J. Differential Equations 250 (2011) 4011–4036

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

The regularized Benjamin–Ono and BBM equations: Well-posedness and nonlinear stability

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ARTICLE INFO

Article history:
Received 2 October 2010
Revised 14 December 2010
Available online 3 January 2011

MSC:
35Q53
35B35
35B10

Keywords:
Periodic traveling waves
Nonlinear stability
Regularized Benjamin–Ono equation
BBM equation

ABSTRACT

Nonlinear stability of nonlinear periodic solutions of the regularized Benjamin–Ono equation and the Benjamin–Bona–Mahony equation with respect to perturbations of the same wavelength is analytically studied. These perturbations are shown to be stable. We also develop a global well-posedness theory for the regularized Benjamin–Ono equation in the periodic and in the line setting. In particular, we show that the Cauchy problem (in both periodic and nonperiodic case) cannot be solved by an iteration scheme based on the Duhamel formula for negative Sobolev indices.

1. Introduction

The principal focus of this work is to study stability properties of periodic traveling wave solutions for two basic nonlinear wave models of the dynamic of fluids, the regularized Benjamin–Ono equation (rBO equation henceforth) and the Benjamin–Bona–Mahony equation (BBM equation henceforth). The rBO equation is given by

$$u_t + u_x + uu_x + \mathcal{H}u_{xt} = 0, \tag{1.1}$$

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doi:10.1016/j.jde.2010.12.016

where $u = u(x,t)$ is a real-valued function with $x, t \in \mathbb{R}$ and $\mathcal{H}$ denotes the Hilbert transform defined via the Fourier transform as

$$\widehat{\mathcal{H}f}(k) = -i\,\mathrm{sgn}(k)\widehat{f}(k),$$

where

$$\mathrm{sgn}(k) = \begin{cases} -1, & k < 0, \\ 1, & k > 0. \end{cases}$$

The regularized Benjamin–Ono equation is a model for the time evolution of long-crested waves at the interface between two immiscible fluids. Some situations in which the equation is useful are the pycnocline in the deep ocean, and the two-layer system created by the inflow of fresh water from a river into the sea, see Kalisch [30]. This equation is formally equivalent to the Benjamin–Ono equation (BO equation henceforth)

$$v_t + v_x + v v_x - \mathcal{H} v_{xx} = 0, \tag{1.2}$$

which was first introduced by Benjamin [14] and later by Ono [36] as a model equation for the same situation as the rBO. More exactly, for suitably restricted initial conditions, the solutions u of (1.1) and v of (1.2) are nearly identical at least for values of t in $[0, T]$ where T is quite large, see Albert and Bona [4] for more details. See also [4] for a more detailed discussion about the advantages and disadvantages of using either equation (1.2) or (1.1) for modeling the propagation of small-amplitude long waves.

This paper is dedicated to an important qualitative aspect of nonlinear dispersive equations, the traveling wave solutions, which depending on the specific boundary conditions on the wave's shape can be either solitary or periodic waves. The existence, the nonlinear stability and the instability of solitary-wave solutions have been discussed in the past two decades from several points of view. Many techniques have been created to find solutions and sufficient conditions have been obtained to insure the stability or instability of this kind of waves, see for instance [2,3,5,13,14,16,23,24,36,40, 41]. In contrast to the study of solitary waves, the periodic traveling wave solutions has received less attention. In recent years some papers in this subject have appeared, see [8,10–12,21,22,27,25,31].

The periodic traveling wave solutions we are interested are solutions of the general form $u(x,t) = \phi(x - ct)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth $2L$-periodic function and $c \neq 1$. Then, by replacing these permanent wave form into (1.1), integrating and considering zero the constant of integration, we obtain

$$c\mathcal{H}\phi_c' + (c-1)\phi_c - \frac{1}{2}\phi_c^2 = 0. \tag{1.3}$$

In the framework of traveling waves of solitary type, it is known the existence of solutions for (1.3) of the form

$$\phi_c(x) = \frac{4(c-1)}{1 + (\frac{c-1}{c}x)^2}, \tag{1.4}$$

where $c > 1$ (see Benjamin [14]). The nonlinear stability for these solitary wave by the flow of the rBO equation was established by Albert, Bona and Henry [3]. Additionally, Kalisch [30] exhibited a periodic family of traveling wave solutions (depending of the speed) with period 2π for the rBO and used it to test the rate of convergence of a numerical scheme, which was introduced in Bona and Kalisch [18] to prove that Eq. (1.1) does not constitute an *infinite-dimensional completely integrable system*.

By using the Benjamin's periodic traveling wave solution profile for the BO equation [14] we obtain the existence of the following smooth curve, $c \to \phi_c$, of positive, even, periodic solutions for Eq. (1.1) with minimal period $2L$,

$$\phi_c(\xi) = \frac{2c\pi}{L} \frac{\sinh(\eta)}{\cosh(\eta) - \cos(\frac{\pi\xi}{L})}, \tag{1.5}$$

with η satisfying

$$\eta(c) = \tanh^{-1}\left(\frac{c\pi}{(c-1)L}\right) \tag{1.6}$$

and $L > \pi$ with $c > \frac{L}{L-\pi}$. It is our purpose here to consider the stability of these latter waveforms. As our general experience with nonlinear, dispersive evolution equations indicates that traveling wave, when they exist, are of fundamental importance in the development of a broad range of disturbance, we expect the issue of stability of periodic waves to be of interest.

We approach the question of stability of the profile ϕ_c in $H^{\frac{1}{2}}_{per}([-L, L])$ by the periodic rBO's flow by extending the classical theory developed by Benjamin [13], Bona [16] and Weinstein [40] to the periodic case.

In our stability theory, we will use that the rBO equation possesses the following conservation laws

$$E(u) = \frac{1}{2}\int \left(u\mathcal{H}u_x - \frac{1}{3}u^3\right)dx \quad \text{and} \quad F(u) = \frac{1}{2}\int \left(u^2 + u\mathcal{H}u_x\right)dx, \tag{1.7}$$

and that ϕ_c is a critical point for the functional $E + (c-1)F$, in other words, we have the equation in (1.3). Moreover, we need to have a specific spectral structure associated to the nonlocal operator

$$\mathcal{L} = c\mathcal{H}\partial_x + c - 1 - \phi_c, \tag{1.8}$$

in a periodic framework. More exactly, $\mathcal{L}$ has a single negative eigenvalue, which is simple; zero is also a simple eigenvalue with eigenfunction ϕ_c' and the remainder of the spectrum is bounded away from zero. In order to get these spectral conditions we use the recent theory developed by Angulo and Natali [11] which is based in positive properties of the Fourier transform of ϕ_c. We note that since we need to obtain the Fourier coefficients of ϕ_c, we use the Poisson Summation theorem for obtaining the profile in (1.5). We believe that this strategy for obtaining periodic profiles has prospects for the study of other similar problems.

Previous results of Spector and Miloh [37] established that a normalized family of periodic solutions of the BO equation with profile determined by (1.5) are *linearly stable*. Their result was obtained by using that the BO equation is completely integrable, so, the inverse scattering transform scheme was applied. In this work we make no use of their technique for studying the operator $\mathcal{L}$ in (1.8).

In the last section of the paper the theory for the rBO equation is extended for a general family of regularized equations. We study a class of equations of the form

$$u_t + u_x + u^p u_x + Hu_t = 0, \tag{1.9}$$

where $p \geqslant 1$ is an integer and H is a differential or pseudo-differential operator in the context of periodic functions. Note that a considerable range of equations with this form arise in practice. For instance, if we consider $H = -\partial_x^2$ we obtain the generalized Benjamin–Bona–Mahony equation, in particular for $p = 1$ we obtain the BBM equation [15], and if $H = \mathcal{H}\partial_x$ we obtain the generalized regularized Benjamin–Ono equation [3]. This kind of generalization in the context of solitary waves have been studied before, see for example Albert, Bona and Henry [3] and Bona and Chen [17]. In the periodic setting Hărăguş [26] studied the spectral stability of periodic traveling wave solutions for the generalized BBM, which are small perturbations of the constant solution $u=(c-1)^{1/p}$, in both $L^2(\mathbb{R})$ and $C_b(\mathbb{R})$. In the case $1\leqslant p\leqslant 2$, she proved spectral stability for $c>1$ and for $p\geqslant 3$, her result says that there exists a critical speed c_p such that the periodic waves are spectrally stable for $c\in(c_p,\frac{p}{p-3})$, and unstable for $c\in(1,c_p)\cup(\frac{p}{p-3},\infty)$.[1] It is worth to note that Hakkaev, Iliev and Kirchev [25] studied the orbital stability of a type of generalized BBM and Camassa–Holm equations. The family of BBM equations that they investigated were of the form

$$u_t+2\omega u_x+3uu_x-u_{xxt}=0,\quad \omega\in\mathbb{R}.$$

They proved the existence solutions of the cnoidal type, but they only proved the orbital stability of this solutions in the case $\omega=0$.

In this paper, we give sufficient conditions to obtain the nonlinear stability by any periodic perturbation, of periodic wave solutions associated to equations of the type (1.9) with the same periodic structure as the underlying wave. As an example we prove that the cnoidal wave solutions of the Benjamin–Bona–Mahony equation (BBM equation), with a profile given by

$$\phi_c(x)=\alpha_1+\alpha_2\,\mathrm{cn}^2(\alpha_3x;k),\tag{1.10}$$

are orbitally stable in $H^1_{per}([0,L])$.

We believe that our approach of stability is of relevance to understand the behavior of systems of dispersive type. Further, our analysis allows a possible numerical simulation in the real world of the behavior of either water waves in the interface of two fluids or gravity water waves in the long-wave regime.

We note that, recently, Johnson [29] studied the stability of a four parameter family of spatially periodic traveling wave solutions $\varphi(\cdot;p)$ of the generalized Benjamin–Bona–Mahoney equation. In particular, he showed that periodic waves of sufficiently long wavelength (in other words, with a fundamental period sufficiently large) are nonlinearly stable for $1\leqq p<4$ by any periodic perturbation on the following smooth submanifold of H^1_{per} of codimension two,

$$\Sigma_p=\left\{f\in H^1_{per}\colon \int f(x)\,dx=\int\varphi(x;p)\,dx,\ \int f^2(x)+f'^2(x)\,dx=\int\varphi^2(x;p)+\varphi'^2(x;p)\,dx\right\}.$$

In an upcoming work (Angulo, Banquet, and Scialom [9]) we will use our approach to obtaining a stability theory for the modified BBM equation ($p=2$) and for the critical BBM equation ($p=4$) by any periodic perturbation.

