has as its first two members the next two members of $D_1$ and as its third element the next element of $D_2$. Continuing inductively in this way we define $\kappa_{k-1}$-many $S_k < S_l < S_t$-triples $t_n$. Now since $k > n_0$, $k - 1 \geq n_0$ and so $\kappa_{k-1} > \gamma$. Thus as $\kappa_{k-1}$ is a cardinal there must be $\alpha < \beta < \kappa_{k-1}$, such that $\alpha_\alpha = \alpha_\beta$. As any two members of $D_1$ are $k-2$-interlaced and any two members of $D_2$ are $l-2$-interlaced we can now use the first shuffling lemma to find an $f$ in $[C]^*$, such that where $H$ denotes $bk_{k-2}(f)$ and $K$ denotes the least $\kappa^{k-2}$-many $\kappa$-sequences in $H$,

$$\{\tilde{K}^{4,4,k-2,1}, \tilde{K}^{4,4,k-2,2}, \tilde{H}^{2,1-2,1}\} = t_0$$

and

$$\{\tilde{K}^{4,4,k-2,3}, \tilde{K}^{4,4,k-2,4}, \tilde{H}^{2,1-2,2}\} = t_\beta.$$ 

By the definition of $G$ and $t_0$ and $t_\beta$, $G(f) = 0$ and so, as $C$ is homogeneous for $G$, $G''[C]^* = \{0\}$. Similarly, $G^*''[C]^* = \{0\}$ for any $G^*$ in $\mathcal{G}$. $\Box$

We now construct our homogeneous set for $F$, that is, a size $\kappa_n$ subset $D$ of $\kappa_n$ such that $\mathcal{F}''[D]^c_{\kappa_n} \subseteq \omega$. We do this by first defining by induction on $n > n_0$ sets $D_n \subseteq \kappa_n$ as follows: $D_{n+1}$ is any size $\kappa_n$ equivalence class under $\equiv_{n+1}$ contained in $S_{\kappa_{n+1}}^{C_{n+1}}/\sim_{n+1}$, and if $D_n$ has been defined let $\alpha_0$ be the least member of $D_n$, let $\alpha$ be the least member of $\kappa_{k+1}$ such that for some $H$ in $S_{\kappa_{k+1}}^{C_{k+1}}$ such that $\tilde{H} = \alpha$, $\tilde{H}_0 = \alpha_0$, and let $D_{k+1}$ be a size $\kappa_k$ equivalence class under $\equiv_{n+1}$ contained in $S_{\kappa_{k+1}}^{C_{k+1}}/\sim_{k+1} - \alpha$. This completes our inductive definition of the $D_n$. Note that for each $n > n_0$, $D_n$ is a size $\kappa_{n-1}$ subset of $\kappa_n$.

Let us now set $D$ equal to the union of the $D_n$. Then clearly $D$ is a size $\kappa_n$ subset of $\kappa_n$.

Claim. $D$ is homogeneous for $F$, that is, $\mathcal{F}''[D]^c_{\kappa_n} \subseteq \omega$.

(Proof. The proof here is similar to the proof of part II appearing in the proof of Lemma G.2. What we show is that all tuples from $D$ of the same “type” are sent by $F$ to the same place. For example, suppose $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ are two
$S_k < S_k < S_l$-triples from $D$. Then $\alpha_1, \alpha_2, \beta_1, \beta_2$ are all in $D_k$ and $\alpha_3$ and $\beta_3$ are in $D_h$. Let $\gamma_1$ and $\gamma_2$ be two members of $D_k$ larger than any of $\alpha_1, \alpha_2, \beta_1, \beta_2$ and let $\gamma_3$ be a member of $D_l$ larger than $\alpha_3$ or $\beta_3$. Then by the first shuffling lemma we can find $f$ and $g$ in $[C]^\kappa$ such that with $H(J)$ denoting $bk_{\kappa-2}(f)$ $(bk_{\kappa-2}(g))$ and $K(L)$ denoting the least $\kappa^{k-2}$-many $\kappa$-sequences in $H(J)$,

$$\{\bar{K}^{4,k-2,1}, \bar{K}^{4,k-2,2}, \bar{H}^{2,l-2,1}\} = \{\alpha_1, \alpha_2, \alpha_3\}$$

and

$$\{\bar{L}^{4,k-2,1}, \bar{L}^{4,k-2,2}, \bar{J}^{2,l-2,1}\} = \{\beta_1, \beta_2, \beta_3\}$$

and

$$\{\bar{K}^{4,k-2,3}, \bar{K}^{4,k-2,4}, \bar{H}^{2,l-2,2}\} \cup \{\bar{L}^{4,k-2,3}, \bar{L}^{4,k-2,4}, \bar{J}^{2,l-2,2}\} = \{\gamma_1, \gamma_2, \gamma_3\}.$$ 

Since for the partition $G$ considered above $G''[C]^\kappa = \{0\}$, the definition of $G$ tells us that

$$F(\{\alpha_1, \alpha_2, \alpha_3\}) = F(\{\gamma_1, \gamma_2, \gamma_3\}) = F(\{\beta_1, \beta_2, \beta_3\}).$$

This argument thus handles type $S_k < S_k < S_l$-triples and in a similar way we handle all types of tuples from $D$. This proves the claim. \qed

Lemma H.3 is thus proved. \qed

References