Application of an idea of Voronoï to John type problems

Peter M. Gruber

Institute for Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstraße 8-10/1046, A-1040 Vienna, Austria

Received 27 August 2007; accepted 18 December 2007
Available online 8 February 2008
Communicated by Michael J. Hopkins

Abstract

Using an idea of Voronoï, many John type and minimum position problems in dimension $d$ can be transformed into more accessible geometric problems on convex subsets of the $\frac{1}{2}d(d + 1)$-dimensional cone of positive definite quadratic forms. In this way, we prove several new John type and minimum position results and give alternative versions and extensions of known results. In particular, we characterize minimum ellipsoidal shells of convex bodies and, in the typical case, show their uniqueness and determine the contact number. These results are formulated also in terms of the circumradius of convex bodies. Next, circumscribed ellipsoids of minimum surface area of a convex body and the corresponding minimum position problem are studied. Then we investigate John type characterizations of minimum positions of a convex body with respect to moments and the product of a moment and the moment of the polar body. The technique used in this context, finally, is applied to obtain corresponding results for the mean width and the surface area.

© 2007 Elsevier Inc. All rights reserved.

MSC: 46B07; 52A20; 52A21; 52A40

Keywords: John type result; Minimum position; Minimum ellipsoidal shell; Contact number; Distance ellipsoid; Minimum moment position; Mean width position; Surface area position; Baire categories; Uniqueness results; Intrinsic volumes

1. Introduction

Let $C$ be a (proper) convex body, i.e. a compact convex subset of $d$-dimensional Euclidean space $E^d$ with non-empty interior and assume that $C$ is symmetric in the origin $o$. John’s [24]...
characterization of the unique inscribed ellipsoid of $C$ of maximum volume has generated a voluminous literature in convex geometry and the asymptotic theory of normed spaces, including a series of applications. Compare, e.g., Lindenstrauss and Milman [26], Praetorius [33], Ball [3] and Giannopoulos and Milman [11].

Among the results obtained is the following Baire type result of the author [15]: For most convex bodies $C$, symmetric in $o$, the inscribed ellipsoid of maximum volume touches the boundary of $C$ at precisely $\frac{1}{2}d(d+1)$ pairs of points $\pm u$. The key idea of the proof was adopted from Voronoï [38–40] who applied it successfully in the geometric theory of positive definite quadratic forms. A version of this idea was used in a recent article [21] with Schuster to give a transparent proof of John’s characterization.

Related to John’s theorem is the following question: consider a real function $F$ on the space $C$ of all convex bodies or on a suitable subspace of it such as the space $C_o$ of all $o$-symmetric convex bodies. Then, given a convex body $C$ in this space and a group of affinities, characterize the images of $C$ under those affinities, for which $F$ is minimum, the minimum $F$-positions of $C$ with respect to the given group. For numerous pertinent results and applications see Milman and Pajor [29], Giannopoulos and Milman [10,11] and Gordon, Litvak, Meyer and Pajor [14].

The standard method of attack for John type and minimum position problems is a variational argument, see, e.g., Giannopoulos and Milman [10]. In contrast, in this article the idea of Voronoï is used in a systematic way to prove John type and minimum position results: First, minimum ellipsoidal shells of convex bodies are investigated. We give a John type characterization and show that for most $o$-symmetric convex bodies $C$ the minimum ellipsoidal shell is unique and touches the boundary of $C$ at precisely $\frac{1}{2}d(d+1) + 1$ pairs of points $\pm u$. This is applied to a question on the Banach–Mazur distance between the norm corresponding to $C$ and the Euclidean norm. The John type characterization of the ellipsoid of minimum volume circumscribed to a convex body and our results on minimum ellipsoidal shells are then formulated in terms of the circumradius of convex bodies. In analogy to John’s theorem we study next ellipsoids which are circumscribed to a convex body $C$ and have minimum surface area. The corresponding minimum positions of $C$ are characterized. Generalizations deal with intrinsic volumes. Then we describe the minimum positions of a convex body with respect to moments and the product of the body and its polar. Using the tools for these characterizations, similar results are proved for the mean width and the surface area. Different, in part weaker, versions of the latter results were known before. The reader will note that the idea of Voronoï makes the proofs and the results more transparent.

A rough description of the basic idea of the proofs of the above results is as follows: ellipsoids in $\mathbb{E}^d$ are identified with the coefficient vectors in $\mathbb{E}_{\frac{d}{2}}^{d(d+1)}$ of the corresponding quadratic forms, and similarly for linear transformations (up to rotations). This translates the problems into geometric questions on subsets of the cone $\mathcal{P}_{d}$ of positive definite quadratic forms in $\mathbb{E}_{\frac{1}{2}}^{d(d+1)}$. For John type and minimum position problems the questions are to show, first, that the subsets are convex and smooth, and, second, that they have a point in common at which they touch. The common point then corresponds to the solution of the problem and the condition that the convex sets touch, properly formulated, is just the John type characterization of the solution. For Baire category results, the questions are to construct dense sets of polytopes which have the desired properties. To make the exposition smooth, we present in general a basic case, for instance the $o$-symmetric one, and state its (technical) extension without proof, or just make a hint to it.

Further applications of Voronoï’s idea will be given in [19,20]. In the first article we study lattice packings of a convex body which are locally of maximum density and lattice packings where the product of the densities of the packing and the dual packing is a local maximum.
The maximum conditions are, in essence, separation conditions for polyhedral convex cones and extend the famous conditions of Voronoi for extreme lattice packings of balls (“perfect and eutactic”). In the second article refined extremum properties of lattice packing and covering of solid circles are investigated.

If \( u, v \in \mathbb{E}^d \), let \( u \otimes v \) denote the matrix \( uv^T \). For (real) \( d \times d \) matrices, \( A = (a_{ik}) \), \( B = (b_{ik}) \) the inner product \( A \cdot B \) is defined to be \( \sum a_{ik}b_{ik} \). The corresponding matrix norm \( \| \cdot \| \) is then \( \| A \| = (\sum a_{ik}^2)^{1/2} \). The dot \( \cdot \) and \( \| \cdot \| \) denote also the usual inner product and the corresponding Euclidean norm in \( \mathbb{E}^d \). Further notions and results of convex geometry will be introduced as needed. See also [17] or [36].

2. Minimum ellipsoidal shells

For \( C \in \mathcal{C}_o \) a pair \( \langle E, \varrho E \rangle \), where \( \varrho \geq 1 \), of (solid) ellipsoids with center \( o \) is a minimum ellipsoidal shell of \( C \), and \( E \) is called a distance ellipsoid of \( C \), if \( E \subseteq C \subseteq \varrho E \) and \( \varrho \) is minimum among all such pairs. If \( \| \cdot \|_C \) is the norm on \( \mathbb{E}^d \) with unit ball \( C \), then the minimum \( \varrho \) is simply the Banach–Mazur distance between \( \| \cdot \|_C \) and the Euclidean norm \( \| \cdot \| \) on \( \mathbb{E}^d \). Our results thus can be expressed in terms of the Banach–Mazur distance, but we prefer a more geometric language. For one exception see Section 2.3. It is interesting to note that our results can also be interpreted in terms of the minimum circumradius, see Section 2.5.

Minimum ellipsoidal shells have been investigated by Maurey (unpublished, but mentioned in [33,34]), Praetorius [34] and others. For some references see Lindenstrauss and Milman [26] and Praetorius [34].

For other ellipsoids which have been investigated in the context of John’s theorem see Lutwak, Yang and Zhang [28].

2.1. Characterization of minimum ellipsoidal shells

A contact point of two convex bodies, one of which is inscribed into the other, is a point of the intersection of their boundaries. In analogy to the well-known characterization of the unique ellipsoid of maximum volume inscribed into a convex body due to John [24] (necessity) and Pełczyński [31] and Ball [2] (sufficiency) and the dual characterization of the circumscribed ellipsoid of minimum volume, we have the following result, where \( B^d \) is the solid Euclidean unit ball of \( \mathbb{E}^d \).

**Theorem 1.** Let \( C \in \mathcal{C}_o \). Then statement (i) holds and, if \( B^d \subseteq C \subseteq \varrho B^d \) with \( \varrho \geq 1 \), then statements (ii) and (iii) are equivalent:

(i) \( C \) has a (not necessarily unique) minimum ellipsoidal shell.

(ii) \( \langle B^d, \varrho B^d \rangle \) is a minimum ellipsoidal shell of \( C \).

(iii) There are contact points \( \pm u_1, \ldots, \pm u_k \in B^d \cap \text{bd} C \) and \( \pm v_1, \ldots, \pm v_l \in C \cap \text{bd} \varrho B^d \) and reals \( \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_l > 0 \), such that,

(a) \( 2 \leq k, l \) and \( k + l \leq \frac{1}{2}d(d+1)+1 \),

(b) \( \sum_{i=1}^k \lambda_i u_i \otimes u_i = \sum_{j=1}^l \mu_j v_j \otimes v_j (= N \neq O, \text{ say}) \),

(c) \( \text{lin}\{u_1, \ldots, u_k\} = \text{lin}\{v_1, \ldots, v_l\} \).
As will be seen below, for the proof of the implication (iii) \implies (ii) it is sufficient to use claim (iii)(b). Tools for the proof are the idea of Voronoï, Radon partitions and a criterion for the separation of convex regions.

A result of Lewis [25] says that for any minimum ellipsoidal shell \((E, \varrho E)\) of a convex body \(C \subset C_o\) the number of contact points \(\pm u \in E \cap \text{bd } C\) is at least 2. More precisely, Praetorius [34] showed that for each \(k = 2, \ldots, d\) there is a convex body \(C \subset C_o\) with precisely \(k\) pairs of contact points \(\pm u \in E \cap \text{bd } C\) for any minimum ellipsoidal shell \((E, \varrho E)\) of \(C\).

**Preliminaries.** Before beginning with the proof of Theorem 1, we collect several results on the cone of positive definite quadratic forms, Radon partitions and separation of convex regions. Some of the results which are stated here, will be needed only later on.

**The cone of positive definite quadratic forms:** A quadratic form on \(\mathbb{R}^d\), say

\[ x \rightarrow A \cdot x \otimes x = x^T A x = \sum_{i,k=1}^{d} a_{ik} x_i x_k \quad \text{for } x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d, \]

can be identified with its symmetric \(d \times d\) coefficient matrix \(A = (a_{ik})\) and also with its coefficient vector \((a_{11}, \ldots, a_{1d}, a_{21}, \ldots, a_{2d}, \ldots, a_{dd})^T \in \mathbb{R}^{d(d+1)}\). In view of this identification, a symmetric \(d \times d\) matrix \(A\) may be called a vector or a point and we also write \(A \in \mathbb{R}^{d(d+1)}\).

The inner product for \(d \times d\) matrices then is a particular inner product on \(\mathbb{R}^{d(d+1)}\), different from the usual one. The set of all coefficient vectors of positive definite quadratic forms on \(\mathbb{R}^d\) is an open convex cone in \(\mathbb{R}^{\frac{1}{2}d(d+1)}\) with apex at the origin \(O\), the *cone of positive definite quadratic forms* \(\mathcal{P}^d\). Its closure \(\mathcal{Q}^d\) is the *cone of positive semidefinite quadratic forms*. \(O\) and \(I\) are the \(d \times d\) zero, resp. unit matrix.

A region or domain \(\mathcal{B}\) in \(\mathbb{R}^{\frac{1}{2}d(d+1)}\) is a set with the property that it is contained in the closure of its interior. It is smooth if its boundary is a surface of class \(C^1\). A region may be convex, but still it need not be closed or bounded. When speaking of a closed set in \(\mathcal{P}^d\) or a neighborhood of \(O\) in \(\mathcal{P}^d\) this is meant with respect to the topology on \(\mathcal{P}^d\), and similarly for a neighborhood of \(O\) in \(\mathcal{Q}^d\).

The cones \(\mathcal{P}^d\) and \(\mathcal{Q}^d\) carry a rich geometric structure, see the author [18]. We describe some properties that will be used in the following. These are either well known, see, e.g. [17], or were proved in [18]. By pos the positive or conical hull is meant.

1. \(D = \{ A \in \mathcal{P}^d: \det A \geq 1 \}\) is an unbounded, closed, strictly convex, smooth region in \(\mathcal{P}^d\) with non-empty interior. \(D\) is disjoint from a suitable neighborhood of \(O\) in \(\mathcal{P}^d\) and \(I\) is an interior normal vector of \(D\) at its boundary point \(I\). (\(\text{bd } D\) is the discriminant surface of algebraic number theory.)
2. Let \(F\) be a face of \(\mathcal{Q}^d\). Then there is a linear subspace \(S\) of \(\mathbb{R}^d\) such that \(F = \text{pos}\{u \otimes u: u \in S\}\). In particular, \(\mathcal{Q}^d = \text{pos}\{u \otimes u: u \in \mathbb{R}^d\}\).
3. Let \(N \in \mathcal{P}^d = \text{int } \mathcal{Q}^d\). Then there are linearly independent points \(u_1, \ldots, u_d \in S^{d-1}\) and reals \(\lambda_1, \ldots, \lambda_d > 0\) such that

\[ N = \lambda_1 u_1 \otimes u_1 + \cdots + \lambda_d u_d \otimes u_d. \]

Only the following proposition requires a proof:
Choose an orthogonal \(d \times d\) matrix \(U\) such that \(S = U E^c\), where \(E^c\) is embedded into \(E^d\) as usual (first \(c\) coordinates). The orthogonal projection of \(I\) into \(Q^c\) is the diagonal matrix \(I_c \in \text{relint} Q^c \subseteq Q^d\), where the first \(c\) diagonal elements of \(I_c\) are all 1 and the remaining ones 0. The mapping \(x \mapsto I_c x\) for \(x \in E^d\) is simply the orthogonal projection of \(E^d\) onto \(E^c\). Consider the linear transformation \(A \mapsto UAU^T\) for \(A \in \mathbb{R}^{\frac{1}{2}d(d+1)}\). It is also orthogonal, maps \(Q^d\) onto 

\[
UQ^dU^T = \{ u \otimes u : u \in E^d \} = \{ uu^T : u \in E^d \}
\]

onto \(Q^c\)

\[
UQ^cU^T = \{ u \otimes u : u \in E^c \} = \{ uu^T : u \in E^c \ = S \}
\]

both by (2), further \(I\) onto \(I\) and \(I_c\) onto \(I_f = U I_c U^T\). Together this shows that \(I_f \in \text{relint} \mathcal{F}\) is the orthogonal projection of \(I\) into \(\mathcal{F}\) and, considered as a transformation of \(E^d\), maps \(Q^d\) onto \(S\). Since

\[
(I - I_f) \cdot u \otimes u = U(I - I_c)U^T \cdot u \otimes u = U(I - I_c) \cdot U^T u \otimes U^T u
\]

\[
= (I - I_c) \cdot v \otimes v = v_{2+1}^2 + \cdots + v_d^2 \geq 0 \quad \text{for } v = U^T u \in E^d,
\]

the halfspace \(\{ X : (I - I_f) \cdot X \geq 0 \} \) supports \(Q^d\), taking into account proposition (2). The proof of (4) is complete.