A second interest of study in this paper is about the well-posedness problem associated to the rBO equation in the periodic Sobolev spaces $H^s_{per}([-L,L])$ or $H^s(\mathbb{R})$. Indeed, we show that the rBO is globally well-posed if $s>1/2$. As far as we known this material is new and not published elsewhere. In our results we improve an estimative of Mammeri in [32] for the periodic rBO equation. In his paper he establishes a global well-posedness result for the equation

$$u_t+u_x+\alpha uu_x+\beta\mathcal{H}u_{xt}=0,\tag{1.11}$$

where α and β are constants such that $0<\alpha,\beta\leqslant 1$. Mammeri also proved that the Cauchy problem associated to Eq. (1.11) is globally well-posed in $H^s_0([-L,L])$, for $s>1/2$, where $H^s_0([-L,L])$ means the elements f of $H^s([-L,L])$ with mean zero. Since we are interested in establishing a stability

[1] Here $\frac{p}{p-3}=\infty$, if $p=3$.

theory for the rBO equation without this type of restrictions the result of Mammeri is not complete. This shows the relevance of our result of well-posedness for the periodic rBO equation.

It is also worth to note that our result of global well-posedness in the continuous case is an improvement of the result obtained by Bona and Kalish [18], where a global well-posedness result for the rBO was proved in $H^s(\mathbb{R})$ with $s \geqslant 3/2$. The conservation laws in (1.7) suggest that the space $H^{\frac{1}{2}}(\mathbb{R})$ (or $H^{\frac{1}{2}}_{per}$) is a good candidate to study the Cauchy problem to Eq. (1.1). This problem is open in $H^s(\mathbb{R})$ (or H^s_{per}) with $s \leqslant 1/2$ and one of the goals of this paper is to present some obstructions to its solution by an iteration methods. More precisely, we prove that the map data-solution cannot be C^2 for $s < 0$, in both, periodic and nonperiodic case. This kind of ill-posedness was studied by Bourgain [19] and Tzvetkov [39] for the KdV equation. Molinet, Saut and Tzvetkov [33,34] did the same for the Benjamin–Ono equation and the Kadomtsev–Petviashvili I (KPI) equation, respectively. So, for the sake of completeness we write an ill-posedness result for the rBO on the periodic and nonperiodic cases.

Finally, this paper is organized as follows: In Section 2 we introduced some notations to be used throughout the whole article. In Section 3, we prove global well-posedness and ill-posedness result in the periodic and nonperiodic setting. In Section 4, we show the existence of periodic traveling waves using the Poisson Summation theorem. The spectral properties needed to obtain the non-linear stability are given in Section 5. In Section 6, we get the stability of the periodic traveling waves based on the ideas in [16,11,13,40]. Finally, we present an extension of the theory for the rBO and then we used it to prove the stability of cnoidal waves associated to the BBM equation in Section 7.

2. Notation and preliminaries

Our notation is the standard one in partial differential equations, for further details see Iorio and Iorio [28]. Let $\mathcal{P} = C^\infty_{per}$ denote the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ which are C^∞ and periodic with period $2L > 0$, and $\mathcal{P}'$ the set of *periodic distributions*. If $\Psi \in \mathcal{P}'$ then we denote the value of Ψ at φ by $\Psi(\varphi) = \langle \Psi, \varphi \rangle$. The Fourier transform of Ψ is the function $\widehat{\Psi} : \mathbb{Z} \to \mathbb{C}$ defined by the formula $\widehat{\Psi}(k) = \frac{1}{2L}\langle \Psi, \Theta_k \rangle$, where $\Theta_k(x) = \exp(\pi i k x/L)$, $k \in \mathbb{Z}$, $x \in \mathbb{R}$. So, if Ψ is a periodic function with period $2L$, we have

$$\widehat{\Psi}(k) = \frac{1}{2L}\int_{-L}^{L} \Psi(x) e^{-\frac{ik\pi x}{L}}\, dx.$$

For $s \in \mathbb{R}$, the Sobolev space of order s, denoted by $H^s_{per}([-L, L])$ is the set of all $f \in \mathcal{P}'$ such that $(1 + |k|^2)^{\frac{s}{2}} \widehat{f}(k) \in l^2(\mathbb{Z})$, with norm $\|f\|^2_{H^s_{per}} = 2L \sum_{k=-\infty}^{\infty} (1 + |k|^2)^s |\widehat{f}(k)|^2$. In the case $s = 0$, H^0_{per} is denoted by L^2_{per}, with $(f, g) = \int_{-L}^{L} f\bar{g}\, dx$ and $\|\cdot\|_{H^0_{per}} = \|\cdot\|_{L^2_{per}}$.

If Y is a Banach space and $T > 0$, then $C([0, T]; Y)$ is the space of continuous mappings from $[0, T]$ to Y and, for $k \geqslant 0$, $C^k([0, T]; Y)$ is the subspace of mappings $t \mapsto u(t)$ such that $\partial_t^j u \in C([0, T]; Y)$ for $0 \leqslant j \leqslant k$, where the derivative is taken in the sense of vector-valued distributions. This space carries the standard norm

$$\|u\|_{C^k([0,T];Y)} = \sum_{j=0}^{k} \max_{0 \leqslant t \leqslant T} \left\|\partial_t^j u(t)\right\|_Y.$$

Finally $\mu(A)$ denotes the Lebesgue measure of the set A.

Next, we establish the Poisson Summation theorem. It will be useful in Sections 4 and 7 to find the periodic traveling wave solutions for the rBO and for the BBM equations, respectively.

Theorem 2.1. *Let $\widehat{f}^{\mathbb{R}}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}\,dx$ and $f(x) = \int_{-\infty}^{\infty} \widehat{f}^{\mathbb{R}}(\xi)e^{2\pi ix\xi}\,d\xi$ satisfy*

$$\bigl|f(x)\bigr| \leqslant \frac{A}{(1+|x|)^{1+\delta}} \quad \textit{and} \quad \bigl|\widehat{f}^{\mathbb{R}}(\xi)\bigr| \leqslant \frac{A}{(1+|\xi|)^{1+\delta}},$$

where $A > 0$ and $\delta > 0$ (then f and $\widehat{f}$ can be assumed continuous functions). Thus, for $L > 0$

$$\sum_{n=-\infty}^{\infty} f(x+2Ln) = \frac{1}{2L}\sum_{n=-\infty}^{\infty} \widehat{f}^{\mathbb{R}}\left(\frac{n}{2L}\right)e^{\frac{\pi inx}{L}}.$$

The two series above converge absolutely.

Proof. See Stein and Weiss [38]. □

3. Well-posedness and ill-posedness results for the rBO

We start our study for the rBO equation by establishing several results associated to the well-posedness problem on the periodic and nonperiodic setting. The periodic case theory will be necessary in our nonlinear stability study of the waveforms solutions of Eq. (1.5).

We say that the initial value problem (IVP) associated to (1.1) is locally well-posed in X (Banach space) if the solution uniquely exists in certain time interval $[-T, T]$ (unique existence), the solution describes a continuous curve in X in the interval $[-T, T]$ whenever the initial data belongs to X (persistence), and the solution varies continuously depending upon the initial data (continuous dependence) i.e., we have the continuity of the application $u_0 \to u(t)$ from X to $C([0, T]; X)$. We say that the IVP associated to (1.1) is globally well-posed in X if the same properties hold for all time $T > 0$. If some property in the definition of locally well-posed fails, we say that the IVP is ill-posed.

3.1. *Global well-posedness in H^s_{per} and $H^s(\mathbb{R})$ with $s > 1/2$*

First, we study the periodic case and for simplicity we consider $L = \pi$. So, we rewrite (1.1) as

$$(1+\mathcal{H}\partial_x)u_t = -\left(u + \frac{1}{2}u^2\right)_x,$$

and since $\mathcal{H}\partial_x \geqslant 0$, formally we have

$$u_t = -\partial_x(1+\mathcal{H}\partial_x)^{-1}\left(u+\frac{1}{2}u^2\right) = K\left(u+\frac{1}{2}u^2\right),$$

where K is such that its Fourier transform satisfies $\widehat{Kf}(n) = \frac{-in}{1+|n|}\widehat{f}(n)$. Integrating and using the initial condition we get

$$u(x,t) = u_0(x) + \int_0^t K\left(u + \frac{1}{2}u^2\right)(x,\tau)\,d\tau, \quad x \in \mathbb{R},\ t > 0.$$

Using the fact that H^s_{per} with $s > 1/2$ is Banach algebra and standard arguments of fixed point type (see also Theorem 2.1 [32]), we obtain the next result.

Theorem 3.1. *Suppose $s > 1/2$, then for all $u_0 \in H^s_{per}$ there exists $T = T(\|u_0\|_{H^s_{per}}) > 0$ and a unique solution of (1.1) on the interval $[-T, T]$, such that $u \in C([-T, T]; H^s_{per})$. Furthermore, for all $T' < T$ there exists a neighborhood V of u_0 in H^s_{per} such that*

$$\mathbb{F} : V \longrightarrow C\big([-T', T']; H^s_{per}\big), \quad \tilde{u}_0 \to \tilde{u}(t),$$

is Lipschitz. Moreover, if $T < \infty$ then $\lim_{t \to T^-} \|u(t)\|_{H^s} = \infty$.

In what follows, for the sake of completeness we prove an estimative of Brezis–Gallouet type.

Lemma 3.2. *Let $f \in H^s_{per}$, with $s > 1/2$. Then, there exists a constant $C > 0$ such that*

$$\|f\|_\infty \leqslant C\big(1 + \sqrt{\log\big(1 + \|f\|_{H^s_{per}}\big)}\|f\|_{H^{\frac{1}{2}}_{per}}\big). \tag{3.1}$$

Proof. Consider $f \in C^\infty_{per}$. Then by Fourier theorem $f(x) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ikx}$. Thus for $R > 0$, where R will be chosen later, we have that

$$\begin{aligned} \big|f(x)\big| &\leqslant \sum_{k\in\mathbb{Z}} \big|\widehat{f}(k)\big| = \sum_{|k|<R} \frac{(1+|k|)^{1/2}}{(1+|k|)^{1/2}}\big|\widehat{f}(k)\big| + \sum_{|k|\geqslant R} \frac{(1+|k|)^s}{(1+|k|)^s}\big|\widehat{f}(k)\big| \\ &\leqslant \bigg(\sum_{|k|<R} \frac{1}{1+|k|}\bigg)^{1/2} \|f\|_{H^{\frac{1}{2}}_{per}} + \frac{C_0}{(R+1)^\epsilon}\|f\|_{H^s_{per}}, \end{aligned}$$

where $\epsilon > 0$ is such that $2(s - \epsilon) > 1$. Note that for $[\cdot]$ denoting the integer part we have

$$\begin{aligned} \sum_{|k|<R} \frac{1}{1+|k|} &= 1 + 2\bigg(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{[R]}\bigg) \leqslant 2e\log 2 + 2e\bigg(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{[R]}\bigg) \\ &\leqslant 2e\int_1^2 \frac{1}{x}\,dx + 2e\int_2^{1+R} \frac{1}{x}\,dx = 2e\log(1+R). \end{aligned}$$

Therefore

$$\|f\|_\infty \leqslant C_1\sqrt{\log(1+R)}\|f\|_{H^{\frac{1}{2}}_{per}} + \frac{C_0}{(R+1)^\epsilon}\|f\|_{H^s_{per}}.$$

Choosing $R = (1 + \|f\|_{H^s_{per}})^{\frac{1}{\epsilon}} - 1$, we get that

$$\|f\|_\infty \leqslant C_1\epsilon^{-\frac{1}{2}}\sqrt{\log(1 + \|f\|_{H^s_{per}})}\|f\|_{H^{\frac{1}{2}}_{per}} + \frac{C_0\|f\|_{H^s_{per}}}{1 + \|f\|_{H^s_{per}}}. \qquad \square$$

We obtain the next corollary following the proof of Theorem 2.3 in [32] and using Lemma 3.2 above.

