Preservation of Nonoscillatory Behavior of Solutions of Second-Order Delay Differential Equations under Impulsive Perturbations

MINGSHU PENG
Department of Mathematics, Beijing Normal University
Beijing 100875, P.R. China

WEIGAO GE
Department of Applied Mathematics, Beijing Institute of Technology
Beijing 100081, P.R. China

QIANLI XU
Department of Mathematics, Yiyang Teachers' College
Yiyang 413049, P.R. China

(Received April 2000; accepted March 2001)

Abstract—We offer new criteria for the preservation of nonoscillatory behavior of solutions of the delay differential equation of second-order

\[ x''(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0 \]

under impulsive perturbations. A technique of direct analysis in this paper is developed. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Impulse, Delay differential equation, Existence and uniqueness, Nonoscillation.

1. INTRODUCTION

Consider the impulsive delay differential equation

\[ y''(t) + p(t)y(t - \tau) = 0, \quad t \neq t_k, \quad k \in \mathbb{N}, \]

\[ y(t_k) - y(t_k^-) = M_k y(t_k^-), \quad y'(t_k) - y'(t_k^-) = L_k y'(t_k^-), \tag{1} \]

where \( p(t) \geq 0 \) is continuous on \([t_0, \infty)\), \( \tau > 0 \), \( M_k, L_k \) are constants, \( \mathbb{N} = \{1, 2, \ldots\} \) is the natural number set, \( 0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots \), and \( \lim_{k \to \infty} t_k = \infty \), and

\[
\begin{align*}
y'(t_k^-) &= \lim_{h \to 0^+} \frac{y(t_k + h) - y(t_k^-)}{h}, \\
y'(t_k^+) &= \lim_{h \to 0^-} \frac{y(t_k + h) - y(t_k)}{h}.
\end{align*}
\]

This work is partially supported by NNSF of P.R. China. The authors are very grateful to the referees for making valuable suggestions and comments on the original manuscript.

0893-9659/02/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved. Typeset by \LaTeX\n
P1: S0893-9659(01)00119-7
By a solution of equation (1), we mean a real-valued function $x(t)$ defined on $[t_0 - \tau, \infty)$, which is piecewise right continuous and differentiable on $(t_0, \infty)$ and satisfies (1).

A solution of equation (1) is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

It is well known that ordinary differential equations with impulses and delay differential equations have been considered by many authors (see [1-4], etc.). The theory of impulsive differential equations is emerging as an important area of investigation, since it is much richer than the corresponding theory of differential equations without impulse effects. Moreover, such equations may exhibit several real world phenomena, such as rhythmical beating, merging of solutions, and noncontinuity of solutions.

In the past ten years, there is an increasing interest on the oscillation/nonoscillation of impulsive delay differential equations, and numerous papers have been published on this class of equations (see [5-17], etc.). Recently, the oscillatory/nonoscillatory behavior of solutions of a second-order linear impulsive delay differential equation of the type

$$y''(t) + p(t)y(t) = 0, \quad t \neq t_k, \quad k \in \mathbb{N},$$

$$y(t_k) - y(t_k^-) = B_ky(t_k^-), \quad y'(t_k) - y'(t_k^-) = D_ky'(t_k^-),$$

was discussed in [12] by Berezansky and Braverman and good results were obtained.

In this paper, a method of direct analysis, which is different from those employed in [12,13], is devoted to the investigation of the preservation of nonoscillatory solutions of the delay differential equation of second order,

$$x''(t) + p(t)x(t - \tau) = 0,$$  \hspace{1cm} (2)

under impulsive perturbations. A new technique, not utilizing the fixed-point theorems, is developed in this paper. As for the study of the persistence of nonoscillatory solutions of the first-order linear delay differential equation under impulsive perturbations, we refer the reader to [7] among many others.

The following is our main result.

**THEOREM 1.** Assume that

$$M_k \geq 0, \quad L_k \geq 0, \quad L_k \geq M_k, \quad k \in \mathbb{N}. \hspace{1cm} (3)$$

If equation (2) is nonoscillatory, then so is equation (1).

**2. THE PROOFS OF MAIN RESULTS**

To show the above results, we need the following lemma, which can be proved by using the method of steps.