Conical Radon partitions: For the usual Radon partitions see, e.g., Eckhoff [8]. In analogy to this notion define the following: Two sets \(U, V\) in a finite dimensional linear space form a conical Radon partition \(\{U, V\}\), if

\[
U, \ V \neq \emptyset, \ U \cap V = \emptyset, \quad \text{and} \quad \text{pos} \ U \cap \text{pos} \ V \text{ contains a ray with endpoint } O,
\]

where \(O\) is the origin of the linear space. A conical Radon partition \(\{U, V\}\) is primitive if there is no conical Radon partition \(\{X, Y\}\) with \(X \subseteq U, \ Y \subseteq V\) where at least one of the inclusions is strict. We need the following simple result; see [8] for the corresponding result for usual Radon partitions.

(5) Each conical Radon partition extends a primitive conical Radon partition. If \(\{U, V\}\) is a primitive conical Radon partition, then \(\text{pos} \ U\) and \(\text{pos} \ V\) are simplicial cones and \(\text{pos} \ U \cap \text{pos} \ V\) is a ray with endpoint \(O\) contained in the relative interiors of both \(\text{pos} \ U\) and \(\text{pos} \ V\). In particular,

\[
\dim \text{pos} \ U + \dim \text{pos} \ V = \dim \text{pos} (U \cup V) + 1.
\]
Touching and separation of convex regions, and normal cones: Two convex regions $\mathcal{B}$ and $\mathcal{C}$ touch at a common boundary point $A$, say, if they can be separated by a common support hyperplane through $A$. The normal cone $N(\mathcal{B}, A)$ of $\mathcal{B}$ at its boundary point $A$ is the closed convex cone of all exterior normal vectors of support hyperplanes of $\mathcal{B}$ at $A$. We state two touching, resp. separation criteria, the first one being trivial.

(6) Let $\mathcal{B}$ and $\mathcal{C}$ be two convex regions with a common boundary point $A$. Then the following statements are equivalent:

(i) $\mathcal{B}$ and $\mathcal{C}$ touch at $A$.

(ii) The cones $N(\mathcal{B}, A)$ and $-N(\mathcal{C}, A)$ have a ray with endpoint $O$ in common.

(7) Let $\mathcal{B}$ and $\mathcal{C}$ be two convex regions with a common boundary point $A$. Then each of the statements (i) and (ii) implies statement (iii):

(i) The cones $N(\mathcal{B}, A)$ and $-N(\mathcal{C}, A)$ have a ray with endpoint $O$ in common which is contained in the interior of each cone.

(ii) Let

$$N(\mathcal{B}, A) = \text{pos}\{x_i \otimes x_i : i = 1, \ldots, s\},$$

$$-N(\mathcal{C}, A) = \text{pos}\{y_j \otimes y_j : j = 1, \ldots, t\}$$

such that $\{x_i \otimes x_i : i = 1, \ldots, s\}, \{y_j \otimes y_j : j = 1, \ldots, t\}$ is a primitive conical Radon partition in $\mathbb{E}^{1\frac{1}{2}d(d+1)}$ with $s + t = \frac{1}{2}d(d + 1) + 1$.

(iii) $\mathcal{B}$ and $\mathcal{C}$ touch precisely at $A$.

**Proof of Theorem 1.** If $C$ is an ellipsoid, Theorem 1 is trivial. We thus may assume that

(8) $C$ is not an ellipsoid.

(i): For $A \in \Omega^d$ let $E_A = \{x: A \cdot x \otimes x = x^T Ax \leq 1\}$ be the corresponding ellipsoid (resp. elliptical cylinder) with center $o$. Define sets $\mathcal{E}^c$ and $\mathcal{E}^i$, representing the ellipsoids (and, in the case of $\mathcal{E}^c$, also elliptical cylinders) with center $o$ which are circumscribed, respectively, inscribed to $C$:

$$\mathcal{E}^c = \mathcal{E}^c(C) = \{A \in \Omega^d: C \subseteq E_A\} = \{A \in \Omega^d: A \cdot v \otimes v \leq 1 \text{ for } v \in \text{bd } C\}$$

$$= \bigcap_{v \in \text{bd } C} \{A \in \mathbb{E}^{1\frac{1}{2}d(d+1)}: A \cdot v \otimes v \leq 1\} \cap \Omega^d (\subseteq \Omega^d),$$

$$\mathcal{E}^i = \mathcal{E}^i(C) = \{A \in \Omega^d: E_A \subseteq C\} = \{A \in \Omega^d: A \cdot u \otimes u \geq 1 \text{ for } u \in \text{bd } C\}$$

$$= \bigcap_{u \in \text{bd } C} \{A \in \mathbb{E}^{1\frac{1}{2}d(d+1)}: A \cdot u \otimes u \geq 1\} \cap \Omega^d (\subseteq \Omega^d).$$

The sets $\mathcal{E}^c$ and $\mathcal{E}^i$ are intersections of families of closed halfspaces in $\mathbb{E}^{1\frac{1}{2}d(d+1)}$, intersected with the closed convex cone $\Omega^d$. Thus $\mathcal{E}^c, \mathcal{E}^i$ are closed convex sets in $\Omega^d$. Consider a ray in $\Omega^d$ starting at $O$. If we move with constant speed along the ray, we are at first in $\mathcal{E}^c$ and leave it in finite positive time. Since this holds for any ray in the closed convex cone $\Omega^d$ starting at $O$, it follows that the closed convex set $\mathcal{E}^c$ contains a neighborhood of $O$ in $\Omega^d$ and is bounded. By (8),
the sets $E^c$ and $E^i$ are disjoint. If the ray meets $E^i$, we stay in $E^i$ after first entering it. In particular, $E^i$ is unbounded. Note also, that $E^i \subseteq \mathcal{P}^d = \text{int} Q^d$. Obviously, $E^c$ and $E^i$ are convex regions. If $(E_A, \sigma E_A)$ is an ellipsoidal shell then $E_A \subseteq C \subseteq \sigma E_A$ holds precisely in case when $A \in E^i$ and $(1/\sigma^2 A \in E^c$, or $) A \in \sigma^2 E^c$. Combining these remarks, we obtain the following proposition which clearly implies (i):

(9) Let $\rho > 1$ be (the unique number) such that the convex regions $\rho^2 E^c$ and $E^i$ touch. Then the minimum ellipsoidal shells of $C$ are precisely the ellipsoidal shells $(E_A, \rho E_A)$, where $A \in \rho^2 E^c \cap E^i$.

(ii) $\Rightarrow$ (iii): (ii) together with (9) implies that the convex regions $\rho^2 E^c$ and $E^i$ touch at $I$. For the normal cones $N^c$ and $N^i$ of these regions at $I$, proposition (6) implies that

(10) $N^c \cap (-N^i)$ contains a ray with endpoint $O$.

The normal cones $N^c$ and $N^i$ are generated by the exterior normal vectors $v \otimes v$, resp. $-u \otimes u$ of those defining halfspaces of $\rho^2 E^c$ and $E^i$, for which $I$ is a boundary point. In the case of $\rho^2 E^c$ this means that $I \cdot v \otimes v = v^2 = \rho^2$, where $v \in \text{bd} C$, or, equivalently, $v \in C \cap \text{bd} \rho^2 B^d$. In the case of $E^i$ this means that $I \cdot u \otimes u = u^2 = 1$, where $u \in \text{bd} C$ or, equivalently, $u \in B^d \cap \text{bd} C$. Hence

(11) $N^c = \text{pos} \{v \otimes v : v \in C \cap \text{bd} \rho B^d\}$, $-N^i = \text{pos} \{u \otimes u : u \in B^d \cap \text{bd} C\}$.

(8) implies that $\rho > 1$. Thus the sets $\{v \otimes v : v \in C \cap \text{bd} \rho B^d\}$ and $\{u \otimes u : u \in B^d \cap \text{bd} C\}$ are disjoint. This, together with (10) and (11), shows that these sets form a conical Radon partition in $\mathbb{R}^{1/2(d+1)}$. By (5), this Radon partition extends a primitive Radon partition. Thus,

(12) there are contact points $\pm u_1, \ldots, \pm u_k \in B^d \cap \text{bd} C$ and $\pm v_1, \ldots, \pm v_l \in C \cap \text{bd} \rho B^d$ and reals $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_l > 0$,

such that the following hold:

(13) $1 \leq k, l$ and $k + l = 1/2 d(d+1) + 1$,

(14) $\sum_{i=1}^{k} \lambda_i u_i \otimes u_i = \sum_{j=1}^{l} \mu_j v_j \otimes v_j$ $(= N \neq O, \text{ say})$.

To see that

(15) $\text{lin} \{u_1, \ldots, u_k\} = \text{lin} \{v_1, \ldots, v_l\}$,

we distinguish two cases. If $\text{lin} \{u_1, \ldots, u_k\} = \text{lin} \{v_1, \ldots, v_l\} = \mathbb{R}^d$, we are finished. If not, we may assume that $\text{lin} \{u_1, \ldots, u_k\} \subseteq \mathbb{R}^d$. To show (15), let $x \in \mathbb{R}^d, x \perp u_1, \ldots, u_k$. Then,
\[
0 = \sum_{i=1}^{k} \lambda_i (u_i \cdot x)^2 = \sum_{i=1}^{k} \lambda_i u_i \otimes u_i \cdot x \otimes x
\]
\[
= \sum_{j=1}^{l} \mu_j v_j \otimes v_j \cdot x \otimes x = \sum_{j=1}^{l} \mu_j (v_j \cdot x)^2,
\]
where we have used the identity \(u \otimes u \cdot x \otimes x = (u \cdot x)^2\) and (14). The inequalities \(\mu_1, \ldots, \mu_l > 0\) then imply \(v_1 \cdot x = \cdots = v_l \cdot x = 0\), i.e., \(x \perp v_1, \ldots, v_l\). In particular, we have, \(\text{lin}\{v_1, \ldots, v_l\} \subseteq \mathbb{E}^d\). An analogous argument shows that \(y \in \mathbb{E}^d, y \perp v_1, \ldots, v_l\) implies \(y \perp u_1, \ldots, u_k\). Together these implications yield the equality (15). For the proof of the inequality

(16) \(2 \leq k, l\)

suppose that, on the contrary, \(k = 1\), say. Since \(\varrho > 1\) by (8), it follows from (12) that \(v_1\) is not a multiple of \(u_1\). Hence \(\text{lin}\{u_1\} \neq \text{lin}\{v_1, \ldots, v_l\}\), a contradiction to (15), concluding the proof of (16).

Having shown statements (12)–(16), the proof of (iii) is complete.

(iii) \(\Rightarrow\) (ii): Since \(B^d \subseteq C \subseteq \varrho B^d\) and \(u_1, \ldots, u_k \in B^d \cap \text{bd} C, v_1, \ldots, v_l \in C \cap \text{bd} \varrho B^d\), we have

\[
1 \leq I \cdot x \otimes x \leq \varrho^2 \quad \text{for } x \in \text{bd} C, \quad I \cdot u_i \otimes u_i = 1, \quad I \cdot v_j \otimes v_j = \varrho^2.
\]

This shows that the unit matrix \(I\) is a common boundary point of the convex regions \(\varrho^2 E^c\) and \(E^i\). For their exterior normal cones \(N^c\) and \(N^i\) at \(I\), (iii)(b) implies \(N^c \cap (-N^i) \supseteq \{N\} \neq \{O\}\). Thus, by (6), the convex regions \(\varrho^2 E^c\) and \(E^i\) touch at \(I\). By (9), the ellipsoidal shell \(\langle B^d, \varrho B^d \rangle\) is a minimum ellipsoidal shell of \(C\), concluding the proof of (ii). \(\square\)

2.2. Most convex bodies have unique minimum ellipsoidal shell

Given a convex body, its circumscribed ellipsoid of minimum volume and its inscribed ellipsoid of maximum volume are both unique. For proofs see Behrend [5] (\(d = 2\)) and Danzer, Laugwitz and Lenz [6] and Zaguskin [41] (general \(d\)). In contrast to these results, Maurey showed that there are convex bodies in \(C_0\) with non-unique minimum ellipsoidal shells, compare the mention in [26,33,34]. If the minimum ellipsoidal shell of the unit ball of a norm is non-unique, this has important analytic consequences for the norm, see, e.g., [33]. It is thus of interest to find out whether the family of such bodies is large or small.

