Constructing an Infinite Family of Cubic 1-Regular Graphs

YAN-QUAN FENG† AND JIN HO KWAK

A graph is 1-regular if its automorphism group acts regularly on the set of its arcs. Miller [J. Comb. Theory, B, 10 (1971), 163–182] constructed an infinite family of cubic 1-regular graphs of order 2p, where \( p \geq 13 \) is a prime congruent to 1 modulo 3. Marušič and Xu [J. Graph Theory, 25 (1997), 133–138] found a relation between cubic 1-regular graphs and tetravalent half-transitive graphs with girth 3 and Alspach et al. [J. Aust. Math. Soc. A, 56 (1994), 391–402] constructed infinitely many tetravalent half-transitive graphs with girth 3. Using these results, Miller’s construction can be generalized to an infinite family of cubic 1-regular graphs of order 2n, where \( n \geq 13 \) is odd such that 3 divides \( \phi(n) \), the Euler function of \( n \). In this paper, we construct an infinite family of cubic 1-regular graphs with order \( 8(k^2 + k + 1)(k \geq 2) \) as cyclic-coverings of the three-dimensional Hypercube.

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1. INTRODUCTION

In this paper we consider an undirected finite connected graph without loops or multiple edges. For a graph \( G \), we denote by \( V(G), E(G), A(G) \) and \( \text{Aut}(G) \) the vertex set, the edge set, the arc set and the automorphism group, respectively. The neighbourhood of a vertex \( v \in V(G) \), denoted by \( N(v) \), is the set of vertices adjacent to \( v \) in \( G \). A graph \( \tilde{G} \) is called a covering of a graph \( G \) with projection \( p : \tilde{G} \to G \) if there is a surjection \( p : V(\tilde{G}) \to V(G) \) such that \( p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v) \) is a bijection for any vertex \( v \in V(G) \) and \( \tilde{v} \in p^{-1}(v) \).

The covering \( G \) is said to be regular (or \( K \)-covering) if there is a semiregular subgroup \( K \) of \( \text{Aut}(G) \) such that the graph \( G \) is isomorphic to the quotient graph \( G/K \), say by \( h \), and the quotient map \( \tilde{G} \to \tilde{G}/K \) is the composition \( ph \) of \( p \) and \( h \). (In this paper, all functions are composed from left to right.) If the regular covering \( \tilde{G} \) is connected, then \( K \) is the covering transformation group.

An \( s \)-arc in a graph \( G \) is an ordered \( (s + 1) \)-tuple \( (v_0, v_1, \ldots, v_{s-1}, v_s) \) of vertices of \( G \) such that \( v_{i-1} \) is adjacent to \( v_i \) for \( 1 \leq i \leq s \), and \( v_{i-1} \neq v_{i+1} \) for \( 1 \leq i \leq s \); in other words, a directed walk of length \( s \) which never includes the reverse of an arc just crossed. A graph \( G \) is said to be \( s \)-arc-transitive if \( \text{Aut}(G) \) is transitive on the set of all \( s \)-arcs in \( G \). In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric.

A graph \( G \) is said to be edge-transitive if \( \text{Aut}(G) \) is transitive on \( E(G) \) and half-transitive if \( G \) is vertex-transitive, edge-transitive, but not arc-transitive. A symmetric graph \( G \) is said to be \( s \)-regular if for any two \( s \)-arcs in \( G \), there is only one automorphism of \( G \) mapping one to the other, i.e., \( \text{Aut}(G) \) acts regularly on the set of \( s \)-arcs in \( G \). A subgroup of the full automorphism group of a graph is said to be \( s \)-regular if the subgroup acts regularly on the set of \( s \)-arcs in the graph. Thus, if a graph is \( s \)-regular, its automorphism group is transitive on the set of all \( s \)-arcs in the graph and the only automorphism fixing an \( s \)-arc is the identity automorphism of \( G \). Tutte [29, 30] showed that every finite cubic symmetric graph is \( s \)-regular for some \( s \), and this \( s \) should be at most five.

