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On the Cauchy Problem for a Class of Hyperbolic Systems of Conservation Laws

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1. INTRODUCTION

For any $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ and $n \geqslant 2$, consider the system of conservation laws

$$u_t + (\varphi(|u|)u)_x = 0, \tag{1.1}$$

to be solved, weakly, by vector-valued functions $u\colon \mathbb{R}^2_+ := \mathbb{R} \times [0, \infty) \to \mathbb{R}^n$ of space $x \in \mathbb{R}$ and time $t \geqslant 0$. The goal of this note is to establish the following properties of solutions.

THEOREM 1. (a) (*Existence*) *System* (1.1) *has a solution for any initial data* $u^0 \in \mathscr{L}^\infty(\mathbb{R}, \mathbb{R}^n)$. (b) (*Uniqueness*) *For any such* u^0, *there is precisely one solution u with the property that* $r := |u|$ *satisfies the additional conservation law* $r_t + (\varphi(r)r)_x = 0$ *and Kruskov's* ([8]) *entropy criterion.* (c) (*Stability*) *The solution* $S(u^0) := u$ *defined by* (b) *depends* $\mathscr{L}^1_{\text{loc}}$-*continuously on the data* u^0. (d) (*Generic regularity*) *For generic smooth* φ *and generic smooth bounded data* u^0, $S(u^0)$ *is piecewise smooth, with locally finitely many shocks that satisfy the Oleinik–Liu* ([10, 14]) *condition* (*E*).

THEOREM 2. (a) (*Transport of angular distribution, in general*) *Consider the solution* $u = S(u^0)$ *corresponding to data* $u^0 \in \mathscr{L}^\infty(\mathbb{R}, \mathbb{R}^n)$. *In* $\mathbb{R}^2_+$ *identify each two points* (x_1, t), (x_2, t) *with* $x_1 < x_2$ *and* $((x_1, x_2) \times \{t\}) \cap \operatorname{spt} u = \emptyset$, *and call the resulting set* Σ (*i.e., essentially,* $\Sigma = \operatorname{spt} u$). *Then,* $\theta = u/|u| \in \mathscr{L}(\Sigma, S^{n-1})$ *and* $\theta^t = \theta(\cdot, t) \in \mathscr{L}(\Sigma^t, S^{n-1})$, $t \geqslant 0$, *are well-defined, with* $\Sigma^t = \{x \mid (x, t) \in \Sigma\}$. *There is a family of continuous transformations* $T^{t,0}\colon \Sigma^t \to \Sigma^0$ *such that*

$$\theta^t = \theta^0 \circ T^{t,0} \qquad \textit{for all} \quad t \geqslant 0. \tag{1.2}$$

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(b) (*Special case: Soliton-like behavior of rotational waves*) *Assume additionally that $\partial^2(\varphi(r)r)/\partial r^2$ exists and is $\neq 0$ for all $r > 0$. Consider data u^0 which are bounded and bounded away from 0 and have asymptotically constant radial part, i.e., $u^0 = r^0\theta^0$ with $r^0 - a \in \mathscr{L}^1(\mathbb{R}, [b_1 - a, b_2 - a])$ for some constants $b_2 \geq a \geq b_1 > 0$. If now $\varphi'(a) \neq 0$ and $r^0 \equiv a$ on some finite interval I, then the behavior of the solution on a corresponding "strip" is characterized by*

$$\lim_{t \to \infty} \sup_{x;\, T^{t,0}(x) \in I} |u(x, t) - u^0(x + \delta - (\varphi(a) + o(t^{-1}))t)| = 0 \tag{1.3}$$

(where the phase shift δ can be computed from r^0). Especially if, moreover, $r^0 - a$ has compact support, then there exists a $t_0 \geq 0$ such that

$$(t \geq t_0 \wedge x + \delta - \varphi(a)t \in I) \Rightarrow u(x, t) = u^0(x + \delta - \varphi(a)t). \tag{1.4}$$