In the following we show that, in the sense of Baire categories, this set is small.

**Theorem 2.** A typical convex body in \(C_0\) has a unique minimum ellipsoidal shell.

We use Baire categories and the usual topology on the space of convex bodies. The main step of the proof of Theorem 2 is to construct in Lemma 1 a dense set of convex polytopes with unique minimum ellipsoidal shell.
Preliminaries. Again, we first put together necessary definitions and tools.

**Baire categories:** A topological space is Baire if any of its meager subsets has dense complement, where a set is meager or of first Baire category if it is a countable union of nowhere dense sets. A version of the Baire category theorem says that each metrically complete or locally compact space is Baire. When speaking of typical or most elements of a Baire space, we mean all elements, with a meager set of exceptions, see [16,30]. For information on Baire type results in convex geometry compare the surveys of Zamfirescu [43] and the author [16].

**Topology on C and C_o:** Let the space $C$ of all (proper) convex bodies in $\mathbb{R}^d$ and thus its subspace $C_o$ of $o$-symmetric convex bodies be endowed with its natural topology. It is induced by, for example, the Hausdorff metric $\delta_H$: for $C, D \in C$ the distance $\delta_H (C, D)$ is the maximum Euclidean distance which a point of one of the bodies $C, D$ can have from the other body. A version of the Blaschke selection theorem says that $C$ and $C_o$ are locally compact and thus Baire. See [16].

**Lemma 1.** Let $P \in C_o$ be a convex polytope. Then there are convex polytopes $S \in C_o$, arbitrarily close to $P$, with unique minimum ellipsoidal shell.

**Proof.** Let $\varepsilon > 0$. First, the following will be shown, where $S^{d-1}$ is the $(d-1)$-dimensional Euclidean unit sphere:

\[(17) \text{ Let } u \in U \subseteq S^{d-1}, \text{ where } U \text{ is a neighborhood of } u \text{ in } S^{d-1}. \text{ Then } u \otimes u \in \text{int pos}\{x \otimes x: x \in U}\].

Note that pos$\{x \otimes x: x \in U\}$ is a convex cone in $\mathbb{R}^{\frac{1}{2}d(d+1)}$ with apex $O$. Thus, if (17) did not hold, there would be a closed halfspace in $\mathbb{R}^{\frac{1}{2}d(d+1)}$, say $\{A: N \cdot A \geq 0\}$ with $N \cdot u \otimes u = 0$ and $N \cdot x \otimes x \geq 0$ for each $x \in U$. This means that the quadric $\{x: N \cdot x \otimes x = 0\}$ in $\mathbb{R}^d$ contains $u \in S^{d-1}$, and all $x \in U \subseteq S^{d-1}$ are on the same side of it. The quadric then touches $S^{d-1}$ at $u$. Since the quadric is symmetric in $o$, it cannot contain the origin, in contradiction to $N \cdot o \otimes o = N \cdot O = 0$. The proof of (17) is complete.

If $Q \in C_o$ is a polytope, a vertex and a facet or two facets of $Q$ are neighbors if they have non-empty intersection. The minimum Euclidean distance between the boundaries of the inner and the outer ellipsoid of the minimum shells of $P$ has a positive lower bound. For all convex polytopes $Q \in C_o$, which are obtained from $P$ by sufficiently small distortions, there is still a positive common lower bound for this distance. Thus, by ‘breaking’ the facets of $P$ into ‘sufficiently small pieces’, we see that there is a polytope $Q \in C_o$ with

\[(18) \; \delta_H (P, Q) \leq \varepsilon,\]

which has the following property: if $\langle E, \varrho E \rangle$ is a minimum ellipsoidal shell of $Q$, then a facet of $Q$ which touches $E$ and a vertex of $Q$ on $\text{bd} \varrho E$, have no common neighbor. Let $\langle E, \varrho E \rangle$ be a minimum ellipsoidal shell of $Q$ (not necessarily unique) and choose a linear transformation $L$ of $\mathbb{R}^d$ such that $LE = B^d$. Then

\[(19) \; \langle B^d, \varrho B^d \rangle \text{ is a minimum ellipsoidal shell of } LQ \text{ and a facet of } LQ \text{ which touches } B^d \text{ and a vertex of } Q \text{ on } \text{bd} \varrho B^d \text{ have no common neighbor.}\]

We now distort $LQ$ slightly to get a polytope $R \in C_o$ which has the following properties:
$$\delta^H(Q, L^{-1}R) \leq \varepsilon,$$

(21) $\langle B^d, \varrho B^d \rangle$ is the unique minimum ellipsoidal shell of $R$.

The distortion can be described as follows: consider the sets of contact points,

$$\{\pm u_1, \ldots, \pm u_k\} = B^d \cap \text{bd} LQ, \{\pm v_1, \ldots, \pm v_l\} = LQ \cap \text{bd} \varrho B^d.$$  

For $i = 1, \ldots, k$ and $j = 1, \ldots, l$ choose points

$$\pm u_{im} \in \text{bd} B^d, \text{ close to } \pm u_i \text{ for } m = 1, \ldots, m_i,$$

$$\pm v_{jn} \in \text{bd} \varrho B^d, \text{ close to } \pm v_j \text{ for } n = 1, \ldots, n_j,$$

such that the following three claims hold: First,

$$u_i \otimes u_j \in \text{int pos}\{u_{im} \otimes u_{im}: m = 1, \ldots, m_i\},$$

$$v_j \otimes v_j \in \text{int pos}\{v_{jn} \otimes v_{jn}: n = 1, \ldots, n_j\}.$$  

This is possible by (17). Second, let $R$ be the convex polytope which is obtained from $LQ$ as follows: For $i = 1, \ldots, k$ replace each pair of support halfspaces of $LQ$ which support $B^d$ at $\pm u_i$ (and thus have exterior normal vectors $\pm u_i$) by the $m_i$ pairs of halfspaces which support $B^d$ at $\pm u_{im}, m = 1, \ldots, m_i$, (and thus have exterior normal vectors $\pm u_{im}$). These halfspaces will be support halfspaces of $R$ and are the only ones which also support $B^d$. For $j = 1, \ldots, l$ replace each pair of vertices $\pm v_j$ of $LQ$ on $\text{bd} \varrho B^d$ by the $n_j$ pairs of points $\pm v_{jn}, n = 1, \ldots, n_j$, on $\text{bd} \varrho B^d$. These pairs will be vertices of $R$ and are the only vertices of $R$ on $\text{bd} \varrho B^d$. The other support halfspaces of $LQ$ determined by facets and the other vertices of $LQ$ are left unchanged.

If the points $u_{im}$ and $v_{jn}$ are chosen sufficiently close to $u_i$ and $v_j$, respectively, which we suppose, then the new vertices and facets of $R$ still have no common neighbors in the above sense, and thus do not affect each other. Third,

$$\delta^H(Q, L^{-1}R) \leq \varepsilon.$$

(23) $\delta^H(Q, L^{-1}R) \leq \varepsilon.$

We now show that $R$ satisfies the properties (20) and (21). Property (20) holds by (23). It remains to prove (21). In order to avoid confusion, we now indicate to what regions and boundary points the sets $E^c, E^i, N^c, N^i$ correspond. It follows from (9) and (19) that the convex regions $\varrho^2 E^c(LQ)$ and $\varrho^i (LQ)$ touch at their common boundary point $I$. By (6) this can be expressed as follows: The cones $N(\varrho^2 E^c(LQ), I)$ and $-N(\varrho^i (LQ), I)$ have a ray with endpoint $O$ in common. Representing the normal cones in terms of the vectors $u_i$ and $v_j$, as in the proof of Theorem 1, this is equivalent to the following statement: the cones $\text{pos}\{v_j \otimes v_j, j = 1, \ldots, l\}$ and $\text{pos}\{u_i \otimes u_i, i = 1, \ldots, k\}$ have a ray with endpoint $O$ in common. By (22) this yields that the cones $\text{pos}\{v_{jn} \otimes v_{jn}, j = 1, \ldots, l, n = 1, \ldots, n_j\}$ and $\text{pos}\{u_{im} \otimes u_{im}, i = 1, \ldots, k, m = 1, \ldots, m_i\}$ have a ray with endpoint $O$ in common which is contained in the interior of each cone. By the construction of $R$, these cones, actually, are the cones $N(\varrho^2 E^c(R), I)$ and $-N(\varrho^i (R), I)$, compare the argument that led to (11). An application of (7) then shows that the convex regions $\varrho^2 E^c(R)$ and $\varrho^i (R)$ touch precisely at $I$. By (9) this means that $\langle B^d, \varrho B^d \rangle$ is the unique minimum ellipsoidal shell of $R$, concluding the proof of (21).
It follows from (21) that $\langle E, \varrho E \rangle$, where $E = L^{-1}B^d$, is the unique minimum ellipsoidal shell of $L^{-1}R \in C_o$. Propositions (18) and (23) imply $\delta^H(P, L^{-1}R) \leq \delta^H(P, Q) + \delta^H(Q, L^{-1}R) \leq 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, the proof of Lemma 1 is finished.

**Proof of Theorem 2.** Let

$$C_n = \left\{ C \in C_o : C \text{ has minimum ellipsoidal shells } \langle E, \varrho E \rangle, \langle F, \varrho F \rangle \text{ where } \delta^H(E, F) \geq \frac{1}{n}, \frac{1}{n}B^d \subseteq E, F \subseteq nB^d, 1 \leq \varrho \leq n \right\} \text{ for } n = 1, 2, \ldots .$$

It is routine to show that

(24) $C_n$ is closed in $C_o$. 

For the proof that

(25) $\text{int} C_n = \emptyset$,

assume the contrary. Since $C_n$ then contains an open set, Lemma 1 shows that there is a polytope $S \in C_o \cap C_n$ with unique minimum ellipsoidal shell, a contradiction, concluding the proof of (25). Propositions (24) and (25) together show that $C_n$ is nowhere dense in $C_o$. Hence

$$\bigcup_{n=1}^{\infty} C_n \text{ is meager in } C_o.$$ 

Noting that, by the definition of $C_n$, this set is the set of all convex bodies in $C_o$ with non-unique minimum ellipsoidal shell, the proof of Theorem 2 is complete.

2.3. **Most convex bodies have precisely $\frac{1}{2}d(d + 1) + 1$ pairs of contact points**

Zamfirescu [42] proved that for most convex bodies $C$ the number of contact points of $C$ and the circumscribed Euclidean ball of minimum radius is precisely $d + 1$. By a result of Zucco [44], for most convex bodies $C$ the number of contact points of a spherical shell of $C$ with minimum difference of radii and $C$ is $d + 2$. In [15] the author showed that for most convex bodies $C$ the number of contact points of $C$ and its circumscribed ellipsoid of minimum volume is precisely $\frac{1}{2}d(d + 3)$, where in the $o$-symmetric case $\frac{1}{2}d(d + 3)$ is replaced by $d(d + 1)$. Similar results hold for inscribed ellipsoids. For alternative proofs see Rudelson [35].

In all these results the contact number in the typical case is precisely the number of points or pairs of points in general position required to determine a ball, a spherical shell, and an ellipsoid, respectively, where one has to distinguish between the general and the $o$-symmetric case.

The following result on ellipsoidal shells complements these results.

**Theorem 3.** A typical convex body $C \in C_o$ has a unique minimum ellipsoidal shell $\langle E, \varrho E \rangle$ and the contact set $(E \cap \text{bd } C) \cup (C \cap \text{bd } \varrho E)$ consists of precisely $\frac{1}{2}d(d + 1) + 1$ pairs of points $\pm u$. Each of the contact sets $E \cap \text{bd } C$ and $C \cap \text{bd } \varrho E$ consists of at least 2 and at most $\frac{1}{2}d(d + 1) - 1$ pairs of points $\pm u$. 
The main step of the proof of Theorem 3 is to construct a dense set of convex polytopes with unique minimum ellipsoidal shells and contact number $\frac{1}{2}d(d + 1) + 1$, see Lemma 2.

A remark on the Banach–Mazur distance. Let $C \in C_\circ$. An inscribed, resp. circumscribed ellipsoid $E$, resp. $F$ of $C$ gives rise to a minimum ellipsoidal shell of $C$, or to the Banach–Mazur distance between the Euclidean norm $\| \cdot \|$ and the norm $\| \cdot \|_C$ with unit ball $C$, if there is $\varrho \geq 1$, such that $(E, \varrho E)$, resp. $(\frac{1}{\varrho} F, F)$ is a minimum ellipsoidal shell of $C$. As a consequence of Theorem 3 and the author’s [15] result that for most convex bodies $C \in C_\circ$ for the inscribed ellipsoid $E$ of maximum volume and the circumscribed ellipsoid of minimum volume the contact sets $E \cap \text{bd}C$ and $C \cap \text{bd}F$ each consists of $\frac{1}{2}d(d + 1)$ pairs of points $\pm u$, we obtain the following negative result:

**Corollary 1.** For a typical convex body $C \in C_\circ$ neither the (unique) inscribed ellipsoid of maximum volume, nor the (unique) circumscribed ellipsoid of minimum volume, gives rise to a minimum ellipsoidal shell of $C$.

**Preliminaries.** The proof of the following result is a refinement of the proof of Lemma 1. It makes use of Baire categories.

**Lemma 2.** Let $P \in C_\circ$ be a convex polytope. Then there are convex polytopes $V \in C_\circ$, arbitrarily close to $P$, with unique minimum ellipsoidal shells, for which the contact sets consist of precisely $\frac{1}{2}d(d + 1) + 1$ pairs of points $\pm u$ and constitute primitive conical Radon partitions.

