On Reflection of Multidimensional Shock Front

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1. INTRODUCTION

In this paper we discuss the problem of reflection of shock waves, which occurs when a shock wave produced by an explosion or by a fast flying projectile hits an obstacle. Because of the importance in application, this problem has attracted attention for a long time. If the shock front and the surface of the obstacle are planes, then the reflection problem was discussed and solved by Courant and Friedrichs in [1]. Later, in the two-dimensional case (steady flow in the two-dimensional case or unsteady flow in the one-space-dimensional case), this problem was solved locally by Gu Chaohao and others (see [2, 3]). However, in the case with more independent variables, such problems have never been precisely discussed until now. Recently, by means of microlocal analysis, A. Majda [4, 5] and G. Metivier [6] studied problems on local existence of shock front solutions and interaction of two shocks for a system of conservation laws. Their work offered a way to deal with the multidimensional problems in fluid dynamics.

Our main purpose in this paper is to prove the local existence of the solutions for the problem of reflection of shock waves in the three-dimensional case. First, in Section 2 we formulate the original problem as a Goursat problem for the system of conservation laws with one free boundary and another fixed characteristic boundary. After reducing it to a nonlinear Goursat problem with two fixed boundaries in Section 4, similar to [6] we find an asymptotic solution as a first approximation and then by Newton's iteration we obtain a convergent sequence. At each step of the iteration including the first we need to solve a linear Goursat problem and derive corresponding estimates. The linearization is given in Section 5, and the estimates are obtained in Sections 8 and 9, while we establish properties of weighted spaces as a preparation for our estimates in Sections 6 and 7. Finally, in Sections 10 and 11 we establish the existence of local solutions for the nonlinear Goursat problem. We remark here that our estimates of solutions for the Goursat problem in a wedge-shaped domain are uniform
with respect to the edge and this allows considerable economy in using the methods of [6].

2. Formulation

Let $\Sigma$ be a given surface, $S$ an incident plane shock front. Suppose that for $t < 0$ on both sides of $S$ the flow fields are constant and $S$ moves towards $\Sigma$ with a constant velocity $V$. At $t = 0$, $S$ meets $\Sigma$ at point 0, and then for $t > 0$ the reflection of the shock wave occurs. In this paper we are going to determine the place of the reflected shock front for $t > 0$ and the flow field behind the reflected shock front.

For notational simplicity we only consider the case of space-dimension 2. The system of conservation laws for inviscid unsteady flow has the form

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho(e + \frac{1}{2} q^2) \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho u(i + \frac{1}{2} q^2) \\ \rho(u(i + \frac{1}{2} q^2)) \end{pmatrix} + \frac{\partial}{\partial y} \begin{pmatrix} \rho v \\ \rho w \\ p + \rho v^2 \\ \rho(v(i + \frac{1}{2} q^2)) \end{pmatrix} = 0, \quad (2.1)$$

where $(u, v)$ are velocity components, $p$, $\rho$, $e$, $i$ represent pressure, density, inner energy, and enthalpy, respectively, $q^2 = u^2 + v^2$. Any discontinuous solution of (2.1) has to satisfy system (2.1) in the region where the solution is smooth, and satisfy the following Rankine–Hugoniot conditions on the shock front,

$$\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho(e + \frac{1}{2} q^2) \end{bmatrix} \psi_t - \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho u(i + \frac{1}{2} q^2) \end{bmatrix} + \begin{bmatrix} \rho v \\ \rho w \\ p + \rho v^2 \\ \rho(v(i + \frac{1}{2} q^2)) \end{bmatrix} \psi_y = 0, \quad (2.2)$$

where $x = \psi(t, y)$ is the equation of the shock front and $\left[ \begin{array}{c} \end{array} \right]$ represents the jump of the corresponding function across the shock front.

System (2.1) can also be written in the form of a symmetric hyperbolic system as

$$\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho(e + \frac{1}{2} q^2) \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \\ p \end{pmatrix} + \begin{bmatrix} \rho u \\ p \\ 1 \\ \rho u \end{bmatrix} \begin{pmatrix} u \\ p \end{pmatrix} + \begin{bmatrix} \rho v \\ 1 \\ \rho v \end{bmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = 0, \quad (2.3)$$
where \( a \) is the sonic speed and \( s \) is entropy. For polytropic gas \( s \) can be taken as \( p/p_0 \).

We choose the coordinates such that point 0 is the origin, the equation of the shock front \( S \) is \( x = Vt \) with \( V < 0 \), the equation of the surface \( \Sigma \) is \( x = \phi(y) \), where \( \phi(y) \) is \( C^\infty \) and \( \phi(0) = \phi'(0) = 0, \phi(y) \leq 0 \). According to our assumptions, for \( t < 0 \) on both sides of \( S \) the flow fields are known as constant: \( u_a, v_a, p_a, \rho_a, s_a \) on the left and \( u_b, v_b, p_b, \rho_b, s_b \) on the right. Obviously, before the shock front \( S \) intersects with the fixed surface \( \Sigma \), the velocity of \( S \) and the flow field on both sides of \( S \) remain constant. Hence the intersection of \( S \) with \( \Sigma \) is

\[
\sigma: x = \phi(y), \quad t = \frac{1}{V} \phi(y). \tag{2.4}
\]

Denoting the equation of the reflected shock front \( S_1 \) issuing from \( 0 \) by \( x = \psi(t, y) \), the expected solution of (2.3) for \( t > (1/V) \phi(y) \) can be written as

\[
U(t, x, y) = \begin{cases} 
U_c(t, x, y), & \phi(y) < x < \psi(t, y) \\
U_b, & x > \psi(t, y),
\end{cases} \tag{2.5}
\]

where \( U \) is the abbreviation of \( (u, v, p, s) \), and \( U_1 \) is a known constant. Since the normal component of the velocity on the rigid wall must be zero, our problem on the reflection of the shock wave can be formulated as to find functions \( \psi(t, y) \) and \( U_c(t, x, y) \), such that

1. \( U_c(t, x, y) \) satisfies (2.3) (or (2.1)) in \( \phi(y) < x < \psi(t, y) \), \( t > (1/V) \phi(y) \),

2. \( U_c(t, x, y) \) and \( U_b \) satisfy the Rankine–Hugoniot conditions (2.2) on \( x = \psi(t, y) \),

3. \( u - \phi_y v = 0 \) on \( x = \phi(y) \).

In the next sections we will prove the local existence of such functions \( \psi \) and \( U_c \).

3. Two Simple Cases

Before we start with our main problem, let us recall two simple cases for our convenience.

Case 1. Assume \( \phi(y) \equiv 0 \), \( u_0 = v_0 = v_1 = 0 \). Then the reflected shock front is also a plane, and the flow field behind the shock front is still constant. The velocity \( v' \) of the reflected shock and the parameters \( u_2, v_2, p_2, \rho_2, s_2 \) of the flow field behind the shock can be determined by solving algebraic equations. For instance, for polytropic gas we have
\[ u_2 = v_2 = 0, \quad v' = u_1 + \frac{a^2_1}{u_1 - V}, \quad \text{(3.1)} \]

where \( \mu^2 = (\gamma - 1)/(\gamma + 1) \).

We remark that near the origin our main problem is merely a perturbation of the above special case.

**Case 2.** Consider a plane shock front moving along a plane surface of an obstacle. The flow pattern may be described in terms of an "incident" and a "reflected" shock front. This case is called regular reflection in [1]. In this paper, if the rigid wall is \( x = (\tan \alpha) y \) with \( \alpha \) being small, then "regular reflection" occurs. Since the incident and reflected shock fronts can be considered as two planes moving with constant velocity along the plane wall, the flow appears to be steady from a frame of reference moving along the wall, and the two shock fronts appear to be stationary.

We assume the equation of the incident shock front is still \( x = Vt \). Then the intersection of the shock front and the rigid wall moves along the wall with the velocity \( V/\sin \alpha \). The reflected shock and the field behind the reflected shock front can be explicitly determined, if \( \alpha \) is small. We establish a moving coordinate system with the origin at the intersection and with x-axis along the wall (see Fig. 1). In the front of the incident shock \( S \), the coming flow has velocity \( (u_0, v_0) = (q_0, 0) = (V/\sin \alpha, 0) \), and the pressure and the density of the flow are all given. Behind the incident shock \( S \), we have

\[ u_1 = u_0[1 - (1 - \mu^2)(\sin^2 \alpha - \sin^2 \alpha_0)] \]
\[ = u_0 \left[ 1 - (1 - \mu^2) \left( 1 - \frac{c_0^2}{V^2} \right) \sin^2 \alpha \right] \quad \text{(3.2)} \]

\[ v_1 = (q_0 - u_1) \cot \alpha \]
\[ = u_0(1 - \mu^2) \left( 1 - \frac{c_0^2}{V^2} \right) \sin \alpha \cos \alpha. \]
and $\rho_1, \rho_1$ can be explicitly expressed as well. Here in (3.2) $c_0$ represents the sound speed, $A_0$ represents the Mach angle.

In order to determine the place of the reflected shock, we introduce the following quantities:

\[
\theta = \arctan \frac{v_1}{u_1},
\]

\[
q_1 = \sqrt{u_1^2 + v_1^2}
\]

\[
\begin{align*}
&= u_0 \left\{ 1 - 2(1 - \mu^2) \left( 1 - \frac{c_0^2}{V^2} \right) \sin^2 \alpha \\
&\quad + (1 - \mu^2)^2 \left( 1 - \frac{c_0^2}{V^2} \right)^2 \sin^2 \alpha \right\}^{1/2},
\end{align*}
\]

\[
c_1^2 = \mu^2 q_0^2 + (1 - \mu^2) c_0^2 = \mu^2 q_1^2 + (1 - \mu^2) c_1^2,
\]

where $\mu^2 = (\gamma - 1)/(\gamma + 1)$, $\gamma$ is the adiabatic exponent, $c_1$ and $c_1$ are the critical sound speed and the sound speed behind $S$.

Denoting the parameters of the flow field behind $S_1$ by $u_2, v_2, p_2, \rho_2$ etc., the angle between the direction $(u_1, v_1)$ and $S_1$ by $\beta_1$, and letting

\[
\begin{align*}
\tilde{u}_1 &= (\mu^2 + (1 - \mu^2) \sin^2 A_1) q_1, \\
U_1 &= (1 - \mu^2) q_1 + \tilde{u}_1
\end{align*}
\]

we have

\[
\begin{align*}
tg \theta_1 &= \frac{q - u_2}{u_2} \sqrt{\frac{u_2 - \tilde{u}_1}{U_1 - \tilde{u}_1}}, \\
u_2 &= q_1 - (1 - \mu^2)(\sin^2 \beta_1 - \sin^2 A_1) q_1, \quad (3.4) \\
v_2 &= (q_1 - u_2) \cot \beta_1.
\end{align*}
\]

Since the flow behind $S_1$ is parallel to the wall again, then $\theta = \theta_1$ and we can determine $u_2, \beta_1, v_2$ from (3.4) successively.

Along with the motion of the intersection of the incident shock front and the rigid wall, the reflected shock $S_1$ forms a plane, the equation of which in the original coordinate system is

\[
\begin{vmatrix}
t & x & y \\
\frac{1}{V} \tan \alpha & \tan \alpha & 1 \\
0 & \tan(\beta_1 - \theta + \alpha) & 1
\end{vmatrix} = 0
\]
or
\[ x = \tan(\beta_1 - \theta + \alpha) y + \frac{\tan \alpha - \tan(\beta_1 - \theta + \alpha)}{\tan \alpha} V_l. \]  
(3.5)

By eliminating the effect of the intermediate moving coordinate system we can obtain all parameters of the field behind the shock (3.5).

We remark that when \( x \to 0 \), the positions of the shock (3.5) and the field behind the shock approach the solution in Case 1. Here we give the proof for the position of the shock. From (3.2) we have
\[ \tan \theta \sim \sin \alpha \cos \alpha (1 - \mu^2) \left(1 - \frac{c_0^2}{V^2}\right) \]
\[ \sim \alpha (1 - \mu^2) \left(1 - \frac{c_0^2}{V^2}\right), \]
where the sign \( \sim \) means equal up to an error of higher order. From (3.4) we have the equality
\[ \tan \beta_1 = \frac{q_1 - u_2}{u_2} \cotg \left(1 - \mu^2 \right) \sin^2 \beta_1 - \sin^2 A_1 q_1 \cotg \theta \]

Therefore, denoting \( \tan \beta_1 \) by \( t_1 \) we have
\[ t_1 \alpha (1 - \mu^2) \left(1 - \frac{c_0^2}{V^2}\right) = (1 - \mu^2) \left(1 + t_1^2 \frac{c_1^2}{V^2} \alpha^2 \right) \]
after neglecting higher order terms. Thus
\[ \left(\frac{t_1}{\alpha}\right)^2 - \frac{t_1}{\alpha} \left(1 - \frac{c_0^2}{V^2}\right) - \frac{c_1^2}{V^2} = 0. \]
(3.6)

By means of Plandtl's equality
\[ c_1^2 = c_0^2 - \mu^2(q_1^2 - q_0^2)(1 - \mu^2)^{-1}, \]
\[ \frac{c_1^2}{V^2} = 1 + (2\mu^2 - 1) \left(1 - \frac{c_0^2}{V^2}\right) - \mu^2(1 - \mu^2) \left(1 - \frac{c_0^2}{V^2}\right). \]

Substituting it into (3.6) yields
\[ \left(\frac{t_1}{\alpha} - 1 - \mu^2 \left(1 - \frac{c_0^2}{V^2}\right) \right) \left(\frac{t_1}{\alpha} + 1 + \mu^2 - \left(1 - \frac{c_0^2}{V^2}\right) \right) = 0. \]
Thus

\[
\frac{t_1}{\alpha} = 1 + \mu^2 \left( 1 - \frac{c_0^2}{V^2} \right). \tag{3.7}
\]

Combining it with (3.5), the limit position of the reflected shock is

\[
x = \lim_{\alpha \to 0} \left( \frac{t_1}{\alpha} - \frac{\theta}{\alpha} \right) Vt;
\]

that is,

\[
x = - \left( 2\mu^2 + (1 - 2\mu^2) \frac{c_0^2}{V^2} \right) Vt. \tag{3.8}
\]

On the other hand, from (3.1) we have

\[
V' = u_1 + \frac{a_1}{u_1 - V}.
\]

Noticing \(u_0 = 0\) and \((u_0 - V)(u_1 - V) = c_0^2 = \mu^2(u_0 - V)^2 + (1 - \mu^2)c_0^2 = \mu^2(u_1 - V)^2 + (1 - \mu^2)c_0^2\), a routine computation gives

\[
V' = - \left( 2\mu^2 + (1 - 2\mu^2) \frac{c_0^2}{V^2} \right) V,
\]

which coincides with (3.8).