Marušič [21] and Malnič et al. [17] constructed two different kinds of infinite families of tetravalent 1-regular graphs. The first cubic 1-regular graph was constructed by Frucht [12] and later two infinite families of cubic 1-regular graphs were constructed by Conder and Praeger [7] with alternating and symmetric groups as their automorphism groups. Miller [25] constructed an infinite family of cubic 1-regular graphs of order \( 2p \), where \( p \geq 13 \) is a prime...
congruent to 1 modulo 3. (They are Cayley graphs on dihedral groups $D_{2p}$ with the generating set $\{1, \rho, \rho \tau, \rho \tau^{r+1}\}$, where $\rho$ is the rotation, $\tau$ one of the reflections and $r \in \mathbb{Z}_p^*$ has order 3.) By Cheng and Oxley’s classification of symmetric graphs of order $2p$ [6], Miller’s construction is actually the all cubic 1-regular graphs of order $2p$. Furthermore, using the result from [24] connecting cubic 1-regular graphs and tetravalent half-transitive graphs with girth 3, Miller’s construction can be generalized to graphs of order $2n$, where $n \geq 13$ is odd such that 3 divides $\phi(n)$, the Euler function of $n$ (see [1, 23]). Recently, Feng et al. [11] classified all cubic 1-regular Cayley graphs on the dihedral groups, and showed that all of such cubic 1-regular Cayley graphs are exactly those graphs generalized by Miller’s construction. Also, as shown in [22], one can see an importance of a study for cubic 1-regular graphs in connection with chiral (that is regular and irreflexible) maps on a surface via tetravalent half-transitive graphs. (See also [4, 13, 26–28, 31]) for some of the relative articles on such maps.

In this paper, we construct an infinite family of cubic 1-regular graphs of order $8(k^2 + k + 1)(k \geq 2)$ which contains a cubic 1-regular graph of order 56. This 1-regular graph is the smallest cubic 1-regular one (see the code 56A in the Foster census [5]), which is not metacirculant so that it cannot belong to any family discussed in the previous paragraph. Actually, this new infinite family of cubic 1-regular graphs consists of cyclic-covering graphs of the Hypercube $Q_k$, which has a larger degree of symmetry (it is 2-regular), and they are also dihedral-coverings of the complete graph $K_k$ of order 4 (see Remark later). However, using results from [9, 18, 19] it may be easily seen that there are no 1-regular cyclic-coverings of $K_k$. (The only arc-transitive cyclic-coverings of $K_k$ are $Q_3$ or the generalized Petersen graph $G(8, 3)$ of order 16.) Furthermore, the authors [10] classified all $s$-regular cyclic-coverings of the bipartite graph $K_{s, s}$ of order 6 for each $s$. However, the same work for $Q_3$ is still elusive even for classifying 1-regular cyclic-coverings.

Let $k$ be a positive integer greater than 1 and let $n = k^2 + k + 1$. The graph $G(k)$ is defined to have vertex set $V(G(k)) = \mathbb{Z}_k \times \mathbb{Z}_n$ and edge set

\[ E(G(k)) = \{(0, i), (1, i), (0, i), (2, i), (0, i), (3, i), (1, i), (4, i), (1, i), (5, i), (2, i), (4, i + 1), (2, i), (6, i + k^2 + 1), (3, i), (5, i + k + 1), (3, i), (6, i + k), (4, i), (7, i + k), (5, i), (7, i), (6, i), (7, i + k) | i = 1, 2, \ldots, n\} \]

where all numbers $i + t(i, t \in \mathbb{Z}_n)$ are taken modulo $n = k^2 + k + 1$.

By showing that all of $G(k), k \geq 2$, are cubic 1-regular graphs, we have the following main theorem. (It can be shown easily that the graph $G(1)$ is 2-arc-transitive, so not 1-regular. In fact, $G(1)$ is 2-regular.)