**Proof.** Let $\varepsilon > 0$. First, proposition (17) will be refined:

(26) Let $u \in U \subseteq S^{d-1}$, where $U$ is a relatively open neighborhood of $u$ in $S^{d-1}$ and let $\{A: N_1 \cdot A = 0\}, \ldots, \{A: N_r \cdot A = 0\}$ be a finite family of linear subspaces of $\mathbb{P}^{d(d+1)}$ through $u \otimes u$, each of codimension 1. Then the set $\{x \otimes x: x \in U\}$ is not contained in the union of these subspaces.

For assume that, on the contrary, $U$ is contained in the union of the closed sets $F_1 = \{x \in S^{d-1}: N_1 \cdot x \otimes x = 0\}, \ldots, F_r = \{x \in S^{d-1}: N_r \cdot x \otimes x = 0\}$. Since $U$ is open relative to $S^{d-1}$, it is locally compact. Thus, by the Baire category theorem, $U$ cannot be the union of finitely many sets which are nowhere dense relative to $S^{d-1}$. Thus not all sets $F_i \cap U$ have empty interior relative to $S^{d-1}$. We may assume that $F_1 \cap U$ has non-empty interior relative to $S^{d-1}$. The quadric $\{x: N_1 \cdot x \otimes x = 0\}$ thus contains a relatively open subset of $S^{d-1}$ and, hence, coincides with $S^{d-1}$. Therefore the quadric cannot contain the origin $o$, a contradiction to $N \cdot o \otimes o = N \cdot O = 0$, concluding the proof of (26).

Let the points $u_{im}, v_{jn}$ and the polytope $R$ be as in the proof of Lemma 1. Then, putting

\[ w_1 = u_{11}, \ldots, w_m = u_{1m}, \quad w_{m+1} = u_{21}, \ldots, w_{m+n} = u_{km}, \]

\[ w_{p+1} = v_{11}, \ldots, w_{p+n} = v_{1n}, \quad w_{p+n+1} = v_{21}, \ldots, w_{p+n+m} = v_{km}, \]

we claim the following:

(27) (a) $\delta^H (P, L^{-1} R) \leq 2\varepsilon.$
(b) \( \pm w_1, \ldots, \pm w_p \) are exterior normal vectors of the facets of \( R \) which touch \( B^d \),
(c) \( \pm w_{p+1}, \ldots, \pm w_q \) are the vertices of \( R \) on \( \partial B^d \),
(d) \( \mathcal{N}(\varepsilon^i(R), I) = \text{pos}\{w_r \otimes w_r: r = p + 1, \ldots, q\} \), \( \mathcal{N}(\varepsilon^i(R), I) = \text{pos}\{w_r \otimes w_r: r = 1, \ldots, p\} \) have a ray with endpoint \( O \) in common which is contained in the interior of each of these cones.

Now \( R \) will be distorted in two steps to get the desired polytope \( V \). In the first step we proceed as follows:

(28) Choose relatively open neighborhoods \( U_r \subseteq \partial B^d \) of \( w_r \) for \( r = 1, \ldots, p \) and \( U_r \subseteq \partial B^d \)

of \( w_r \) for \( r = p + 1, \ldots, q \) which are so small that by replacing \( w_r \) by any \( \overline{w}_r \in U_r \) for \( r = 1, \ldots, q \), the polytope \( S \in \mathcal{C}_o \) which is constructed from \( \overline{w}_1, \ldots, \overline{w}_q \) as \( R \) from \( w_1, \ldots, w_q \), see the proof of Lemma 1, has the following properties:

(a) \( \delta^H(L^{-1} R, L^{-1} S) \leq \varepsilon \),
(b) \( \pm \overline{w}_1, \ldots, \pm \overline{w}_p \) are exterior normal vectors of the facets of \( S \) which touch \( B^d \),
(c) \( \pm \overline{w}_{p+1}, \ldots, \pm \overline{w}_q \) are the vertices of \( S \) on \( \partial B^d \),
(d) \( \mathcal{N}(\varepsilon^i(S), I) = \text{pos}\{\overline{w}_r \otimes \overline{w}_r: r = p + 1, \ldots, q\} \), \( \mathcal{N}(\varepsilon^i(S), I) = \text{pos}\{\overline{w}_r \otimes \overline{w}_r: r = 1, \ldots, p\} \) have a ray with endpoint \( O \) in common which is contained in the interior of each of these cones.

We now choose such points \( \overline{w}_r \in U_r, r = 1, \ldots, q \), by induction which have the property that

(29) any \( \frac{1}{2} d(d+1) \) among the vectors \( \overline{w}_1 \otimes \overline{w}_1, \ldots, \overline{w}_q \otimes \overline{w}_q \) are linearly independent.

The first part of the proof of (29) is to show by induction,

(30) for \( r = 1, \ldots, \frac{1}{2} d(d+1) \) there is \( \overline{w}_r \in U_r \) such that the vectors \( \overline{w}_1 \otimes \overline{w}_1, \ldots, \overline{w}_r \otimes \overline{w}_r \) are linearly independent.

For \( r = 1 \), simply take \( \overline{w}_1 = w_1(\neq o) \). Assume now that \( 1 \leq r < \frac{1}{2} d(d+1) \) and that \( \overline{w}_1, \ldots, \overline{w}_r \) have already been chosen as to satisfy (30). Choose a linear subspace of \( \mathbb{E}^{\frac{1}{2} d(d+1)} \) of codimension 1, which contains \( \overline{w}_1 \otimes \overline{w}_1, \ldots, \overline{w}_r \otimes \overline{w}_r \). Take \( \overline{w}_{r+1} \in U_{r+1} \subseteq S^{d-1} \) such that \( \overline{w}_{r+1} \otimes \overline{w}_{r+1} \) is not contained in this subspace. This is possible by (26). The induction is complete, concluding the proof of (30).

The second and final part of the proof of (29) is to show, again by induction,

(31) for \( r = \frac{1}{2} d(d+1), \ldots, q \), there is \( \overline{w}_r \in U_r \) such that any \( \frac{1}{2} d(d+1) \) among the vectors \( \overline{w}_1 \otimes \overline{w}_1, \ldots, \overline{w}_r \otimes \overline{w}_r \) are linearly independent.

For \( r = \frac{1}{2} d(d+1) \) the point \( \overline{w}_r \) has already been chosen according to (30) and for this value of \( r \) proposition (31) holds by the case \( r = \frac{1}{2} d(d+1) \) of (30). Assume now that \( \frac{1}{2} d(d+1) \leq r < q \) and that \( \overline{w}_1, \overline{w}_2, \ldots, \overline{w}_r \) have already been chosen such that (31) holds. Consider the finitely many linear subspaces of \( \mathbb{E}^{\frac{1}{2} d(d+1)} \) of codimension 1, each of which is spanned by \( \frac{1}{2} d(d+1) - 1 \) (necessarily linearly independent) vectors among \( \overline{w}_1 \otimes \overline{w}_1, \ldots, \overline{w}_r \otimes \overline{w}_r \). By (26) we can choose a point \( \overline{w}_{r+1} \in U_{r+1} \) such that \( \overline{w}_{r+1} \otimes \overline{w}_{r+1} \) is not contained in any of these subspaces. Together with the induction hypothesis, this implies that any \( \frac{1}{2} d(d+1) \) vectors...
among $\bar{w}_1 \otimes \bar{w}_1, \ldots, \bar{w}_{r+1} \otimes \bar{w}_{r+1}$ are linearly independent. This concludes the induction and thus the proof of (31). Proposition (29) now is an immediate consequence of (30) and (31).

Let $S \in \mathcal{C}_o$ be the convex polytope corresponding to the chosen vectors $\bar{w}_1, \ldots, \bar{w}_q$. Then

\begin{equation}
\delta^H(P, L^{-1}S) \leq \delta^H(P, L^{-1}R) + \delta^H(L^{-1}R, L^{-1}S) \leq 2\varepsilon + \varepsilon = 3\varepsilon
\end{equation}

by (27)(a) and (28)(a). By (28)(d) $\{\bar{w}_r \otimes \bar{w}_r: r = p + 1, \ldots, q\}, \{\bar{w}_r \otimes \bar{w}_r: r = 1, \ldots, p\}$ is a conical Radon partition. By (5), it extends a primitive conical Radon partition, say $\{[x_i \otimes x_i: i = 1, \ldots, s]\}, \{[y_j \otimes y_j: j = 1, \ldots, t]\}$, where

\begin{equation}
\text{pos}\{x_i \otimes x_i: i = 1, \ldots, s\}, \text{pos}\{y_j \otimes y_j: j = 1, \ldots, t\}
\end{equation}

have precisely one ray with endpoint $O$ in common which is in the relative interior of each cone and $s + t \leq \frac{1}{2}d(d + 1) + 1$.

Next, the following equality will be shown:

\begin{equation}
s + t = \frac{1}{2}d(d + 1) + 1.
\end{equation}

By (33), there are $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t > 0$, such that

$$\lambda_1 x_1 \otimes x_1 + \cdots + \lambda_s x_s \otimes x_s = \mu_1 y_1 \otimes y_1 + \cdots + \mu_t y_t \otimes y_t.$$ 

If $s + t \leq \frac{1}{2}d(d + 1)$, the equality shows that the $s + t \leq \frac{1}{2}d(d + 1)$ vectors $x_1 \otimes x_1, \ldots, y_t \otimes y_t$ are linearly dependent, a contradiction to (29), concluding the proof of (34).

Finally, let $T \in \mathcal{C}_o$ be a convex polytope which is obtained from $S$ by slightly pushing outside the facets of $S$ which touch $B^d$ and with exterior normal vectors different from $\pm x_1, \ldots, \pm x_s$, and by slightly pushing inside the vertices of $S$ on $\partial \varrho_B d$ different from $\pm y_1, \ldots, \pm y_t$. Clearly, this can be done such that

\begin{equation}
\delta^H(L^{-1}S, L^{-1}T) \leq \varepsilon.
\end{equation}

It follows from (31), (33) and (34) that

$$\{[x_i \otimes x_i: i = 1, \ldots, s]\}, \{[y_j \otimes y_j: j = 1, \ldots, t]\} \text{ is a primitive conical Radon partition with } s + t = \frac{1}{2}d(d + 1) + 1.$$ 

By the construction of $T$,

the regions $\varrho^2E_c(T)$ and $E_i(T)$ have $I$ in common and

$\mathcal{N}(\varrho^2E_c(T), I) = \text{pos}\{y_j \otimes y_j: j = 1, \ldots, t\}$,

$-\mathcal{N}(E_i(T), I) = \text{pos}\{x_i \otimes x_i: i = 1, \ldots, s\}$.

These propositions together with (7) show that the convex regions $\varrho^2E_c(T)$ and $E_i(T)$ have precisely $I$ in common. (9) then yields that $\langle B^d, \varrho B^d \rangle$ is the unique ellipsoidal shell of $T$. By
(34) and the construction of $T$ the contact set consists precisely of the $\frac{1}{2}d(d+1)+1$ pairs of points $\pm x_1, \ldots, \pm x_s, \pm y_1, \ldots, \pm y_t$. Since by (32) and (35),

$$\delta^H(P, L^{-1}T) \leq (\delta(P, L^{-1}S) + \delta(L^{-1}S, L^{-1}T) \leq 3\epsilon + \epsilon = 4\epsilon,$$

and $\epsilon > 0$ was chosen arbitrarily, the proof of Lemma 2 is complete. \qed

**Proof of Theorem 3.** Let

$$\mathcal{D}_n = \left\{ C \in \mathcal{C}_o : C \text{ has a minimum ellipsoidal shell } \langle E, \varrho E \rangle \text{ where} \right\}$$

$$\frac{1}{n}B^d \subseteq E \subseteq nB^d, 1 \leq \varrho \leq n, \text{ and the contact set contains points}$$

$$\pm w_1, \ldots, \pm w_{\frac{1}{2}d(d+1)+2}, \text{ such that } \|w_i \pm w_j\| \geq \frac{1}{n} \text{ for } i \neq j \right\}, \; n = 1, 2, \ldots$$

By routine arguments,

$$\mathcal{D}_n \text{ is closed and } \text{int } \mathcal{D}_n = \emptyset \text{ in } \mathcal{C}_o,$$

where in the proof that $\text{int } \mathcal{D}_n = \emptyset$, Lemma 2 is used. Thus,

(36) $\bigcup_{n=1}^{\infty} \mathcal{D}_n$ is meager in $\mathcal{C}_o$.

Since this set contains all convex bodies in $\mathcal{C}_o$ for which the contact set has more than $\frac{1}{2}d(d+1)+1$ pairs of points $\pm u$, we get from (36) and Theorem 2 the following result:

(37) Most convex bodies in $\mathcal{C}_o$ have a unique minimum ellipsoidal shell and the contact set consists of at most $\frac{1}{2}d(d+1)+1$ pairs of points $\pm u$.

Slightly more difficult is the proof that

(38) the set of convex bodies in $\mathcal{C}_o$ which have a (not necessarily unique) minimum ellipsoidal shell, such that the contact set consists of at most $\frac{1}{2}d(d+1)$ pairs of points $\pm u$, is meager in $\mathcal{C}_o$.

For the proof of this it suffices to show,

(39) the set of convex bodies in $\mathcal{C}_o$ which have a minimum ellipsoidal shell $\langle E, \varrho E \rangle$, such that the contact set extends a primitive conical Radon partition of at most $\frac{1}{2}d(d+1)$ points, is meager in $\mathcal{C}_o$.