4. Reduction to a Goursat Problem with Fixed Boundary

As mentioned in Section 2 the problem of reflection of shock waves can be formulated as a Goursat problem for system (2.1) with fixed boundary \(\Sigma\) and moving boundary \(S_1\). Now we will transform it into a new Goursat problem with two fixed boundaries by a coordinate transformation involving an unknown function describing the moving boundary \(S_1\).

The moving boundary \(S_1\) issuing from \(\sigma\) is described by \(x = \psi(t, y)\) as in Section 2. It satisfies

\[
\phi(y) = \psi \left( \frac{1}{V} \phi(y), y \right). \tag{4.1}
\]

The derivatives of \(\psi\) on \(\sigma\) can be determined as follows. At time \(t_0 \geq 0\), the incident shock front hits the boundary \(\Sigma\) at \(P(t_0)\): \((1/V) \phi(y_0), \phi(y_0), y_0\), where \(y_0\) satisfies \(t_0 = (1/V) \phi(y_0)\). Locally we can regard \(\Sigma\) as a plane
\( x - \varphi(y_0) = \varphi'(y_0)(y - y_0) \), and determine the place of the reflected shock front as

\[ x = \varphi(y_0) = g_{i_0} t + h_{i_0} y \tag{4.2} \]

according to Case 1 \((t_0 = 0)\) or Case 2 \((t_0 > 0)\) in Section 3. Hence \(\nabla \psi\) at \(P(t_0)\) can be given as \((g_{i_0}, h_{i_0})\). Moreover, the parameters of the flow field at \(P(t_0)\) behind \(S_1\) can also be obtained as in Section 3. This gives the first approximation of our problem. We notice that if \(\varphi(y) \in C^\infty\), then \(\nabla \psi|_{\sigma}\) and the flow field on \(\sigma\) are also \(C^\infty\).

Before introducing coordinate transformation let us make two simplifications. First, replacing \(t - \varphi(y)\) by \(t\), we may assume \(\varphi(y) = 0\). (This does not mean we go back to Case 1 in Section 3.) Second, since we only need to solve our problem locally near the origin, we may assume all functions which appear in our problem are periodic with respect to \(y\) with period \(2d\) and denote the period region of \(y\) by \(\mathcal{D}\). Such an assumption is reasonable due to the property of finite domain of dependence for hyperbolic systems.

Under the above two simplifications we introduce the coordinate transformation:

\[
\begin{align*}
y &= y, \\
x_1 &= t \cdot \frac{x - \varphi(y)}{\psi(t, y) - \varphi(y)} \\
x_2 &= t \cdot \frac{\psi(t, y) - x}{\psi(t, y) - \varphi(y)}.
\end{align*}
\tag{4.3}
\]

By transformation (4.3) the region with wedge shape \(\varphi(y) < x < \psi(t, y), t > 0\), is transformed into the quadrant \(x_1 > 0, x_2 > 0\), while the surface \(x = \varphi(y)\), the reflected shock front \(x = \psi(t, y)\), and the intersection \(\sigma\) are transformed to \(x_1 = 0, x_2 = 0,\) and \(x_1 = x_2 = 0\), respectively.

The Jacobian of transformation (4.3) is equal to

\[
\begin{vmatrix}
d(x_1, x_2, y) \\
\frac{\partial(t, x, y)}{\partial(t, x)}
\end{vmatrix}
= \begin{vmatrix}
\frac{\partial(x_1, x_2)}{\partial(t, x)}
\end{vmatrix}

= \begin{vmatrix}
(\psi - \varphi)^{-1} (x - \varphi - x_1 \psi_1) & t(\psi - \varphi)^{-1} \\
(\psi - \varphi)^{-1} (\psi - x - x_2 \psi_2 + t \psi_1) & -t(\psi - \varphi)^{-1}
\end{vmatrix}

= \begin{vmatrix}
(\psi - \varphi)^{-1} (x - \varphi - x_1 \psi_1) & t(\psi - \varphi)^{-1} \\
1 & 0
\end{vmatrix}

= -t(\psi - \varphi)^{-1}.
\tag{4.4}
\]
By (4.1), (4.2) we have

\[ \psi - \varphi |_{t=0} = 0, \]
\[ \nabla(\psi - \varphi)|_{t=0} \neq 0; \]

thus transformation (4.3) is nonsingular, and it is a \( C^\infty \) diffeomorphism, if \( \varphi, \psi \) are \( C^\infty \).

Let us rewrite system (2.3) in the form

\[ M \frac{\partial U}{\partial t} + N \frac{\partial U}{\partial x} + Q \frac{\partial U}{\partial y} = 0. \]  

(4.5)

Under transformation (4.3), it is changed to

\[ A \frac{\partial U}{\partial x_1} + B \frac{\partial U}{\partial x_2} + Q \frac{\partial U}{\partial y} = 0, \]  

(4.6)

where

\[ A = \frac{\partial x_1}{\partial t} M + \frac{\partial x_1}{\partial x} N + \frac{\partial x_1}{\partial y} Q, \]
\[ B = \frac{\partial x_2}{\partial t} M + \frac{\partial x_2}{\partial x} N + \frac{\partial x_2}{\partial y} Q. \]

We emphasize here that \( A, B \) depend not only on \( U \) but also on the unknown function \( \psi \) and \( \nabla \psi \).

Now our original Goursat problem in Section 2 is transformed to another Goursat problem for (4.5) in the domain \( x_1 > 0, x_2 > 0 \) with fixed boundary \( x_1 = 0 \) and \( x_2 = 0 \). For our convenience in the sequel we rewrite this nonlinear Goursat problem in an abstract form as

\[ L(U, \psi) U = 0, \quad x_1 > 0, \quad x_2 > 0 \]  

(4.7)

\[ lU = 0, \quad x_1 = 0 \]  

(4.8)

\[ \mathcal{F}(x_1, y, U, \psi, \nabla \psi) = 0, \quad x_2 = 0; \quad \psi(t, y)|_{t=0} = 0. \]  

(4.9)

Here \( L \) is a first-order differential operator with coefficients depending on \( U, \psi, \nabla \psi \). \( l \) is a vector \( (1, -\varphi_y, 0, 0) \) defined on \( x_1 = 0 \), \( \mathcal{F} \) is a \( C^\infty \) nonlinear function of its arguments. The variable \( t \) of \( \psi \) must be replaced by \( x_1 + x_2 \).

We point out two facts on (4.7)–(4.9). First, condition (4.9) satisfies uniform stability of shock fronts in the sense of [4], as it is verified there. Second, since \( U \) satisfies (4.8) or \( x_1 = 0 \), the boundary \( x_1 = 0 \) is a characteristic boundary for (4.7). To verify it we check the matrix \( A \) on \( x_1 = 0 \). Obviously, \( \partial x_1 / \partial t = 0 \) on \( x_1 = 0 \); hence
\[
A = \frac{t}{\psi - \phi} (N - \phi, Q)
\]

\[
= \frac{t}{\psi - \phi} \begin{pmatrix}
\rho(U - \phi, v) & 1 \\
1 & -\phi_y \\
-\phi_y & a^{-2} \rho^{-1}(u - \phi, v) \\
\end{pmatrix}
\]

By condition (4.8),

\[
A = \frac{t}{\psi - \phi} \begin{pmatrix}
0 & 1 \\
0 & -\phi_y \\
1 - \phi_y & 0 \\
1 & 0
\end{pmatrix}
\]

on \( x_1 = 0 \); hence \( \text{rank } A = 2 \) on \( x_1 = 0 \) and \( \det |A| = 0 \). This certainly means that \( x_1 = 0 \) is characteristic.

5. Linearization

From (4.7)-(4.9) we may derive a corresponding linearization as follows

\[
L \delta U = A \frac{\partial \delta U}{\partial x_1} + B \frac{\partial \delta U}{\partial x_2} + Q \frac{\partial \delta U}{\partial y} = f, \quad x_1 > 0, x_2 > 0 \quad (5.1)
\]

\[
l_{\gamma_1} \delta U = 0, \quad x_1 = 0 \quad (5.2)
\]

\[
F(\gamma_2 \delta U, \delta \psi) = g, \quad x_2 = 0, \quad \delta \psi|_{t=0} = 0, \quad (5.3)
\]

where \( \gamma_1 \delta U, \gamma_2 \delta U \) represent the trace of \( \delta U \) on \( x_1, x_2 = 0 \). For our convenience, \( F(\gamma_2 \delta U, \delta \psi) \) is the Fréchet derivative of \( \mathcal{F} \) with respect to \( \gamma_2 U, \psi, \) and \( \nabla \psi \).

For given \( U, \psi \) satisfying (4.8) and \( \rho > 0 \), system (5.1) is hyperbolic. Moreover we have the following conclusion.

**Lemma 5.1.** If \( U \in C^1, \psi \in C^2, U \) satisfies (4.8), and \( \rho > 0 \), then there are four characteristic surfaces through \( x_1 = x_2 = 0 \), two coinciding with \( x_1 = 0 \), and the other two placed in \( x_1 < 0, 0 < x_2 < -x_1/\delta \) and \( x_2 < 0, 0 < x_1 < -(1/\delta)x_2 \) with small \( \delta \), respectively. Furthermore, near \( x_1 = 0 \) there exists a family of characteristic surfaces, on which \( \text{rank } A = 2 \).

**Proof.** Since \( L \) is a given operator for given \( U, \psi \), so we can go back to (4.5) and prove the corresponding properties for system (4.5). The
characteristic matrix for (4.5) is \( n_t M + n_x N + n_y Q \), where \( (n_t, n_x, n_y) \) is the normal direction at a given point. By computation

\[
det(n_t M + n_x N + n_y Q) = 
\begin{vmatrix}
\rho(n_t + un_x + vn_y) & n_x \\
\rho(n_t + un_x + vn_y) & n_y \\
n_x & n_y \\
a^{-2}p^{-1}(n_t + un_x + vn_y)
\end{vmatrix}
= a^{-2}\rho(n_t + un_x + vn_y)^2 \left[ (n_t + un_x + vn_y)^2 - a^2(n_x^2 + n_y^2) \right].
\]

(5.4)

On \( x = 0 \) (i.e., \( x_1 = 0 \)) we have \( n_t = 0 \), and condition (4.8) indicates \( n_t + un_x + vn_y = 0 \). Therefore, \( x = 0 \) is a characteristic surface with multiplicity 2. For the other two characteristic surfaces, by \( n_t = 0 \) we have

\[
(n_t + (u + a)n_x)(n_t + (u - a)n_x) = 0.
\]

(5.5)

Meanwhile, the Rankine–Hugoniot conditions in the case \( n_y \) are

\[
\begin{bmatrix}
\rho \\
\rho u \\
\rho v \\
\rho(e + \frac{1}{2}q^2)
\end{bmatrix} n_t^{(s)} + 
\begin{bmatrix}
\rho u \\
p + pu^2 \\
pw \\
\rho u(i + \frac{1}{2}q^2)
\end{bmatrix} n_x^{(s)} = 0,
\]

(5.6)

where \( n_t^{(s)}, n_x^{(s)} \) are the components of the normal direction to the shock front. Since (5.6) can be regarded as shock relations in one space variable, then the computation in [1] indicates

\[
u - a < u < \frac{n_t^{(s)}}{n_x^{(s)}} < u + a \quad \text{on } \sigma.
\]

(5.7)

This means that the other two characteristic surfaces through \( \sigma \) are placed on both sides of the wedge-shaped region \( \varphi(y) < x < \psi(t, y) \). Therefore, by the coordinate transformation, we obtain the first conclusion in the lemma.

To prove the second conclusion we only need to find a family of surfaces, on which \( n_t + un_x + vn_y = 0 \). Since \( (1, u, v) \) is a given smooth vector field, the required family can be obtained by finding a family of integral manifolds of the vector field. This can be done by solving Cauchy problems for a first-order partial differential equation. Here we omit the details.

Later, in constructing approximate solutions to the nonlinear Goursat problem, we need also to consider another linear problem which is deduced from (5.1)–(5.3), omitting the terms containing the derivatives with respect to \( y \):
\[ L_0 \delta U \overset{\text{def}}{=} A \frac{\partial \delta U}{\partial x_1} + B \frac{\partial \delta U}{\partial x_2} = f; \quad (5.8) \]

\[ l_{\gamma_1} \delta U = 0, \quad (5.9) \]

\[ F_0(\gamma_2 \delta U, \delta \psi) \overset{\text{def}}{=} p \frac{d \delta \psi}{dt} + h \delta \psi + m \delta \gamma_2 u = g; \quad \delta \psi |_{t=0} = 0. \quad (5.10) \]

From Lemma 5.1 we can easily derive the next lemma.

**Lemma 5.2.** Under the assumptions in Lemma 5.1, system (5.8) is hyperbolic. Through \( x_1 = x_2 = 0 \) there are four characteristic lines, two coinciding with \( x_1 = 0 \), and the other two placed in \( x_1 < 0, 0 < x_2 < -x_1/\delta \) and \( x_2 < 0, 0 < x_1 < -(1/\delta) x_2 \) with small \( \delta \), respectively.

6. **SPACES OF WEIGHT FUNCTIONS**

In this section we introduce some spaces of functions as a preparation for establishing various estimates.

We use the following notations: \( X = (x_1, x_2, y), \quad \Omega = \{ x; \ x_1 > 0, \ x_2 > 0, \ y \in \mathcal{D} \} \), \( \Omega_T = \Omega \cap \{ x_1 + x_2 < T \}, \quad \omega = \{ (t, y); \ t > 0, \ y \in \mathcal{D} \} \), \( \omega_T = \omega \cap \{ 0 < t < T \}, \quad \tilde{\psi}(x_1, x_2, y) = \psi(x_1 + x_2, y), \quad V_1 = x_1 \partial x_1, \quad V_2 = \partial x_2, \quad V_3 = \partial y, \quad D = \partial x_1, \) and multi-indices \( \alpha = (\alpha_{x_1}, \alpha_{x_2}, \alpha_y) = (\alpha_1, \alpha_2, \alpha_y), \quad \beta = (\beta_{x_1}, \beta_{x_2}, \beta_y) = (\beta_1, \beta_2, \beta_y), \quad \gamma = (\gamma_1, \gamma_2). \) Moreover, we introduce the following spaces and corresponding norms:

\[ L_x^2(\Omega_T) = \{ u; (x_1 + x_2)^{-k} u \in L^2(\Omega_T) \} \]

\[ L_x^2(\omega_T) = \{ f; t^{-\gamma} f \in L^2(\omega_T) \} \]

\[ H_x^r(\Omega_T) = \{ u; \partial^\alpha u \in L_x^2(\Omega_T) \}, \quad |\alpha| \leq r \]

\[ H_{x, \gamma}^{r,k}(\Omega_T) = \{ u; V^\beta D^s u \in L_x^2(\Omega_T) \}, \quad s + |\beta| \leq r + k, \ s \leq r \]

\[ \| U \|_{H_x^r(\Omega_T)} = \left\{ \sum_{|\beta| \leq k + r} \lambda^{2(r + k - s - |\beta|)} \| V^\beta D^s u \|_{L_x^2(\Omega_T)}^2 \right\}^{1/2} \]

\[ H_{x, \gamma}^r(\omega_T) = \{ f; \partial^\gamma f \in L_x^2(\omega_T) \}, \quad |\gamma| \leq k \]

\[ \| f \|_{H_{x, \gamma}^r(\omega_T)} = \left( \sum_{|\gamma| \leq k} \lambda^{2\gamma} \| \partial^\gamma f \|_{L_x^2(\omega_T)}^2 \right)^{1/2} \]

\[ B_x^k(\Omega_T) = \bigcap_{\gamma \leq k/2} H_{x, \gamma}^{r,k-2\gamma}(\Omega_T), \]

\[ \| U \|_{k, \gamma, \omega_T} = \| U \|_{B_x^k(\Omega_T)} = \left( \sum_{r \leq k/2} \| U \|_{H_{x, \gamma}^{r,k-2\gamma}(\Omega_T)}^2 \right)^{1/2} \].
Denoting the set of $C^\infty$ functions with compact support, vanishing in a neighborhood of the origin by $\hat{C}^\infty(\Omega_T)$ or $\hat{C}^\infty(\omega_T)$, we have

**Lemma 6.1.** $\hat{C}^\infty(\Omega_T)$ is dense in $H^s_\lambda(\Omega_T)$ and $B^s_\lambda(\Omega_T)$, and $\hat{C}(\omega_T)$ is dense in $H^s_\lambda(\omega_T)$.