**LEMMA 1.** Let $\phi(t) \in PC[t_0]$, where $PC[\sigma] = \{\phi(t) : [\tau_\sigma, \sigma] \rightarrow R, \phi(t) \text{ is piecewise right continuous on } [\tau_\sigma, \sigma]\}$. Then the initial problem,

$$x''(t) + p(t)x(t) = 0, \quad t \geq t_0,$$

$$x(t) = \phi(t), \quad t \in [t_0 - \tau, t_0],$$

$$x'(t_0^+) = A,$$

has a unique solution $x(t) : [t_0 - \tau, +\infty) \rightarrow R$ satisfying $x(t) \in C^1([t_0, +\infty), R), x(t) = \phi(t)$ for $t_0 - \tau \leq t \leq t_0, x'(t) \in C([t_0, +\infty), R)$ and $x'(t_0^+) = x'(t_0) = A$. 


PROOF. We assume that there exist $t_0 - \tau = \delta_1 < \delta_2 < \cdots < \delta_n < \delta_{n+1} = t_0$ such that

$$
\phi(t) = \begin{cases} 
\phi_1(t), & \delta_1 \leq t < \delta_2, \\
\phi_2(t), & \delta_2 \leq t < \delta_3, \\
\vdots \\
\phi_n(t), & \delta_n \leq t < \delta_{n+1}, \\
\phi(t_0), & t = t_0,
\end{cases}
$$

where $\phi_i(t)$ is continuous on $(\delta_i, \delta_{i+1})$, $i = 1, 2, \ldots, n$. Define

$$
X_1(t) = \begin{cases} 
A - \int_{t_0}^{t} p(s)\phi_1(s - \tau) \, ds, & t_0 \leq t < \delta_2 + \tau, \\
A - \int_{t_0}^{\delta_2 + \tau} p(s)\phi_1(s - \tau) \, ds & \\
- \int_{\delta_2 + \tau}^{t} p(s)\phi_2(s - \tau) \, ds, & \delta_2 + \tau \leq t < \delta_3 + \tau, \\
\vdots \\
A - \sum_{i=1}^{n-1} \int_{\delta_i + \tau}^{\delta_{i+1} + \tau} p(s)\phi_i(s - \tau) \, ds & \\
- \int_{\delta_{n} + \tau}^{t} p(s)\phi_n(s - \tau) \, ds, & \delta_n + \tau \leq t < \delta_{n+1} + \tau, \\
\lim_{t \to \infty} X_1(t), & t = t_0 + \tau,
\end{cases}
$$

and

$$
x_1(t) = \begin{cases} 
\phi(t), & t_0 - \tau \leq t < t_0, \\
\phi(t_0) + \int_{t_0}^{t} X_1(s) \, ds, & t_0 \leq t < t_0 + \tau.
\end{cases}
$$

Clearly, $X_1(t)$ is continuous on $[t_0, t_0 + \tau]$ and $x_1(t) \in C^1[t_0, t_0 + \tau]$. It is easily verified that $x_1(t)$ satisfies equation (2) for $t \in [t_0, t_0 + \tau]$ with $x_1(t) = \phi(t)$ for $t_0 - \tau \leq t \leq t_0$ and $x_1'(t_0) = A$.

Now consider the new initial condition problem

$$
x''(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0 + \tau,
x(t) = x_1(t), \quad x'(t_0 + \tau) = x_1'(t_0 + \tau), \quad t_0 \leq t \leq t_0 + \tau.
$$

Since $X_1(t)$ is continuous on $[t_0, t_0 + \tau]$ and $x_1(t) \in C^1[t_0, t_0 + \tau]$, it follows that the above equation has a unique solution $x_2(t)$ on $[t_0, +\infty)$ and satisfies $x_2(t) = x_1(t)$, $x_2'(t_0 + \tau) = x_1'(t_0 + \tau)$ for $t \in [t_0, t_0 + \tau]$. Clearly,

$$
x_3(t) = \begin{cases} 
x_1(t), & t_0 - \tau \leq t \leq t_0 + \tau, \\
x_2(t), & t_0 \leq t < +\infty,
\end{cases}
$$

is a unique solution of equation (2) satisfying $x_3(t) = \phi(t)$ for $t_0 - \tau \leq t \leq t_0$ and $x_3'(t_0) = A$.

PROOF OF THEOREM 1. Since $x(t)$ is a nonoscillatory solution of equation (2), we may assume that $x(t)$ is eventually positive. Then $x'(t)$ is also eventually positive. Let $N$ be an integer such that $x(t) > 0$ for $t \geq t_N - \tau$.

Now consider the initial function

$$
\phi_1(t) = \begin{cases} 
x(t), & t_N - \tau \leq t < t_N, \\
x(t_N) + M_N(x(t_N)), & t = t_N,
\end{cases}
$$

and

$$
A_1 = (1 + L_N)x'(t_N),
$$

with property (3). It is clear that $\phi_1(t)$ is piecewise right continuous on $[t_N - \tau, t_N]$. Thus, by Lemma 1, we know that equation (2) has a unique solution $x_1(t) : [t_N - \tau, +\infty) \to R$, which is
continuous and differentiable on \([t_N - \tau, +\infty)\) and satisfies \(x_1(t) = \phi_1(t)\) for \(t_N - \tau \leq t \leq t_N\) and \(x'_1(t_N) = A_1\).