**Theorem 1.1.** For any $k \geq 2$, there exists a cubic 1-regular graph of order $8(k^2 + k + 1)$.

### 2. Derived Coverings and a Lifting Problem

Every edge of a graph $G$ gives rise to a pair of opposite arcs. By $e^{-1}$, we mean the reverse arc to an arc $e$. Let $K$ be a finite group. An ordinary voltage assignment (or, $K$-voltage assignment) of $G$ is a function $\phi : A(G) \rightarrow K$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in A(G)$. The values of $\phi$ are called voltages, and $K$ is called the voltage group. The ordinary derived graph $G \times_{\phi} K$ from an ordinary voltage assignment $\phi : A(G) \rightarrow K$ has vertex set $V(G) \times K$ and edge set $E(G) \times K$, so that an edge $(e, g)$ of $G \times_{\phi} K$ joins a vertex $(u, g)$ to $(v, \phi(e)g)$ for $e = uv \in A(G)$ and $g \in K$. The first coordinate projection $p_{\phi} : G \times_{\phi} K \rightarrow G$ is a regular covering of $G$ since $K$ is semiregular on $V(G \times_{\phi} K)$.
Let \( p : \tilde{G} \to G \) be a \( K \)-covering. If \( \alpha \in \text{Aut}(G) \) and \( \tilde{\alpha} \in \text{Aut}(\tilde{G}) \) satisfy \( \tilde{\alpha} p = p \alpha \), we call \( \tilde{\alpha} \) a lift of \( \alpha \), and \( \alpha \) the projection of \( \tilde{\alpha} \). For a subgroup \( H \) of the automorphism group \( \text{Aut}(G) \),

\[
\tilde{H} = \{ \tilde{\alpha} \mid \tilde{\alpha} \text{ is a lift of } \alpha \text{ for some } \alpha \in H \}
\]

is called a lift of the subgroup \( H \). Clearly, the lift \( \tilde{H} \) is a subgroup of \( \text{Aut}(\tilde{G}) \). In particular, if the covering graph \( \tilde{G} \) is connected, then the covering transformation group \( K \) is the lift of the identity of \( G \). Gross and Tucker [14] showed that every \( K \)-covering of a graph \( G \) can be derived from a \( K \)-voltage assignment which assigns the identity voltage 1 to the arcs on an arbitrary fixed spanning tree of \( G \).

Let \( G \times_{\phi} K \to G \) be a connected \( K \)-covering, where \( \phi = 1 \) on the arcs of a spanning tree of \( G \). Then, the covering graph \( G \times_{\phi} K \) is connected if and only if the voltages on the cotree arcs generate the voltage group \( K \).

The problem whether an automorphism \( \alpha \) of \( G \) lifts can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Define a mapping \( \alpha^* \) from the set of voltages of fundamental closed walks based at a vertex \( v \) of the graph \( G \) to the voltage group \( K \) as the following:

\[
(\phi(C))^\alpha^* = \phi(C^\alpha),
\]

where \( C \) ranges over all fundamental closed walks at \( v \), and \( \phi(C) \) and \( \phi(C^\alpha) \) are the voltages of \( C \) and \( C^\alpha \), respectively. Note that if \( K \) is abelian, \( \alpha^* \) does not depend on the choice of the base vertex, and the fundamental closed walks at \( v \) can be substituted by the fundamental cycles generated by the cotree edges of \( G \).

**Proposition 2.1 ([20]).** Let \( G \times_{\phi} K \to G \) be a connected \( K \)-covering. Then, an automorphism \( \alpha \) of \( G \) lifts if and only if \( \alpha^* \) extends to an automorphism of \( K \).

For more results on the lifts of automorphisms of \( G \), we refer the reader to [2, 3, 8, 15, 16].

3. **Proof of Theorem 1.1**

Let \( k \geq 2 \) be an integer and \( n = k^2 + k + 1 \). By \( Q_3 \), we denote the three-dimensional Hypercube and by \( \mathbb{Z}_n = \{0, 1, 2, \ldots, n - 1\} \) the cyclic (additive) group of order \( n \). Clearly, \( k \) is prime to \( n \) and so the map \( 1 \to k \) induces an automorphism of the group \( \mathbb{Z}_n \). Identify the vertex set of \( Q_3 \) with \( \mathbb{Z}_8 = \{0, 1, 2, \ldots, 7\} \). To construct a \( \mathbb{Z}_n \)-covering of \( Q_3 \), we define an ordinary voltage assignment \( \phi \) of \( Q_3 \) as illustrated in Figure 1, in which the dark lines represent a spanning tree \( T \) of \( Q_3 \). Note that \( \phi = 0 \) on \( T \).