Corollary 3.3. *The periodic initial value problem associated to* (1.1) *is globally well-posed in* H^s_{per} *for* $s > 1/2$.

Proof. Let $u_0 \in H^s_{per}$ and $t \in [-T, T]$. Initially, we have the Duhamel formula

$$u(t) = S(t)u_0 + \frac{1}{2}\int_0^t S(t-\tau)\big(Ku^2\big)(\tau)\,d\tau,$$

with $S(t)f \equiv \sum_{n\in\mathbb{Z}} e^{-it\frac{k}{1+|k|}}\widehat{f}(k)e^{ikx}$. So, from Lemma 3.2 we obtain

$$\begin{aligned}\big\|u(t)\big\|_s &\leqq \|u_0\|_s + \frac{1}{2}\int_0^t \big\|u(\tau)\big\|_\infty \big\|u(\tau)\big\|_s\,d\tau \\ &\leqq \|u_0\|_s + C_0\int_0^t \Big(1+\sqrt{\log\big(1+\big\|u(\tau)\big\|_s\big)}\Big)\big\|u(\tau)\big\|_s\,d\tau \equiv \Phi(t)\end{aligned}$$

with C_0 depending only on $\|u_0\|_{\frac{1}{2}}$-norm. Using standard argument we have that there exists $C_1 > 0$ such that

$$\frac{d}{dt}\log\big(1+\log\big(1+\Phi(t)\big)\big) \leqslant C_1.$$

Therefore we deduce that there exist $C_2 > 0$ and $C_3 > 0$ such that for all $t \in [-T, T]$, $\|u(t)\|_s \leqq e^{C_2 e^{C_3 t}}$. In particular, $\|u(t)\|_s$ remains bounded on every finite time interval and the solution can be extended in time at all $\mathbb{R}$. $\square$

It is well known that in the nonperiodic case, the estimative (3.1) is also valid. In this case

$$\begin{aligned}\big|f(x)\big| &\leqslant \int_{\mathbb{R}} \big|\widehat{f}(\xi)\big|\,d\xi = \int_{|\xi|<R} \frac{(1+|\xi|)^{1/2}}{(1+|\xi|)^{1/2}}\big|\widehat{f}(\xi)\big|\,d\xi + \int_{|\xi|\geqslant R} \frac{(1+|\xi|)^s}{(1+|\xi|)^s}\big|\widehat{f}(\xi)\big|\,d\xi \\ &\leqslant \bigg(\int_{|\xi|<R} \frac{d\xi}{1+|\xi|}\bigg)^{1/2} \|f\|_{H^{\frac{1}{2}}(\mathbb{R})} + \frac{C_0}{(R+1)^\epsilon}\|f\|_{H^s(\mathbb{R})} \\ &= \sqrt{2}\sqrt{\log(1+R)}\|f\|_{H^{\frac{1}{2}}(\mathbb{R})} + \frac{C_0}{(R+1)^\epsilon}\|f\|_{H^s(\mathbb{R})}.\end{aligned}$$

Therefore, a similar analysis as in the periodic case lead to the next corollary which improved the result of Bona and Kalish [18].

Corollary 3.4. *The Cauchy problem associated to the rBO equation is globally well-posed in* $H^s(\mathbb{R})$, *for* $s > 1/2$.

3.2. Ill-posedness in H^s_{per} and $H^s(\mathbb{R})$ with $s<0$

In this subsection we show that the map data-solution for the Cauchy problem associated to the rBO equation is not C^2 at the origin for initial data in H^s_{per} (or $H^s(\mathbb{R})$), with $s<0$. Therefore, we cannot apply the Contraction Principle to solve the integral equation (3.2), see below.

First, we analyze the problem on the periodic setting. For simplicity we consider functions of period 2π. We know that the linear problem associated to (1.1) with initial datum ψ has solution

$$u(x,t)=S(t)\psi(x)=\sum_{n=-\infty}^{+\infty} e^{inx-\frac{in}{1+|n|}t}\widehat{\psi}(n).$$

Now, if u is solution of (1.1), then by Duhamel principle we have that

$$u(x,t)=S(t)\psi(x)-\int_0^t S(t-\tau)\Lambda\big[u(x,\tau)u_x(x,\tau)\big]\,d\tau, \tag{3.2}$$

where $\widehat{\Lambda u}(n)=(1+|n|)^{-1}\widehat{u}(n)$.

The following theorem is the principal result of this section.

Theorem 3.5. *Let $s<0$ and T a positive number. Then there does not exist a space X_T continuously embedded in $C([-T,T];H^s_{per})$ such that there exist $c_0>0$ satisfying*

$$\big\|S(t)\psi\big\|_{X_T}\leqslant c_0\|\psi\|_{H^s_{per}},\quad \forall\psi\in H^s_{per} \tag{3.3}$$

and

$$\left\|\int_0^t S(t-\tau)\Lambda\big[u_x(\tau)u(\tau)\big]\,d\tau\right\|_{X_T}\leqslant c_0\|u\|^2_{X_T},\quad \forall u\in X_T. \tag{3.4}$$

Proof. Suppose by contradiction that there exists such a space. Consider $\psi\in H^s_{per}$ and define $u:=S(t)\psi$. Then, from (3.3) we have that $u\in X_T$ and since $X_T\hookrightarrow C([-T,T];H^s_{per})$, we get from (3.4) that

$$\left\|\int_0^t S(t-\tau)\Lambda\big[S(t)\psi\big(S(t)\psi\big)_x\big]\,d\tau\right\|_{H^s_{per}}\leqslant c_0\|\psi\|^2_{H^s_{per}}. \tag{3.5}$$

Next we prove that choosing ψ, appropriately, (3.5) does not hold. In fact, consider

$$\psi(x):=N^{-s}\cos(Nx),\quad \text{with } N\in\mathbb{N},\ N\gg 1.$$

It easy to see that $S(t)\psi(x)=N^{-s}\cos(Nx-\frac{N}{1+N}t)$. Then,

$$\begin{aligned}\varphi(x,t)&:=\int_0^t S(t-\tau)\Lambda\big[S(t)\psi(x)\big(S(t)\psi(x)\big)_x\big]\,d\tau\\ &=-\frac{1}{2}N^{-2s+1}\int_0^t S(t-\tau)\Lambda\left[\sin\left(2Nx-\frac{2N}{1+N}\tau\right)\right]d\tau.\end{aligned}$$

Now, using the specific form of Λ we obtain that

$$\int_0^t S(t-\tau)\Lambda\left[\sin\left(2Nx-\frac{2N}{1+N}\tau\right)\right]d\tau = -\frac{1}{2(1+2N)\gamma_N}\left[e^{i(2Nx-\frac{2N}{1+2N}t)}-e^{i(2Nx-\frac{2N}{1+N}t)}\right] + \frac{1}{2(1+2N)\gamma_N}\left[e^{-i(2Nx-\frac{2N}{1+N}t)}-e^{-i(2Nx-\frac{2N}{1+2N}t)}\right],$$

where $\gamma_N = \frac{2N^2}{(1+N)(1+2N)}$. Therefore

$$\varphi(x,t) = \frac{1}{2}N^{-2s+1}\frac{1}{\gamma_N(1+2N)}\left[\cos\left(2Nx-\frac{2N}{1+2N}t\right)-\cos\left(2Nx-\frac{2N}{1+N}t\right)\right].$$

Hence,

$$\left\|\varphi(\cdot,t)\right\|^2_{H^s_{per}} \sim N^{-4s}\left|e^{-i\frac{2N}{1+2N}t}-e^{-i\frac{2N}{1+N}t}\right|^2\left(1+4N^2\right)^s \sim N^{-2s}\left(1-\cos(\gamma_N t)\right).$$

Note that $\|\psi\|^2_{H^s_{per}} \sim 1$, then for all $t \in (0,T)$ we have

$$\frac{\|\varphi(\cdot,t)\|_{H^s_{per}}}{\|\psi\|^2_{H^s_{per}}} \sim N^{-s}\left(1-\cos(\gamma_N t)\right)^{\frac{1}{2}}.$$

Without loss of generality we can suppose $0 < T < 2\pi$. For $s < 0$ fixed, we obtain that

$$\frac{\|\varphi(\cdot,t)\|_{H^s_{per}}}{\|\psi\|^2_{H^s_{per}}} \longrightarrow +\infty,$$

as $N \to +\infty$, for all $0 < t < T$, which contradict (3.5). $\square$

As a consequence we get the next result.

Corollary 3.6. *Fix $s < 0$. There does not exist a $T > 0$ such that (1.1) admits a unique local solution defined on the interval $[-T,T]$ and such that for any fixed $t \in [-T,T]$ the map*

$$\psi \longmapsto u(t)$$

is C^2 differentiable at zero from H^s_{per} to H^s_{per}.

Proof. Consider the Cauchy problem

$$\begin{cases} u_t + u_x + uu_x + \mathcal{H}u_{xt} = 0, \\ u(x,0) = \psi_\gamma(x), \quad 0 < \gamma \ll 1, \end{cases} \tag{3.6}$$

where $\psi_\gamma(x) := \gamma\psi(x)$. Suppose that $u(\gamma,t,x)$ is a local solution of (3.6) and the map data-solution is C^2 at the origin from H^s_{per} to H^s_{per}. Then

$$\left.\frac{\partial u}{\partial\gamma}(\gamma,t,x)\right|_{\gamma=0} = S(t)\psi(x)$$

and

$$\left.\frac{\partial^2 u}{\partial \gamma}(\gamma, t, x)\right|_{\gamma=0} = -2\int_0^t S(t-\tau)\Lambda\big[\big(S(\tau)\psi\big)\big(S(\tau)\psi\big)_x\big]\,d\tau.$$

Using the assumption, we have

$$\left\|\int_0^t S(t-\tau)\Lambda\big[\big(S(\tau)\psi\big)\big(S(\tau)\psi\big)_x\big]\,d\tau\right\|_{H^s_{per}} \leqslant c_0\|\psi\|^2_{H^s_{per}}.$$

The last estimative is the same as in (3.5), which has been shown to fail in the last theorem. □

Now we establish the same type of previous results in the nonperiodic setting. Recall that in this case we have

$$S(t)\psi(x) = \int_{\mathbb{R}} \widehat{\psi}(\xi) e^{i(\xi x - \frac{\xi}{1+|\xi|}t)}\,d\xi$$

and $\widehat{\Lambda u}(\xi) = (1+|\xi|)^{-1}\widehat{u}(\xi)$, for $\xi \in \mathbb{R}$.