**Lemma 6.2.** For any pair of integers $(N_1, N_2)$, there exists a constant $K$, such that for any $T > 0$, we can find two linear extension operators $E_T$ and $E'_T$

$$
E_T : H^s_\lambda(\Omega_T) \to H^s_\lambda(\Omega), \\
E'_T : H^s_\lambda(\omega_T) \to H^s_\lambda(\omega)
$$

with norm less than $K$ for any $s$, $k$, and $\lambda$ satisfying $0 \leq s \leq N_1$, $0 \leq k \leq N_2$, and $\lambda \in \mathbb{R}_+$. Moreover, supp $E_T u \subset \Omega_2$, supp $E'_T f \subset \omega_2$ for $u \in H^s_\lambda(\Omega_T)$ and $f \in H^s_\lambda(\omega_T)$.

The proof of Lemma 6.1 is easy, and for the proof of Lemma 6.2 we refer the reader to [6].

Similar to [6], on $\Omega$ we introduce a coordinate transformation

$$
\begin{cases}
t = x_1 + x_2 \\
\theta = \frac{x_1}{x_1 + x_2} \\
y = y,
\end{cases}
$$

which maps $\Omega_T$ to $\hat{\Omega}_T = (0, T) \times (0, 1) \times \mathcal{O}$. Correspondingly, we denote

$$
J_\lambda : u(x_1, x_2, y) \mapsto J_\lambda u(t, \theta, y) = t^{-\lambda}(u t, t(1 - t), y), \\
\psi(t, y) \mapsto J_\lambda \psi(t, y) = t^{-\lambda} \psi(t, y).
$$

and

$$
\hat{H}^{r,k}(\Omega_T) = \{ v; (t \partial_t)^j (\theta \partial_\theta)^m \partial_{\theta}^m \partial_y^r v \in L^2(\hat{\Omega}_T), \\
\text{where } j + m_1 + m_2 + l \leq r + k, m_2 \leq r \}
$$

$$
\|v\|_{\hat{H}^{r,k}(\Omega_T)} = \left( \sum_{j + m_1 + m_2 + l \leq r + k \text{ such that } m_2 \leq r} \lambda^{2(r+k-j-m_1-m_2-l)} \times \|(t \partial_t)^j (\theta \partial_\theta)^m \partial_{\theta}^m \partial_y^r v\|_{L^2(\hat{\Omega}_T)}^2 \right)^{1/2},
$$

$$
\hat{H}^{k}(\omega_T) = \{ \psi; (t \partial_t)^j \partial_y^l \psi \in L^2(\omega_T), \text{ where } j + l \leq k \},
$$

$$
\|\psi\|_{\hat{H}^{k}(\omega_T)} = \left( \sum_{j + l \leq k} \lambda^{2(k-j-l)} \|(t \partial_t)^j \partial_y^l \psi\|_{L^2(\omega_T)}^2 \right)^{1/2}.
$$
Lemmas 6.3. \( J_\lambda \) is an isomorphic mapping \( H^{s+k}_{\lambda+1/2}(\Omega_\tau) \to \hat{H}^{s+k}_{\lambda}(\hat{\Omega}_\tau) \) or \( H^s_\lambda(\omega_\tau) \to \hat{H}^s_\lambda(\omega_\tau) \), and there is a constant \( K \), such that
\[
K^{-1} \| u \|_{H^{s+k}_{\lambda+1/2}(\Omega_\tau)} \leq \| J_\lambda u \|_{\hat{H}^{s+k}_{\lambda}(\hat{\Omega}_\tau)} \leq K \| u \|_{H^{s+k}_{\lambda+1/2}(\Omega_\tau)} \quad (6.4)
\]
\[
K^{-1} \| \phi \|_{H^s_\lambda(\omega_\tau)} \leq \| J_\lambda \phi \|_{\hat{H}^s_\lambda(\omega_\tau)} \leq K \| \phi \|_{H^s_\lambda(\omega_\tau)}. \quad (6.5)
\]

Proof. We only prove (6.4). First, \( J_\lambda \) is an isomorphic mapping from \( L^2_{\lambda+1/2}(\Omega_\tau) \) to \( L^2(\hat{\Omega}_\tau) \) because of \( dt \, dt \, dy = -t \, dx_1 \, dx_2 \, dy \). From (6.3) we have
\[
\partial_t J_\lambda u = t^{-\lambda} \left( \frac{x}{t} \partial_x u + \frac{y}{t} \partial_y u \right) - \lambda t^{-\lambda-1} u,
\]
\[
t \partial_t J_\lambda u = t^{-\lambda} (x \partial_x u + y \partial_y u) - \lambda t^{-\lambda} u
\]
\[
\partial_\sigma J_\lambda u = t^{-\lambda+1} \partial_x u - t^{-\lambda+1} \partial_y u
\]
\[
\theta \partial_\sigma J_\lambda u = t^{-\lambda} x \partial_x u - t^{-\lambda} x \partial_y u
\]
\[
\partial_\zeta J_\lambda u = t^{-\lambda} \partial_\zeta u.
\]
Therefore
\[
| (t \partial_t)^j (\theta \partial_\sigma)^m \partial_{\sigma}^m \partial_{\zeta}^l J_\lambda u | \leq
\]
\[
t^{-\lambda} C \sum_{b+|\beta| \leq j+m_1+m_2+l} | \partial_{x_1}^b (x_1 \partial_{x_1})^\beta_1 \partial_{x_2}^\beta_2 \partial_{y}^\beta_3 u | \lambda^{j+m_1+m_2+l-|\beta|} t^{b+\beta_1+\beta_2},
\]
and for \( j + m_1 + m_2 + l \leq r + k, m_2 + r \), the estimate
\[
\lambda^{r+k+j-m_1-m_2-l} (t \partial_t)^j (\theta \partial_\sigma)^m \partial_{\sigma}^m \partial_{\zeta}^l J_\lambda u |
\]
\[
\leq C \sum_{b+|\beta| \leq r+k} | \partial_{x_1}^b (x_1 \partial_{x_1})^\beta_1 \partial_{x_2}^\beta_2 \partial_{y}^\beta_3 u | \lambda^{r+k-b-|\beta|} t^{-\lambda-\beta+b+\beta_2}
\]
holds. This yields the right inequality in (6.4).
On the other hand, we have
\[
\partial_t u = t^\lambda \partial_x J_\lambda u
\]
\[
(t \partial_t - \theta \partial_\sigma) J_\lambda u = t^{-\lambda+1} \partial_x u - \lambda t^{-\lambda} u
\]
\[
\partial_{x_1} u = t^{\lambda-1} (t \partial_t - \theta \partial_\sigma) J_\lambda u + \lambda t^{-\lambda-1} J_\lambda u,
\]
\[
\partial_{x_2} u = t^{\lambda-1} \partial_\sigma J_\lambda u + t^{\lambda-1} (t \partial_t - \theta \partial_\sigma) J_\lambda u + \lambda J_\lambda u t^{\lambda-1},
\]
\[
x_1 \partial_{x_1} u = t^{\lambda} \left( \frac{x_1}{t} (t \partial_t - \theta \partial_\sigma) J_\lambda u + \frac{x_1}{t} \lambda J_\lambda u + \frac{x_1}{t} \partial_{\sigma} J_\lambda u \right);
\]
MULTIDIMENSIONAL SHOCK FRONT

Therefore,

\[
|2^r + b - \beta|(x, \partial_{x_1})^\beta_1 \partial_{x_2} \partial_{x_3} \partial_{x_4} |t^{-\lambda - \beta_1} | \leq C \sum_{m_2 \leq r} |\partial_{\theta_d}(\theta \partial_{\theta_d})^{m_1} (t \partial_t)^j \lambda^h \partial_{\theta_d}^{m_2} \partial_{\theta_d}^{m_3} J_{\lambda} u| t^{\lambda - m_1 - m_2 - j - h - \beta_1} \]

If \( b + |\beta| \leq r + k, b \leq r \), the right-hand side does not exceed

\[
C \sum_{m_1 + m_2 + j + l \leq r + k} |(t \partial_t)^j (\theta \partial_{\theta_d})^{m_1} \partial_{\theta_d}^{m_2} \partial_{\theta_d}^{m_3} J_{\lambda} u| \lambda^{r + k - m_1 - m_2 - j - l}.
\]

Thus the left half of (6.4) is obtained.

Set \( \chi \in C^\infty_0(\mathbb{R}^1) \), such that \( \text{supp} \chi \subset (\frac{1}{2}, 2) \) and

\[
\sum_{j = -\infty}^{\infty} \chi(2^j t) = 1, \quad t \geq 0.
\]

Moreover, let

\[
\begin{align*}
&v_j(t, \theta, y) = \chi(2^j t) v(t, \theta, y), \quad \psi_j(t, y) = \chi(2^j t) \psi(t, y), \\
&\tilde{v}_j(t, \theta, y) = 2^{-j/2} v_j(2^{-j} t, \theta, y), \quad \tilde{\psi}_j(t, y) = 2^{-j/2} \psi_j(2^{-j} t, y), \\
&T_j = \min(2, 2^j T), \\
&\mathcal{O}_T = (-\infty, T) \times (0, 1) \times \mathcal{O}, \quad \mathcal{O}_T = (-\infty, T) \times \mathcal{O}, \\
&H^{r,k} (\mathcal{O}_T) = \{w; \partial^j (\theta \partial_{\theta_d})^{m_1} \partial_{\theta_d}^{m_2} \partial_{\theta_d}^{m_3} w \in L^2(\mathcal{O}_T), \text{ where } j + m_1 + m_2 + l \leq r + k, m_2 \leq r \}.
\end{align*}
\]

We have

\[
\|w\|_{(r,k), \lambda} = \left( \sum_{j + l + m_1 + m_2 \leq r + k} \lambda^{2(r + k - j - m_1 - m_2 - l)} \cdot \left( \sum_{m_2 \leq r} \|\partial^j (\theta \partial_{\theta_d})^{m_1} \partial_{\theta_d}^{m_2} \partial_{\theta_d}^{m_3} w \|_{L^2(\mathcal{O}_T)}^2 \right) \right)^{1/2},
\]

\[
\|\mathcal{F}\|_{(r,k), \lambda} = \left( \sum_{j + l \leq k} \lambda^{2(j - l)} \|\partial^j \partial_{\theta_d}^l \mathcal{F} \|_{L^2(\mathcal{O}_T)}^2 \right)^{1/2}.
\]

**Lemma 6.4.** (i) If \( v \in \tilde{H}^{r,k}(\mathcal{O}_T) \), then \( \tilde{v}_j \in H^{r,k}(\mathcal{O}_T) \), and

\[
\sum \|\tilde{v}_j\|_{(r,k), \lambda} \leq C \|v\|_{\tilde{H}^{2,k}(\mathcal{O}_T)}^2.
\]
(ii) If there exists a sequence \( \{w_j\} \) satisfying \( w_j \in H^{r,k}(\Omega_T) \), \( \text{supp } w_j \subset \{ T_j \geq t \geq \gamma \} \) with \( \gamma > 0 \), and \( \Sigma \|w_j\|_{(r,k), \lambda}^2 < \infty \), then \( v = \sum_j 2^{j/2} w_j(2^j t, \theta, z) \in \tilde{H}^{r,k}(\tilde{\Omega}_T) \), and
\[
\|v\|_{H^{r,k}(\tilde{\Omega}_T)} \leq C (\Sigma \|w_j\|_{(r,k), \lambda}^2)^{1/2}.
\]

(iii) If \( \psi \in H^k(\tilde{\omega}_T) \), then \( \tilde{\psi}_j \in H^k(\tilde{\omega}_{T_j}) \), and
\[
\sum_{2^j T_j > 1} \|\tilde{\psi}_j\|_{H^k, \lambda}^2 \leq C \|\psi\|_{H^k(\omega_T)}^2.
\]

(iv) If there exists a sequence \( \{\phi_j\} \) satisfying \( \phi_j \in H^k(\tilde{\omega}_{T_j}) \), \( \text{supp } \phi_j \subset \{ T_j \geq t \geq \gamma \} \) with \( \gamma > 0 \), and \( \Sigma \|\phi_j\|_{(r,k), \lambda}^2 < \infty \), then \( \psi = \sum_j 2^{j/2} \phi_j(2^j t, y) \in \tilde{H}^k(\tilde{\omega}_T) \), and
\[
\|\psi\|_{\tilde{H}^k(\omega_T)} \leq C (\Sigma \|\phi_j\|_{(r,k), \lambda}^2)^{1/2}.
\]