Since the voltages \( \{1, k, k + 1, k^2 + 1\} \) generate the voltage group \( \mathbb{Z}_n \), the covering graph \( Q_3 \times_{\phi} \mathbb{Z}_n \) is connected. It is not difficult to show the following lemma.

![Figure 1. The cube $Q_3$ with voltage assignment $\phi$.](image-url)
GG \gamma \gamma and Z contrary to the hypothesis that V can be extended to an automorphism of Z, and by Table 1 we have that 1^γ = k + 1 and (k + 1)^γ = 1. It follows that 1 = (k + 1)^γ = (k + 1)(k + 1) = k \mod(n) and so k = 1, contrary to the hypothesis that k ≥ 2.

\[ \gamma \]

\section*{TABLE 1.}

Fundamental cycles and their images with corresponding voltages.

<table>
<thead>
<tr>
<th>C</th>
<th>φ(C)</th>
<th>C = γ</th>
<th>φ(C)</th>
<th>C = γ+1</th>
<th>φ(C)</th>
<th>C = γ+2</th>
<th>φ(C)</th>
<th>C = γ+3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0241</td>
<td>1</td>
<td>1420</td>
<td>k^2 + k</td>
<td>2014</td>
<td>k^2 + k</td>
<td>3675</td>
<td>k^2 + k</td>
<td></td>
</tr>
<tr>
<td>0263</td>
<td>k^2 + 1</td>
<td>1475</td>
<td>k</td>
<td>2036</td>
<td>k</td>
<td>3620</td>
<td>k</td>
<td></td>
</tr>
<tr>
<td>0351</td>
<td>k + 1</td>
<td>1530</td>
<td>k^2</td>
<td>2674</td>
<td>k^2</td>
<td>3015</td>
<td>k^2</td>
<td></td>
</tr>
<tr>
<td>1475</td>
<td>k</td>
<td>0263</td>
<td>k^2 + 1</td>
<td>4157</td>
<td>k^2 + 1</td>
<td>5741</td>
<td>k^2 + 1</td>
<td></td>
</tr>
<tr>
<td>036751</td>
<td>k</td>
<td>157630</td>
<td>k^2 + 1</td>
<td>263574</td>
<td>k^2 + 1</td>
<td>302415</td>
<td>k^2 + 1</td>
<td></td>
</tr>
</tbody>
</table>

\section*{LEMMA 3.1.}
The covering graph Q₃ × φ Zₙ and the graph G(k) defined before Theorem 1.1 are isomorphic.

It is well known that Aut(Q₃) \cong S₄ × Z₂. Let α₁ = (01) (24) (35) (67), α₂ = (02) (14) (36) (57), α₃ = (03) (15) (26) (47), β = (123) (465) and γ = (23) (45), as permutations on V(Q₃). It is easy to check that α₁, α₂, α₃, β and γ are actually automorphisms of Q₃. First, we prove the following lemma.

\section*{LEMMA 3.2.}
(1) Aut(Q₃) = α₁, α₂, α₃, β, γ and the subgroup of Aut(Q₃) generated by the four automorphisms α₁, α₂, α₃ and β is 1-regular.

(2) The automorphisms α₁, α₂, α₃ and β lift to automorphisms of the covering graph Q₃ × φ Zₙ, but γ cannot.

\section*{PROOF.}
(1) Clearly, (α₁, α₂, α₃) \cong Z₂ × Z₂ × Z₂ is transitive on V(Q₃). The automorphism β is a product of two 3-cycles and its factor (123) is a permutation on the neighbourhoods of the vertex 0 in Q₃. Since β fixes 0, (α₁, α₂, α₃, β) acts arc-transitively on Q₃. Since α₁ β = β⁻¹α₁β = α₂, α₂ = α₁, α₂ = α₁, α₁, α₂, α₃) is a normal subgroup of (α₁, α₂, α₃, β). Hence, the group (α₁, α₂, α₃, β) has order 3 × 8 = 24 and so is a 1-regular subgroup of Aut(Q₃). Noting that γ fixes 0 and 1, we have γ \not\in (α₁, α₂, α₃, β). Then, the 2-regularity of Q₃ implies that Aut(Q₃) = α₁, α₂, α₃, β, γ.