Theorem 3.7. *Fix $s < 0$. There does not exist a $T > 0$ such that (1.1) admits a unique local solution defined on the interval $[-T, T]$ and such that for any fixed $t \in [-T, T]$ the map*

$$\psi \longmapsto u(t)$$

is C^2 differentiable at zero from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.

The following lemma is found in Molinet and Saut [33].

Lemma 3.8.

$$\int_0^t S(t-\tau)\Lambda\big[\big(S(\tau)\psi\big)\big(S(\tau)\psi\big)_x\big]\,d\tau = c_0\int_{\mathbb{R}^2} e^{i(\xi x - p(\xi)t)}\frac{\xi}{1+|\xi|}\widehat{\psi}(\eta)\widehat{\psi}(\xi-\eta)\frac{e^{-it\chi(\xi,\eta)}-1}{\chi(\xi,\eta)}\,d\eta\,d\xi,$$

where $p(\xi) = \frac{\xi}{1+|\xi|}$ and $\chi(\xi,\eta) = p(\eta) + p(\xi-\eta) - p(\xi)$.

Proof of Theorem 3.7. We define

$$\varphi(x,t) := \int_0^t S(t-\tau)\Lambda\big[\big(S(\tau)\psi\big)\big(S(\tau)\psi\big)_x\big]\,d\tau.$$

Then, using the last lemma we have

$$\widehat{\varphi}(\xi,t) = c_0\frac{\xi}{1+|\xi|}e^{-ip(\xi)t}\int_{\mathbb{R}}\widehat{\psi}(\eta)\widehat{\psi}(\xi-\eta)\frac{e^{-it\chi(\xi,\eta)}-1}{\chi(\xi,\eta)}\,d\eta. \tag{3.7}$$

In this case we consider

$$\widehat{\psi}(\xi) := N^{-s}\chi_{[N,N+1]}(\xi), \quad \text{with } N \in \mathbb{N},\ N \gg 1,$$

where χ_A denotes the characteristic function of A. Note that $\|\psi\|_{H^s(\mathbb{R})} \sim 1$ and using (3.7) we obtain

$$\widehat{\varphi}(\xi,t) = c_0 \frac{\xi}{1+|\xi|} e^{-p(\xi)t} N^{-2s} \int_{\Omega_\xi} \frac{e^{-it\chi(\xi,\eta)}-1}{\chi(\xi,\eta)}\, d\eta,$$

with $\Omega_\xi = \{\eta\colon\ \eta \in \operatorname{supp}\widehat{\psi} \text{ and } \xi-\eta \in \operatorname{supp}\widehat{\psi}\}$. Since $s<0$, we can choose $\epsilon>0$ such that $-s-\epsilon>0$. Now, consider $t = N^{-\epsilon}$ and note that for $\xi \in (2N+\frac{1}{2}, 2N+1)$ we have $\mu(\Omega_\xi) \gtrsim 1$. It is easy to see that

$$\chi(\xi,\eta) = \frac{\eta(\xi-\eta)(2+\xi)}{(1+\eta)(1+\xi-\eta)(1+\xi)} \leqslant 3, \quad \forall \eta, \xi-\eta \in [N,N+1].$$

Then, for N big enough we compute that

$$\begin{aligned}
\|\varphi(\cdot,t)\|^2_{H^s(\mathbb{R})} &\gtrsim \int_{2N+\frac{1}{2}}^{2N+1} \left(1+|\xi|^2\right)^s N^{-4s} \frac{|\xi|^2}{(1+|\xi|)^2} |t|^2 \left| \int_{\Omega_\xi} \frac{e^{-it\chi(\xi,\eta)}-1}{t\chi(\xi,\eta)}\, d\eta \right|^2 d\xi \\
&\gtrsim \int_{2N+\frac{1}{2}}^{2N+1} \left(1+|\xi|^2\right)^s N^{-4s} \frac{|\xi|^2}{(1+|\xi|)^2} |t|^2 \left| \int_{\Omega_\xi} \frac{\sin(t\chi(\xi,\eta))}{t\chi(\xi,\eta)}\, d\eta \right|^2 d\xi \\
&\gtrsim N^{-4s} N^{2s} t^2.
\end{aligned}$$

Hence $1 \sim \|\psi\|_{H^s(\mathbb{R})} \gtrsim \|\varphi(\cdot,t)\|_{H^s(\mathbb{R})} \gtrsim N^{-s-\epsilon}$, which is a contradiction for $N \gg 1$. This completes the proof in the nonperiodic case. □

4. Periodic traveling wave solutions for the rBO

The aim of this section is to obtain the existence of a smooth curve of periodic traveling wave solutions for (1.3) via the Poisson Summation theorem. In fact, the equation

$$w\mathcal{H}\varphi_w' + (w-1)\varphi_w - \frac{1}{2}\varphi_w^2 = 0$$

determines the following solitary traveling wave solutions

$$\varphi_w(x) = \frac{4(w-1)}{1+(\frac{w-1}{w}x)^2}, \quad x \in \mathbb{R},\ w>1,$$

such that its Fourier transform is given by $\widehat{\varphi}_w^{\mathbb{R}}(\xi) = 4\pi w e^{-2\pi|\frac{w}{w-1}\xi|}$. Now, using the Poisson Summation theorem and some manipulations we get the following periodic function

$$\psi_w(x) := \frac{2\pi w}{L} \sum_{n=-\infty}^{\infty} e^{-\frac{2\pi w|n|}{2(w-1)L}} e^{\frac{\pi i n x}{L}} = \frac{2\pi w}{L} \sum_{n=0}^{\infty} \epsilon_n \, e^{-\frac{2\pi w n}{2(w-1)L}} \cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2\pi w}{L} \left(\frac{\sinh(\frac{2\pi w}{2(w-1)L})}{\cosh(\frac{2\pi w}{2(w-1)L}) - \cos(\frac{\pi}{L} x)} \right), \tag{4.1}$$

where we used the Fourier expansion 1.89 in Oberhettinger [35] and $\epsilon_n = 1$ if $n = 0$ and $\epsilon_n = 2$ if $n = 1, 2, 3, \ldots$, to obtain the last identity. Next, we find the right solitary-wave velocity w such that ψ_w becomes a periodic traveling wave solution for the rBO. To that, we consider ϕ_c with $c \neq 1$ a smooth periodic solution of Eq. (1.3). We express ϕ_c and ϕ_c^2 as

$$\phi_c(x) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{i\pi n x}{L}} \quad \text{and} \quad \phi_c^2(x) = \sum_{n=-\infty}^{\infty} b_n e^{\frac{i\pi n x}{L}}. \tag{4.2}$$

Replacing (4.2) in (1.3) we obtain that

$$c a_n \left[1 + \frac{\pi}{L} |n| \right] - a_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} a_m a_{n-m}, \quad \forall n \in \mathbb{Z}. \tag{4.3}$$

Inspired by (4.1), we choose $a_n = \frac{2\pi c}{L} e^{-\eta|n|}$, with $\eta > 0$, for all $n \in \mathbb{Z}$, then we obtain

$$\sum_{m=-\infty}^{\infty} a_m a_{n-m} = \frac{4\pi^2 c^2}{L^2} e^{-\eta|n|} \left[|n| + 1 + 2 \sum_{k=1}^{\infty} e^{-2\eta k} \right] = \frac{4\pi^2 c^2}{L^2} e^{-\eta|n|} \big(|n| + \coth \eta \big).$$

Therefore, we conclude from (4.3) that

$$c \left[1 + \frac{\pi}{L} |n| \right] - 1 = \frac{\pi c}{L} \big(|n| + \coth \eta \big), \quad \forall n \in \mathbb{Z}. \tag{4.4}$$

Then, from (4.1) by denoting $\eta = \frac{2\pi w}{2(w-1)L}$ and considering c such that $0 < \frac{c}{c-1} < \frac{L}{\pi}$, we choose $w = w(c) > 1$ such that $\tanh(\eta) = \frac{\pi c}{(c-1)L}$. Then, from (4.4) we have that $\phi_c = \frac{c}{w} \psi_{w(c)}$. Therefore we obtain that ϕ_c has the form established in (1.5) with $\eta > 0$ satisfying $\tanh(\eta) = \frac{\pi c}{(c-1)L}$.

Remark 4.1.

(1) Note that, from the fact that $c \neq 1$ satisfies $0 < \frac{c}{c-1} < \frac{L}{\pi}$ we have three cases to consider:
   (a) If $L = \pi$, then $c \in (-\infty, 0)$.
   (b) If $L < \pi$, then $c \in (\frac{L}{L-\pi}, 0)$.
   (c) If $L > \pi$, then $c \in (-\infty, 0) \cup (\frac{L}{L-\pi}, +\infty)$.

(2) Observe that the sign of the solution ϕ_c depends on the sign of c. Since we are interested in positive solutions (to apply the theory in [11]) we will suppose that $L > \pi$ and $c > \frac{L}{L-\pi}$.

(3) If we consider $c = 1$ in (1.3), then the unique real smooth solution that we obtain is $\phi \equiv 0$. In fact, in this case ϕ satisfies $\mathcal{H}\phi' - \frac{1}{2}\phi^2 = 0$. Taking Fourier transform we arrive at

$$\frac{2\pi}{L} |n| \widehat{\phi}(n) - \sum_{k=-\infty}^{+\infty} \widehat{\phi}(n-k) \widehat{\phi}(k) = 0, \quad \forall n \in \mathbb{N}.$$

In particular for $n=0$, $\sum_{k=-\infty}^{+\infty}\widehat{\phi}(-k)\widehat{\phi}(k)=0$. Since ϕ is a real solution we have that $\widehat{\phi}(-k)=\overline{\widehat{\phi}(k)}$. Thus using the smoothness of ϕ we get that $\phi\equiv 0$.

Lastly, since $\eta(c)=\tanh^{-1}(\frac{c\pi}{(c-1)L})$ is a differentiable function if $c\neq 1$, we have the next result.

Proposition 4.2. *Let $L>\pi$. Then the curve $c\in(\frac{L}{L-\pi},+\infty)\to\phi_c\in H_{per}^{\frac{1}{2}}([-L,L])$ is of class C^n, where ϕ_c is given by* (1.5)*. Furthermore, since $c>0$, we have that $\phi_c>0$.*

5. Spectral analysis for the rBO

This section is dedicated to study specific spectral properties associated to the linear operator $\mathcal{L}=c\mathcal{H}\partial_x-1+c-\phi_c$, where ϕ_c is the periodic solution (1.5) given by Proposition 4.2 with fundamental period $2L$, $L>\pi$ and $c>\frac{L}{L-\pi}$. This information will be basic in our stability theory for the rBO equation. Our analysis focuses on the periodic eigenvalue problem considered on $[-L,L]$

$$\begin{cases}\chi\in H_{per}^1,\\ \mathcal{L}\chi=\lambda\chi.\end{cases}\tag{5.1}$$

We will show that problem (5.1) determines exactly the existence of a single negative eigenvalue, which is simple; zero is also a simple eigenvalue with eigenfunction ϕ_c' and the remainder of the spectrum is bounded away from zero.