Proof. We prove conclusions (i) and (ii) only. Since the dyadic partition of unity and dilation are taken along the \( t \) direction, we only indicate the estimates of derivatives in the \( t \) direction. From the expression of \( \tilde{v}_j \) we have
\[
\int_{\Omega_{t_j}} |\tilde{\partial}_t \tilde{v}_j(t, \theta, y)|^2 \, dt \, d\theta \, dy
\]
\[
= \int_{\Omega_{T_j}} 2^{-j/2} 2^{2j} |(\tilde{\partial}_t v_j)(2^{-j} t, \theta, y)|^2 \, dt \, d\theta \, dy
\]
\[
= \int_{2^{-j} \Omega_{T_j}} 2^{-2j} |\tilde{\partial}_t v_j(t_1, \theta, y)|^2 \, dt_1 \, d\theta \, dy,
\]
where \( t_1 = 2^{-j} t \). By virtue of \( \frac{1}{2} < 2^{-j} t < 2 \) on \( \text{supp } v_j \), the inequality \( 2^{-2j} < 4 t_1^2 \) holds. Hence
\[
\int_{\Omega_{t_j}} |\tilde{\partial}_t \tilde{v}_j(t, \theta, y)|^2 \, dt \, d\theta \, dy
\]
\[
\leq C \int_{2^{-(j+1)} < t < 2^{-j+1}} |t \tilde{\partial}_t v_j(t, \theta, y)|^2 \, dt \, d\theta \, dy,
\]
\[
\Sigma \int_{\Omega_{T_j}} |\tilde{\partial}_t \tilde{v}_j(t, \theta, y)|^2 \, dt \, d\theta \, dy
\]
\[
\leq C \int |t \tilde{\partial}_t v_j(t, \theta, y)|^2 \, dt \, d\theta \, dy.
\]
A similar procedure yields (6.7).
Conversely, assume that \( \{w_j\} \) is given as in (ii). Set \( w_j^*(t, \theta, y) = 2^{j/2}w_j(t, \theta, y) \). Then \( \text{supp } w_j^* \subset (2^{-j}, 2^{-j+1}) \), and

\[
\int_{\Omega_{r_j}} |t \partial_r w_j^*(t, \theta, y)|^2 \, dt \, d\theta \, dy
\]

\[
= \int_{\Omega \cap \{r \leq 2^j \leq r_j\}} |t \partial_r (2^{j/2}w_j(2^j t, \theta, y))|^2 \, dt \, d\theta \, dy
\]

\[
\leq \int_{\Omega_{r_j}} |t^2 2^{2j}(\partial_r w_j)(t_1, \theta, y)|^2 \, dt_1 \, d\theta \, dy
\]

\[
\leq C \int_{\Omega_{r_j}} |\partial_r w_j(t, \theta, y)|^2 \, dt \, d\theta \, dy;
\]

therefore

\[
\int_{\Omega_{r_j}} |t \partial_r v|^2 \, dt \, d\theta \, dy \leq \sum_j \int_{\Omega_{r_j}} |t \partial_r w_j^*(t, \theta, y)|^2 \, dt \, d\theta \, dy
\]

\[
\leq C \sum_j \int_{\Omega_{r_j}} |\partial_r w_j(t, \theta, y)|^2 \, dt \, d\theta \, dy.
\]

A similar procedure yields (6.8).  

As in the beginning of this section we define

\[
\hat{B}^k(\hat{\Omega}_T) = \bigcap_{r \leq k/2} H^{r,k-2r}(\hat{\Omega}_T), \quad B^k(\hat{\Omega}_T) = \bigcap_{r \leq k/2} H^{r,k-2r}(\hat{\Omega}_T),
\]

\[
\begin{align*}
\|v\|_{\hat{B}^k_2}(\hat{\Omega}_T) &= \left( \sum_{r \leq k/2} \|v\|_{\hat{H}^{r,k-2r}(\hat{\Omega}_T)}^{2} \right)^{1/2}, \\
\widetilde{v}^{2}_{\hat{B}^k_2}(\hat{\Omega}_T) &= \left( \sum_{r \leq k/2} \langle v \rangle_{(r,k-2r),2} \right)^{1/2}.
\end{align*}
\]

The following conclusions are valid.

**Lemma 6.5.** \( J_\hat{\lambda} \) is an isomorphic from \( B^k_{\lambda + 1/2}(\Omega_T) \) to \( \hat{B}^k(\hat{\Omega}_T) \) and there is a constant \( C \), such that

\[
C^{-1} \|u\|_{B^k_{\lambda + 1/2}(\Omega_T)} \leq \|J_\hat{\lambda} u\|_{\hat{B}^k_2(\hat{\Omega}_T)} \leq C \|u\|_{B^k_{\lambda + 1/2}(\Omega_T)}. \tag{6.11}
\]

**Lemma 6.6.** (i) If \( v \in \hat{B}^k_2(\hat{\Omega}_T) \), \( \hat{v}_j \) is defined as above, then \( \hat{v}_j \in B^k(\hat{\Omega}_{r_j}) \), and

\[
\sum_j \|\hat{v}_j\|_{\hat{B}^k_2} \leq C \|v\|_{\hat{B}^k_2(\hat{\Omega}_T)}. \tag{6.12}
\]
(ii) If there exists a sequence \( \{w_j\} \) satisfying \( w_j \in B^k(\tilde{\Omega}_T) \), \( \text{supp } w_j \subset \{ T_j \supset t \supset \gamma \} \) with \( \gamma > 0 \), and \( \Sigma \|w_j\|_{B^k_T}^2 < \infty \), then \( v = \Sigma_j 2^{-j/2} w_j(2^j t, \theta, \gamma) \in B^k(\tilde{\Omega}_T) \), and
\[
\|v\|_{B^k(\Omega_T)}^2 \leq C (\Sigma \|w_j\|_{B^k_T}^2)^{1/2}. \tag{6.13}
\]

7. NONLINEAR COMPOSITION

**Lemma 7.1.** If \( k \geq 8 \), \( k_1 \leq k, \; \tilde{v}_1 \in B^{k_1}(\tilde{\Omega}_T), \; \tilde{v}_L \in B^k(\tilde{\Omega}_T) \), then \( \tilde{v}_1 \tilde{v}_2 \in B^{k_1}(\tilde{\Omega}_T) \) and
\[
\|\tilde{v}_1 \tilde{v}_2\|_{B^k} \leq C \|\tilde{v}_1\|_{B^{k_1}} \|\tilde{v}_2\|_{B^k} \quad (\lambda \geq 1). \tag{7.1}
\]

**Proof.** In view of \( B^{k_1} = \bigcap_s 2r = k^1 H^{r,s} \) we consider
\[
\sum_{\lambda + m_1 + m_2 + r \leq r+s \atop m_2 \geq r} \lambda^{2(r+s - j - m_1 - m_2 - 1)} \|\partial^j_x (\partial \partial \partial)^m_1 \partial^m_2 \partial^r y (\tilde{v}_1 \tilde{v}_2)\|_{L^2},
\]
and each term in the sum can be written as
\[
\lambda^{2(r+s - j - m_1 - m_2 - 1)} \|\partial^j_x (\partial \partial \partial)^m_1 \partial^m_2 \partial^r y \tilde{v}_1\|_{L^2} \times \|\partial^j_y - \partial^j \partial \partial \partial^m_1 \partial^m_2 \partial^r y - \tilde{v}_2\|_{L^2}. \tag{7.2}
\]

Denoting \( w_1 = \partial^j_x (\partial \partial \partial)^m_1 \partial^m_2 \partial^r y \tilde{v}_1, \; w_2 = \partial^j_y - \partial^j \partial \partial \partial^m_1 \partial^m_2 \partial^r y - \tilde{v}_2, \) we have
\[
|w_1| \leq C (\|w_1\|_{L^2} + \|w_1\|_{H^{r,s}}) \leq C \|\tilde{v}_1\|_{B^{r+s+m_1+m_2+r}}, \tag{7.3}
\]
\[
|w_2| \leq C (\|w_2\|_{L^2} + \|w_2\|_{H^{r,s}}) \leq C \|\tilde{v}_2\|_{B^{(j-r) + (m_1 - m_2) + (l-r) + 2(m_2 - m_2') + s}}. \tag{7.4}
\]

The sum of the indices of these two \( B \)-spaces is \( j + m_1 + l + 2m_2 + 8 \leq 2r + s + 8 \leq k + k \); therefore at least one inequality among \( j' + m_1' + l' + 2m_2' + 4 \leq k_1 \) and \( (j - j') + (m_1 - m_1') + (l - l') + 2(m_2 - m_2') + 4 \leq k \) holds. If \( r + s \geq 2 \) and the first inequality holds, then in view of
\[
\lambda^{-2(j' + m_1' + m_2' + l')} \|w_1\|_{B^{j',1}}^2
\]
\[
= \Sigma \lambda^{-(j' + m_1' + m_2' + l')} \partial^{j'} (\partial \partial \partial)^m_1 \partial^m_2 \partial^r y \tilde{v}_1 \|_{L^2},
\]
with \( j'' + m''_1 + l'' \leq j' + l' + 1, \; m''_2 \leq m_2' + 1, \) we have
\[
 j'' + m''_1 + m''_2 + l'' \leq r + s + j' + m_1' + m_2' + l'
\]
and

\[
\lambda^{-2(U' + m_1 + m_2 + l')} |w_1|^2 \\
\leq C \sum \|2^{-j - s - j' - m_1 - m_2 - l'} \frac{\partial^j \partial^s}{\partial y \partial \vartheta} \frac{\partial^{j'} \partial^{s'}}{\partial x \partial \vartheta} \bar{v}_1 \|_{L^2}^2 \\
\leq C \|\bar{v}_1\|_{H^k_x}^2,
\]

which yields (7.1). If \( r + s \leq 1 \) or \((j - j') + (m_1 - m_1') + (l - l') + 2(m_2 - m_2') + 4 \leq k\), then we may use (7.4) instead of (7.3) to obtain (7.1).

**Lemma 7.2.** If \( k \geq 8, k_1 \geq k, \lambda \geq 1, u_1 \in B^k_{\lambda}(\Omega_T), u_2 \in B^k_{\lambda}(\Omega_T)\), then \( u_1, u_2 \in B^{k_1}_{2\lambda-1}(\Omega_T)\), and

\[
\|u_1 u_2\|_{B^{k_1}_{2\lambda-1}(\Omega_T)} \leq C \|u_1\|_{B^k_{\lambda}(\Omega_T)} \|u_2\|_{B^k_{\lambda}(\Omega_T)}.
\]

(7.5)

Moreover, if \( \varphi_1 \in H^{k_1}_{\lambda}(\Omega_T), \varphi_2 \in H^{k}_{\lambda}(\Omega_T)\), then \( \varphi_1 \varphi_2 \in H^{k}_{2\lambda-1/2}(\Omega_T)\), and

\[
\|\varphi_1 \varphi_2\|_{H^{k}_{2\lambda-1/2}(\Omega_T)} \leq C \|\varphi_1\|_{H^{k_1}_{\lambda}(\Omega_T)} \|\varphi_2\|_{H^{k}_{\lambda}(\Omega_T)}.
\]

(7.6)

**Proof.** Let \( v_1 = J_{\lambda - 1/2} u_1, v_2 = J_{\lambda - 1/2} u_2 \). Then

\[
J_{2\lambda - 3/2}(v_1, v_2) = t^{-2\lambda + 3/2} t^{2\lambda - 1} v_1 v_2 = t^{1/2} v_1 v_2.
\]

Replacing \( v_1, v_2 \) by \( \Sigma v_{1j}, \Sigma v_{2j} \) (see (6.6)), and noticing the property of \( v_{1j}, v_{2j} \) about their support, we have

\[
t^{1/2} v_1 v_2 = \sum_j \left( v_{1j} \sum_{|j - j'| \leq 1} t^{1/2} v_{2j'} \right).
\]

Setting \( w_j = v_{1j} \sum_{|j - j'| \leq 1} t^{1/2} v_{2j'}\),

\[
\tilde{w}_j = 2^{-j/2} w_j(2^{-j}, \theta, y)
\]

\[
= 2^{-j/2} v_{1j} 2^{-j/2} \sum_{|j - j'| \leq 1} 2^{j/2} t^{1/2} v_{2j'}
\]

\[
= \tilde{v}_{1j} \cdot \tilde{v}_{2j},
\]

where \( \tilde{v}_{2j}(t, \theta, y) = 2^{-j/2} \sum_{|j - j'| \leq 1} 2^{j/2} t^{1/2} v_{2j'}(2^{-j}, \theta, y)\), we can prove as in Lemmas 6.4 and 6.6

\[
\sum_j \|\tilde{\nu}_{2j}\|_{L^2(\Omega_T, L^2_x)} \leq C \|v_2\|_{L^2(\Omega_T, L^2_x)}
\]

\[
\sum_j \|\tilde{\nu}_{2j}\|_{L^2(\Omega_T, L^2_x)}^2 \leq C \|v_2\|_{L^2(\Omega_T, L^2_x)}^2.
\]
Therefore by Lemma 7.1

$$
\sum_j |\hat{w}_j|^{\frac{2^j}{2^j_1}} \leq C \left( \sum_j |\hat{v}_1|^{\frac{2^j}{2^j_1}} \right) \left( \sum_j |\hat{v}_2|^{\frac{2^j}{2^j_1}} \right)
$$

$$
\leq C \|v_1\|^{\frac{2^j}{2^j_1}(\Omega_T)} \|v_2\|^{\frac{2^j}{2^j_1}(\Omega_T)}.
$$

Using Lemmas 6.3 and 6.4 we obtain (7.5). The proof of (7.6) is similar. 

**Lemma 7.3.** Assume that $f$ is a $C^\infty$ function on $\mathbb{R}^1$, $f(0) = 0$, $T_0 > 0$, $K > 0$, $k \geq 8$, $\lambda \geq k + 1$. If a real function $u \in B^k_\lambda(\Omega_{T_0})$ satisfies

$$
\|u\|_{B^k_\lambda(\Omega_{T_0})} \leq K,
$$

then $f(u) \in B^k_\lambda(\Omega_{T_0})$, and for $t \leq T_0$

$$
\|f(u)\|_{B^k_\lambda(\Omega_{T_0})} \leq C(K) \|u\|_{B^k_\lambda(\Omega_{T_0})}.
$$

**Proof.** First, for $|\beta| \leq 1$, $b \leq 1$ or $|\beta| \leq 3$, $b = 0$ we have

$$
\|V^\beta D^b u\|_{L^\infty(\Omega_{T_0})} \leq C(\|u\|_{L^2(\Omega_{T_0})} + \|u\|_{H^3(\Omega_{T_0})}) + \|u\|_{H^\lambda(\Omega_{T_0})})
$$

$$
\leq C \|u\|_{B^k_\lambda(\Omega_{T_0})} \leq K.
$$

By $f(0) = 0$, $f(u)$ can be written as $ug(u)$ with $g \in C^\infty$, and $\|u\|_{L^\infty(\Omega_{T_0})} \leq K$ yields $\|g(u)\|_{L^\infty(\Omega_{T_0})} \leq C(K)$. Moreover

$$
\lambda^k \|f(u)\|_{L^2} \leq \lambda^k \|ug(u)\|_{L^2} \leq C(K) \lambda^k \|u\|_{L^2} \leq C(K) \|u\|_{H^\lambda} \leq C(K).
$$

To estimate $\|f(u)\|_{B^k_\lambda(\Omega_{T_0})}$, we expand its expression, and estimate

$$
\lambda^r + s - b - |\beta| \|V^\beta D^b f(u)\|_{L^2} \leq C \lambda^k \|u\|_{L^2}.
$$

Expanding $v^\beta D^b f(u)$, we have

$$
V^\beta D^b f(u) = \sum f^{(\beta)}(u) V^{\beta(1)} D^{b(1)} u \ldots V^{\beta(q)} D^{b(q)} u.
$$

where $\sum_{j=-1}^{q} |\beta_{(j)}| \leq |\beta|$, $\sum_{j=-1}^{q} b_{(j)} \leq b$, $|\beta_{(j)}| + \alpha_{(j)} > 0$.