To see this, let
\[ E_n = \{ C \in C_o : \text{there is a non-singular linear transformation } L \text{ of } \mathbb{R}^d \text{ such that } \langle B^d, \varrho B^d \rangle \text{ is a minimum ellipsoidal shell of } LC \text{ where} \]

the contact set contains pairs of points \( \pm u_1, \ldots, \pm u_k \in B^d \cap \partial C, \pm v_1, \ldots, \pm v_l \in C \cap \partial \varrho B^d \) such that the following claims hold:

(a) \( \frac{1}{n} \leq \det L, \| L \| \leq n, \)
(b) \( 1 \leq \varrho \leq n, \)
(c) \( 2 \leq k, l \) and \( k + l \leq \frac{1}{2}d(d + 1), \)
(d) \( \{ [u_1 \otimes u_1, \ldots, u_k \otimes u_k], [v_1 \otimes v_1, \ldots, v_l \otimes v_l] \} \) is a primitive conical Radon partition,
(e) \( \{ u_1 \otimes u_1, \ldots, u_k \otimes u_k \} \) and \( \{ v_1 \otimes v_1, \ldots, v_l \otimes v_l \} \) span \( k \)-, resp. \( l \)-dimensional parallelotopes of \( k \)-, resp. \( l \)-dimensional volume \( \geq \frac{1}{n} \),
(f) Let \( L = \text{lin} [u_1 \otimes u_1, \ldots, u_k \otimes u_k], M = \text{lin} [v_1 \otimes v_1, \ldots, v_l \otimes v_l] \) and
\[ \text{cone}(\mathcal{R}, \frac{2}{n}) \cap \mathcal{L} \subseteq \text{pos} [u_1 \otimes u_1, \ldots, u_k \otimes u_k] \subseteq \text{cone}(\mathcal{R}, \pi - \frac{2}{n}) \cap L, \]
\[ \text{cone}(\mathcal{R}, \frac{2}{n}) \cap M \subseteq \text{pos} [v_1 \otimes v_1, \ldots, v_l \otimes v_l] \subseteq \text{cone}(\mathcal{R}, \pi - \frac{2}{n}) \cap M, \]
\[ \text{dist}(\mathcal{L} \cap \mathcal{R} \cap S^{\frac{1}{2}d(d+1)-1}, M \cap \mathcal{R} \cap S^{\frac{1}{2}d(d+1)-1}) \geq \frac{1}{n}. \]

Here, \( \text{cone}(\mathcal{R}, \frac{2}{n}) \) means the cone with apex \( O \), axis \( \mathcal{R} \), and angle \( \frac{2}{n} \). It is easy to see that

\[ (40) \ E_n \text{ is closed in } C_o. \]

To prove that

\[ (41) \ \text{int} E_n = \emptyset, \]

assume the contrary. Then, by Lemma 2, there is a convex polytope in \( E_n \) with unique minimum ellipsoidal shell such that the contact set consists of precisely \( \frac{1}{2}d(d + 1) + 1 \) pairs of points \( \pm u \) and gives rise to a primitive Radon partition. This is incompatible with the definition of \( E_n \) and thus concludes the proof of (41). By (40) and (41),

\[ \bigcup_{n=1}^{\infty} E_n \text{ is meager in } C_o. \]

Since this set consists of all convex bodies which have a minimum ellipsoidal shell such that the contact set extends a primitive conical Radon partition of at most \( \frac{1}{2}d(d + 1) \) points, the proof of (39) and thus of (38) is complete.

Theorem 3, finally, follows from (37) and (38). \( \square \)

2.4. The non-symmetric case

The above results, suitably modified, hold also for convex bodies which are not necessarily symmetric. The proofs are technically more involved than in the symmetric case, but the basic ideas are the same. So, we prefer to state the more general results without proofs.
Given a convex body \( C \in \mathbb{C} \), an ellipsoidal shell \( \langle E + c, \varrho E + c \rangle \) is a minimum (concentric) ellipsoidal shell of \( C \) if \( E \) is an ellipsoid with center \( o \), such that \( E + c \subseteq C \subseteq \varrho E + c \) and \( \varrho \geq 1 \) is minimum among all such ellipsoidal shells.

**Theorem 4.** Let \( C \in \mathbb{C} \). Then statement (i) holds and, assuming \( B^d \subseteq C \subseteq \varrho B^d \), where \( \varrho \geq 1 \), then statements (ii) and (iii) are equivalent:

(i) \( C \) has a (not necessarily unique) minimum ellipsoidal shell.

(ii) \( \langle B^d, \varrho B^d \rangle \) is a minimum ellipsoidal shell of \( C \).

(iii) There are contact points \( u_1, \ldots, u_k, v_1, \ldots, v_l \in B^d \cap \text{bd} C, v_1, \ldots, v_l \in C \cap \text{bd} \varrho B^d \) and reals \( \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_l > 0 \), such that the following claims hold:

\begin{align*}
(\text{a}) & \quad 2 \leq k, l \text{ and } k + l \leq \frac{1}{2}d(d + 3) + 1, \\
(\text{b}) & \quad \sum_{i=1}^{k} \lambda_i u_i \otimes u_i = \sum_{j=1}^{l} \mu_j v_j \otimes v_j \text{ and } \sum_{i=1}^{k} \lambda_i u_i = \sum_{j=1}^{l} \mu_j v_j, \\
(\text{c}) & \quad \text{lin}\{u_1, \ldots, u_k\} = \text{lin}\{v_1, \ldots, v_l\}.
\end{align*}

For the proof of proposition (i) we may assume that \( C \) is far away from the origin \( o \), such that \( o \) is not contained in the outer ellipsoid of any minimum ellipsoidal shell of \( C \). An ellipsoid which does not contain \( o \) may be represented in the form

\[ \{x: A \cdot x \otimes x + a \cdot x \leq -1\}, \quad \text{where } (A, a) \in \mathbb{P}^d \times \mathbb{E}^d \subseteq \mathbb{E}^{\frac{1}{2}d(d+3)}. \]

The center of this ellipsoid is \( c = -\frac{1}{2}A^{-1}a \) and the homothetic ellipsoids with the same center and disjoint from \( \{o\} \) are of the form

\[ \{x: \lambda A \cdot x \otimes x + \lambda a \cdot x \leq -1\}, \quad \text{where } \lambda > 0. \]

Instead of the sets \( \mathcal{E}^{c} \) and \( \mathcal{E}^{i} \) in the proof of Theorem 1, here the following sets may be used:

\[ \{(A, a) \in \mathcal{Q}^{d} \times \mathbb{E}^d : A \cdot v \otimes v + a \cdot v \leq -1 \text{ for } v \in \text{bd } C\}, \]

\[ \{(A, a) \in \mathcal{Q}^{d} \times \mathbb{E}^d : A \cdot u \otimes u + a \cdot u \geq -1 \text{ for } u \in \text{bd } C\}. \]

**Theorem 5.** A typical convex body \( C \in \mathbb{C} \) has a unique minimum ellipsoidal shell \( \langle E + c, \varrho E + c \rangle \) and the contact set \( ((E + c) \cap \text{bd } C) \cup (C \cap \text{bd}(\varrho E + c)) \) consists of precisely \( \frac{1}{2}d(d + 3) + 1 \) points.

**2.5. Minimum circumradius position**

As remarked earlier, John’s theorem and the above results, including their generalizations, can be formulated in terms of the Banach–Mazur distance or generalizations of it. They can also be formulated in terms of the circumradius or the inradius. Among these results we present those dealing with the circumradius of general convex bodies. For \( C \in \mathbb{C} \) the circumradius \( R(C) \) of \( C \) is the minimum radius of a (solid Euclidean) ball which contains \( C \). For the notion of minimum position with respect to volume preserving affinities, see the Introduction.

The first two results are reformulations of the dual of a version of John’s theorem (for the latter compare [21]) and of a result of the author [15]. The simple proofs of the results of this section are omitted.
Theorem 6. Let $C \in C$. Then statement (i) holds and, if $C \subseteq RB^d$, then statements (ii) and (iii) are equivalent:

(i) Up to rigid motions of $\mathbb{R}^d$ the convex body $C$ has a unique minimum $R$-position with respect to volume preserving affinities.

(ii) $C$ is in minimum $R$-position and $RB^d$ is the corresponding circumscribed ball.

(iii) There are contact points $\pm u_1, \ldots, \pm u_k \in C \cap bd RB^d$ and reals $\lambda_1, \ldots, \lambda_k > 0$ such that

(a) $d \leq k \leq \frac{1}{2}d(d + 3)$,

(b) $I = \sum_{i=1}^k \lambda_i u_i \otimes u_i$ and $o = \sum_{i=1}^k \lambda_i u_i$.

Theorem 7. For a typical convex body $C \in C$ the minimum $R$-position with respect to volume preserving affinities is unique up to rigid motions. If $C$ is in minimum $R$-position and $RB^d$ the corresponding circumscribed ball, then the contact set $C \cap bd RB^d$ consists of precisely

$$\frac{1}{2}d(d + 3)$$

points.

Note that these results are not in contradiction with the result of Zamfirescu [42]. In his result, for each convex body $C$ the circumball is considered, while in our results we choose a volume preserving affine image of $C$ for which the circumball is as small as possible.

In the next two results, which are reformulations of Theorems 1, 2, and 3, we consider for $C \in C_0$ its polar body $C^* = \{y: x \cdot y \leq 1 \text{ for } x \in C\}$. Minimization is with respect to all linear transformations, i.e., we minimize $R(AC)R((AC)^*)$ where $A$ ranges over all non-singular $d \times d$ matrices.

By Maurey’s result, $C$ may have minimum $RR^*$-positions which are not equivalent via rigid motions. This is expressed by saying that the minimum $RR^*$-positions of $C$ are non-unique.

In principle, it is also possible to express Theorems 4 and 5 in terms of the circumradius, but then we must use polarity with respect to arbitrary points different from the origin $o$. Since this makes the formulations a bit technical, we have preferred, not to state these results.

Theorem 8. Let $C \in C_0$. Then statement (i) holds and, if $B^d \subseteq C \subseteq RB^d$, then statements (ii) and (iii) are equivalent:

(i) $C$ has a (not necessarily unique) minimum $RR^*$-position with respect to non-singular linear transformations.

(ii) $C$ is in minimum $RR^*$-position where $RB^d$ and $B^d$ are the corresponding balls circumscribed to $C$ and $C^*$, respectively.

(iii) There are contact points $\pm u_1, \ldots, \pm u_k \in C \cap bd RB^d$ and $\pm v_1, \ldots, \pm v_l \in C^* \cap bd B^d$ and reals $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_l > 0$ such that

(a) $2 \leq k, l$ and $k + l \leq \frac{1}{2}d(d + 1) + 1$,

(b) $\sum_{i=1}^k \lambda_i u_i \otimes u_i = \sum_{j=1}^l \mu_j v_j \otimes v_j$,

(c) $\text{lin}\{u_1, \ldots, u_k\} = \text{lin}\{v_1, \ldots, v_l\}$.

Theorem 9. For a typical convex body $C \in C_0$ the minimum $RR^*$-position with respect to non-singular linear transformations is unique up to rigid motions. If $C$ is in minimum $RR^*$-position and $RB^d$ and $B^d$ are the corresponding circumscribed balls of $C$ and $C^*$, the contact set $(C \cap bd RB^d) \cup (C^* \cap B^d)$ consists of precisely

$$\frac{1}{2}d(d + 1) + 1$$

points. The contact set $C \cap B^d$ contains at least 2 and at most

$$\frac{1}{2}d(d + 1) - 1$$

points and similarly for the contact set $C^* \cap B^d$. 
3. Circumscribed ellipsoids of minimum surface area

In the light of John’s theorem and its dual, the following question naturally arises: given a convex body \( C \), characterize the inscribed and circumscribed ellipsoids of maximum, resp. minimum surface area or, more generally, of maximum, respectively minimum \( i \)th quermassintegral, \( i = 0, \ldots, d - 1 \). Moreover, what are the corresponding maximum and minimum positions of \( C \) with respect to volume preserving linear transformations.

In this section we study these questions for circumscribed ellipsoids of \( o \)-symmetric convex bodies. The results for the surface area are proved, their extensions to all quermassintegrals are stated without proof since their proofs make use of the same ideas. It turns out that for all quermassintegrals, except the ordinary volume, the minimizing positions of the convex body coincide and the corresponding ellipsoids are the same.

3.1. Circumscribed ellipsoids of minimum surface area are unique

The proofs that the circumscribed and inscribed ellipsoids of a given convex body of minimum, resp. maximum volume, are unique, are comparatively simple. For references see Section 2.2.

Our proofs of the corresponding results, where the volume is replaced by the surface area are complicated and await simplification. In the following we consider only circumscribed ellipsoids.

**Theorem 10.** Let \( C \in \mathcal{C}_o \). Then there is a unique ellipsoid containing \( C \) of minimum surface area.

In the proof of Theorem 10 and Lemma 3, we use projection bodies, Cauchy’s surface area formula, Minkowski’s determinant inequality, Alexandrov’s projection theorem, and the fact that the mappings \( A \rightarrow A^{-1} \) and \( A \rightarrow A^2 \) of \( \mathcal{P}^d \) onto itself are diffeomorphisms.

In general, the minimizing ellipsoid \( E_A = \{ x : A \cdot x \otimes x \leq 1 \} \) will not be a ball. In principle it is possible to give a John type characterization of \( E_A \). For this characterization we need a normal vector at \( A \) of the smooth convex surface in \( \mathcal{P}^d \) corresponding to ellipsoids with surface area equal to that of \( E_A \). Since no simple explicit expression for such a normal vector seems to be known, this characterization, at present, is of little value and, thus, will not be given here.

**Preliminaries.** We first collect some tools and then show that to the family of all \( o \)-symmetric ellipsoids with surface area less than or equal to a given constant there corresponds a smooth and strictly convex subset of \( \mathcal{P}^d \).