Next, from Eq. (1.3) we obtain immediately that $\mathcal{L}\phi_c'=0$. The theory of compact self-adjoint operators applied to (5.1) implies that the spectrum of $\mathcal{L}$ is a countable infinity set of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ with $\lambda_0<\lambda_1\leqslant\lambda_2\leqslant\cdots$, where $\lambda_n\to\infty$ as $n\to\infty$ (see for instance Proposition 3.1 in [11]). Now, for the sake of completeness we present here a summary of the theory developed by Angulo and Natali [11]. In their work, it was studied the existence and the nonlinear stability of periodic traveling wave solutions for the family of equations

$$u_t+u^pu_x-(Mu)_x=0,\tag{5.2}$$

where $p\geqslant 1$ is an integer and M is a differential or pseudo-differential operator in the context of periodic functions. The operator M was defined through Fourier multipliers as $\widehat{Mg}(k)=\zeta(k)\widehat{g}(k)$, for all $k\in\mathbb{Z}$, whose symbol ζ is a real, measurable, locally bounded and even function satisfying

$$A_1|n|^{m_1}\leqslant\left|\zeta(n)\right|\leqslant A_2\left(1+|n|\right)^{m_2},\tag{5.3}$$

for $1\leqslant m_1\leqslant m_2$, $|n|\geqslant n_0$, $\zeta(n)>b$ for all $n\in\mathbb{Z}$ and $A_i>0$, $i=1,2$. The main result in [11] reads as follows.

Theorem 5.1. *Consider the self-adjoint operator $\mathcal{L}_0:D(\mathcal{L}_0)\to L_{per}^2([-L,L])$ given by*

$$\mathcal{L}_0u=(M+c)u-\varphi^pu,$$

where $D(\mathcal{L}_0)$ is dense in $L_{per}^2([-L,L])$ and φ_c is a periodic traveling wave solution of Eq. (5.2)*. Suppose that φ_c is a positive even solution of* (5.2) *such that $\widehat{\varphi_c}>0$ and $\widehat{\varphi_c^p}\in PF(2)$ discrete (see* (5.4) *below). Then,*

(a) *$\mathcal{L}_0$ has a unique negative eigenvalue λ, and it is simple;*
(b) *the eigenvalue 0 is simple.*

We say that a sequence $\alpha = (\alpha_n)_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ belongs to $PF(2)$ discrete if

$$\begin{aligned} &\text{(i)} \quad \alpha_n > 0 \quad \text{for all } n \in \mathbb{Z}, \\ &\text{(ii)} \quad \alpha_{n_1-m_1}\alpha_{n_2-m_2} - \alpha_{n_1-m_2}\alpha_{n_2-m_1} > 0 \quad \text{for } n_1 < n_2 \text{ and } m_1 < m_2. \end{aligned} \tag{5.4}$$

Although the rBO does not have the form (5.2), we still can apply Theorem 5.1 to get the required spectrum information for the operator in (1.8). Indeed, assume $L > \pi$, $c > \frac{L}{L-\pi}$ and define $M := c\mathcal{H}\partial_x - 1$, then $\mathcal{L} = (M + c) - \phi_c$. In this case $\widehat{Mf}(k) = \zeta(k)\widehat{f}(k)$, where $\zeta(k) \equiv c|k| - 1$, for all $k \in \mathbb{Z}$. Then for $A_1 = c/2$, $A_2 = c + 1$ and $m_1 = m_2 = 1$, it easy to see that there exists $N_0 \in \mathbb{N}$ such that ζ satisfies (5.3) for all $k \geqslant N_0$.

From the analysis in Section 4, we have that the Fourier transform of ϕ_c in (1.5) is given by $a_n = \widehat{\phi_c}(n) = \frac{2\pi c}{L} e^{-\eta|n|} > 0$, $n \in \mathbb{N}$.

Before continuing with the study of the spectrum of $\mathcal{L}$, we note that Theorem 5.1 is still valid if we replace (ii) in (5.4) by the weaker condition

$$\text{(ii}'\text{)} \quad \begin{cases} \alpha_{n_1-m_1}\alpha_{n_2-m_2} - \alpha_{n_1-m_2}\alpha_{n_2-m_1} \geqslant 0, & \text{for all } n_1 < n_2 \text{ and } m_1 < m_2, \\ \alpha_{n_1-m_1}\alpha_{n_2-m_2} - \alpha_{n_1-m_2}\alpha_{n_2-m_1} > 0, & \text{if } n_1 < n_2,\ m_1 < m_2,\ n_2 > m_1, \text{ and } n_1 < m_2. \end{cases}$$

In order to prove that $\widehat{\phi_c} \in PF(2)$ discrete, we just have to show that $a_n = \frac{2c\pi}{L} e^{-\eta|n|}$ satisfies (ii$'$), which is equivalent to prove that

$$\begin{aligned} &\text{(a)} \quad |n_1 - m_1| + |n_2 - m_2| \leqslant |n_1 - m_2| + |n_2 - m_1|, \quad \text{if } n_1 < n_2 \text{ and } m_1 < m_2, \text{ and} \\ &\text{(b)} \quad |n_1 - m_1| + |n_2 - m_2| < |n_1 - m_2| + |n_2 - m_1|, \quad \text{if } n_1 < n_2,\ m_1 < m_2,\ n_2 > m_1 \\ &\qquad \text{and } n_1 < m_2. \end{aligned}$$

As the proofs of (a) and (b) are easy, so we skip the details. Then as a consequence of this analysis we have the next result.

Proposition 5.2. *Let ϕ_c be the periodic wave solution given by Proposition* 4.2, *with $c > \frac{L}{L-\pi}$ and $L > \pi$. Then, the linear operator $\mathcal{L}$ define by* (1.8) *with domain $H^1_{per}([-L, L])$ has its first two eigenvalues simple with zero being the second one. Moreover, the remainder of the spectrum is formed by a discrete sequence of eigenvalues which converge to $+\infty$.*

6. Stability of periodic traveling wave solutions for the rBO in $H^{\frac{1}{2}}_{per}$

In this section we extend the Lyapunov method developed by Benjamin [13], Bona [16] and Weinstein [40] for studying the nonlinear stability of the periodic solutions ϕ_c given by Proposition 4.2. The notion of stability that we will use is the *orbital stability*, see Definition 7.2 below. So, in the case of the rBO equation we take $m_2 = 1$. Since we do not have a global well-posedness result in $H^{\frac{1}{2}}_{per}$, we need to choose $s_0 > 1/2$.

Before establishing our main stability result we will prove a useful lemma.

Lemma 6.1. *Let ϕ_c be the wave solution given by Proposition* 4.2 *with $c \in (\frac{L}{L-\pi}, +\infty)$ and $L > \pi$. Then, the linear operator $\mathcal{L} = c\mathcal{H}\partial_x - 1 + c - \phi_c$ satisfies*

$$\text{(a)} \quad \alpha := \inf\bigl\{(\mathcal{L}f, f)\colon\ \|f\|_{L^2_{per}} = 1 \text{ and } \bigl(f, \phi_c + \mathcal{H}\phi_c'\bigr) = 0\bigr\} = 0, \tag{6.1}$$

$$\text{(b)} \quad \beta := \inf\bigl\{(\mathcal{L}f, f)\colon\ \|f\|_{L^2_{per}} = 1,\ \bigl(f, \phi_c + \mathcal{H}\phi_c'\bigr) = 0 \text{ and } \bigl(f, \phi_c\phi_c'\bigr) = 0\bigr\} > 0. \tag{6.2}$$

Proof. (a) Because ϕ_c is bounded, it is inferred that α is finite. Since $(\phi, \phi_c + \mathcal{H}\phi_c') = 0$ and $\mathcal{L}\phi_c' = 0$ it follows that $\alpha \leqslant 0$. Next we show that the inf in (6.1) is attained. In fact, since α is finite, there exists a sequence $\{f_j\}_{j=0}^{\infty} \subset H_{per}^{\frac{1}{2}}$ with $\|f_j\|_{L^2_{per}} = 1, (f_j, \phi_c + \mathcal{H}\phi_c') = 0$ and $(\mathcal{L}f_j, f_j) \to \alpha$ as $j \to \infty$. It follows that $\|f_j\|_{H_{per}^{\frac{1}{2}}}$ is uniformly bounded as j varies. So, there exists a subsequence of f_j, which we denote $\{f_j\}$ again and a function $f^* \in H_{per}^{\frac{1}{2}}$ such that $f_j \rightharpoonup f^*$ in $H_{per}^{\frac{1}{2}}$. Since the embedding $H_{per}^{\frac{1}{2}} \hookrightarrow L^2_{per}$ is compact we obtain that $(f^*, \phi_c + \mathcal{H}\phi_c') = 0$ and $(\phi_c f_j, f_j) \to (\phi_c f^*, f^*)$ when $j \to \infty$. Thus $f^* \neq 0$ and since the weak convergence is lower continuous we obtain for $D_x^{\frac{1}{2}}$ defined as $\widehat{D_x^{\frac{1}{2}} f}(k) = |k|^{1/2}\widehat{f}(k)$, that

$$\left\|D_x^{\frac{1}{2}} f^*\right\|^2_{L^2_{per}} \leqslant \liminf_{j\to\infty} \left\|D_x^{\frac{1}{2}} f_j\right\|^2_{L^2_{per}}.$$

Now, define $f = \frac{f^*}{\|f^*\|_{L^2}}$, then $(f, \phi_c + \mathcal{H}\phi_c') = 0$, $\|f\|_{L^2_{per}} = 1$ and

$$\alpha \leqslant (\mathcal{L}f, f) \leqslant \frac{\alpha}{\|f^*\|^2_{L^2_{per}}} \leqslant \alpha.$$

Therefore, α is a minimum. Next, we show that $\alpha \geqslant 0$. For this purpose, we apply Lemma E1 in Weinstein [40] (which works in the periodic setting) with $A = \mathcal{L}$ and $R = \phi_c + \mathcal{H}\phi_c'$. In fact, from Proposition 5.2, $\mathcal{L}$ has the spectral required properties in [40]. Next, we need to find χ such that

$$\mathcal{L}\chi = \phi_c + \mathcal{H}\phi_c', \quad \text{and} \quad \left(\chi, \phi_c + \mathcal{H}\phi_c'\right) \leqslant 0.$$

From Proposition 4.2 we have that the mapping $c \in (\frac{L}{L-\pi}, +\infty) \mapsto \phi_c \in H_{per}^{\frac{1}{2}}([-L, L])$ is of class C^1, so by differentiating (1.3) with regard to c we obtain that $\chi = -\frac{d}{dc}\phi_c$ satisfies $\mathcal{L}\chi = \phi_c + \mathcal{H}\phi_c'$. Now, we observe that

$$\begin{aligned}
\left(\chi, \phi_c + \mathcal{H}\phi_c'\right) &= -L\frac{d}{dc}\sum_{n=-\infty}^{\infty}\left(1+|n|\right)\left|\widehat{\phi_c}(n)\right|^2 \\
&= -\frac{8\pi^2 c}{L}\sum_{n=-\infty}^{\infty}\left(1+|n|\right)e^{-4\pi|n|\eta} + \frac{8\pi^2c^2}{L}\frac{d\eta}{dc}\sum_{n=-\infty}^{\infty}\left(1+|n|\right)|n|e^{-4\pi|n|\eta}.
\end{aligned}$$

Then, from (1.6) and using the fact that $c > \frac{L}{L-\pi}$, we obtain

$$\frac{d\eta}{dc} = \frac{d}{dc}\left(\tanh^{-1}\left(\frac{c\pi}{(c-1)L}\right)\right) = -\frac{\pi}{(c-1)^2L}\left(1 - \left(\frac{c\pi}{(c-1)L}\right)^2\right)^{-1} < 0.$$

Therefore $(\chi, \phi_c + \mathcal{H}\phi_c') < 0$, which gives us that $\alpha \geqslant 0$. This finishes the proof of (6.1).