For $q = 1$, in view of $|f^{(\beta)}(u)| \leq C(K)$, the estimate

$$
\lambda^r + s - b - |\beta| \|V^{\beta(1)} D^{b(1)} u\|_{L^2} \leq C \lambda^k \|u\|_{L^2}.
$$

is valid. Hence the terms in (7.10) corresponding to $q = 1$ are dominated.

When $q \geq 2$, among all factors $V^{\beta(1)} D^{b(1)} u$ in the left-hand side of (7.10) we only need estimate the $L^2$-norm for one of them, and estimate the $L^\infty$
norm for the remaining factors. In fact, if one of the sums $|\beta(j)| + 2b(j)$ is greater than 4, for instance $|\beta(j)| + 2b(j) > 4$, then $|\beta(j)| + 2b(j) \leq k - 4$ for $j \geq 2$, and

$$
\| V^{\beta(j)} D^{b(j)} u \|_{L^2(\Omega_{T_0})} \leq C \| V^{\beta(j)} D^{b(j)} u \|_{H^2(\Omega_{T_0})}
$$

$$
\leq C \| u \|_{B^{\beta(j) + 2b(j)+4}_{\infty}(\Omega_{T_0})}
$$

$$
\leq C \| u \|_{B^k(\omega_{T_0})} \leq C(K). \quad (7.12)
$$

Furthermore, we can estimate $\| V^{\beta(j)} D^{b(j)} u \|_{L^2}$ as in (7.11). Now if $|\beta(j)| + 2b(j) \leq 4$ holds for all $j$, then $|\beta(j)| + 2b(j) \leq k - 4$ by $k \geq 8$ also holds for all $j$. Thus the above argument is still effective. Summing up, we have

$$
\lambda^{r+s-b-|\beta|} \| V^\beta D^s f(u) \|_{L^2_{\lambda+b-s}(\Omega_T)} \leq C(K) \| u \|_{B^k(\omega_T)}, \quad (7.13)
$$

whence (7.7).

**Corollary.** Under the assumptions of Lemma 7.3, set $v = f(u) - uf'(u)$, then $v \in B^{k}_{2k-1}(\Omega_{T_0})$, and for $T \leq T_0$

$$
\| v \|_{B^{k}_{2k-1}(\Omega_T)} \leq C(K) \| u \|_{B^k(\omega_T)}. \quad (7.14)
$$

**Proof.** We write $f(t) = f'(t)$ as $\xi g(\xi)$. Then $g \in C^\infty(\mathbb{R}')$ and $g(0) = 0$. Hence the conclusions in Lemmas 7.2 and 7.3 yield (7.14) immediately.

Similarly, for the space $H^k(\omega_T)$ we have the following proposition, the proof of which is similar to and simpler than Lemma 6.3.

**Lemma 7.4.** Assume that $f$ is a $C^\infty$ function on $\mathbb{R}$, $f(0) = 0$, $T_0 > 0$, $K > 0$, $k \geq 2$, $\lambda \geq k + 1$. If a real function $\varphi \in H^k(\omega_{T_0})$ satisfies $\| \varphi \|_{H^k(\omega_{T_0})} \leq K$, then $f(\varphi) \in H^k(\omega_{T_0})$, and for $T \leq T_0$

$$
\| f(\varphi) \|_{H^k(\omega_T)} \leq C(K) \| \varphi \|_{H^k(\omega_T)}. \quad (7.15)
$$

Moreover, $f(\varphi) - \varphi f'(\varphi) \in H^k(\omega_{T_0})$, and for $T \leq T_0$

$$
\| f(\varphi) - \varphi f'(\varphi) \|_{H^k(\omega_T)} \leq C(K) \| \varphi \|_{H^k(\omega_T)}. \quad (7.16)
$$

8. **Estimates for Linear Hyperbolic Systems with Characteristic Boundary**

In order to construct the sequence of approximate solutions and to determine the first term of this sequence, we need some estimates for the
solutions of linear hyperbolic systems. First, we consider the case with two independent variables.

Let $\Omega_0^{(0)}$ be the domain $x_1 > 0, x_2 > 0$ on the plane $0_{x,y}$, $\Omega_0^{(0)} = \Omega^{(0)} \cap \{x_1 + x_2 < T\}$. On $\Omega_0^{(0)}$ we give a boundary value problem (P) for a first-order hyperbolic system,

$$L_0 U \equiv A \frac{\partial u}{\partial x_1} + B \frac{\partial u}{\partial x_2} + Gu = f, \quad l_1 u = 0,$$

which is just problem (5.8)- (5.10) except adding the term $Gu$ in the system. We assume all conditions for the coefficients in (5.8)-(5.10) are satisfied for (8.1), $G \in C^0(\Omega_0^{(0)})$. Denoting the coefficients in (8.1) by $A^*, B^*, \ldots, m^*$, when these coefficients are frozen at the origin, and setting

$$\varepsilon(p) = \max(\|A - A^*\|_{L^\infty(\Omega_0^{(0)})}, \ldots, \|m - m^*\|_{L^\infty(\Omega_0^{(0)}, T_0)}),$$

we have

**Theorem 8.1.** Under the assumptions there exists $\varepsilon_1 > 0$, such that for $\varepsilon(p) \leq \varepsilon_1$, $T \leq T_0$, problem (8.1) has a unique solution $(u, \varphi) \in C^0(\Omega_0^{(0)}) \times C^1([0, T])$ for each pair $(f, g) \in C^0(\Omega_0^{(0)}) \times C^0([0, T])$ and

(i) for $\tau > 0$

$$\|e^{-\tau(x_1 + x_2)}u\|_{L^\infty(\Omega_0^{(0)})} + \left\| \frac{d\varphi}{dt} \right\|_{L^\infty([0, T])} \leq C \left(\frac{1}{\tau} \|e^{-\tau(x_1 + x_2)}f\|_{L^\infty(\Omega_0^{(0)})} + \|e^{-\tau}g\|_{L^\infty([0, T])}\right),$$

(ii) for $\tau > 0, \lambda \geq 1$

$$\|(x_1 + x_2)^{-\tau(x_1 + x_2)}f\|_{L^\infty(\Omega_0^{(0)})} + \left\| t^{-\lambda}e^{-\tau} \frac{d\varphi}{dt} \right\|_{L^\infty([0, T])} \leq C(\|\lambda(x_1 + x_2)^{\lambda - 1} + \tau(x_1 + x_2)\|_{L^\infty(\Omega_0^{(0)})} + \|e^{-\tau(x_1 + x_2)}f\|_{L^\infty(\Omega_0^{(0)})} + \|e^{-\tau}g\|_{L^\infty([0, T])});$$

(iii) if the coefficients in (8.1) are $C^k$, $(f, g) \in C^k(\Omega_0^{(0)}) \times C^k([0, T])$, then $(u, \varphi) \in C^k(\Omega_0^{(0)}) \times C^{k+1}([0, T])$, and for sufficiently large $\tau$,
Furthermore, if the coefficients and the right-hand side in (8.1) continuously depend on a parameter \( y \in \mathcal{D} \), then denoting \( \Omega_T^{(0)} \times \mathcal{D} \) by \( \Omega_T \), \([0, T] \times \mathcal{D} \) by \( \omega_T \), we have

**Theorem 8.2.** Suppose that the assumptions of Theorem 8.1 are satisfied uniformly with respect to \( y \). Then there exists \( \varepsilon_1 > 0 \) such that for \( \sup_y \varepsilon(p) \leq \varepsilon_1 \), \( T \leq T_0 \), problem (8.1) has a unique solution \( (u, \varphi) \in C^0(\Omega_T) \times C^{(1,0)}(\omega_T) \), and

(i) for \( \tau > 0 \), \( \lambda \geq 1 \), the inequalities (8.3), (8.4) are still valid after replacing \( L^\infty(\Omega_T^{(0)}) \), \( L^\infty([0, T]) \) by \( L^\infty(\Omega_T) \), \( L^\infty(\omega_T) \), respectively.

(ii) if the coefficients in (8.1) are \( C^k \), and \( (f, g) \in C^k(\overline{\Omega}_T) \times C^k(\omega_T) \), then \( (u, \varphi) \in C^k(\overline{\Omega}_T) \times C^{(k+1,k)}(\omega_T) \). Moreover, for sufficiently large \( \tau \)

\[
\sum_{|\alpha| \leq k} \tau^{k-|\alpha|} \| e^{-\tau(x_1 + x_2)} \partial^\alpha u \|_{L^\infty(\Omega_T)} \\
+ \sum_{i+j \leq k+1} \tau^{k+1-i} \| e^{-\tau \partial_i^j \varphi} \|_{L^\infty(\omega_T)} \\
\leq C \left( \frac{1}{\tau} \sum_{|\alpha| \leq k} \tau^{k-|\alpha|} \| e^{-\tau(x_1 + x_2)} \partial^\alpha f \|_{L^\infty(\Omega_T)} \\
+ \sum_{i+j \leq k+1} \tau^{k-i} \| e^{-\tau \partial_i^j g} \|_{L^\infty(\omega_T)} \right),
\]

where \( C^{(r,s)}(\omega_T) = \{ \varphi; \partial_i^r \partial_j^s \varphi \in C^0(\omega_T), i + j \leq r, j \leq s \} \).

The proof of Theorems 8.1 and 8.2 can be derived by the method of integrating along characteristic curves. Here we omit the details.

Next we consider the hyperbolic symmetric system in the three-dimensional case. Let

\[
A_0 \frac{\partial u}{\partial t} + A_1 \frac{\partial u}{\partial x} + A_2 \frac{\partial u}{\partial y} + \lambda u + Bu = f
\]
be a linear hyperbolic symmetric system in $Q_T = [0, T] \times [0, 1] \times D$, where $D$ is a periodic interval of variable $y$. Assume that the coefficients in (8.7) are periodic functions with respect to $y$, $A_0, A_1, A_2$ are symmetric $N \times N$ matrices, and $A_0 > 0$. $\lambda$ is a large constant.

We also assume that $x = 1$ is a noncharacteristic boundary and that $x = 0$ is a regular characteristic boundary. The latter means that there exists a family of surfaces $g(t, x, y) = s$ with $\nabla g \neq 0$, such that $g(t, 0, y) \equiv 0$ and by the transformation $t' = t, x' = g(t, x, y), y' = y$, system (8.7) can be changed to the form

$$A_0 \frac{\partial u}{\partial t'} + \left( \frac{\partial g}{\partial t} A_0 + \frac{\partial g}{\partial x} A_1 + \frac{\partial g}{\partial y} A_2 \right) \frac{\partial u}{\partial x'} + A_2 \frac{\partial u}{\partial y'} + \lambda u + Bu = f$$

(8.8)

with $\text{rank}((\partial g/\partial t) A_0 + (\partial g/\partial x) A_1 + (\partial g/\partial y) A_2) - N_1$ being invariant.

The initial and boundary conditions for (8.7) are given as

$$u - u_0 \quad t = 0$$

(8.9)

$$l_1 u = 0 \quad x = 0$$

(8.10)

$$l_2 u = 0 \quad x = 1.$$  

(8.11)

We assume that (8.10) (resp. (8.11)) are maximum nonnegative subspaces for the quadratic form $-uA_1 u$ (resp. $uA_1 u$).

As in Section 6 we denote $\partial_t, \partial_x, \partial_x$ by $V, \partial_x$ by $D$, and we introduce

$$H^{r,k} = \{u; V^\beta D^b u \in L^2 \text{ for } |\beta| \leq k, b \leq r\},$$

$$B^k = \bigcap_{r \leq k/2} H^{r,k-2r},$$

$$\|u\|_{H^{r,k}} = \left( \sum_{|\beta| \leq k, b \leq r} \lambda^{2(k + r - b - |\beta|)} \left\| V^\beta D^b u \right\|^2_{L^2} \right)^{1/2},$$

$$\|u\|_{B^k} = \left( \sum_{r \leq k/2} \|u\|^2_{H^{r,k-2r}} \right)^{1/2}.$$

**Theorem 8.3.** If the coefficients of (8.7), (8.10), (8.11) are smooth, $u_0 = 0, f \in B^k(Q_T)$, the trace of $\partial_t f$ with $j \leq k - 1$ on $t = 0$ is 0, and $\lambda$ is sufficiently large, then there exists a unique solution for the above-mentioned initial boundary value problem (8.7), (8.9), (8.10), (8.11), and for $0 \leq k' \leq k$

$$\lambda \|u\|_{H^{r,k}_{x'}} \leq C \|f\|_{H^{r,k}_{x'}},$$

$$\lambda \|u\|_{B^k} \leq C \|f\|_{B^k}.$$  

(8.12)
Proof. We only derive the a priori estimates here, since the existence and uniqueness of $L^2$ solutions are known by [8], and the smoothness of the solution can be improved by using a priori estimates and the mollifier operator.