*The projection body and Cauchy’s surface area formula:* Given a convex body \( C \in \mathcal{C} \), its projection body \( \Pi C \) is a convex body with support function defined by

\[
h_{\Pi C}(u) = v(C | u^\perp) \quad \text{for} \quad u \in S^{d-1}.
\]

Here \( v(\cdot) \) is the volume in \( d - 1 \) dimensions and \( \cdot | u^\perp \) stands for the orthogonal projection of \( \mathbb{R}^d \) onto \( u^\perp \), where \( u^\perp \) is the linear subspace of \( \mathbb{R}^d \) of codimension 1 orthogonal to \( u \). For the following well-known formula, see, e.g., Petty [32] and Lutwak [27]:
(43) $\Pi(AC) = (\det A)A^{-1}\Pi C$ for $A \in \mathcal{P}^d$.

A simple argument which makes use of the definition of support functions, yields the identity

(44) $h_{AC}(u) = h_C(Au)$ for $A \in \mathcal{P}^d, u \in S^{d-1}$.

The surface area formula of Cauchy for a convex body is as follows, where $\sigma$ and $S$ denote the usual surface area measure in $\mathbb{E}^d$:

(45) $S(C) = \frac{1}{v(B^{d-1})} \int_{S^{d-1}} v(C|u^\perp) \, d\sigma(u) = \frac{1}{v(B^{d-1})} \int_{S^{d-1}} h_{\Pi C}(u) \, d\sigma(u)$ for $C \in \mathcal{C}$.

**Volume of an ellipsoid:** We state the formula in dimension $d - 1$; the strange notation is in view of the proof of Lemma 3 below:

(46) $v(E_{[A]}) = \frac{v(B^{d-1})}{(\det [A])^{\frac{1}{2}}}$ for $E_{[A]} = \{ x \in \mathbb{E}^{d-1} : [A] \cdot x \otimes x \leq 1 \}, [A] \in \mathcal{P}^{d-1}$.

**Diffeomorphisms of $\mathcal{P}^d$:** The fact that a symmetric, positive definite matrix has a unique square root (see, e.g., [23, p. 187]), and a version of the inverse function theorem together yield the following result:

(47) Each of the mappings $A \mapsto A^{-1}, A \mapsto A^2$ for $A \in \mathcal{P}^d$ is a bijective diffeomorphism of the locally compact space $\mathcal{P}^d$ onto itself. Locally at $I$, the mappings $A \mapsto A^{-1}$ and $A \mapsto A^2$ are a reflection in $I$, resp. a dilatation with center $I$ and ratio 2.

**Alexandrov’s projection theorem** [1] says the following:

(48) Let $E, F \in \mathcal{C}_o$ and assume that $v(E|u^\perp) = v(F|u^\perp)$ for $u \in S^{d-1}$. Then $E = F$.

**Lemma 3.** The set $A = \{ A \in \mathcal{P}^d : S(E_A) \leq S(B^d) \}$ is a closed, unbounded, smooth and strictly convex region in $\mathcal{P}^d$ and $I$ is an interior normal vector of $A$ at its boundary point $I$.

**Proof.** The function $A \mapsto S(E_A)$ for $A \in \mathcal{P}^d$ is continuous and on each ray in $\mathcal{P}^d$ with endpoint $O$ it decreases strictly from $+\infty$ to 0 if we move away from $O$. Being a level set of this function, $bdA$ is a continuous surface in $\mathcal{P}^d$ and

(49) $A$ is a closed, unbounded region.

The main step of the proof is to show that the region

(50) $A$ is strictly convex.

In order to apply Cauchy’s formula, we study orthogonal projections of ellipsoids onto linear subspaces of codimension 1. First, a special case is considered. Let $\mathbb{E}^{d-1}$ be embedded into $\mathbb{E}^d$ as usual (first $d - 1$ coordinates) and denote the orthogonal projection of $\mathbb{E}^d$ onto $\mathbb{E}^{d-1}$ by $\cdot | \mathbb{E}^{d-1}$.
For $a \in \mathbb{E}^d$ the point $a \in \mathbb{E}^{d-1}$ is obtained from $a$ by deleting the last coordinate. Similarly, for $A \in \mathbb{P}^d$ let the symmetric $(d - 1) \times (d - 1)$ matrix $\tilde{A} \in \mathbb{P}^{d-1}$ be $A$ with the last row and column omitted. Define

$$[A] = \tilde{A} - \frac{1}{a_{dd}} a_d \otimes a_d$$

for $A \in \mathbb{P}^d$, where $a_d$ is the last column of $A$. Let det stand for determinant of a $(d - 1) \times (d - 1)$ matrix.

The proof of proposition (50) is divided into three parts:

First, we prove

$$E_A|\mathbb{E}^{d-1} = E[A] \text{ for } A \in \mathbb{P}^d.$$  

Let $b = (0, 0, \ldots, 0, 1)^T$. If $x = (x_1, \ldots, x_{d-1}, 0)^T \in \text{relbd}(EA|\mathbb{E}^{d-1})$, there is a unique $t \in \mathbb{R}$ with

$$(x + tb)^T A(x + tb) = 1.$$  

The quadratic equation for $t$,

$$t^2 b^T A b + 2tb^T A x + x^T A x - 1 = t^2 a_{dd} + 2ta_d \cdot x + x^T A x - 1 = 0,$$

then has a unique (double) solution or, equivalently,

$$(a_d \cdot x)^2 - a_{dd} (x^T A x - 1) = 0.$$  

This yields the following equation for $\text{relbd}(E_A|\mathbb{E}^{d-1})$, concluding the proof of proposition (51):

$$x^T A x - \frac{1}{a_{dd}} x^T a_d \otimes a_d x = 1, \text{ or } x^T [A] x = 1.$$  

Second,

$$E_{[(1-\lambda)A + \lambda B]} \subseteq E_{(1-\lambda)[A] + \lambda [B]} \text{ for } A, B \in \mathbb{P}^d, \ 0 \leq \lambda \leq 1.$$  

Since the ellipsoids in (52) can be represented in the form

$$x^T (1 - \lambda) A + \lambda B) x - \frac{1}{(1 - \lambda)a_{dd} + \lambda b_{dd}} (((1 - \lambda) a_d + \lambda b_d) \cdot x)^2 \leq 1,$$

$$x^T (1 - \lambda) A + \lambda B) x - \frac{1}{(1 - \lambda)a_{dd} + \lambda b_{dd}} ((1 - \lambda) a_d \cdot x)^2 - \frac{1}{\lambda b_{dd}} ((\lambda b_d \cdot x)^2 \leq 1,$$

it is sufficient for the proof of the inclusion in (52), to show,

$$\frac{1}{(1 - \lambda)a_{dd} + \lambda b_{dd}} (((1 - \lambda) a_d + \lambda b_d) \cdot x)^2 \leq \frac{1}{(1 - \lambda)a_{dd}} ((1 - \lambda) a_d \cdot x)^2 + \frac{1}{\lambda b_{dd}} ((\lambda b_d \cdot x)^2,$$

or
\[ a_{dd}b_{dd}((1 - \lambda)^2(a_d \cdot x)^2 + 2\lambda(1 - \lambda)(a_d \cdot x)(b_d \cdot x) + \lambda^2(b_d \cdot x)^2) \lesssim ((1 - \lambda)a_{dd} + \lambda b_{dd})((1 - \lambda)b_{dd}(a_d \cdot x)^2 + \lambda a_{dd}(b_d \cdot x)^2), \]

or

\[ 2\lambda(1 - \lambda)a_{dd}b_{dd}(a_d \cdot x)(b_d \cdot x) \leq \lambda(1 - \lambda)b_{dd}^2(a_d \cdot x)^2 + \lambda(1 - \lambda)a_{dd}^2(b_d \cdot x)^2, \]

or

\[ 2a_{dd}b_{dd}(a_d \cdot x)(b_d \cdot x) \leq b_{dd}^2(a_d \cdot x)^2 + a_{dd}^2(b_d \cdot x)^2 \text{ for } x \in \mathbb{R}^d. \]

The latter inequality holds by the geometric-arithmetic mean inequality, concluding the proof of the inclusion in (52).

Third, we show,

\[ \text{(54) the function } S(E(\cdot)) : A \rightarrow S(E_A) \text{ for } A \in \mathcal{P}_d \text{ is strictly convex.} \]

The first step of the proof is to show the inequality

\[ \text{(55) } \det[(1 - \lambda)A + \lambda B]^\frac{1}{d-1} \geq (1 - \lambda)\det[A]^\frac{1}{d-1} + \lambda \det[B]^\frac{1}{d-1} \text{ for } A, B \in \mathcal{P}_d, \ 0 < \lambda < 1. \]

To see this, note that (52) implies that \( v(E((1 - \lambda)A + \lambda B)) \leq v(E((1 - \lambda)[A] + \lambda[B])). \) By the formula (46) for the volume of ellipsoids, this is equivalent to the inequality

\[ \det[(1 - \lambda)A + \lambda B] \geq \det((1 - \lambda)[A] + \lambda[B]). \]

Minkowski’s determinant inequality says that

\[ \det((1 - \lambda)[A] + \lambda[B]) \geq ((1 - \lambda)\det[A]^\frac{1}{d-1} + \lambda \det[B]^\frac{1}{d-1})^{d-1}. \]

Together, these two inequalities yield (55).

(55) and the simple fact that the function \( t \rightarrow \frac{1}{t^{\frac{d-1}{2}}} \) for \( t > 0 \) is strictly convex, yield the next inequality

\[ \frac{1}{\det[(1 - \lambda)A + \lambda B]^\frac{1}{2}} = \frac{1}{\{ \det[(1 - \lambda)A + \lambda B]^\frac{1}{d-1} \}^\frac{d-1}{2}} \leq \frac{1}{\{ (1 - \lambda)\det[A]^\frac{1}{d-1} + \lambda \det[B]^\frac{1}{d-1} \}^\frac{d-1}{2}} < (1 - \lambda)\frac{1}{\det[A]^\frac{1}{2}} + \lambda \frac{1}{\det[B]^\frac{1}{2}}, \]

where equality holds, if at all, only in case \( \frac{1}{\det[A]^\frac{1}{2}} = \frac{1}{\det[B]^\frac{1}{2}}. \)

This, together with (46) shows that
\begin{align*}
v\left(E_{(1-\lambda)A+\lambda B}|\mathbb{E}^{d-1}\right) & \leq (1-\lambda)v\left(E_A|\mathbb{E}^{d-1}\right) + \lambda v\left(E_B|\mathbb{E}^{d-1}\right), \\
\text{where equality holds, if at all, only if } v\left(E_A|\mathbb{E}^{d-1}\right) &= v\left(E_B|\mathbb{E}^{d-1}\right).
\end{align*}

Clearly, this holds not only for the particular subspace $\mathbb{E}^{d-1}$ of $\mathbb{E}^d$, but for any linear subspace $u^\perp$ of $\mathbb{E}^d$ where $u \in S^{d-1}$. Thus

\begin{align*}
v\left(E_{(1-\lambda)A+\lambda B}|u^\perp\right) & \leq (1-\lambda)v\left(E_A|u^\perp\right) + \lambda v\left(E_B|u^\perp\right) \\
\text{where equality holds, if at all, only if } v\left(E_A|u^\perp\right) &= v\left(E_B|u^\perp\right) \text{ for } u \in S^{d-1}.
\end{align*}

Cauchy’s area formula (45) then implies the convexity of the function $S(E(\cdot))$:

\begin{align*}
S(E_{(1-\lambda)A+\lambda B}) & \leq (1-\lambda)S(E_A) + \lambda S(E_B) \quad \text{for } A, B \in \mathbb{P}^d, \quad 0 < \lambda < 1, \\
\text{where equality holds, if at all, only if } v\left(E_A|u^\perp\right) &= v\left(E_B|u^\perp\right) \text{ for } u \in S^{d-1}.
\end{align*}

If there is equality, then we have $v(E_A|u^\perp) = v(E_B|u^\perp)$ for all $u \in S^{d-1}$. Since $E_A, E_B \in C_o$, Alexandrov’s projection inequality (48) then yields $E_A = E_B$. In other words, $S(E(\cdot))$ is strictly convex, concluding the proof of (54) and thus of proposition (50).

The next step of the proof is to show that

(56) \quad $A$ is a smooth region in $\mathbb{P}^d$ and $-I$ is an exterior normal vector of $A$ at its boundary point $I$.