(b) We infer from part (a) that $\beta \geqslant 0$. Suppose that $\beta = 0$. Then we can find a function f such that $\|f\|_{L^2_{per}} = 1$ and $(f, \phi_c + \mathcal{H}\phi_c') = (f, \phi_c\phi_c') = (\mathcal{L}f, f) = 0$. Thus, there exist γ, θ, ν such that

$$\mathcal{L}f = \gamma f + \theta\left(\phi_c + \mathcal{H}\phi_c'\right) + \nu\phi_c\phi_c',$$

bringing on $\gamma = \nu = 0$. Now, consider $\chi = -\frac{d}{dc}\phi_c$, it follows that $\mathcal{L}(f - \theta\chi) = 0$, then $(f - \theta\chi, \phi_c + \mathcal{H}\phi_c') = 0 = (f, \phi_c + \mathcal{H}\phi_c') - \theta(\chi, \phi_c + \mathcal{H}\phi_c')$. Hence, $\theta = 0$, because $(\chi, \phi_c + \mathcal{H}\phi_c') \neq 0$, therefore

$\mathcal{L}f = 0$, and so there exists a $\lambda \in \mathbb{R} - \{0\}$ such that $f = \lambda\phi_c'$, and hence ϕ_c' is orthogonal to $\phi_c\phi_c'$, which is a contradiction. Therefore $\beta > 0$ and the proof is completed. □

We note that from (6.2) and the specific form of $\mathcal{L}$ that if $(f, \phi_c + \mathcal{H}\phi_c') = 0$ and $(f, \phi_c\phi_c') = 0$, then there exists $\beta_0 > 0$ such that

$$(\mathcal{L}f, f) \geqslant \beta_0 \|f\|^2_{H^{\frac{1}{2}}_{per}}.$$

The main result of this section reads as follows.

Theorem 6.2. *Let $L > \pi$ and ϕ_c be the periodic wave solution for the rBO equation given by Proposition 4.2 with $c \in (\frac{L}{L-\pi}, +\infty)$. Then the orbit $\mathcal{O}_{\phi_c}$ is nonlinear stable with regard to the periodic flow generated by the rBO equation.*

Proof. The proof is based in the ideas developed in [11,13,16,41]. We shall give only an outline of the proof. Initially, we note from (1.7) that $F'(u) = u + \mathcal{H}u_x$ and $E'(u) = \mathcal{H}u_x - \frac{1}{2}u^2$, then ϕ_c is a critical point of the functional $\mathcal{B} := E + (c-1)F$. Additionally, since $F''(u) = 1 + \mathcal{H}\partial_x$ and $E''(u) = \mathcal{H}\partial_x - u$, we have

$$E''(\phi_c) + (c-1)F''(\phi_c) = c\mathcal{H}\partial_x + (c-1) - \phi_c = \mathcal{L}.$$

Now, define for $r \in [-L, L]$ and $t \in \mathbb{R}$,

$$\Omega_t(r) := \left\| D^{\frac{1}{2}}u(\cdot + r, t) - D^{\frac{1}{2}}\phi_c \right\|^2_{L^2_{per}} + \frac{c-1}{c}\left\| u(\cdot + r, t) - \phi_c \right\|^2_{L^2_{per}}.$$

Then, using standard arguments (see [13,16]) there exists an interval of time $I = [0, T]$ such that the $\inf_{r\in\mathbb{R}} \Omega_t(r)$ is attained in $\gamma = \gamma(t)$ for every $t \in I$. Hence,

$$\Omega_t\bigl(\gamma(t)\bigr) = \inf_{r\in\mathbb{R}} \Omega_t(r). \tag{6.3}$$

Consider the perturbation of the periodic traveling wave ϕ_c

$$u(x + \gamma, t) = \phi_c(x) + v(x, t) \tag{6.4}$$

for $t \in [0, T]$ and $\gamma = \gamma(t)$ determined by (6.3). Then, differentiating $\Omega_t(r)$ with respect to r, evaluating at values that minimize $\Omega_t(r)$ and using (6.4) we obtain that v satisfies the compatibility relation

$$\int_{-L}^{L} \phi_c'(x)\phi_c(x)v(x, t)\,dx = 0, \tag{6.5}$$

for all $t \in [0, T]$. Next, using that E and F are conserved quantities, the representation (6.4), the embedding $H^{\frac{1}{2}}_{per}([-L, L]) \hookrightarrow L^r([-L, L])$ for all $r \geqslant 2$, and the fact that ϕ_c satisfies (1.3), we conclude

$$\Delta\mathcal{B}(t) = \mathcal{B}(u_0) - \mathcal{B}(\phi_c) = \mathcal{B}\bigl(\phi_c + v(\cdot, t)\bigr) - \mathcal{B}(\phi_c) \geqslant \frac{1}{2}(\mathcal{L}v, v) - c_0\|v\|^3_{H^{\frac{1}{2}}_{per}}, \tag{6.6}$$

where c_0 is a positive constant. To obtain our result we need to establish a suitable bound for the quadratic form in (6.6). Initially, we consider the normalization $F(u_0) = F(\phi_c)$, then

$$\int_{-L}^{L} u^2(t) + \left(D^{\frac{1}{2}} u(t)\right)^2 dx = \int_{-L}^{L} \phi_c^2 + \left(D^{\frac{1}{2}} \phi_c\right)^2 dx$$

for all $t \in [0, T]$. By (6.4) it follows

$$-2\left(v, \phi_c + \mathcal{H}\phi_c'\right) = \left\| v(t) \right\|^2_{L^2_{per}} + \left\| D^{\frac{1}{2}} v \right\|^2_{L^2_{per}}.$$

Without loss of generality, we suppose that $\|\phi_c + \mathcal{H}\phi_c'\|_{L^2_{per}} = 1$. Define $v_{\parallel}$ and $v_{\perp}$ as $v_{\parallel} = (v, \phi_c + \mathcal{H}\phi_c')(\phi_c + \mathcal{H}\phi_c')$ and $v_{\perp} = v - v_{\parallel}$. So, $(v_{\perp}, \phi_c + \mathcal{H}\phi_c') = 0$ and $(v_{\perp}, \phi_c \phi_c') = 0$. By (6.5) and Lemma 6.1 it follows that

$$(\mathcal{L}v_{\perp}, v_{\perp}) \geqslant \beta \|v_{\perp}\|^2_{L^2_{per}} \geqslant \beta \|v\|^2_{L^2_{per}} - \tilde{\beta}_3 \|v\|^4_{H^{\frac{1}{2}}_{per}} \tag{6.7}$$

with $\beta, \tilde{\beta}_3 > 0$. Again, without loss of generality suppose that $(\mathcal{L}(\phi_c + \mathcal{H}\phi_c'), \phi_c + \mathcal{H}\phi_c') < 0$, then

$$(\mathcal{L}v_{\parallel}, v_{\parallel}) \geqslant -\tilde{\beta}_4 \|v\|^4_{H^{\frac{1}{2}}_{per}}. \tag{6.8}$$

Furthermore, using the Cauchy–Schwarz inequality we get

$$(\mathcal{L}v_{\parallel}, v_{\perp}) \geqslant -\tilde{\beta}_2 \left\| v(t) \right\|^3_{H^{\frac{1}{2}}_{per}}, \tag{6.9}$$

where $\tilde{\beta}_j > 0$, for $j = 3, 4$. Now, using (6.7), (6.8), (6.9) and the specific form for $\mathcal{L}$ we arrive at

$$(\mathcal{L}v, v) \geqslant \beta_0 \left\| v(t) \right\|^2_{H^{\frac{1}{2}}_{per}} - \beta_1 \left\| v(t) \right\|^3_{H^{\frac{1}{2}}_{per}} - \beta_2 \left\| v(t) \right\|^4_{H^{\frac{1}{2}}_{per}}, \tag{6.10}$$

where $\beta_j > 0$, for $j = 0, 1, 2$. Hence, from (6.3), (6.6) and (6.10) it follows that for all $t \in [0, T]$

$$\Delta\mathcal{B}(t) \geqslant g\left(\left\| v(t) \right\|_{\frac{1}{2}, c}\right), \tag{6.11}$$

where $\|f\|^2_{\frac{1}{2}, c} := \|D^{\frac{1}{2}} f\|^2_{L^2_{per}} + \frac{c-1}{c} \|f\|^2_{L^2_{per}}$ and $g(s) = \eta s^2 - \sum_{k=3}^{4} d_k(c) s^k$, with $\eta, d_k > 0$. The essential properties of g are $g(0) = 0$ and $g(s) > 0$ for all s small. The stability result is an immediately consequence of (6.11). In fact, let $\epsilon > 0$ small enough such that $g(\epsilon) > 0$. Then using the properties that $\mathcal{B}$ is uniformly continuous on $S := \{u \in H^{\frac{1}{2}}_{per}\colon\ F(u) = F(\phi_c)\}$, $\Delta\mathcal{B}(t)$ is constant in time and $t \mapsto \|v(t)\|^2_{\frac{1}{2}, c}$ is a continuous function, we have that there is $\delta(\epsilon) > 0$ such that if $v \in S$ and $\|v - \phi_c\|_{\frac{1}{2}, c} < \delta$ then for $t \in [0, T]$,

$$g\left(\left\| v(t) \right\|_{\frac{1}{2}, c}\right) \leqslant \Delta\mathcal{B}(0) \leqslant \left| \Delta\mathcal{B}(0) \right| < g(\epsilon) \quad \Rightarrow \quad \left\| v(t) \right\|_{\frac{1}{2}, c} < \epsilon. \tag{6.12}$$

Thus, it shows that $\mathcal{O}_{\phi_c}$ is orbitally stable in $H^{\frac{1}{2}}_{per}([-L,L])$ relative to small perturbations which preserve the $H^{\frac{1}{2}}_{per}$ norm. The inequality (6.12) is still true for all $t > 0$, this is an immediately consequence of the fact that the mapping $t \mapsto \inf_{r\in\mathbb{R}} \Omega_t(r)$ is continuous (see Bona [16]).