Next, we shall prove that
\[ x_1(t) \geq x(t), \quad t \geq t_N - \tau, \tag{4} \]
and
\[ x'_1(t) \geq x'(t), \quad t \geq t_N. \tag{5} \]
It is easy to see that (4) and (5) hold for \(t_N - \tau \leq t \leq t_N\). For \(t \in [t_N, t_N + \tau]\), we have
\[
x'_1(t) = x'_1(t_N) - \int_{t_N}^{t} p(s)x_1(s - \tau(s)) \, ds
= x'_1(t_N) - \int_{t_N}^{t} p(s)\phi_1(s - \tau(s)) \, ds
= x'_1(t_N) - \int_{t_N}^{t} p(s)x(s - \tau(s)) \, ds
= x'_1(t_N) + x'(t) - x'(t_N)
= x'(t) + L_N x'(t_N),
\]
i.e.,
\[ x'_1(t) = x'(t) + L_N x'(t_N), \quad t \in [t_N, t_N + \tau], \tag{6} \]
which in view of \(x'(t) > 0\) for \(t \geq t_N\), shows that (5) and so (4) holds for \(t \in [t_N, t_N + \tau]\). It follows from equation (2) and \(x(t) > 0\) that \(x'(t)\) is decreasing for \(t \geq t_N - \tau\). Therefore, we have \(x'(t) \geq x'(t_N + \tau)\) for \(t_N \leq t \leq t_N + \tau\). Hence, we obtain
\[
\frac{x'_1(t)}{x'_1(t_N + \tau)} = \frac{x'(t) + L_N x'(t_N)}{x'(t_N + \tau) + L_N x'(t_N)} \leq \frac{x'(t)}{x'(t_N + \tau)},
\]
i.e.,
\[
\frac{x'_1(t)}{x'(t)} \leq \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)} \tag{7}
\]
holds for \(t_N \leq t \leq t_N + \tau\). It follows from (7) that
\[
x'_1(t) \leq \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)} x'(t),
\]
which by integrating from \(t_N\) to \(t\), leads to
\[
x_1(t) - x_1(t_N) \leq \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)} (x(t) - x(t_N)).
\]
Therefore,
\[
x_1(t) \leq x_1(t_N) + \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)} (x(t) - x(t_N))
\leq x(t_N) \left( \frac{x_1(t_N)}{x(t_N)} - \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)} \right) + \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)} x(t),
\]
for \(t_N \leq t \leq t_N + \tau\). From (7), we obtain
\[
\frac{x'_1(t_N)}{x'(t_N)} \leq \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)}.
\]
Hence, in view of (3), we have
\[
x_1(t) \leq x(t_N) \left( \frac{x_1(t_N)}{x(t_N)} - \frac{x_1'(t_N + \tau)}{x'(t_N + \tau)} \right) + \frac{x_1'(t_N + \tau)}{x'(t_N + \tau)} x(t)
\]
\[
= x(t_N) (M_N - L_N) + \frac{x_1'(t_N + \tau)}{x'(t_N + \tau)} x(t)
\]
\[
\leq \frac{x_1'(t_N + \tau)}{x'(t_N + \tau)} x(t),
\]
i.e.,
\[
\frac{x_1(t)}{x(t)} \leq \frac{x_1'(t_N + \tau)}{x'(t_N + \tau)}.
\]
for \( t_N \leq t \leq t_N + \tau \). Thus, by using (8), we get
\[
x_1'(t) = x_1'(t_N + \tau) - \int_{t_N+\tau}^{t} p(s)x_1(s - \tau) \, ds
\]
\[
\geq x_1'(t_N + \tau) - \int_{t_N+\tau}^{t} p(s)x_1'(t_N + \tau) \frac{x(s - \tau)}{x'(t_N + \tau)} \, ds
\]
\[
= x_1'(t_N + \tau) - \frac{x_1'(t_N + \tau)}{x'(t_N + \tau)} \int_{t_N+\tau}^{t} p(s)x(s - \tau) \, ds
\]
\[
= x_1'(t_N + \tau) + \frac{x_1'(t_N + \tau)}{x'(t_N + \tau)} (x'(t) - x'(t_N + \tau))
\]
\[
= \frac{x_1'(t_N + \tau)}{x'(t_N + \tau)} x'(t)
\]
\[
\geq x'(t),
\]
which shows that (5) and so (4) holds for \( t_N + \tau \leq t \leq t_N + 2\tau \). Now, we shall prove that (4) and (5) hold for \( t \in [t_N + 2\tau, t_N + 3\tau] \). To this end, we shall prove
\[
\frac{x_1'(t)}{x'(t)} \leq \frac{x_1'(t_N + 2\tau)}{x'(t_N + 2\tau)}, \quad \text{for } t \in [t_N + \tau, t_N + 2\tau].
\]
It suffices to show that
\[
\left( \frac{x_1'(t)}{x'(t)} \right)' \geq 0, \quad t \in [t_N + \tau, t_N + 2\tau].
\]
Since
\[
\left( \frac{x_1'(t)}{x'(t)} \right)' = p(t) \frac{-x_1(t - \tau) x'(t) + x_1'(t) x(t - \tau)}{(x'(t))^2},
\]
it follows that if (10) does not hold, then there exists some \( t^* \in [t_N + \tau, t_N + 2\tau] \) such that
\[
x_1'(t^*) x(t^* - \tau) < x'(t^*) x_1(t^* - \tau),
\]
or
\[
\frac{x_1'(t^*)}{x'(t^*)} < \frac{x_1(t^* - \tau)}{x(t^* - \tau)}. \quad (11)
\]
But, by (8) we see that (10) holds at \( t = t_N + \tau \). Therefore, we find that \( t^* > t_N + \tau \). It follows from equation (2) and (8) that