The estimate (8.12) in case $k' = 0$ can be obtained by standard energy methods. Multiplying (8.7) by $2u$, integrating both sides, and noticing the dissipative property of the boundary conditions, we have

$$
\int_{\mathcal{D}} u A_0 u \, dx \, dy + \int_{\mathcal{T}} u \left( 2\lambda I + 2B - \frac{\partial A_0}{\partial t} - \frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial y} \right) u \, dt \, dx \, dy
$$

$$
\leq \int_{\mathcal{D}} u A_0 u \, dx \, dy + \int_{\mathcal{T}} \lambda u^2 + \frac{1}{\lambda} f^2 \, dt \, dx \, dy.
$$

Since $A_0 > 0$, then for large $\lambda$ we have

$$
\lambda \| u \|^2_{L^2(\Omega_T)} \leq \frac{1}{\lambda} \| f \|^2_{L^2(\Omega_T)}.
$$

(8.13)

In order to obtain the estimates for derivatives of the solution, we introduce a $C^\infty$ function $\eta(x)$ with $\eta - 1$ in $0 \leq x \leq \frac{1}{2}$ and $\text{supp} \, \eta \subseteq [0, \frac{1}{2}]$. Obviously, $\eta u$ and $(1 - \eta) u$ can be estimated separately, and for $\eta u$ (resp. $(1 - \eta) u$) we only need consider one boundary condition (8.10) (resp. (8.11)), because near another boundary the function $\eta u$ (resp. $(1 - \eta) u$) vanishes. Since the boundary $x = 1$ is noncharacteristic, and hence the estimation for $(1 - \eta) u$ and its derivatives is simpler, here we only proceed the estimation for $\eta u$. In the sequel we assume that the system has been changed to

$$
\frac{\partial (\eta u)}{\partial t} + \left( \frac{\partial^2}{\partial t} A_0 + \frac{\partial^2}{\partial x} A_1 + \frac{\partial^2}{\partial y} A_2 \right) \frac{\partial (\eta u)}{\partial x'} + A_2' \frac{\partial (\eta u)}{\partial y'} + \lambda \eta u
$$

$$
+ \left( B - A_0' \frac{\partial \eta}{\partial t} - \left( \frac{\partial^2}{\partial t} A_0 + \frac{\partial^2}{\partial x} A_1 + \frac{\partial^2}{\partial y} A_2 \right) \frac{\partial \eta}{\partial x'} - A_2' \frac{\partial \eta}{\partial y'} \right) u = \eta f,
$$

(8.14)

and to avoid the notational burden we rewrite it as (8.7) again,

$$
L'u \overset{\text{def.}}{=} A_0' \frac{\partial u}{\partial t} + A_1' \frac{\partial u}{\partial x} + A_2' \frac{\partial u}{\partial y} + \lambda u + B' u = f',
$$

(8.15)

where $A_0' > 0$, $A_1' = \begin{pmatrix} R \end{pmatrix}$ with $R$ being a nonsingular $N_1 \times N_1$ matrix. Correspondingly, the boundary condition on $x = 0$ is

$$
u_1 = \cdots = u_{N_2} = 0, \quad N_2 \leq N_1
$$

(8.16)

and near $x = 1$, all components of $u$ are 0.
Applying the differential operators $\partial_t$, $x\partial_x$, $\partial_y$ on (8.15), we have

$$\begin{align*}
L' Vu + \left[ V, L' \right] J u &= Vf \\
Vu_1 &= \cdots = Vu_{N_2} = 0, \quad \text{on } x = 0.
\end{align*}$$

(8.17) (8.18)

According to (8.13)

$$\lambda \| Vu \|_{L^2} \leq C(\| Vf \|_{L^2} + \| \left[ V, L' \right] u \|_{L^2}).$$

Since $\| \left[ V, L' \right] u \|_{L^2} \leq C(\lambda \| u \|_{L^2} + \| u \|_{H^{0,1}})$,

$$\lambda \| u \|_{H^{0,1}} \leq C \| f \|_{H^{0,1}}.$$

This is (8.12) in the case $k' = 1$.

For general $k'$ we prove (8.12) by induction. Suppose the proposition is true for all $k' < k$. We prove it is also true for $k' + 1$.

First by the assumption of the induction, we can obtain the estimate

$$\lambda \| Vu \|_{H^{0,k}} \leq C(\| Vf \|_{H^{0,k}} + \| \left[ V, L' \right] u \|_{H^{0,k}})$$

from (8.17). By virtue of

$$\| \left[ V, L' \right] u \|_{H^{0,k}} \leq C(\lambda \| u \|_{H^{0,k}} + \| u \|_{H^{0,k+1}})$$

we have

$$\lambda \| u \|_{H^{0,k+1}} \leq C \| f \|_{H^{0,k+1}}$$

(8.19)

for large $\lambda$. Thus this first estimate in (8.12) is proved.

In order to estimate $\| u \|_{B^{k+1}_\infty}$, we use

$$\| u \|_{B^{k+1}_\infty} \leq \| Vu \|_{B^{k}_\infty} + \lambda \| u \|_{B^{k}_\infty} + \| u \|_{H^{1,k+1/2}_F,0}.$$  (8.20)

If $k' + 1$ is odd, then the last term in (8.20) appears in the expression of $\| u \|_{B^{k}_\infty}$. Therefore we may use the same method as above to prove (8.12). If $k' + 1$ is even, we have to estimate $\| u \|_{H^{1,k+1/2}_F,0}$. To do that we denote $u_a = (u_1, ..., u_{N_1})$, $u_b = (u_{N_1+1}, ..., u_N)$, and split system (8.15) to

$$\begin{align*}
\frac{\partial u_a}{\partial x} &= R^{-1} \left( f'_a - (A'_0)_a \frac{\partial u}{\partial t} - (A'_2)_a \frac{\partial u}{\partial y} - \lambda u - B'_a u \right) \\
(A'_0)_{bb} \frac{\partial u_b}{\partial t} + (A'_2)_{bb} \frac{\partial u_b}{\partial y} + \lambda u_b + B'_{bb} u_b \\
= f'_b - (A'_0)_{ba} \frac{\partial u_a}{\partial t} - (A'_2)_{ba} \frac{\partial u_a}{\partial y} - \lambda - B'_{ba} u_a.
\end{align*}$$

(8.21) (8.22)
where the subscripts $a, b$ indicate the corresponding blocks. In the following we denote the right-hand sides of (8.21), (8.22) by $F_a, F_b$, respectively. Since (8.22) is still a symmetric hyperbolic system for $u_b$ without boundary condition assigned at $x = 0$, we can regard $x$ as a parameter, and differentiate (8.22) with respect to $x$,

$$(A_0^b)_{bb} \frac{\partial (u_b)_x}{\partial t} + (A_2^b)_{bb} \frac{\partial (u_b)_x}{\partial y} + \lambda (u_b)_x + B_{bb} (u_b)_x = \tilde{F}_b,$$

where

$$\tilde{F}_b = (F_b)_{x} - (\partial/\partial x) (A_0^b)_{bb} (\partial u_b/\partial t) - (\partial/\partial x) (A_2^b)_{bb} (\partial u_b/\partial y) - (\partial B_{bb}/\partial x) u_b.$$

By using the assumption of induction, we have

$$\lambda \left\| Du_b \right\|_{H^{k'-1/2}, 0} \leq \lambda \left\| Du_b \right\|_{B^{k'-1}} \leq C \left\| \tilde{F}_b \right\|_{B^{k'-1}} \leq C (\| Vu_b \|_{B^{k'-1}} + \| Du_a \|_{B^{k'-1}} + \| VD u_a \|_{B^{k'-1}}). \quad (8.23)$$

On the other hand, (8.21) indicates

$$\lambda \left\| Du_a \right\|_{H^{k'-1/2}, 0} \leq C\lambda (\| f \|_{B^{k'-1}} + \| Vu \|_{H^{k'-1/2}, 0}) \leq C (\| f \|_{B^{k'-1}} + \| u \|_{B^{k'}}) \leq C (\| f \|_{B^{k+1}} + \| u \|_{B^{k'+1}}) \quad (8.24)$$

$$\| VD u_a \|_{B^{k'-1}} \leq C (\| Vf \|_{B^{k'-1}} + \| VVu \|_{B^{k'-1}}) \leq C (\| f \|_{B^{k}} + \| u \|_{B^{k+1}}). \quad (8.25)$$

Combining (8.23)–(8.25) we have

$$\lambda \left\| u \right\|_{H^{k'+1/2}, 0} \leq \lambda \left\| Du \right\|_{H^{k'-1/2}, 0} + \lambda^2 \||u||_{H^{k'-1/2}, 0} \leq C (\| f \|_{B^{k+1}} + \| u \|_{B^{k'+1}}).$$

Substituting that into (8.20) we obtain (8.12) for $k' + 1$. Therefore the conclusion is valid by induction. \[\]

**Remarks.** The requirement on the smoothness of coefficients in Theorem 8.3 can be weakened. In fact we only need that the coefficients of (8.7) belong to $B^k(\Omega_T)$ with $k \geq 10$. To verify it we notice that in the proof we often estimate $\|(VE)w\|_{B^{k'-1}}$ or $\|(DE)w\|_{B^{k'-2}}$, where $k' \leq k$, $E$ represents some coefficient, and $w$ represents $u$ or its derivatives. Following the process in the proof of Lemma 7.1, these terms can be estimated by $\|w\|_{B^{k'-1}}$ or $\|w\|_{B^{k'-2}}$, respectively. Therefore, the proof of Theorem 8.3 still works.
9. Solution to the Linear Goursat Problem

Let us come back to the linear Goursat problem (5.1)-(5.3). Without loss of generality we assume that I is a constant vector. Let N be an integer, \( N \geq 10 \).

\[
\varepsilon(L, F) = \max(\| A - A(0) \|_{L^\infty(\Omega_{T_0})}, \ldots, \| m - m(0) \|_{L^\infty(\Omega_{T_0})}),
\]

\[
\| L, F \|_N = \| A \|_{\mathcal{B}^\infty(\Omega_{T_0})} + \cdots + \| Q \|_{\mathcal{B}^\infty(\Omega_{T_0})} + \| p \|_{H^\infty(\omega_{T_0})} + \cdots + \| m \|_{H^\infty(\omega_{T_0})}.
\]

\[
W^k_{\lambda, T} = \{(u, \varphi); u \in B^k_{\lambda+1/2}(\Omega_T), \gamma_1 u = 0, \gamma_2 u \in H^k_{\lambda}(\Omega_T), \varphi \in H^k_{\lambda+1}(\omega_T)\},
\]

\[
\| (u, \varphi) \|_{k, \lambda, T} = \| (u, \varphi) \|_{W^k_{\lambda, T}}
\]

\[
= \{ \lambda \| u \|_{B^k_{\lambda+1/2}(\Omega_T)} + \| \varphi \|_{H^k_{\lambda}(\Omega_T)} \}^{1/2},
\]

\[
W'_{\lambda, T} = \{(f, g), f \in B^k_{\lambda-1/2}(\Omega_T), g \in H^k_{\lambda}(\omega_T)\},
\]

\[
\| (f, g) \|_{k, \lambda, T} = \| (f, g) \|_{W'_{\lambda, T}} = \left\{ \frac{1}{\lambda} \| f \|_{B^k_{\lambda-1/2}(\Omega_T)} + \| g \|_{H^k_{\lambda}(\omega_T)} \right\}^{1/2}.
\]

We have

**Theorem 9.1.** There exist \( \varepsilon_0 > 0 \) and \( \lambda_0(K), c_0(K) \), such that under the conditions

\[
k < N, \quad \varepsilon(L, F) \leq \varepsilon_0, \quad \| L, F \|_N \leq K, \quad (9.1)
\]

the problem (5.1)-(5.3) has a unique solution \((u, \varphi) \in W^k_{\lambda, T}\) for any \( \lambda > \lambda_0(K) \), \( T \leq T_0 \), \((f, g) \in W^k_{\lambda, T}\). Moreover,

\[
\| (u, \varphi) \|_{k, \lambda, T} \leq C_0(K) \| (f, g) \|_{k, \lambda, T}. \quad (9.2)
\]

**Proof.** Since \( \mathcal{C}^\infty(\Omega_T) \times \mathcal{C}^\infty(\omega_T) \) is dense in \( W^k_{\lambda, T} \) we only need establish existence, uniqueness, and energy estimates for \((f, g) \in \mathcal{C}^\infty(\Omega_T) \times \mathcal{C}^\infty(\omega_T)\).

As in Section 6, we introduce a polar coordinate system by (6.2), which implies

\[
\partial_t = \theta \partial_{x_1} + (1 - \theta) \partial_{x_2},
\]

\[
t \partial_t = t \theta (\partial_{x_1} - \partial_{x_2}) + t \partial_{x_2},
\]

\[
\partial_{\theta} = t (\partial_{x_1} - \partial_{x_2}).
\]

Hence

\[
\frac{A}{\partial_{x_1}} + \frac{B}{\partial_{x_2}} = A \left[ \frac{1}{t} (-\theta \partial_{\theta} + t \partial_t) + \frac{\partial_{\theta}}{t} \right] + B \left[ \frac{1}{t} (-\theta \partial_{\theta} + t \partial_t) \right]
\]

\[
= (A + B) \partial_t + \frac{1}{t} [(1 - \theta) A - \theta B] \partial_{\theta}.
\]
Denote $\hat{u} = J_\lambda u$, $\hat{\phi} = J_{\lambda + 1} \phi$ as in (6.3). Then (5.1)–(5.3) can be reduced to

\[
\begin{align*}
\hat{L}_\lambda \hat{u} &\overset{\text{def.}}{=} (\hat{A} + \hat{B}) \left( t \frac{\partial \hat{u}}{\partial t} + \lambda \hat{u} \right) + [(1 - \theta) \hat{A} - \theta \hat{B}] \frac{\partial \hat{u}}{\partial \theta} + t \hat{Q} \frac{\partial \hat{u}}{\partial y} = J_{\lambda - 1} f \\
\hat{L}_{\gamma_1} \hat{u} &= 0 \\
\hat{F}_\lambda (\gamma_2 \hat{u}, \hat{\phi}) &\overset{\text{def.}}{=} \left( t \frac{\partial \hat{\phi}}{\partial t} + (\lambda + 1) \hat{\phi} \right) p + t \left( q \frac{\partial \hat{\phi}}{\partial y} + h\hat{\phi} + m\varphi_2 \hat{u} \right) = J_{\lambda} g \\
\hat{\phi} |_{t = 0} &= 0,
\end{align*}
\]

(9.3)

where $\hat{A}(t, \theta, z) = A(t\theta, t(1 - \theta), z)$, and similarly for $\hat{B}$, $\hat{Q}$.

In view of $(f, g) \in C^\infty(\Omega_T) \times C^\infty(\omega_T)$, there is $\delta > 0$ such that $J_{\lambda - 1} f \equiv 0$, $J_{\lambda} g \equiv 0$ for $t < \delta$. Therefore, we may add an initial datum $\hat{u} |_{t < \delta} = 0$ on (9.3) to form an initial-boundary value problem. When $t \geq \delta$, the operator $t(\partial/\partial t)$ is nondegenerate. Hence the conclusion of the theorem is equivalent to the existence and uniqueness of a solution for (9.3), adding $\hat{u} |_{t < \delta} = 0$, and the energy estimate

\[
\begin{align*}
\lambda \| \hat{u} \|_{\hat{B}^2_2(\Omega_T)} + \| \gamma_2 \hat{u} \|_{\hat{H}^2_2(\omega_T)} + \| \hat{\phi} \|_{\hat{H}^2_{\lambda + 1}(\omega_T)} &\leq C_0(K) \left( \frac{1}{\lambda} \| \hat{L}_\lambda \hat{u} \|_{\hat{B}^2_{\lambda - 1}(\Omega_T)} + \| \hat{F}_\lambda (\gamma_2 \hat{u}, \hat{\phi}) \|_{\hat{H}^2_2(\omega_T)} \right).
\end{align*}
\]

(9.4)

Since the boundary $\theta = 0$ is regular characteristic and the boundary condition on $\theta = 1$ satisfies uniform Lopatinski conditions in the sense of Majda, the technique of elliptic regularization introduced in [4] is available and we only need to establish the estimate (9.4).