To prove stability to general perturbations, we use that the mapping $c \in (\frac{L}{L-\pi}, +\infty) \mapsto \phi_c \in H^{\frac{1}{2}}_{per}([-L,L])$ is continuous, the mapping $c \in (\frac{L}{L-\pi}, +\infty) \mapsto F(\phi_c)$ is strictly increasing, the above theory and the triangle inequality (see [16,41,6]). Then, Theorem 6.2 is proved. □

7. Stability criterium for BBM-type equations

In this section we extend the theory developed for the rBO equation to the family of Eq. (1.9), with H defined as

$$\widehat{Hu}(n) = \alpha(n)\widehat{u}(n), \quad \forall n \in \mathbb{Z}.$$

The symbol α is assumed to be a real, mensurable, locally bounded, even function on $\mathbb{R}$ and satisfying the conditions in (5.3). The traveling wave solutions ϕ_c of (1.9) satisfy

$$cH\phi_c + (c-1)\phi_c - \frac{1}{p+1}\phi_c^{p+1} = 0. \tag{7.1}$$

As it is well-known Eq. (1.9) has the following two conservation laws

$$E(u) = \frac{1}{2}\int_{-L}^{L} uHu - \frac{2}{(p+1)(p+2)}u^{p+2}\,dx \quad \text{and} \quad F(u) = \frac{1}{2}\int_{-L}^{L} uHu + u^2\,dx,$$

and so using them we have from Eq. (7.1) that the periodic solution ϕ_c satisfies $E'(\phi_c) + (c-1) \times F'(\phi_c) = 0$. Now, define

$$\mathcal{L} := E''(\phi_c) + (c-1)F''(\phi_c) = cH + (c-1) - \phi_c^p. \tag{7.2}$$

Then the operator $\mathcal{L} : D(\mathcal{L}) \to L^2_{per}([-L,L])$ is linear, closed, not bounded and self-adjoint defined on a dense subset of $L^2_{per}([-L,L])$. Also it is easy to see that $\mathcal{L}\phi_c' = 0$. Following the proof of stability for the rBO equation we obtain the following main conditions (see also Grillakis, Shatah and Strauss [23]):

(C_0) there is a nontrivial smooth curve of periodic solutions for (7.1) of the form $c \in I \subset \mathbb{R} \to \phi_c \in H^{m_2}_{per}([-L,L])$;

(C_1) $\mathcal{L}$ has an unique negative eigenvalue and it is simple;

(C_2) the eigenvalue zero is simple;

(C_3)
$$\frac{d}{dc}\int_{-L}^{L} \left[\phi_c H\phi_c + \phi_c^2\right] dx > 0. \tag{7.3}$$

Next, we give sufficient conditions to obtain conditions (C_1) and (C_2) for the operator $\mathcal{L}$ associated to the problem (1.9). The principal stability criterium is the following.

Theorem 7.1. *Let ϕ_c be a positive even solution of* (7.1)*. Assume that $\widehat{\phi_c} > 0$ and $\widehat{\phi_c^p} \in PF(2)$ discrete, then* (C_1) *and* (C_2) *in* (7.3) *hold for the operator $\mathcal{L}$ in* (7.2)*.*

Proof. Note that the operator $\mathcal{L}$ can be written as $\mathcal{L}u = (M+c)u - \phi_c^p u$, where $M = cH - 1$. The symbol of M is $\zeta(n) = c\alpha(n) - 1$. So, it is easy to see that for all $c \neq 0$ there exists $N_0 \in \mathbb{N}$ such that

$$B_1|n|^{m_1} \leqslant \left|\zeta(n)\right| \leqslant B_2\big(1+|n|\big)^{m_2}, \quad \forall n \geqslant N_0,$$

where $B_1 = \frac{cA_1}{2}$ and $B_2 = cA_2 + 1$. Then we can apply Theorem 5.1 to obtain that (C_1) and (C_2) hold for the operator $\mathcal{L}$. $\square$

7.1. Stability of cnoidal waves to the BBM equation in H^1_{per}

Here, we apply Theorem 7.1 to obtain the orbital stability of the periodic traveling wave solutions of cnoidal type associated to the BBM equation. Concerning the global well-posedness problem in H^s_{per}, with $s \geqslant 1$, we refer to Benjamin, Bona and Mahoney [15] and Albert [1]. For a global well-posedness theory in L^2_{per} we refer to Chen [20]. Next, we present the definition of stability.

Definition 7.2. Let ϕ_c be a periodic traveling-wave solution with period $2L$ of (7.1). We define the set $\mathcal{O}_{\phi_c} \subset H^{\frac{m_2}{2}}_{per}$, called the orbit generated by ϕ_c, as

$$\mathcal{O}_{\phi_c} = \big\{f\colon\ f = \phi_c(\cdot + r) \text{ for some } r \in \mathbb{R}\big\},$$

and, for any $\gamma > 0$, the set $U_\gamma \subset H^{\frac{m_2}{2}}_{per}$ by

$$U_\gamma = \Big\{f\colon\ \inf_{g \in \Omega_{\phi_c}} \|f - g\|_{H^{\frac{m_2}{2}}_{per}} < \gamma\Big\}.$$

With this terminology, we say that ϕ_c is (orbitally) stable in $H^{\frac{m_2}{2}}_{per}$ by the flow generated by (1.9) if the following hold:

(i) There is s_0 such that $H^{s_0}_{per} \subset H^{\frac{m_2}{2}}_{per}$ and the initial value problem associated to (1.9) is globally well-posed in $H^{s_0}_{per}$.
(ii) For every $\epsilon > 0$, there is $\delta > 0$ such that, for all $u_0 \in U_\delta \cap H^{s_0}_{per}$, the solution u of (1.9) with $u(0,x) = u_0(x)$ satisfies $u(t) \in U_\epsilon$ for all $t > 0$. Otherwise, we say that ϕ_c is unstable in $H^{\frac{m_2}{2}}_{per}$.

Remark 7.3. The choice of a second space $H^{s_0}_{per}$ in Definition 7.2 is because the local-well posedness problem may not be easy to be obtained in the energy space $H^{\frac{m_2}{2}}_{per}$. An example of this situation is the case of the rBO equation.

The proof of the next general stability theorem follows the ideas used in the rBO stability theorem.

Theorem 7.4. *Let ϕ_c be a periodic traveling-wave solution of* (7.1)*, and suppose that part* (i) *of the definition of stability holds. Suppose also that the operator $\mathcal{L}$ defined previously in* (7.2) *has properties* (C_1) *and* (C_2) *in* (7.3)*. Choose $\chi \in L^2_{per}$ such that $\mathcal{L}\chi = \phi_c + H\phi_c$, and define $I = (\chi, \phi_c + H\phi)_{L^2_{per}}$. If $I < 0$ then ϕ_c is stable.*

Remark 7.5. In our cases the function χ in Theorem 7.4 is $\chi = -\frac{d}{dc}\phi_c$.

Next, we apply Theorem 7.4 to the BBM equation. The periodic traveling wave solutions that we will study are the cnoidal wave-profile given by the formula

$$\phi_c(x) = \beta_2 + (\beta_3 - \beta_2)\,\mathrm{cn}^2\left(\sqrt{\frac{\beta_3 - \beta_1}{12c}}x; k\right), \tag{7.4}$$

which satisfy

$$c\phi_c'' - (c-1)\phi_c + \frac{1}{2}\phi_c^2 = 0. \tag{7.5}$$

Since is not immediate to find the Fourier transform of the cnoidal profile ϕ_c, we will use the Poisson Summation theorem. Indeed, the solitary-wave solution for (7.5) is

$$\varphi_w(x) = 3(w-1)\,\mathrm{sech}^2\left(\sqrt{\frac{w-1}{w}}\frac{x}{2}\right),$$

with $w > 1$, whose Fourier transform is given by $\widehat{\varphi}_w^{\mathbb{R}}(\xi) = 12\pi\xi w\,\mathrm{csch}(\sqrt{\frac{w}{w-1}}\pi\xi)$. Then we obtain the L-periodic function

$$\psi_w(\xi) := \frac{12w}{L}\sqrt{\frac{w-1}{w}} + \frac{24\pi w}{L^2}\sum_{n=1}^{\infty} n\,\mathrm{csch}\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right)\cos\left(\frac{2\pi n\xi}{L}\right), \tag{7.6}$$

where $w > 1$ will be chosen in such way that ψ_w (may be with some scaling) becomes a periodic traveling wave for the BBM equation in the form (7.4).

First, we consider the Fourier expansion of the *dnoidal* Jacobi elliptic function (see Oberhettinger [35]) of period L, namely,

$$K^2\left[\mathrm{dn}^2\left(\frac{2K\xi}{L}; k\right) - \frac{E}{K}\right] = 2\pi\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos\left(\frac{2\pi n\xi}{L}\right),$$

where $K = K(k)$ and $E = E(k)$ denote the complete elliptic integrals of the first and second type, respectively, $q = e^{-(\frac{\pi K'}{K})}$, $K'(k) = K(\sqrt{1-k^2})$ and

$$\frac{q^n}{1-q^{2n}} = \frac{1}{2}\,\mathrm{csch}\left(\frac{n\pi K'}{K}\right).$$

Therefore

$$K^2\left[\mathrm{dn}^2\left(\frac{2K\xi}{L}; k\right) - \frac{E}{K}\right] = \pi\sum_{n=1}^{\infty} n\,\mathrm{csch}\left(\frac{n\pi K'}{K}\right)\cos\left(\frac{2\pi n\xi}{L}\right). \tag{7.7}$$

Then, motivated by the form of the series that determines ψ_w, we consider $\phi_c(x) = a + b[\mathrm{dn}^2(d\xi; k) - \frac{E}{K}]$ a periodic traveling wave solution for (7.5) with period L. Thus, the following non-linear system is obtained:

$$\begin{cases} \dfrac{b^2}{2} - 6cbd^2 = 0, \\ 4bd^2c\big(1+k'^2\big) + ab - b^2\dfrac{E}{K} - (c-1)b = 0, \\ \dfrac{a^2}{2} - ab\dfrac{E}{K} + \dfrac{b^2}{2}\left(\dfrac{E}{K}\right)^2 - (c-1)a + (c-1)b\dfrac{E}{K} - 2cbd^2k'^2 = 0. \end{cases}$$

Since ϕ_c is periodic of period L it follows that $d = \frac{2K}{L}$. Then, the first equation of the system above implies $b = \frac{48cK^2}{L^2}$. Substituting those values at the second equation we get

$$aL^2 = 48cKE - 16K^2c\big(2-k^2\big) + cL^2 - L^2.$$

Plugging the values of aL^2, d and b in the third equation we arrived at the following quadratic equation

$$\big[256K^4\big(1-k^2+k^4\big) - L^4\big]c^2 + 2cL^4 - L^4 = 0. \tag{7.8}$$

From the last identity we get that $L^2 = \frac{16c\sqrt{1-k^2+k^4}K^2(k)}{c-1}$. Since $c > 1$, we have that L satisfies $L > 2\pi$. Solving Eq. (7.8) for c, we obtain the solutions

$$c = \frac{L^2}{L^2 + 16K^2\sqrt{1-k^2+k^4}} \quad \text{and} \quad c = \frac{L^2}{L^2 - 16K^2\sqrt{1-k^2+k^4}}.$$

Using again that $c > 1$, we choose c as

$$c = \frac{L^2}{L^2 - 16K^2\sqrt{1-k^2+k^4}}. \tag{7.9}$$

Note that $c > 1$ implies that exists $k_L \in (0,1)$ such that $L^2 - 16K^2\sqrt{1-k^2+k^4} > 0$ for all $k \in (0, k_L)$. Now, from (7.9) we have that for $L > 2\pi$ fixed, the function $k \mapsto c(k)$ is an increasing function on $(0, k_L)$ (see Fig. 1), therefore $c \in (c^*, +\infty)$, for all $k \in (0, k_L)$, where $c^* = \frac{L^2}{L^2 - 4\pi^2}$.