(2) We denote by i₁, i₂, ..., iₙ a cycle which has vertex set \{i₁, i₂, ..., iₙ\} and edge set \{(i₁, i₂), (i₂, i₃), ..., (iₙ−1, iₙ), (iₙ, i₁)\). There are five fundamental cycles 0241, 0263, 0351, 1475 and 036751 in Q₃, which are generated by the five cotree edges. Each cycle maps to a cycle of same length under the actions of α₁, α₂, α₃, β, and γ. We list all these cycles and their voltages in Table 1, in which C denotes a fundamental cycle of Q₃ and φ(C) is the voltage on the cycle C.

Consider the mapping β^* from the set of voltages of the five fundamental cycles of Q₃ to the cyclic group Zₙ, defined by φ(C)β^* = φ(Cβ), where C ranges over the five fundamental cycles. From Table 1, we have that β^* can be extended to an automorphism of Zₙ deduced by 1 \to k.

Similarly, one can define α₁^*, α₂^*, α₃^* and γ^* and show that α₁^*, α₂^* and α₃^* can be extended to an automorphism of Zₙ deduced by 1 \to −1. By Proposition 2.1, all of α₁, α₂, α₃ and β can lift. Now, we show that γ cannot lift. Suppose to the contrary that γ lifts. By Proposition 2.1, γ^* can be extended to an automorphism of Zₙ, say \tilde{γ}, and by Table 1 we have that 1\tilde{γ} = k + 1 and (k + 1)^\tilde{γ} = 1. It follows that 1 = (k + 1)^\tilde{γ} = (k + 1)(k + 1) = k \mod(n) and so k = 1, contrary to the hypothesis that k ≥ 2. \qed
Let $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$, and $\tilde{\beta}$ denote the lifts of $\alpha_1, \alpha_2, \alpha_3$ and $\beta$, respectively. Noting that $\langle \alpha_1, \alpha_2, \alpha_3, \beta \rangle \trianglelefteq (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3$, we have $\langle \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\beta}, Z_n \rangle \cong \mathbb{Z}_n \rtimes ((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3)$. Since $\langle \alpha_1, \alpha_2, \alpha_3, \beta \rangle$ is a 1-regular subgroup of $\text{Aut}(Q_3)$, $\langle \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\beta}, Z_n \rangle$ is also a 1-regular subgroup of $\text{Aut}(Q_3 \times \phi Z_n)$. Thus, we have the following lemma.

**Lemma 3.3.** The subgroup $\langle \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\beta}, Z_n \rangle$ acts regularly on the arc set of the covering graph $Q_3 \times \phi Z_n$.

Now, we are ready to prove Theorem 1.1. By Lemma 3.1 it suffices to prove that $Q_3 \times \phi Z_n$ is 1-regular, where $n = k^2 + k + 1$ for some $k \geq 2$.

Let $A = \text{Aut}(Q_3 \times \phi Z_n)$. Denote by $A_{(0,0)}$ the stabilizer of $(0, 0) \in V(Q_3 \times \phi Z_n)$ in $A$ and by $A \times (0,0)$ the subgroup of $A_{(0,0)}$ fixing $(1, 0), (2, 0)$ and $(3, 0)$, which are the neighbours of $(0, 0)$ in $Q_3 \times \phi Z_n$. It is easy to see that there are exactly three cycles of length 6 passing through the vertex $(0, 0)$ of $Q_3 \times \phi Z_n$, that is $(0, 0)(1, 0)(4, 0)(7, k)(6, 0)(3, 0)$, $(2, 0)(5, 0)(7, 0)(6, k^2 + 1)(2, 0)$ and $(0, 0)(2, 0)(4, 1)(7, k + 1)(5, k + 1)(3, 0)$. Then, one of these three cycles passes through any two given edges in the set $\{(0, 0), (1, 0), ((0, 0), (2, 0)), ((0, 0), (3, 0))\}$. This implies that each element in $A \times (0, 0)$ fixes the three cycles pointwise and so all vertices adjacent to $(1, 0), (2, 0)$ and $(3, 0)$. By the arc-transitivity of $Q_3 \times \phi Z_n$, $A \times (0, 0) = 1$, implying that $Q_3 \times \phi Z_n$ is at most 2-regular. Suppose to the contrary that $\text{Aut}(Q_3 \times \phi Z_n)$ is 2-regular, that is $A$ is 2-regular. Let $B = \langle \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\beta}, Z_n \rangle$. By Lemma 3.3, $B$ is 1-regular. Thus, $|A : B| = 2$ and $B \leq A$. Clearly, $n$ must be odd because $n = k^2 + k + 1 (k \geq 2)$. If $3 \nmid n$ then $Z_n$ is a characteristic subgroup of $B$ because $|B| = 24n$ and $Z_n \not\leq B$. It follows that $Z_n \leq A$ and so the full automorphism group of $Q_3$ lifts, contrary to Lemma 3.2 (2). Thus, we may assume that $3 | n$.