Systems of the form (1.1) were first studied in [7, 12]. They arise in various situations in continuum mechanics. Especially, setting $\varphi(|u|) = |u|^2$ makes (1.1) a model of any generically rotationally degenerate hyperbolic system of conservation laws ([2]), yielding a qualitatively and quantitatively good approximation in a regime corresponding to small values of $|u|$, see [1, 3]. Note that, for generic systems (1.1), the vanishing viscosity method is not an appropriate way to establish a solution theory which would satisfy Hadamard's classical requirements of existence, uniqueness, and continuous dependence (see [4]): for "appropriate" data u^0, upon adding dissipative terms to (1.1), there typically are solutions which are far from the solution $S(u^0)$ constructed here. In part (b) of Theorem 2, the word "soliton-like" is used only to indicate that the rotational wave considered there for the purpose of illustration finally emerges as a traveling wave of unchanged shape, no matter which other (radial) waves may have crossed its way in between; for more realistic "Alfvén solitons" see rather [13], which is based on a system that can be viewed as an extension of (1.1) with $\varphi(|u|) = |u|^2$, $n = 2$, by dispersive terms. The intention in proving Theorems 1 and 2 is to give and to illustrate (respectively) a mathematically satisfying self-consistent solution theory for hyperbolic systems (1.1). Systems (1.1) are typically non-strictly hyperbolic so that also the classical theory of Lax [9], Glimm [5], and Liu [11] does not apply immediately. For a study of the Cauchy problem for a related but different class of non-strictly hyperbolic systems, see [16] and [17]. The results of the very recent paper [17] ([17] and the present paper have been written independently) are partly similar in spirit to the results presented here; the different methods used in either paper do not readily cover the situations treated in the other paper.

Part (a) of Theorem 1 is only slightly more general than an existence theorem that Liu and Wang gave in [12], using Glimm's scheme. Liu and

Wang also employed the natural polar coordinates (r, θ) for u, and, in particular, the extra conservation law $r_t + (\varphi(r)r)_x = 0$. This equation clearly holds, *per constructionem*, for their solution, which is thus identical with ours (in the case which they have considered). Note that it is not valid for general weak solutions of (1.1), as, e.g., certain solutions with antiparallel Riemann data show. The key idea of the proofs below of Theorems 1 and 2 is to treat an extended system, consisting of (1.1) plus the extra conservation law for r, with Wagner's transformation theory [18]. This latter means introducing new independent ("Lagrangian") variables. Note that different transformations of this type are possible (cf. [18]), depending on which conserved quantity is interpreted as a "particle" density; the crucial point in the transformation used here is that we take the quantity r for this purpose, which is not even a conserved quantity for arbitrary weak solutions of the original system (1.1).

Parts (a) and (b) of Theorem 1 are proved in the following Section 2, part (c) and Theorem 2 are proved in Section 3, and part (d) of Theorem 1—by then an easy corollary of the work of Schaeffer [15] and Guckenheimer [6]—is shown in the short final Section 4.

2. Existence and Uniqueness

This whole paper is based on the following corollary of Wagner's work:

LEMMA 1. *For any $m \in \mathbb{N}$, there is a one-to-one correspondence between (equivalence classes of) bounded Lebesgue measurable solutions $(r, r\theta)$: $\mathbb{R}^2_+ \to [0, \infty) \times \mathbb{R}^m$ to the system*

$$r_t + (\varphi(r)r)_x = 0 \tag{2.1}$$

$$(r\theta)_t + (\varphi(r)\, r\theta)_x = 0 \tag{2.2}$$

which satisfy

$$\int_{-\infty}^{0} r(x, t)\, dx = \int_{0}^{\infty} r(x, t)\, dx = \infty \tag{2.3}$$

and (equivalence classes of) weak solutions $(\tau, \tilde{\theta})$ to the system

$$\tau_t - (\varphi(1/\tilde{\tau}))_y = 0, \tag{2.4}$$

$$\tilde{\theta}_t = 0, \tag{2.5}$$

in which τ is a Radon measure on $\mathbb{R}^2_+$ which dominates Lebesgue (outer) measure λ_2 (i.e., $\tau \geq k\lambda_2$ for some $k > 0$), $\tilde{\tau}$ is the density of the absolutely continuous part of τ with respect to λ_2 (so that $1/k \geq 1/\tilde{\tau} \in \mathscr{L}^\infty$), and