\[
x'_1(t^*) = x'_1(t_N + \tau) - \int_{t_N + \tau}^{t^*} p(s) x'_1(s - \tau) \, ds \geq x'_1(t_N + \tau) - \int_{t_N + \tau}^{t^*} p(s) \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)} x(s - \tau) \, ds
\]

\[
= x'_1(t_N + \tau) - \int_{t_N + \tau}^{t^*} p(s) x(s - \tau) \, ds
\]

\[
= x'_1(t_N + \tau) + \int_{t_N + \tau}^{t^*} p(s) x'(t^*) - x'(t_N + \tau) \, ds
\]

\[
= x'_1(t_N + \tau) x'(t),
\]

i.e.,

\[
\frac{x'_1(t^*)}{x'(t^*)} \geq \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)}.
\]

In view of (8) and noting that \( t_N < t^* - \tau \leq t_N + \tau \), we have

\[
\frac{x'_1(t^* - \tau)}{x'(t^*)} \leq \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)},
\]

which together with (11), yields

\[
\frac{x'_1(t^*)}{x'(t^*)} < \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)},
\]

which contradicts (12). This shows that (10) holds and so (9) holds. It follows from (9) that

\[
x'_1(t) \leq \frac{x'_1(t_N + 2\tau)}{x'(t_N + 2\tau)} x'(t),
\]

which by integrating from \( t_N + \tau \) to \( t \), leads to

\[
x_1(t) - x_1(t_N + \tau) \leq \frac{x'_1(t_N + 2\tau)}{x'(t_N + 2\tau)} (x(t) - x(t_N + \tau)).
\]

Hence, we get

\[
x_1(t) \leq x_1(t_N + \tau) + \frac{x'_1(t_N + 2\tau)}{x'(t_N + 2\tau)} (x(t) - x(t_N + \tau))
\]

\[
\leq x(t_N + \tau) \left( \frac{x_1(t_N + \tau)}{x'(t_N + \tau)} - \frac{x'_1(t_N + 2\tau)}{x'(t_N + 2\tau)} \right) + \frac{x'_1(t_N + 2\tau)}{x'(t_N + 2\tau)} x(t),
\]

for \( t_N + \tau \leq t \leq t_N + 2\tau \). From (8) and (9), we obtain

\[
\frac{x_1(t_N + \tau)}{x(t_N + \tau)} \leq \frac{x'_1(t_N + \tau)}{x'(t_N + \tau)} \leq \frac{x'_1(t_N + 2\tau)}{x'(t_N + 2\tau)},
\]