We prove (9.4) by induction. For $k = 0$ it reduces to

\[
\begin{align*}
\lambda \| \hat{u} \|_{\hat{L}^2_2(\Omega_T)} + \| \gamma_2 \hat{u} \|_{\hat{L}^2_2(\omega_T)} + \| \hat{\phi} \|_{\hat{H}^{2\lambda + 1}_{\lambda + 1}(\omega_T)} &\leq C_0(K) \left( \frac{1}{\lambda} \| \hat{L}_\lambda \hat{u} \|_{\hat{L}^2_2(\Omega_T)} + \| \hat{F}_\lambda (\gamma_2 \hat{u}, \hat{\phi}) \|_{\hat{L}^2_2(\omega_T)} \right).
\end{align*}
\]

(9.5)

Denoting $\hat{u}$, $\hat{\phi}$ by $v \cdot \psi$ and introducing $v_j$, $\psi_j$, $\tilde{v}_j$, $\tilde{\psi}_j$ by dyadic partition of unity and dilation, as we did in Section 6, we have

\[
\begin{align*}
\hat{L}_\lambda v_j &= \chi(2^j t) \hat{L}_\lambda v + w_j, \\
\hat{F}_\lambda (\gamma_2 v_j, \psi_j) &= \chi(2^j t) \hat{F}_\lambda (\gamma_2 v, \psi) + g_j,
\end{align*}
\]

(9.6)

where

\[
\begin{align*}
w_j &= (\hat{A} + \hat{B}) \cdot t \cdot \chi'(2^j t) 2^j v, \\
g_j &= t\chi'(2^j t) 2^j \psi p.
\end{align*}
\]
Obviously, \( \| w_j \|_{L^2(\Omega_T)}^2 \leq C \int_{t \in (2^{-j}t, 2^{-j+1}t)} \| v \|^2 dt \, d\theta \, dt \), \( \| g_j \|_{L^2(\Omega_T)}^2 \leq C \int_{t \in (2^{-j-1}t, 2^{-j}t)} \| \psi \|^2 dt \, d\theta \). Therefore, once we establish (9.5) for \((v_j, \psi_j)\), by Lemma 6.4 we obtain (9.5) for \((\hat{u}, \phi)\) immediately.

By means of (6.6), \( \bar{L}_\lambda v_j, \bar{F}_\lambda(\gamma_2 v_j, \psi_j) \) can be written as \( \bar{L}_\lambda w_j \) and \( \bar{F}_\lambda(\gamma_2 \bar{w}_j, \bar{\psi}_j) \) such that

\[
\bar{L}_\lambda w_j = (\bar{A}_j + \bar{B}_j) \left( t \frac{\partial w_j}{\partial t} + \lambda w_j \right) + \left( (1 - \theta) \bar{A}_j - \theta \bar{B}_j \right) \frac{\partial w_j}{\partial \theta} + t \bar{Q}_j \frac{\partial w_j}{\partial y},
\]

\[
\bar{F}_\lambda(\gamma_2 w_j, \chi) = \left( t \frac{\partial \chi}{\partial t} + (\lambda + 1) \chi \right) \bar{p}_j + t \left( \frac{\partial \chi}{\partial y} \bar{q}_j + \chi \bar{h}_j + \bar{m}_j \gamma_2 w_j \right),
\]

where \( \bar{A}_j(t, \theta, y) = \hat{A}(2^{-j}t, \theta, y), \bar{p}_j(t, y) = \hat{p}(2^{-j}t, y) \), etc. Correspondingly, (9.5) for \((v_j, \psi_j)\) is equivalent to

\[
\lambda \| w \|_{L^2(\Omega_T)}^2 + \| \gamma_2 w \|_{L^2(\Omega_T)}^2 + \| \chi \|_{H^1(\Omega_T)}^2 + \| \chi \|_{L^2(\Omega_T)}^2 \leq C_0(K) \left\{ \frac{1}{\lambda} \| \bar{L}_\lambda \omega \|_{L^2(\Omega_T)}^2 + \| \bar{F}_\lambda(\gamma_2 \omega, \chi) \|_{L^2(\Omega_T)}^2 \right\}. \tag{9.7}
\]

Let \( \eta(t) \) be a \( C^\infty \) function with \( \eta = 1 \) on \( 0 \leq \theta \leq \frac{1}{3} \) and \( \text{supp} \ \eta \subset [0, \frac{3}{10}] \). The estimate (9.7) for \( \eta w \) is given in Section 8, and the estimate (9.7) for \((1 - \eta) w \) and \( \chi \) is a result in [4]. Summing up and taking \( \lambda \) sufficiently large we obtain (9.7). Noticing that \( \text{supp} \ w \) and \( \text{supp} \ \chi \) are in \( t > T/2 \), we confirm that \( C_0(K) \) is independent of \( \delta \). Thus (9.4) for \( k = 0 \) is proved.

Suppose (9.4) is valid for the index less than \( k \). We give two problems for \( \eta \hat{u} \) and \((1 - \eta) \hat{u} \) as follows,

\[
\hat{L}_\lambda \eta \hat{u} = \eta \hat{L}_\lambda \hat{u} + E \hat{u},
\]

\[
l_1 \gamma_1 \eta \hat{u} = 0, \tag{9.8}
\]

\[
\eta \hat{u} = 0, \quad \theta \geq \frac{1}{3} \quad \text{or} \quad t < \delta
\]

\[
\hat{L}_\lambda (1 - \eta) \hat{u} = (1 - \eta) \hat{L}_\lambda \hat{u} - E \hat{u},
\]

\[
\bar{F}_\lambda(\gamma_2 (1 - \eta) \hat{u}, \phi) = \bar{F}_\lambda(\gamma_2 \hat{u}, \phi), \tag{9.9}
\]

\[
(1 - \eta) \hat{u} = 0, \quad \theta \leq \frac{1}{3} \quad \text{or} \quad t < \delta,
\]

where \( E = ((1 - \theta) \hat{A} - \theta \hat{B}) \eta \). Obviously,

\[
\| E \hat{u} \|_{L^2(\Omega_T)} \leq C \| \hat{u} \|_{L^2(\Omega_T)},
\]

\[
\| E \hat{u} \|_{H^2(\Omega_T)} \leq C \| \hat{u} \|_{H^2(\Omega_T)}.
\]
where $C$ is independent of $\delta$. Hence (9.4) can be derived by the assumption of induction and the following two equalities:

$$
\lambda^2 \| \eta \hat{u} \|_{B^2_{\omega}(\theta)} \leq C_0(K)(\| \eta \hat{p}_x \|_{B^2_{\omega}(\theta)} + \| \hat{u} \|_{B^2_{\omega}(\theta)}).
$$

(9.10)

Here we have replaced $\| (1 - \eta) \hat{u} \|_{H^1_{\omega}(\theta)}$ by $\| (1 - \eta) \hat{u} \|_{H^1_{\omega}(\theta)}$ because $(1 - \eta) \hat{u} \equiv 0$ near $\theta = 0$. By dyadic partition of unity and dilation as indicated in the case $k = 0$, (9.10), (9.11) can be reduced to the estimates for the functions with support on $t \geq T/2$. Again using the result in [4] and Section 8 we establish the required estimates.  

10. ASYMPTOTIC SOLUTIONS OF THE NONLINEAR PROBLEM

In order to prove the existence of the solution for the nonlinear problem (4.7)-(4.9) we proceed in two steps. First, we construct asymptotic solutions, which let the left side of (4.7)-(4.9) be small quantities of higher order. Then, taking the asymptotic solution as the first term, we obtain the solution by iteration.

As we indicated in Section 4, the quantities $U, \psi, \nabla \psi$ on the wedge $x_1 = x_2 = 0$ are given. This means that we can find $V(y)$ and $\tau(y)$ such that

$$
IV(y) = 0,
$$

$$
\mathcal{F}(0, y, V(y), 0, \tau(y)) = 0.
$$

(10.1)

In the sequel we always assume $N \geq 10$ and use the abbreviation

$$
\tilde{\psi}(x_1, x_2, y) = \psi(x_1 + x_2, y), \quad X = (x_1, x_2, y)
$$

$$
H\psi(X) = \left( \tilde{\psi}(X), \left( \frac{\tilde{\psi}}{t} \right)(X), \nabla \tilde{\psi}(X), x_1 \left( \frac{\nabla \psi}{t} \right)(X), x_2 \left( \frac{\nabla \psi}{t} \right)(X) \right)
$$

$$
\mathcal{F}(\gamma_2 U, \psi) = \mathcal{F}(t, y, \gamma_2 U, \psi, \nabla \psi).
$$

THEOREM 10.1. For any $\varepsilon > 0$, there exist $M > 0$ and a $C^\infty$ sequence $\{U_j, \psi_j\}$ such that


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\[
f_j \overset{\text{def.}}{=} L(U_j, \psi_j) U_j = O((x_1 + x_2)^t) \quad (10.2)
\]
\[
g_j \overset{\text{def.}}{=} \mathcal{F}(\gamma_2 U_j, \psi_j) = O(t^{j+1}), \quad (10.3)
\]
\[
h_j U_j = 0, \quad (10.4)
\]
\[
\varepsilon_{T_0}(u_j, \psi_j) \overset{\text{def.}}{=} \|U_j - V(0)\|_{L^\infty} + \|\psi_j - \tau(0) t\|_{L^\infty} + \|\nabla(\psi_j - \tau(0) t)\|_{L^\infty}
\]
\[
+ \|H(\psi_j - \tau(0) t)\|_{L^\infty} < \epsilon, \quad (10.5)
\]
\[
\|(U_j, \psi_j)\|_{W_{T_0}} \overset{\text{def.}}{=} \|U_j\|_{B^\infty(\Omega_{T_0})} + \|\gamma_2 U_j\|_{H^\infty(\Omega_{T_0})} + \|\psi_j - \tau(0) t\|_{H^{n+1}(\Omega_{T_0})}
\]
\[
+ \|H(\psi_j - \tau(0) t)\|_{B^\infty(\Omega_{T_0})} \leq M. \quad (10.6)
\]

**Proof.** As the first term of the asymptotic sequence we take \(U_0 = V(y), \quad \psi_0 = \tau(y) t\). We assume that the periodic interval \(\mathcal{D}\) of the variable \(y\) has been chosen sufficiently small that (10.2)--(10.6) are valid.

Now we define successively \(U_j, \psi_j\) by

\[
I_0 V_j = -L(U_j, \psi_j) U_j
\]
\[
h_j V_j = 0
\]
\[
F_0(\gamma_2 V_j, \chi_j) = \mathcal{F}(\gamma_2 U_j, \psi_j), \quad \chi_j|_{t=0} = 0
\]
\[
U_{j+1} = U_j + V_j, \quad \psi_{j+1} = \chi_j + \psi_j,
\]

where

\[
L_0 V_j = A(U_0, \psi_0) \frac{\partial V_j}{\partial x_1} + B(U_0, \psi_0) \frac{\partial V_j}{\partial x_2},
\]
\[
F_0(\gamma_2 V_j, \chi_j) = F(\gamma_2 U_0, \psi_0)(\gamma_2 V_j, \chi_j)
\]
\[
= p(\gamma_2 U_0, \psi_0) \frac{\partial \chi_j}{\partial t} + h(\gamma_2 U_0, \psi_0) \chi_j + m(\gamma_2 U_0, \psi_0) \gamma_2 V_j.
\]

Regarding \(y\) as a parameter, to determine \((V_n, \chi_n)\) from \((U_n, \psi_n)\) we just need to solve a linear Goursat problem with two variables. Hence Theorems 8.1 and 8.2 are available.

It is convenient to add two conditions

\[
U_j - U_0 = O(x_1 + x_2), \quad \psi_j - \psi_0 = O(t^2) \quad (10.9)
\]
to (10.2)--(10.6). Obviously, (10.2)--(10.6) and (10.9) are valid for \(j = 0\). Now suppose they are valid for \(j \leq n\); we prove they are also valid for \(j = n + 1\).
By using Theorems 8.1 and 8.2 from (10.2), (10.3) for $j = n$, we have

$$V_n = O((x_1 + x_2)^{n+1}), \quad \frac{d\chi_n}{dt} = O(t^{n+1}). \quad (10.10)$$

Since $\chi_n|_{t=0} = 0$, then $\chi_n = O(t^{n+2})$, which implies $H\chi_n = O((x_1 + x_2)^{n+1})$.

Moreover,

$$f_{n+1} - (L(U_{n+1}, \psi_{n+1}) - L(U_n, \psi_n)) U_{n+1} + (L(U_n, \psi_n) - L_0) V_n, \quad (10.11)$$

and $L(U_{n+1}, \psi_{n+1}) - L(U_n, \psi_n)$ can be written as a first-order partial differential operator with coefficients containing $U_{n+1} - U_n$ or $H(\psi_{n+1} - \psi_n)$. Therefore,

$$(L(U_{n+1}, \psi_{n+1}) - L(U_n, \psi_n)) U_{n+1} = O((x_1 + x_2)^{n+1}). \quad (10.12)$$

Similarly, $I(U_n, \psi_n) - I_0$ can be written as a sum of a first-order operator with respect to $y$ and a first-order operator with respect to $x_1, x_2$ with coefficients containing the factor $U_n - U_0$ or $H(\psi_n - \psi_0)$. Therefore, by $\nabla_{x_1, x_2} V_n = O((x_1 + x_2)^n)$, $\partial V_n/\partial y = O((x_1 + x_2)^{n+1})$, and (10.12) we have $f_{n+1} = O((x_1 + x_2)^{n+1})$.

As for $g_{n+1}$ we write it as

$$g_{n+1} = F(\gamma_2 U_n + \gamma_2 V_n, \psi_n + \chi_n)$$

$$= F(\gamma_2 U_n, \psi_n) + F_{(\gamma_2 U_n, \psi_n)}(\gamma_2 V_n, \chi_n) + O(|\gamma_2 V_n|^2 + |\chi_n|^2)$$

$$= (F(\gamma_2 U_n, \psi_n) - F(\gamma_2 U_n, \chi_n)) + O(|\gamma_2 V_n|^2 + |\chi_n|^2).$$

From (10.9) we know $\gamma_2 U_n - \gamma_2 U_0 = O(t)$, $\psi_n - \psi_0 = O(T^2)$. Then by $\gamma_2 V_n = O(t^{n+1})$, $\chi_n = O(t^{n+2})$, and $\partial \chi_n/\partial y = O(t^{n+2})$ we know $g_{n+1} = O(t^{n+2})$.

Equality (10.4) is satisfied due to (10.7).