$\tilde{\theta}\colon \mathbb{R}^2_+ \to \mathbb{R}^m$ is bounded and Lebesgue measurable. This correspondence is established through transformations $T\colon (x, t) \mapsto (y(x, t), t)$ defined relative to any bounded measurable solution of (2.1) *by*

$$\frac{\partial y}{\partial x}(x, t) = r(x, t), \qquad \frac{\partial y}{\partial t}(x, t) = -\varphi(r(x, t))\, r(x, t), \qquad y(0, 0) = 0, \tag{2.6}$$

namely setting

$$\tau = \lambda_2 \circ T^{-1}, \tag{2.7}$$

$$\tilde{\theta} = \theta \circ T^{-1}. \tag{2.8}$$

Proof of Lemma 1. Identical with Wagner's proof of Theorems 1 and 2 in [18]! x and y play the roles of "Eulerian" and "Lagrangian" coordinates (i.e., spatial coordinate and particle marker, respectively), r and τ those of mass density (referring to x) and specific volume (referring to y) of a fictitious gas. In this analogy, $\varphi \circ r$ corresponds to speed as a function of x and t, and, as Wagner's remarks after the proof of Lemma 2 in [18] immediately imply, the corresponding Lagrangian density is given by $\varphi(1/\tilde{\tau})$.

Let now any data $u^0 \in \mathscr{L}^\infty(\mathbb{R}, \mathbb{R}^n)$ for (1.1) be given. Consider $r^0 \in \mathscr{L}^\infty(\mathbb{R}, [0, \infty))$, $\theta^0 \in \mathscr{L}(\mathbb{R}, S^{n-1})$ with the property

$$u^0 = r^0 \theta^0. \tag{2.9}$$

Let r be the unique Kruskov ([8]) type solution of the scalar conservation law (2.1) with data r^0. In case condition (2.3) should not already hold, it is clear that for arbitrarily given compact subsets K^0 and K of $\mathbb{R}$ and $\mathbb{R}^2_+$, respectively, one can change r^0 outside K^0 in such a way that (2.3) becomes satisfied and at the same time r remains unaffected inside K. We will thus without loss of generality assume from now on that (2.3) holds. Applying now Lemma 1 with $m = 0$, i.e., to (2.1), (2.4) without (2.2), (2.5), we see that (2.7) defines a solution τ of (2.1) which is a Radon measure dominating λ_2. Define $\tilde{\theta}^0 \in \mathscr{L}(\mathbb{R}, S^{n-1})$ through

$$\tilde{\theta}^0(y(x, 0)) = \theta^0(x, 0) \qquad \text{for all} \quad x \in \mathbb{R}, \tag{2.10}$$

$\tilde{\theta} \in \mathscr{L}(\mathbb{R}^2_+, S^{n-1})$ through

$$\tilde{\theta}(y, t) = \tilde{\theta}^0(y) \qquad \text{for all} \quad t \geqslant 0,\ y \in \mathbb{R}, \tag{2.11}$$

and find $\theta \in \mathscr{L}(\mathbb{R}^2_+, S^{n-1})$ such that (2.8) holds; note that (2.8) determines θ on spt r. Since $\tilde{\theta}$ trivially solves (2.5), Lemma 1 implies that the pair (r, θ) solves system (2.1), (2.2), with now $m = n$. Obviously, $u = r\theta$ solves (1.1), which is identical with (2.2). We have just proved assertion (a) of Theorem 1, as well as part of assertion (b).

To complete the proof of (b), it only remains to show that the solution u we have constructed is the only one satisfying the additional conservation law (2.1). This, however, becomes immediately clear from reversing the above argumentation; it is obvious that the only apparent ambiguity, i.e., the fact that θ^0 and θ are arbitrary outside spt r^0 and spt r, respectively, does not affect the desired uniqueness of u.

3. Stability and Transport Property

Here we prove Theorem 2 and part (c) of Theorem 1. We start with some observations on the solution we have constructed. First, note that while *a priori* r, as a weak solution, has to be defined only almost everywhere, Kruskov's theory [8] implies that $t \mapsto r^t := r(\cdot, t)$, $t \in [0, \infty)$, actually is an $\mathscr{L}^1_{\text{loc}}$-continuous curve in $\mathscr{L}^\infty(\mathbb{R}, \mathbb{R})$. Thus, the coordinate y that is defined through (2.6) can be represented as

$$y(x, t) = \int_{x_*(t)}^{x} r^t(\bar{x})\, d\bar{x} \qquad \text{for all} \quad (x, t) \in \mathbb{R}^2_+ \tag{3.1}$$