Therefore,

\[
x_1(t) \leq x(t_N + \tau) \left( \frac{x_1(t_N + \tau)}{x'(t_N + \tau)} - \frac{x'_1(t_N + 2\tau)}{x'(t_N + 2\tau)} \right) + \frac{x'_1(t_N + 2\tau)}{x'(t_N + 2\tau)} x(t)
\]

\[
\leq \frac{x'_1(t_N + 2\tau)}{x'(t_N + 2\tau)} x(t),
\]
Preservation of Nonoscillatory Behavior

i.e.,

\[
x_1(t) \leq \frac{x'_1(tN + 2\tau)}{x'(tN + 2\tau)},
\]

for \( tN + \tau \leq t \leq tN + 2\tau \). Therefore, for \( t \in [tN + 2\tau, tN + 3\tau] \), by using (13), we obtain

\[
x'_1(t) = x'_1(tN + 2\tau) - \int_{tN+2\tau}^{t} p(s)x_1(s - \tau)\,ds
\]

\[
\geq x'_1(tN + 2\tau) - \int_{tN+2\tau}^{t} \frac{x'_1(tN + 2\tau)}{x'(tN + 2\tau)} x'(s - \tau)\,ds
\]

\[
= x'(tN + \tau) + \frac{x'_1(tN + 2\tau)}{x'(tN + 2\tau)} [x'(t) - x'(tN + 2\tau)]
\]

\[
= \frac{x'_1(tN + 2\tau)}{x'(tN + 2\tau)} x'(t)
\]

\[
\geq x'(t),
\]

which shows that (5) and so (4) hold for \( t \in [tN + 2\tau, tN + 3\tau] \). In general, by using mathematical induction, we can show that (5) and (4) hold for \( t \in [tN + n\tau, tN + (n + 1)\tau] \), \( n = 0, 1, 2, \ldots \), and so (5) and (4) hold for all \( t \geq tN - \tau \).

Next, we consider the second initial function

\[
\phi_2(t) = \begin{cases}
x_1(t), & t_{N+1} - \tau \leq t < t_{N+1}, \\
x_1(t_{N+1}) + M_{N+1}(x_1(t_{N+1})), & t = t_{N+1},
\end{cases}
\]

and

\[
A_2 = x'_1(t_{N+1})(1 + L_{N+1})
\]

and with property (3). It is clear that \( \phi_2(t) \) is piecewise right continuous and differentiable on \([t_{N+1} - \tau, t_{N+1}]\). Then, by Lemma 1, equation (2) has a unique solution \( x_2(t) : [t_{N+1} - \tau, +\infty) \rightarrow R \) such that \( x_2(t) \) is continuous and differentiable on \([t_{N+1}, +\infty) \) and \( x_2(t) = \phi_2(t) \) for \( t_{N+1} - \tau \leq t \leq t_{N+1} \) and \( x'_2(t_{N+1}) = A_2 \). Similarly, we can show that

\[
x_2(t) \geq x_1(t), \quad \text{for } t \geq t_{N+1} - \tau.
\]

Thus, we can obtain a sequence of solutions \( \{x_k(t)\} \) of equation (2), which has the following properties:

(i) \( x_0(t) = x(t) \);

(ii) \( x_k(t) \) is a solution of equation (2) defined on

\([t_{N+k-1} - \tau, +\infty) \quad \text{and} \quad x_k(t) \in C^1([t_{N+k-1}, +\infty), R)\)

and satisfies the initial conditions

\[
x_1(t) = \phi_1(t), \quad x'_1(tN) = A_1, \quad t_{N} - \tau \leq t \leq t_{N},
\]

\[
\phi_k(t) = \begin{cases}
x_{k-1}(t), & t_{N+k-1} - \tau \leq t < t_{N+k-1}, \\
x_{k-1}(t_{N+k-1}) + M_{N+k-1}(x_{k-1}(t_{N+k-1})), & t = t_{N+k-1}, \quad k \in \mathbb{N}, \quad k \geq 2,
\end{cases}
\]

and

\[
x_k(t) \equiv \phi_k(t), \quad \text{for } t_{N+k-1} - \tau \leq t \leq t_{N+k-1},
\]

\[
x_k(t) = (1 + L_{N+k-1})(x'_{k-1}(t_{N+k-1}))
\]

with property (3);

(iii) \( x_k(t) \geq x_{k-1}(t), \quad t \geq t_{N+k-1} - \tau; \)

(iv) \( x_k(t) > 0, \quad t \geq t_{N+k-1} - \tau. \)
Finally, we define

\[ y(t) = \begin{cases} 
  x_0(t), & t_N - \tau \leq t < t_N, \\
  x_1(t), & t_N \leq t < t_{N+1}, \\
  \vdots & \\
  x_i(t), & t_{N+i-1} \leq t < t_{N+i}, \quad i = 1, 2, \ldots 
\end{cases} \]

It is easy to show that \( y(t) \) is positive and piecewise right continuous on \([t_N - \tau, +\infty)\), and it is a solution of equation (1) on \([t_N, +\infty)\). The proof is complete.

REFERENCES