For the proof of (56), we represent ellipsoids in a different form: if $E_A$ is represented in the form $\{x: A \cdot x \otimes x \leq 1\}$ where $A \in \mathbb{P}^d$, then $E_A = BB^d$, where $B \in \mathbb{P}^d$ is such that $A = B^{-2}$. Then

\begin{align*}
\text{bd}A &= \left\{A \in \mathbb{P}^d: S(E_A) = S(B^d)\right\} = \mathbb{T}^2, \quad \text{say, where} \\
\mathbb{T} &= \left\{B^{-1} \in \mathbb{P}^d: S(BB^d) = S(B^d)\right\} = \left\{B \in \mathbb{P}^d: S(B^{-1}B^d) = S(B^d)\right\}.
\end{align*}

For $\mathbb{T}$ we have

(57) \quad $\mathbb{T} = \left\{B \in \mathbb{P}^d: S(B^{-1}B^d) = \frac{1}{v(B^{-1})} \int_{S^{d-1}} h_{BB^{-1}}(u) d\sigma(u) \right\}$

\begin{align*}
= \frac{\det B^{-1}}{v(B^{-1})} \int_{S^{d-1}} h_{BB^{-1}}(Bu) d\sigma(u) = \frac{1}{\det B} \int_{S^{d-1}} \|Bu\| d\sigma(u) = S(B^d)
\end{align*}

is a smooth surface in $\mathbb{P}^d$ and $I$ is a normal vector of $\mathbb{T}$ at $I \in \mathbb{T}$,

where for the representation of $\mathbb{T}$ we have applied Cauchy’s formula (45) and propositions (42), (43) and (44). The definition of determinant and a version of Leibniz’ rule for the differentiation of parameter integrals yield

\begin{align*}
\text{grad } \det B|_{B=I} &= B^{-T} \det B|_{B=I} = I, \\
\text{grad } (Bu)^2|_{B=I} &= 2Bu \otimes u|_{B=I} = 2u \otimes u,
\end{align*}
and thus
\[
\frac{1}{\det B} \frac{\partial}{\partial B} \left( \int_{S^{d-1}} \|B\| \, d\sigma(u) \right)_{B=I}
\]
\[
= -\frac{1}{(\det B)^{3/2}} (\frac{\partial}{\partial B}) \frac{\partial}{\partial B} \left( \int_{S^{d-1}} \|B\| \, d\sigma(u) \right)_{B=I}
\]
\[
+ \frac{1}{\det B} \int_{S^{d-1}} \left( \frac{\partial}{\partial B} (B u)^{2/3} \right) \, d\sigma(u) \bigg|_{B=I}
\]
\[
= -S(B^d) I + \int_{S^{d-1}} \frac{1}{2} \|B\| \frac{1}{2} (B u \otimes u) \, d\sigma(u) \bigg|_{B=I}
\]
\[
= -S(B^d) I + \int_{S^{d-1}} u \otimes u \, d\sigma(u) = -S(B^d) \left( 1 - \frac{1}{d} \right) I,
\]
where the integral of a $d \times d$ matrix is the $d \times d$ matrix of the integrals of its entries. Thus the gradient of the function $B \to S(B^{-1}B^d)$ at $B = I$ is a multiple of $-I$. This shows that $-I$ is a normal vector of the surface $\mathcal{T}$ at $I \in \mathcal{T}$. Now use (47) to see that the $-I$ is an exterior normal vector of $\mathcal{A}$ at $I$. The gradient of the function $B \to \mathcal{S}(B^{-1}B^d)$ exists on all of $\mathcal{P}^d$. Since $S((tB)^{-1}B^d) = t^{1-d} S(B^{-1}B^d) \neq 0$ for $t > 0$, the directional derivative in the direction of $B$ is not equal to 0. Hence the gradient is different from $O$ on $\mathcal{P}^d$. A version of the implicit function theorem then yields that $\mathcal{T}$ is a smooth surface, concluding the proof of (57). An application of (47) finally yields (56).

Clearly, $\mathcal{A}$ is unbounded. Having proved (49), (50) and (56), the proof of Lemma 3 is complete.

**Proof of Theorem 10.** An elementary compactness argument yields the existence of an ellipsoid $E_B$ which contains $C$ and has minimum surface area. We show that the regions

\[(58) \quad \mathcal{E}^c = \{ A \in \mathcal{P}^d : C \subseteq E_A \} \quad \text{and} \quad \mathcal{S} = \{ A \in \mathcal{P}^d : \mathcal{S}(E_A) \leq \mathcal{S}(E_B) \}\]

touch precisely at their common boundary point $B$.

By the proof of Theorem 1, the region $\mathcal{E}^c$ is convex and the region $\mathcal{S}$ is smooth and strictly convex by Lemma 3. If these regions overlap, there is $A \in \partial \mathcal{E}^c \cap \text{int} \mathcal{S}$. Then $C \subseteq E_A$ and $\mathcal{S}(E_A) < \mathcal{S}(E_B)$, a contradiction to our choice of $B$. Hence $\mathcal{E}^c$ and $\mathcal{S}$ touch at $B$. Since $\mathcal{S}$ is strictly convex, they touch precisely at $B$, concluding the proof of (58). Proposition (58) implies that $E_B$ is the unique ellipsoid circumscribed to $C$ of minimum surface area.

**3.2. Minimum position with respect to the surface area of circumscribed ellipsoids**

Let $S_m$ be the function which assigns to each $C \in \mathcal{C}_0$ the minimum surface area among the ellipsoids which are circumscribed to $C$. Given $C \in \mathcal{C}_0$, it is our aim to characterize the minimum $S_m$-positions of $C$. Instead of orthogonal transformation we say also rotation.
Theorem 11. Let $C \in C_o$. Then statement (i) holds and, if $C \subseteq B^d$, then statements (ii) and (iii) are equivalent:

(i) Up to rotations of $\mathbb{E}^d$, the body $C$ has a unique minimum $S_m$-position with respect to volume preserving linear transformations. The corresponding circumscribed ellipsoid of minimum surface area is unique and, necessarily, a (solid Euclidean) ball.

(ii) $C$ is in minimum $S_m$-position and $B^d$ is the unique circumscribed ellipsoid of minimum surface area.

(iii) There are contact points $\pm u_1, \ldots, \pm u_k \in C \cap \text{bd} B^d$ and reals $\lambda_1, \ldots, \lambda_k > 0$ such that hold

(a) $d \leq k \leq \frac{1}{2}d(d + 1)$,
(b) $I = \sum_{i=1}^{k} \lambda_i u_i \otimes u_i$,
(c) $\text{lin}(u_1, \ldots, u_k) = \mathbb{E}^d$.

The equivalence of propositions (ii) and (iii) in Theorem 11 and the corresponding equivalence in the John type characterization of the circumscribed ellipsoid of minimum volume, yield the following result:

Corollary 2. Let $C \in C_o$ and assume that $C \subseteq B^d$. Then the following claims (i) and (ii) are equivalent:

(i) $B^d$ is the unique circumscribed ellipsoid of $C$ of minimum volume.

(ii) $C$ is in minimum $S_m$-position with respect to volume preserving linear transformations and $B^d$ is the unique circumscribed ellipsoid of $C$ with minimum surface area.

Proof of Theorem 11. (i): Since for each convex body there is a unique circumscribed ellipsoid of minimum volume, we may choose a volume preserving linear transformation $L$ of $\mathbb{E}^d$ such that

(59) the unique ellipsoid of minimum volume circumscribed to the position $LC$ of $C$ is a ball $B$ with center $o$.

Now it will be shown that

(60) $LC$ is a minimum $S_m$-position of $C$ and $B$ is the corresponding circumscribed ellipsoid of minimum surface area.

Let $MC$ be a minimum $S_m$-position of $C$ and $E$ the corresponding ellipsoid of minimum surface area. We show that $E$ is a ball. If not, there is a volume preserving linear transformation $N$ such that $NE$ is a ball and thus $S(NE) < S(E)$. Since $NMC \subseteq NE$ this contradicts the assumption that $MC$ is a minimum $S_m$-position of $C$ with corresponding ellipsoid $E$. By the assumption in (59) we have $V(B) \leq V(E)$. Since $B$ and $E$ are balls, this shows that $S(B) \leq S(E)$. Finally, noting that $MC$ is a minimum $S_m$-position of $C$ with corresponding ellipsoid $E$, we obtain (60).

Next, we have the following:

(61) A position $MC$ of $C$ is minimum with respect to $S_m$ if and only if $M = RL$ where $R$ is a rotation of $\mathbb{E}^d$. 
Clearly, all positions of the form $MC$, where $M = RL$, are minimum. To show the converse, assume that $MC$ is a minimum position. Then a circumscribed ellipsoid of $MC$ of minimum surface area is a ball and thus unique. This ball has the same surface area as $B$ and thus coincides with $B$. By (59), $L^{-1}B$ and $M^{-1}B$ are ellipsoids of minimum volume containing $C$ and thus coincide, $L^{-1}B = M^{-1}B$, or $ML^{-1}B = B$. Hence $ML^{-1} = R$, or $M = RL$, where $R$ is a suitable rotation, concluding the proof of (61).

Having proved (60) and (61), the proof of (i) is complete.

(ii) ⇒ (iii): If (ii) holds, then the regions

$$(62) \mathcal{E}^c = \{ A \in \mathcal{P}^d : C \subseteq E_A \} \text{ and } \mathcal{A} = \{ A \in \mathcal{P}^d : S(E_A) \leq S(B^d) \}$$

touch precisely at their common boundary point $I$.

The proof of (62) is verbatim the same as that of (58) with $S, S(E_B), B$ replaced by $\mathcal{A}, S(B^d), I$.

By the proof of Theorem 1, the region $\mathcal{E}^c$ is convex and its exterior normal cone at $I$ is

$$(63) \mathcal{N}^c = \text{pos}\{ u \otimes u : u \in C \cap \text{bd} B^d \}.$$  

By Lemma 3, the region $\mathcal{A}$ is smooth and strictly convex and $-I$ is an exterior normal vector at its boundary point $I$. This together with (62) and (6) yields $I \in \mathcal{N}^c$. By (63) and a version of Carathéodory’s theorem for cones, there are contact points $\pm u_1, \ldots, \pm u_k \in C \cap \text{bd} B^d$ and reals $\lambda_1, \ldots, \lambda_k > 0$, such that

$$k \leq \frac{1}{2}d(d+1), \quad I = \sum_{i=1}^{k} \lambda_i u_i \otimes u_i.$$  

Thus (b) holds. If (c) did not hold, choose $x \in \mathbb{R}^d \setminus \{0\}$, such that $x \perp u_1, \ldots, u_k$. Then

$$0 = \sum_{i=1}^{k} \lambda_i (u_i \cdot x)^2 = \sum_{i=1}^{k} \lambda_i u_i \otimes u_i \cdot x \otimes x = I \cdot x \otimes x = \|x\|^2 > 0,$$

a contradiction. This proves (c). (c) implies $d \leq k$, concluding the proof of (a).

(iii) ⇒ (ii): If (iii) holds, then the regions

$\mathcal{E}^c$ and $\mathcal{A}$ have $I$ as a common boundary point.

It follows from (iii)(b) and (63) that $I \in \mathcal{N}^c$. Thus $I$ is an exterior normal vector of the convex region $\mathcal{E}^c$ at $I$. By Lemma 3, $-I$ is an exterior normal vector of the smooth and strictly convex region $\mathcal{A}$ at its boundary point $I$. Hence, by (6), the regions $\mathcal{E}^c$ and $\mathcal{A}$ touch at $I$ and, since $\mathcal{A}$ is strictly convex, they touch precisely at $I$. This is equivalent to (ii). □

3.3. Extension to quermassintegrals

John’s theorem or, rather, its dual, and Theorem 11 above deal with the problem to minimize the volume, respectively the surface area of ellipsoids which are circumscribed to a given convex body. In view of these results, it is a natural question to investigate the corresponding problems for the other quermassintegrals. For the latter see, e.g., [17,36].
Let $W_{im}, i = 1, \ldots, d - 1,$ be the function which assigns to each $C \in \mathcal{C}_0$ the minimum $i$th quermassintegral of an ellipsoid circumscribed to $C$. Then we have the following extension of Theorem 11. The proof is similar to the proof of Theorem 11, where one has to iterate the projection argument in the proof of Theorem 11. In particular, instead of Cauchy’s surface area formula, formulae of Kubota are used.

**Theorem 12.** Let $C \in \mathcal{C}_0$ and $i = 1, \ldots, d - 1$. Then statement (i) holds and, assuming $C \subseteq B^d$, statements (ii) and (iii) are equivalent:

(i) Up to rotations of $\mathbb{R}^d$, the body $C$ has a unique minimum $W_{im}$-position with respect to volume preserving linear transformations. The corresponding circumscribed ellipsoid of minimum $i$th quermassintegral is unique and a ball.

(ii) $C$ is in minimum $W_{im}$-position and $B^d$ is the unique circumscribed ellipsoid of minimum $i$th quermassintegral.

(iii) There are contact points $\pm u_1, \ldots, \pm u_k \in C \cap \partial B^d$ and reals $\lambda_1, \ldots, \lambda_k > 0$, such that

(a) $d \leq k \leq \frac{1}{2}d(d + 1),$

(b) $I = \sum_{i=1}^k \lambda_i u_i \otimes u_i,$

(c) $\text{lin}\{u_1, \ldots, u_k\} = \mathbb{R}^d.$

Comparing this result and the dual of a version of John’s theorem, yields the following:

**Corollary 3.** Let $C \in \mathcal{C}_0, i = 1, \ldots, d - 1,$ and assume that $C \subseteq B^d$. Then the following claims (i) and (ii) are equivalent:

(i) $B^d$ is the unique circumscribed ellipsoid of $C$ of minimum volume.

(ii) $C$ is in minimum $W_{im}$-position with respect to volume preserving linear transformations and $B^d$ is the unique circumscribed ellipsoid of minimum $i$th quermassintegral.

**Further extensions.** There are corresponding results in the non-symmetric case and for inscribed ellipsoids.

4. Minimum moment position

Let $C \in \mathcal{C}$ be a proper convex body and $f : [0, +\infty) \to [0, +\infty)$ a non-decreasing function. The (polar) $f$-moment $M(C, f)$ of (the Lebesgue measure on) $C$ with respect to $f$ is defined by

$$M(C, f) = \int_C f(\|x\|) \, dx.$$ 

If $f(t) = t^2$, this is the polar moment of inertia of $C$.

In the following we characterize for convex $f$ the minimum polar $f$-moment positions of $C$ and then describe the minimum positions of $C$ for the product $M(AC, t^2)M((AC)^*, t^2)$, where $C^*$ is the polar body of $C$.