with a curve $x_* \in C^0([0, \infty), \mathbb{R})$, $x_*(0) = 0$ (see [18]). For any $t \geqslant 0$, let Σ^t be $\mathbb{R}$ modulo the identification of each two points $x_1 < x_2$ with the property that $r^t = 0$ a.e. in (x_1, x_2), and define a continuous bijection $T^t: \Sigma^t \to \mathbb{R}$ unambiguously through

$$T^t(x) = y(x, t) \qquad \text{for all} \quad x \in \Sigma^t. \tag{3.2}$$

Setting

$$T^{t,0} := (T^0)^{-1} \circ T^t, \tag{3.3}$$

we have already proved part (a) of Theorem 2. To prove part (b) of Theorem 2, note first that under its assumptions, $r(x, t) \to a$, uniformly in x, as $t \to \infty$, by virtue of Theorem 6.2 in [9]. Thus,

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |u(x, t) - a\theta^0(T^{t,0}(x))| = 0. \tag{3.4}$$

By (1.2), (3.3), and since $T^{0\prime} \equiv a$ on I, it suffices to show that the function z defined by

$$z(x, t) = y(x, t) - a(x - \varphi(a) t) \tag{3.5}$$

satisfies

$$\lim_{t \to \infty} \sup_{x;\, T^{t,0}(x) \in I} |z(x, t) - c| = 0 \qquad \text{for some} \quad c \in \mathbb{R}. \tag{3.6}$$

By (2.6), z satisfies $z_x = r - a$, $z_t = -\varphi \circ r r + \varphi(a)a$, $z(0, 0) = 0$ and is thus given by

$$z(x, t) = c_\pm + \int_{\pm\infty}^{x} (r^t(\bar{x}) - a)\, d\bar{x} \qquad \text{with} \quad c_\pm = \int_0^{\pm\infty} (r^0(\bar{x}) - a)\, d\bar{x}. \tag{3.7}$$

z is constant along curves of speed $-z_t/z_x$, which, for $t \to \infty$ and uniformly in x, approaches the value $\varphi(a) + \varphi'(a)a$. This is different from the asymptotic speed $\lim_{t\to\infty}(-y_t/y_x) = \varphi(a)$ of the curves along which y and θ are constant. Therefore, (3.6) holds with

$$c = \lim_{x \to -\infty} z(x, 0) = c_- \quad [c = \lim_{x \to \infty} z(x, 0) = c_+] \tag{3.8}$$

if $\varphi'(a) > 0$ $[\varphi'(a) < 0]$. If (in these respective cases) x_0 is the left [right] endpoint of I, then the phase shift δ is given by

$$\delta = \int_{-\infty}^{x_0} (1 - (r^0(\bar{x})/a))\, d\bar{x} \quad \left[\delta = \int_{x_0}^{\infty} ((r^0(\bar{x})/a) - 1)\, d\bar{x}\right]; \tag{3.9}$$

this is clear from the special case $x_0 = 0$, in which $c = a\delta$. Note finally that in case $r^0 - a$ is compactly supported, the sup in (3.6) will be identically zero after a finite time t_0, with which (1.4) will hold. The proof of Theorem 2 is now complete.

We turn to part (c) of Theorem 1. Observe first that (3.1)–(3.3) imply that, with λ_1 denoting one-dimensional Lebesgue (outer) measure, $r^0\lambda_1$ is the pull-back measure of $r^t\lambda_1$ with respect to $T^{t,0}$, i.e.,

$$\int_{\mathbb{R}} f(T^{t,0}(x))\, r^t(x)\, dx = \int_{\mathbb{R}} f(x^0)\, r^0(x^0)\, dx^0 \qquad \text{for all test functions } f. \tag{3.10}$$

Let now $C, T > 0$ and $u^\infty \in \mathbb{R}^n \setminus \{0\}$ be arbitrary constants and denote by $\mathscr{S}$ the set of bounded measurable solutions, of the above type, with the additional technical properties that $|u| \leqslant C$ in all of $\mathbb{R}^2_+$ and $u(x, t) = u^\infty$ for all $(x, t) \in \mathbb{R}^2_+$ with $|x| \geqslant C$, $0 \leqslant t \leqslant T$. Continue to write $u = r\theta$, and $u^0 = r^0\theta^0$ for the corresponding initial values. Part (c) will be proved once we can show that for any elements u_n $(n \in \mathbb{N})$, $u_* \in \mathscr{S}$, convergence $u_n^0 \to u_*^0$ in $\mathscr{L}^1(\mathbb{R}, \mathbb{R}^n)$ implies convergence $u_n \to u_*$ in $\mathscr{L}^1(\mathbb{R} \times [0, T], \mathbb{R}^n)$. Before showing this, we observe the following remarkable property:

LEMMA 2. *For solutions with coinciding radial part, $\mathscr{L}^1$-distance is a time invariant; i.e., if $u_a = r_a\theta_a$, $u_b = r_b\theta_b \in \mathscr{S}$ satisfy $r_a^0 = r_b^0$, and thus $r_a = r_b$, then*

$$\int_{\mathbb{R}} |u_a(x, t) - u_b(x, t)|\, dx = \int_{\mathbb{R}} |u_a^0(x) - u_b^0(x)|\, dx \qquad \text{for all} \quad t \geqslant 0.$$

Proof of Lemma 2. In this case, with $r_a = r_b =: r$,

$$\begin{aligned}\int_{\mathbb{R}} |u_a(x,t) - u_b(x,t)|\,dx &= \int_{\mathbb{R}} |\theta_a(x,t) - \theta_b(x,t)|\, r(x,t)\,dx \\ &= \int_{\mathbb{R}} |\tilde{\theta}_a(T^t(x),t) - \tilde{\theta}_b(T^t(x),t)|\, r^t(x)\,dx \\ &= \int_{\mathbb{R}} |\tilde{\theta}_a(T^t(x),0) - \tilde{\theta}_b(T^t(x),0)|\, r^t(x)\,dx \\ &= \int_{\mathbb{R}} |\tilde{\theta}_a(T^0(x^0),0) - \tilde{\theta}_b(T^0(x^0),0)|\, r^0(x^0)\,dx^0 \\ &= \int_{\mathbb{R}} |\theta_a(x^0) - \theta_b(x^0)|\, r^0(x^0)\,dx^0 \\ &= \int_{\mathbb{R}} |u_a^0(x^0) - u_b^0(x^0)|\,dx^0.\end{aligned}$$

Now, in considering the above-mentioned sequence, we use Lemma 2 by assuming without loss of generality that $\theta_n^0 = \theta_*^0$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}&\int_0^T \int_{-\infty}^{\infty} |u_n(x,t) - u_*(x,t)|\,dx\,dt \\ &\quad\leqslant \int_0^T \int_{-\infty}^{\infty} |(r_n(x,t) - r_*(x,t))|\,dx\,dt \\ &\qquad + \int_0^T \int_{-\infty}^{\infty} r_*(x,t)\,|\theta_n(x,t) - \theta_*(x,t)|\,dx\,dt.\end{aligned}$$

When $n \to \infty$, the first term tends to 0 according to Kruskov's theory. Since, by assumption, $\theta_n^0 = \theta_*^0$, the second term is dominated by

$$\begin{aligned}C \iint_{([-C,C]\times[0,T])\cap \operatorname{spt} r_*} &|\theta_*^0((T_n^0)^{-1}(y_n(x,t)) \\ &- \theta_*^0((T_*^0)^{-1}(y_*(x,t))|\,dx\,dt,\end{aligned}$$

with y_n, y_*, T_n^0, T_*^0 defined as in (3.1), (3.2) relative to r_n and r_*, respectively. As

$$(T_n^0)^{-1}(y_n(x,t)) \to (T_*^0)^{-1}(y_*(x,t)) \qquad \text{for a.e.} \quad (x,t) \in \operatorname{spt} r_*,$$

the convergence of the second term follows from Lebesgue's theorem. The proof of part (c) of Theorem 1 is complete.

4. Regularity

The proof of part (d) of Theorem 1 is now very easy. Generically, smooth initial data $u^0 = r^0\theta^0$ satisfy $r^0 \in C^\infty(\mathbb{R}, (0, \infty))$. According to [6], for generic smooth r^0, the (Kruskov type) solution r is piecewise smooth, with locally finitely many shocks that satisfy condition (E). The transformation T of $\mathbb{R}^2_+$ given by (2.6) is (not only Lipschitz, which it is in general, but also) piecewise smooth in this case. Since $r > 0$ everywhere, the same holds for T^{-1}. Thus, (2.8) implies that θ is piecewise smooth if $\tilde{\theta}$ is. However, due to (2.5), (2.8), (2.10), (2.11), $\tilde{\theta}$ is as regular as θ^0 is, and thus smooth in our case.

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