Now, to show that $9 \nmid n$, let $k = 3\ell, 3\ell + 2$ or $3\ell + 1$, so that $n = k^2 + k + 1 = 9\ell^2 + 3\ell + 1$, $9\ell^2 + 15\ell + 7$ or $3(3\ell^2 + 3\ell + 1)$, respectively. For the first two cases, $3 \nmid n$ and for the last case, $9 \nmid n$. Thus, we always have $9 \nmid n$.

Let $O = O_3(B)$ be the largest normal 3-subgroup of $B$. Since $|B| = 24n$, $9 \nmid n$ and $Z_n \not\leq B$, we have that $|O| = 3$ or $9$. Moreover, if $|O| = 3$ then $O$ is a subgroup of $Z_n$. Note that $O$ is a characteristic subgroup of $B$ and so normal in $A$. Suppose $|O| = 9$. Then, $O$ is a Sylow 3-subgroup of $A$. Since $A$ is 2-regular, the stabilizer $A_0$ of $u \in V(Q_3 \times \phi Z_n)$ in $A$ has order 6. This forces that the stabilizer $O_0$ of $u \in O$ is isomorphic to $\mathbb{Z}_3$ and so each orbit of $O$ on $V(Q_3 \times \phi Z_n)$ contains three elements. Let $W_1$ and $W_2$ be two orbits of $O$ such that $u \in W_1$ and there is a $v \in W_2$ with $uv \in E(Q_3 \times \phi Z_n)$. Since $O_0 \cong \mathbb{Z}_3$, the induced subgraph $(W_1 \cup W_2)$ of $W_1 \cup W_2$ in $Q_3 \times \phi Z_n$ must be $K_{3, 3}$, the bipartite graph of order 6. It follows that $Q_3 \times \phi Z_n \cong K_{3, 3}$ which is impossible, so that $|O| = 3$. Since $3 | n$ and $9 \nmid n$ we may let $Z_n = O \times N$, where $N$ is a subgroup of $Z_n$ with $3 \nmid |N|$. Then $N$ is a characteristic subgroup of $B$ and $N \not\leq A$. It follows that $Z_n = O \times N \not\leq A$ and the full automorphism group of $Q_3$ lifts, a contradiction. Thus, $Q_3 \times \phi Z_n$ is 1-regular.

**Remark.** We have proved that $Z_n \rtimes ((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3)$ is the full automorphism group of $Q_3 \times \phi Z_n$, where $n = k^2 + k + 1$ for some integer $k \geq 2$. It is easy to see that $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes Z_3$ has an involution in its centre and $\mathbb{Z}_n \rtimes ((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3)$ has a normal subgroup of order $2n$ containing $Z_n$. Thus, $Q_3 \times \phi Z_n$ is also a regular covering of the complete graph $K_4$ of order 4. In fact, using results from [9, 18, 19] one may prove that $Q_3 \times \phi Z_n$ is isomorphic to the following graph $G(k, D_{2n})$. Let $n = k^2 + k + 1$ with $k \geq 2$ and let $D_{2n} = \langle a, b | a^2 = b^k = 1, b^k = b^{-1} \rangle$. The graph $G(k, D_{2n})$ is defined to have vertex set $V(G(k, D_{2n})) = \mathbb{Z}_n \times D_{2n}$ and edge set $E(G(k, D_{2n})) = \{(0, c), (1, c), ((0, c), (2, c)), ((0, c), (3, c)), ((1, c), (2, ac)), ((1, c), (3, abk+1 c)), (\{c \in D_{2n}, 1\} \cap D_{2n}, 1\}$. 
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