To prove (10.5) and (10.6), we modify the sequence $\{(U_j, \psi_j)\}$. Let $\zeta(t) \in C_0^\infty$ be a cut-off function, equal to 1 near the origin. $\delta_J$ is a constant, which will be determined later. For $j \leq J$, set

$$U_j^* = \zeta\left(\frac{x_1 + x_2}{\delta_J}\right)(U_j - U_0) + U_0, \quad \psi_j^* = \zeta\left(\frac{t}{\delta_J}\right)(\psi_j - \psi_0) + \psi_0. \quad (10.13)$$

Obviously, replacing $(U_j, \psi_j)$ by $(U_j^*, \psi_j^*)$ does not affect the validity of (10.2)–(10.4). In light of (10.9) and the boundedness of $t\chi'(t/\delta_J)$, we may choose $\delta_J$ sufficiently small such that finite pairs $(U_j, \psi_j)$ for $j \leq J$ satisfy (10.5). Besides, let $M = 1 + \max_{j \leq J} \|(U_j, \psi_j)\|$. Equality (10.6) is certainly true.
For \( j > J \), set

\[
U^*_j = U^*_j + \zeta \left( \frac{x+y}{\delta_j} \right) (U_j - U_j),
\]

\[
\psi^*_j = \psi^*_j + \zeta \left( \frac{t}{\delta_j} \right) (\psi_j - \psi_j).
\]

(10.14)

where \( U^*_j, \psi^*_j \) are defined in (10.13). As mentioned above the equalities

(10.2)-(10.4) are valid, and taking \( \delta_j \) sufficiently small we have

\[
\| U^*_j - U^*_j, \psi^*_j - \psi^*_j \|_{W^*_0} < \varepsilon, \quad \| U^*_j - U^*_j, \psi^*_j - \psi^*_j \|_{W^*_T} < 1
\]

by virtue of

\[
U_j - U_j = O((x_1 + x_2)^{t/2+1}), \quad \psi_j - \psi_j = O(t^{t/2}).
\]

Hence (10.5), (10.6) are valid for all \((U^*_j, \psi^*_j)\).

Remark. Let \( \varepsilon_0 \) and \( \varepsilon_1 \) be the constants in Theorem 9.1 and Theorem 8.2, and let \( \varepsilon_2 \) be another small number such that \( \varepsilon_0(U, \psi) < \varepsilon_2 \) ensures

\( \varepsilon(L, F) \leq \varepsilon_0 \) in Theorem 9.1 and \( \varepsilon(p) \leq \varepsilon_1 \) in Theorem 8.2. Take \( \varepsilon = \varepsilon_2/2 \), and determine \( M \) according to the proof of Theorem 10.2. Thus from

\[
\| (U, \psi) \|_{W^*_0} \leq 2M
\]

we have \( \| L(U, \psi), F(U, \psi) \|_N \leq K \). Therefore \( \lambda_0(K) \) can be determined by Theorem 9.1. Fix \( j_0 > \lambda_0(K) \). We will choose \((U^*_j, \psi^*_j)\) by

Theorem 10.1 as the first term in an iteration for solving a nonlinear Goursat problem in the next section.

11. EXISTENCE

In this section we use Newton's iteration scheme to construct a convergent sequence of approximate solutions (refer to [5, 6]). First we take

\((U_{j_0}, \psi_{j_0})\) obtained in the end of Section 10 as \((U_0, \psi_0)\), \((W_0, \theta_0) = (0, 0)\).

To determine the general terms we use induction. If \((U_n, \psi_n) \in W^N_{\lambda, T}(\Omega_{T_0})\) has been determined, we take \((W_{n+1}, \theta_{n+1})\) as the solution of

\[
L(\bar{V}_n, \bar{\psi}_n) W_{n+1} = -L(\bar{V}_n, \bar{\psi}_n) \theta_{n+1}, \quad h \int W_{n+1} = 0
\]

\[
F_{\gamma_2 \bar{V}_n, \psi_n}(\gamma_2 W_{n+1}, \theta_{n+1}) = -\mathcal{F}(\gamma_2 \bar{V}_n, \bar{\psi}_n) + F_{\gamma_2 \bar{V}_n, \psi_n}(\gamma_2 V_n, \chi_n),
\]

(11.1)

\[
\theta_{n+1} \big|_t = 0 = 0
\]

on \((\Omega_T, \omega_T)\), and extend \((W_{n+1}, \theta_{n+1})\) to \((\Omega_{T_0}, \omega_{T_0})\) by (6.1). Let

\[
\bar{V}_{n+1} = E_T W_{n+1}, \quad \bar{\psi}_{n+1} = E_T \psi_{n+1},
\]

(11.2)

and denote the map \((\bar{U}_n, \bar{\psi}_n) \rightarrow (\bar{U}_{n+1}, \bar{\psi}_{n+1})\) by \( \pi \). Next we only need to indicate that \( \pi \) is well defined and contractive.
THEOREM 11.1. There is $T_1 \in (0, T_0)$, such that for $T < T_1$, by means of
(11.1), (11.2), and the given first term $(U_0, \psi_0)$ we can define sequences
\{$(\bar{U}_n, \bar{\psi}_n)$\}, \{$(\bar{W}_n, \theta_n)$\}, which satisfy
\[
\varepsilon_{T_0}(\bar{W}_n, \theta_n) \leq \varepsilon_2/2, \quad \| (\bar{W}_n, \theta_n) \|_{N, \lambda, T_0} \leq 1,
\]  
(11.3)
where $\varepsilon_2$ is a number given at the end of Section 10.

Proof. Equation (11.3) is valid for $n = 0$. Now if it is valid for index $n$, we
prove the validity for index $n + 1$.

Write $-L(\bar{U}_n, \bar{\psi}_n) U_0$ as $-L(\bar{U}_0, \bar{\psi}_0) U_0 + (L(\bar{U}_0, \bar{\psi}_0) - L(\bar{U}_n, \bar{\psi}_n)) U_0$.

Lemma 7.3 and (11.3) imply
\[
\| A(\bar{U}_n, \bar{\psi}_n) - A(\bar{U}_0, \bar{\psi}_0) \|_{B^{N}_{1/2}(\omega_T)} \leq C \{ \| W_n \|_{B^{N}_{1/2}(\omega_T)} + \| H\lambda_n \|_{B^{N}_{1/2}(\omega_T)} \}.
\]
Noticing
\[
\| H\lambda_n \|_{B^{N}_{1/2}(\omega_T)} \leq C \| \lambda_n \|_{H^{N+1}(\omega_T)},
\]  
(11.4)
we have
\[
\| -L(\bar{U}_n, \bar{\psi}_n) \bar{U}_0 \|_{B^{N}_{1/2}(\omega_T)} \leq C \{ \| L(\bar{U}_0, \bar{\psi}_0) \bar{U}_0 \|_{B^{N}_{1/2}(\omega_T)} + \| W_n, \theta_n \|_{N, \lambda, T_0} \}.
\]  
(11.5)
Since $L(\bar{U}_0, \bar{\psi}_0) \bar{U}_0 = O((x_1 + x_2)^{j_0})$ and $j_0 > \lambda_0(K)$ according to the remark in Section 10, we derive from (10.3), (10.5)
\[
\| -L(\bar{U}_n, \bar{\psi}_n) U_0 \|_{B^{N}_{1/2}(\omega_T)} \leq C, \quad T.
\]  
(11.6)

On the other hand, from (11.1)
\[
F_{\gamma_2} \epsilon_n, \phi_n(\gamma_2 W_n, \theta_n) = F(\gamma_2 \bar{U}_0, \bar{\psi}_0) - F(\gamma_2 \bar{U}_n, \bar{\psi}_n) - F(\gamma_2 \bar{U}_n, \bar{\psi}_0) - F(\gamma_2 \epsilon_n, \phi_n(\gamma_2 W_n, \theta_n)),
\]
and Lemma 7.4 implies
\[
\| F(\gamma_2 \bar{U}_n, \bar{\psi}_n) - F(\gamma_2 \bar{U}_0, \bar{\psi}_0) - F(\gamma_2 \epsilon_n, \phi_n(\gamma_2 W_n, \theta_n)) \|_{H^{N}_{\lambda}(\omega_T)}
\leq C \{ \| \gamma_2 W_n \|_{H^{N}_{\lambda}(\omega_T)} + \| \theta_n \|_{H^{N+1}_{\lambda}(\omega_T)} \}.
\]  
(11.7)
Therefore, in the case $2\lambda - 1 > \lambda + 1$, (11.7) and (10.3) yield
\[
\| F_{\gamma_2} \epsilon_n, \phi_n(\gamma_2 W_n, \theta_n) \|_{H^{N}_{\lambda}(\omega_T)} \leq C \varepsilon_2 T.
\]  
(11.8)

Now let us check if the coefficients in (11.1) satisfy the conditions in
Theorem 9.1. From (10.5) and (11.3) we know $\varepsilon_{T_0}(\bar{U}_n, \bar{\psi}_n) < \varepsilon_2$, which
implies \( \varepsilon(L, F) \leq \varepsilon_0 \) by the remark in Section 10. Moreover, from the expression of the norms of \( W_{\lambda,T} \) and \( W^N_{\lambda,T} \), we have

\[
\| (\tilde{W}_n, \tilde{\nu}_n) \|_{W_{\lambda,T}} \leq \mu_2 \| (\tilde{W}_n, \tilde{\nu}_n) \|_{N, \lambda, T_0}, \quad \text{for } \lambda > N,
\]

where \( \mu_2 \) is a given constant, independent of \( \tilde{W}_n \) and \( \tilde{\nu}_n \). Certainly, we may choose \( M > \mu_2 \). Hence \( \| (\tilde{W}_n, \tilde{\nu}_n) \|_{W_{\lambda,T}} < M \), and in view of (10.6) we have \( \| (U_n, \psi_n) \|_{W_{\lambda,T}} < 2M \). Again using the remark in Section 10 we obtain

\[
\| L(U_n, \psi_n), \tilde{F}_{(y_2, U_n, \psi_n)} \| \leq K.
\]

Since the assumptions in Theorem 9.1 are verified, by this theorem and (11.6), (11.8) we obtain \( (W_{n+1, \theta_{n+1}}) \in W^N_{\lambda,T} \), and

\[
\| (W_{n+1, \theta_{n+1}}) \|_{N, \lambda, T} \leq C_3 T,
\]

which implies

\[
\| (\tilde{W}_{n+1, \tilde{\nu}_{n+1}}) \|_{N, \lambda, T_0} \leq C_3 KT.
\]

When \( N \geq 10 \), we can use the embedding theorem to obtain

\[
\varepsilon_{T_0} (\tilde{W}_{n+1, \tilde{\nu}_{n+1}}) \leq C_5 C_4 KT.
\]

Taking \( T_1 = \min(1/C_3 K, \varepsilon/2C_3 C_4 K) \), (11.3) is valid for \( (\tilde{W}_{n+1, \tilde{\nu}_{n+1}}) \). Thus the sequences \( \{ (\tilde{U}_n, \tilde{\psi}_n) \} \), \( \{ (\tilde{W}_n, \tilde{\nu}_n) \} \) can be determined successively.

**THEOREM 11.2.** There exists a constant \( C_0 \) such that for \( T < T_1 \), the sequence \( (\tilde{U}_n, \tilde{\psi}_n) \) defined in Theorem 11.1 satisfies

\[
\| (\tilde{U}_{n+2} \tilde{U}_{n+1}, \tilde{\psi}_{n+2} - \tilde{\psi}_{n+1}) \|_{N-1, \lambda, T}
\leq C_0 T \| (\tilde{U}_{n+1} - \tilde{U}_n, \tilde{\psi}_{n+1} - \tilde{\psi}_n) \|_{N-1, \lambda, T},
\]

and for small \( T \) the map \( \pi \) is contractive in the space \( W^N_{\lambda,T} \).

**Proof.** Subtracting (11.1) with index \( n \) from the equality with index \( n+1 \), we have

\[
L(\tilde{U}_{n+1}, \tilde{\psi}_{n+1})(\tilde{U}_{n+1} - \tilde{U}_{n+1}) = (L(\tilde{U}_n, \tilde{\psi}_n) - L(\tilde{U}_{n+1}, \tilde{\psi}_{n+1})) \tilde{U}_{n+1},
\]

\[
F_{\gamma_2 \tilde{U}_{n+1}, \tilde{\psi}_{n+1}}(\gamma_2 (\tilde{U}_{n+1} - \tilde{U}_{n+1}), \tilde{\psi}_{n+1} - \tilde{\psi}_{n+1})
\]

\[
= -\mathcal{F}(\gamma_2 \tilde{U}_{n+1}, \tilde{\psi}_{n+1}) + \mathcal{F}(\gamma_2 \tilde{U}_n, \tilde{\psi}_n)
\]

\[
+ F_{\gamma_2 \tilde{U}_n, \tilde{\psi}_n}(\gamma_2 (\tilde{U}_{n+1} - \tilde{U}_n), \tilde{\psi}_{n+1} - \tilde{\psi}_n),
\]

\[
\tilde{\psi}_{n+2} - \tilde{\psi}_{n+1}, \tilde{U}_n = 0,
\]

\[
h_{\gamma_1}(\tilde{U}_{n+2} - \tilde{U}_{n+1}) = 0.
\]
Denoting the right sides of the first two equalities by $b_n$, $\beta_n$, respectively, and noting $N-1 \geq 8$, we may use Lemma 7.3 and 7.4 to obtain
\begin{equation}
\| b_n \|_{L^{N-1}((\gamma, \tau))} \leq C_1 T \| (\bar{U}_{n+1} - \bar{U}_n, \bar{Y}_{n+1} - \bar{Y}_n) \|_{N-1, \lambda-1, \tau},
\end{equation}
\begin{equation}
\| B_n \|_{H^{N-1}((\gamma, \tau))} \leq C_2 \| (\bar{U}_{n+1} - \bar{U}_n, \bar{Y}_{n+1} - \bar{Y}_n) \|_{N-1, \lambda-1, \tau}.
\end{equation}

Taking $2\lambda - 1 > \lambda + 1$, in light of the uniform boundedness of $\| (\bar{U}_n, \bar{Y}_n) \|_{N-1, \lambda, T_0}$, we have
\begin{equation}
\| \beta_n \|_{H^{N-1}((\gamma, \tau))} \leq C_2 T \| (\bar{U}_{n+1} - \bar{U}_n, \bar{Y}_{n+1} - \bar{Y}_n) \|_{N-1, \lambda, \tau}.
\end{equation}

Since the conclusion of Theorem 9.1 shows
\begin{equation}
(11.11)
\end{equation}
is obtained by fixing $\lambda$. 

Now we can use the limit process to obtain the solution of problem (4.7)-(4.9). In fact, set $T = \min(T_1, 1/2C_0)$. Since the map $\pi$ is a contraction we can take the limit (11.1). The limit $(\bar{U}, \bar{Y})$ of $(\bar{U}_n, \bar{Y}_n)$ satisfies (4.7)-(4.9). Using the Banach–Saks theorem $(\bar{U}, \bar{Y}) \in W^{N}_\lambda(\Omega_T)$. Hence $\bar{Y} \in C^\lambda(\bar{\Omega}_T)$, $\bar{U} \in C^3(\Omega_T) \cap C^4(\Omega_T \setminus \{x_1 = 0\})$.

Going back to the original physical space $(t, x, y)$, we obtain the local existence of smooth functions $U_n(t, x, y)$ and $\psi(t, y)$. Hence the solution (2.5) for the problem with a reflected shock front is obtained.

**Remark.** Our discussion is also valid in the case when the coming flow is cylindrical or spherical, because in this case we still have an explicit expression for the shock front and the parameters of the flow field on both sides of the front before the reflection occurs.

**REFERENCES**