From above analysis we can write ϕ_c in terms of the Jacobi elliptic function *cnoidal* in the form (7.4), where

$$\beta_2 = \frac{16cK^2(2k'^2-1)}{L^2} + c - 1, \qquad \beta_3 = \frac{16cK^2}{L^2}\big(1+k^2\big) + c - 1$$

and β_1 is such that $\beta_3 - \beta_1 = \frac{48cK^2}{L^2}$. Then, by making a similar analysis as in Angulo [7] (i.e., using the Implicit Function Theorem), we obtain a smooth curve of positive cnoidal waves with the same period L in the form

$$c \in \big(c^*, +\infty\big) \longmapsto \phi_c \in H^n_{per}\big([0,L]\big),$$

for all $n \in \mathbb{N}$, and such that $k := k(c)$ is a strictly increasing smooth function of c.

Next, we choose the speed w of the solitary-wave solution φ_w in such way that this will become ψ_w in (7.6) in a periodic traveling wave solution for the BBM equation. In fact, define for $c \in (c^*, +\infty)$, $w = w(c)$ as

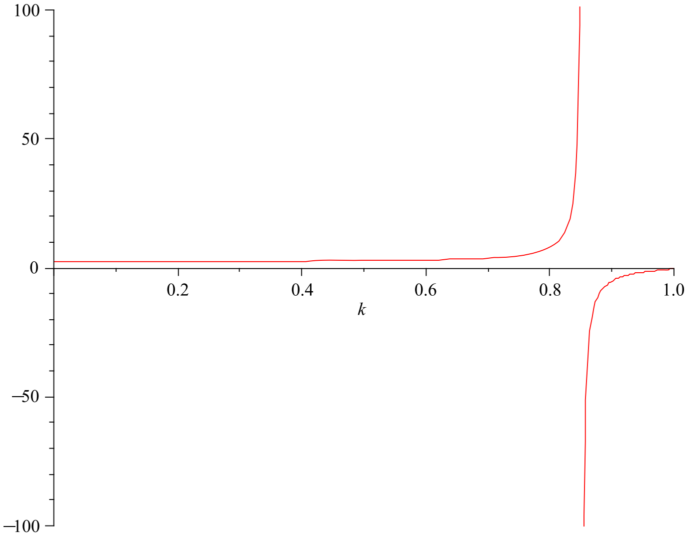

Fig. 1. Graphic of $c(k)$ with $L = 8$.

$$w(c) := \frac{16c\sqrt{k^4 - k^2 + 1}K'^2(k)}{16c\sqrt{k^4 - k^2 + 1}K'^2(k) - c + 1},$$

where $k = k(c) \in (0, k_L)$. Using the definition of w and (7.9) we obtain $\sqrt{\frac{w}{w-1}} = \frac{LK'}{K}$. Then, from (7.6) and (7.7) we arrive at the cnoidal profile

$$\psi_{w(c)}(\xi) = \frac{12w}{L}\sqrt{\frac{w-1}{w}} + \frac{24K^2w}{L^2}\left[\mathrm{dn}^2\left(\frac{2K\xi}{L}; k\right) - \frac{E}{K}\right]. \tag{7.10}$$

Since $\frac{K(k)}{K'(k)} \in (0, L)$, for all $k \in (0, k_L)$, then for $c \in (c^*, +\infty)$ we obtain $w(c) \in (1, +\infty)$. Therefore, we get that the map

$$c \in \left(c^*, +\infty\right) \longmapsto \psi_{w(c)} \in H^n_{per}\left([0, L]\right)$$

is a smooth curve for all $n \in \mathbb{N}$.

The stability result for the BBM equation reads.

Theorem 7.6. *Assume $L > 2\pi$ fixed. If $c > \frac{L^2}{L^2 - 4\pi^2}$, then the periodic traveling wave solution ϕ_c in (7.4) is stable by the flow of the BBM equation.*

Proof. From (7.10), we get that $\phi_c = a(k(c)) - \frac{24c}{L}\sqrt{\frac{w-1}{w}} + \frac{2c}{w}\psi_{w(k(c))}$, where

$$a(k) = \frac{16cK}{L^2}\left[3E - \left(1 + k'^2\right)K\right] + c - 1.$$

Thus, $\phi_c(x) = s(k(c)) + \frac{2c}{w}\psi_{w(k(c))}(x)$, with $s(k(c)) := a(k(c)) - \frac{24c}{L}\sqrt{\frac{w-1}{w}}$. Then we obtain easily that the Fourier coefficients of ϕ_c are for $n \in \mathbb{Z}$,

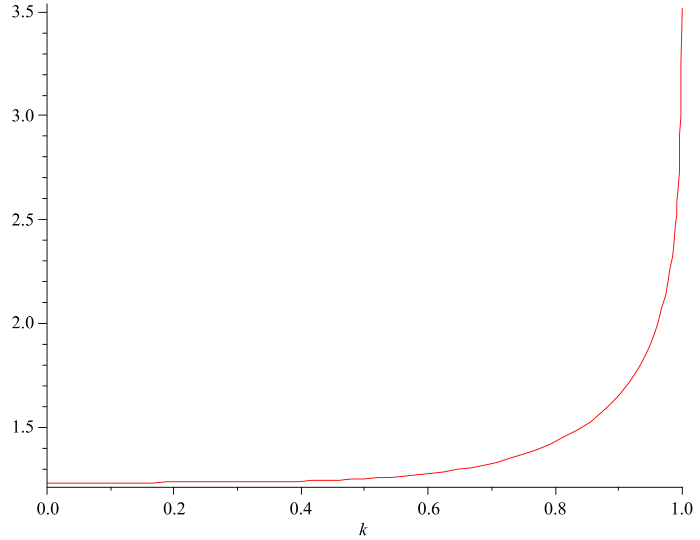

Fig. 2. Graphic of $\tilde{a}(k)$ with $L = 8$.

$$\widehat{\phi_c}(n) = \begin{cases} a(k), & n = 0, \\ \frac{12c\pi}{L^2} n \operatorname{csch}(\sqrt{\frac{w}{w-1}} \frac{\pi n}{L}), & n \neq 0. \end{cases}$$

Now, by using that $\frac{c-1}{c} = \frac{16K^2\sqrt{1-k^2+k^4}}{L^2}$ we obtain

$$s(k) = c\left[\frac{16K^2}{L^2}\left(\sqrt{1-k^2+k^4} - 2 + k^2 + 3\frac{E}{K}\right) - \frac{24}{L^2}\frac{K(k)}{K(k')}\right] =: c\tilde{s}(k)$$

and

$$a(k) = \frac{16K^2c}{L^2}\left[3\frac{E}{K} - 2 + k^2 + \sqrt{1-k^2+k^4}\right] =: c\tilde{a}(k).$$

Since the function $\tilde{s}$ is a positive function in $(0,1)$ and $a(k)$ is a positive strictly increasing function in $(0, k_L)$ (because $\tilde{a}$ is strictly increasing, see Fig. 2), we conclude that $\widehat{\phi_c} \in PF(2)$ discrete (see Angulo and Natali [11]).

Next, we prove (C_3) in (7.3). In fact, it is easy to see that $\chi = -\frac{d}{dc}\phi_c$ satisfies $\mathcal{L}\chi = \phi_c - \phi_c''$. Then by Parseval theorem, it follows that $I = -\frac{L}{2}\frac{d}{dc}\|(1+|\cdot|^2)^{\frac{1}{2}}\widehat{\phi_c}\|_{l^2}^2$. But,

$$\begin{aligned} &\frac{d}{dc}\left\|(1+|\cdot|^2)^{\frac{1}{2}}\widehat{\phi_c}\right\|_{l^2}^2 \\ &\quad = 2a(k)\frac{da}{dk}\frac{dk}{dc} + c_1\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}(1+|n|^2)n^2\operatorname{csch}^2\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right) \\ &\qquad + c_2\left((w-1)^3w\right)^{-1/2}\frac{dw}{dk}\frac{dk}{dc}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}(1+|n|^2)n^3\operatorname{csch}^2\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right)\coth\left(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}\right), \end{aligned}$$

where $c_1 = c_1(L, c) > 0$ and $c_2 = c_2(L, c) > 0$. To prove that $I < 0$ we only need to show that $\frac{dw}{dk} > 0$, because $b_n = (1 + |n|^2)n^3 \operatorname{csch}^2(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L}) \coth(\sqrt{\frac{w}{w-1}}\frac{\pi n}{L})$ is a positive sequence and $k = k(c)$ is a strictly increasing function. So, from the equality

$$\frac{dw}{dk} = \frac{2L^2 K' K[k' \frac{dK}{dk} - K \frac{dK'}{dk}]}{(L^2 K'^2 - K^2)^2}, \quad \forall k \in (0, k_0),$$

we have that $\frac{dw}{dk} > 0$ since $\frac{dK}{dk} > 0$ and $\frac{dK'}{dk} < 0$. Therefore, the positive cnoidal waves ϕ_c are stable in $H^1_{per}([0, L])$ by the periodic flow of the BBM equation. □

Acknowledgments

J. Angulo was supported partially by grant from CNPq/Brazil and by Edital Universal MCT/CNPq, 14/2009. C. Banquet was supported by CNPq/Brazil doctoral fellowship at State University of Campinas. C. Banquet is also grateful to the University of Córdoba, Montería, Colombia for its financial support. The authors are grateful to the reviewers for their fruitful remarks.

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