The tools developed in this section, in particular Lemma 5, will yield simple proofs of minimum position results with respect to the mean width and the surface area in the next section.
4.1. Characterization of minimum moment position

A result of Milman and Pajor [29] says that a convex body $C \in C_o$ is in minimum polar $t^2$-moment or polar moment of inertia position with respect to volume preserving linear transformations, if and only if the following holds: (The Lebesgue measure on) $C$ is in isotropic position, that is,

$$\int_C x \otimes x \, dx = \lambda I$$

for suitable $\lambda > 0$.

Equivalently, the Legendre ellipsoid of (the Lebesgue measure on) $C$ is a ball with center $o$. Up to a dilatation it is the ellipsoid

$$\left\{ y \in \mathbb{E}^d : \int_C (x \cdot y)^2 \, dx = \int_C x \otimes x \cdot y \otimes y \leq 1 \right\}.$$

See Milman and Pajor [29] for further pertinent results and applications. Compare also the article of Dar [7].

The aim of this section is to give a transparent proof of the following result.

**Theorem 13.** Let $C \in C_o$ and assume that $f : [0, +\infty) \to [0, +\infty)$ is convex and $f(t) = 0$ precisely for $t = 0$. Then statement (i) holds and statements (ii) and (iii) are equivalent:

(i) Up to rotations of $\mathbb{E}^d$, the body $C$ has a unique minimum polar $f$-moment position with respect to volume preserving linear transformations.

(ii) $C$ is in minimum polar $f$-moment position.

(iii) $I = \lambda \int_C f'(|x|) \frac{x \otimes x \, dx}{|x|}$ for suitable $\lambda > 0$.

Tools for the proof of the theorem and the lemma below are differentiability properties of convex functions, properties of matrices and Lebesgue’s bounded convergence theorem.

If $f$ is differentiable, the proof of Theorem 13 is essentially simpler.

The particular case of this theorem where $f(t) = t^2$ has a mechanical interpretation: up to rotations $C$ has a unique minimum polar moment of inertia position. $C$ is in minimum position if and only if its Legendre ellipsoid is a ball with center $o$.

**Preliminaries.** We first state some well-known results.

**Differentiability of convex functions:** Convex functions have the following property, see, e.g. [17, Theorem 2.5].

(64) Let $F : \mathcal{C} \to \mathbb{R}$ be a convex function, where $\mathcal{C}$ is an open convex set in $\mathbb{E}^{d(d+1)}$. If all partial derivatives of $F$ exist on $\mathcal{C}$, then $F$ is of class $C^1$.

**Representation of matrices:** As a consequence of remarks in [9], we have the following proposition:
(65) Let $L$ be a non-singular $d \times d$ matrix. Then $L$ can be represented in the form $L = RA$, where $A = (L^T L)^{1/2} \in \mathcal{P}_d$ and $R = LA^{-1}$ is orthogonal.

Given $C$ and $f$, define a function $F : \mathcal{P}_d \to \mathbb{R}$ by

$$F(AC) = \frac{M(AC,f)}{\det A} = \frac{1}{\det A} \int_A f(\|x\|) \, dx = \int_C f(\|Ax\|) \, dx \quad \text{for } A \in \mathcal{P}_d.$$ 

**Lemma 4.** Let $C \in \mathcal{C}$ and assume that $f : [0, +\infty) \to [0, +\infty)$ is convex and $f(t) = 0$ precisely for $t = 0$. Then $\mathcal{F} = \mathcal{F}(C) = \{ A \in \mathcal{P}_d : F(AC) \leq F(C) \}$ is a closed convex region, smooth in a neighborhood of its boundary point $I$ and

$$\int_C f'(\|x\|) \frac{x \otimes x}{\|x\|} \, dx \neq O$$

is an exterior normal vector of $\mathcal{F}$ at $I$.

**Proof.** The definition of $F$ implies that $F$ is strictly increasing from 0 to $+\infty$ on each ray $\{ tA : t \geq 0 \}$ in $\mathcal{P}_d$ with endpoint $O$. Together with the continuity of $F$, this shows that

(66) $\mathcal{F}$ is a closed region in $\mathcal{P}_d$.

For the proof that the region

(67) $\mathcal{F}$ is convex

it is sufficient to show that the function $F$ is convex, which can be seen as follows: let $A, B \in \mathcal{P}_d$ and $0 \leq \lambda \leq 1$. Then

$$F( ((1 - \lambda)A + \lambda B) C) = \int_C f( ((1 - \lambda)A + \lambda B) x) \, dx$$

$$\leq \int_C f( (1 - \lambda)\|Ax\| + \lambda\|Bx\|) \, dx \quad \text{(since $f$ is non-decreasing)}$$

$$\leq \int_C ((1 - \lambda)f(\|Ax\|) + \lambda f(\|Bx\|)) \, dx \quad \text{(since $f$ is convex)}$$

$$= (1 - \lambda)F(AC) + \lambda F(BC).$$

Next, we show,

(68) $F$ is of class $C^1$ and $\text{grad } F(AC) = \int_C \frac{f'(\|x\|)}{\|x\|} x \otimes x \, dx (\neq O)$. 
Since $F$ is convex, for the proof that it is of class $C^1$, it is sufficient by (64), to show that for each positive definite $d \times d$ matrix $A$ the partial derivatives of $F$ with respect to the entries $a_{ik}$ of $A$ exist. It is not clear that $f(\|Ax\|)$ has continuous partial derivatives with respect to the entries of $A$. Thus, one cannot simply use Leibniz’ rule as in the proof of Lemma 3. We therefore proceed as follows. Clearly,

\begin{equation}
\text{(69) if } \lim_{h \to 0} \int_C \frac{f(\|A(h)x\|) - f(\|Ax\|)}{h} \, dx \text{ exists, it equals } \frac{\partial F(AC)}{\partial a_{ik}},
\end{equation}

where the matrix $A(h)$ has the same entries as $A$, with the exception that instead of $a_{ik}$ we have $a_{ik} + h$. A version of the Cauchy–Schwarz inequality implies,

\begin{equation}
\|A(h)x - Ax\| \leq \|A(h) - A\| \|x\| \leq |h| \|x\| \quad \text{for } x \in \mathbb{R}^d.
\end{equation}

The function $f: [0, +\infty) \to [0, +\infty)$ is convex and $f(0) = 0$. Thus, $f$ is Lipschitz on each bounded set in $\mathbb{R}$. Proposition (70) then shows,

\begin{equation}
\text{(71) } \frac{f(\|A(h)x\|) - f(\|Ax\|)}{h} \text{ is bounded on } C \text{ for } |h| \leq 1.
\end{equation}

Being a convex function, $f$ is differentiable for almost every $t \in [0, +\infty)$, see, e.g., [17]. Since $A$ is non-singular, $f$ is differentiable at $t = \|Ax\|$ for almost every $x \in C$. An elementary calculation then implies,

\begin{equation}
\text{(72) } \lim_{h \to 0} \frac{f(\|A(h)x\|) - f(\|Ax\|)}{h} = f'(\|Ax\|)\left((a_{k1}x + \cdots)x_k + (a_{k1} + \cdots)x_i\right)
\end{equation}

for almost every $x \in C \setminus \{o\}$.

Propositions (69), (71) and (72), together with the theorem of Lebesgue on bounded convergence, yield the following expression for the partial derivative of $F$ with respect to $a_{ik} (= a_{ki})$: \[ \frac{\partial F(AC)}{\partial a_{ik}} = \int_C \frac{f'(\|Ax\|)}{\|Ax\|} \left((a_{k1}x + \cdots)x_k + (a_{k1} + \cdots)x_i\right) \, dx \quad \text{for } A \in \mathcal{P}^d. \]

Since, thus, all partial derivatives of $F$ exist and $F$ is convex by the proof of (67), it follows from (64) that $F$ is of class $C^1$. Now put $A = I$ to get the desired expression for $\text{grad } F(IC)$ which, in particular, shows that $\text{grad } F(IC) \neq O$. The proof of (68) is complete.

Propositions (66)–(68), together with a version of the implicit function theorem, finally, yield Lemma 4. $\square$

**Proof of Theorem 13.** (i): By elementary arguments, $C$ has a minimum position. Without loss of generality, assume that $C$ is already in minimum position. For the proof that this position is unique up to rotations, it is, by (65), sufficient to prove its uniqueness within $\mathcal{P}^d$. To see the latter, we first show the regions

\begin{equation}
\text{(73) } \mathcal{D} \text{ and } \mathcal{F} \text{ touch precisely at their common boundary point } I.
\end{equation}
The definitions of \(D\) and \(F\) in (1), resp. Lemma 4, show that \(I\) is a common boundary point. By (1), the region \(D\) is smooth and strictly convex and by Lemma 4, the region \(F\) is smooth in a neighborhood of \(I\) and convex. Thus, if (73) did not hold, the regions \(D\) and \(F\) overlap. Then there is a \(d \times d\) matrix \(A \in \text{bd} D \cap \text{int} F\). The definitions of \(F\) and \(D\) and the assumption that \(C\) is in minimum position imply
\[
M(C, f) > F(AC) = \frac{M(AC, f)}{\det A} = M(AC, f) \geq M(C, f).
\]
This is a contradiction, concluding the proof of (73). Proposition (73) clearly implies that \(C\) is the unique minimum position of \(C\), concluding the proof of (i).

(ii) \(\Rightarrow\) (iii): Let \(C\) be in minimum position. Then (73) holds as before. Since \(D\) and \(F\) are smooth close to \(I\) and convex, the exterior normal vector \(\text{grad} F(IC)\) of \(F\) at \(I\) is a multiple of the interior normal vector of \(D\) at \(I\). Now, to obtain (iii), use the expressions for these vectors, in Lemma 4 and proposition (1).

(iii) \(\Rightarrow\) (ii): By (1) and Lemma 4, \(I\) is a common boundary point of \(D\) and \(F\). By (iii), (1) and Lemma 4 the exterior normal vectors at \(I\) of the smooth and strictly convex set \(D\) and the convex set \(F\) which is smooth close to \(I\), point in opposite directions. Thus \(D\) and \(F\) have precisely \(I\) in common. As shown in the proof of (i), this implies (ii). \(\square\)

4.2. Characterization of minimum \(MM^*\)-position

In analogy to the investigations of Giannopoulos and Milman [10] and Bastero and Romance [4] on minimum \(WW^*\)- and \(AA^*\)-positions, where \(W\) and \(A\) are the mean width and the surface area, respectively, we minimize the product \(M(AC, t^2)M((AC)^*, t^2)\), where \(A\) is a non-singular \(d \times d\) matrix.

**Theorem 14.** Let \(C \in \mathcal{C}_o\). Then statement (i) holds and statements (ii) and (iii) are equivalent:

(i) Up to similarities of \(\mathbb{E}^d\) with center \(o\), the convex body \(C\) has a unique minimum \(MM^*\)-position with respect to non-singular linear transformations.
(ii) \(C\) is in minimum \(MM^*\)-position.
(iii) \(\int_C x \otimes x \, dx = \lambda \int_{C^*} x \otimes x \, dx\) for suitable \(\lambda > 0\).

Statement (iii) means that in minimum position the Legendre ellipsoids of \(C\) and \(C^*\) coincide up to a dilatation with center \(o\).

The main step of the proof of this result is the proof of the Lemma 5, which says that the image of a certain cone in \(\mathbb{P}^d\) under the mapping \(A \rightarrow A^{-1}\) is convex. Tools for its proof are a convexity criterion of Tietze and the convergence of the geometric series of a matrix with small norm.

**Preliminaries.** A theorem of Tietze [37] is as follows:

\((74)\) Let \(\mathcal{E}\) be a closed, connected region in \(\mathbb{E}^{d(d+1)}\), such that for each \(B \in \text{bd} \mathcal{E}\) there are a neighborhood \(N\) of \(B\) and a closed halfspace \(\mathcal{H}\) where \(B \in \text{bd} \mathcal{H}\) and such that
\[
\mathcal{E} \cap N \subseteq \mathcal{H} \quad \text{and} \quad \mathcal{E} \cap N \cap \text{bd} \mathcal{H} = \{B\}.
\]
That is, \( E \) is \textit{locally strictly supported} at each of its boundary points. Then \( E \) is strictly convex.

Using this result, we prove the following statement which will be used in this and also in later sections.

**Lemma 5.** Let \( \mathcal{K} = \{ A \in \mathcal{P}_d : A \cdot N \leq I \cdot N \} \) where \( N \in \mathcal{P}_d \). Then \( \mathcal{K}^{-1} \) is an unbounded, strictly convex, smooth region in \( \mathcal{P}_d \). \( N_1 = N \) is an interior normal vector of \( \mathcal{K}^{-1} \) at its boundary point \( I \). In particular,

\[
\mathcal{K}^{-1} \subseteq \left\{ A \in \mathcal{P}_d : A \cdot N > I \cdot N \right\} \cup \{I\}.
\]

If \( N \in \text{bd} \mathcal{Q}_d \), then \( \mathcal{K}^{-1} \) is an unbounded, convex, smooth region in \( \mathcal{P}_d \).

In each case, \( \mathcal{K}^{-1} \) is the part of \( \mathcal{P}_d \) on the far side of the unbounded, convex, smooth surface \( \mathcal{B}^{-1} \) where \( \mathcal{B} = \{ A \in \mathcal{P}_d : A \cdot N = I \cdot N \} \) is the base of the cone \( \mathcal{K} \).

**Proof of Lemma 5.** Assume first that \( N \in \mathcal{P}_d \). Each ray in \( \mathcal{P}_d \) with endpoint \( O \) meets \( \mathcal{K}^{-1} \) in a halfline with endpoint at \( \mathcal{B}^{-1} \). Its boundary in \( \mathcal{P}_d \) is the unbounded smooth surface \( \mathcal{B}^{-1} \). Thus,

(75) \( \mathcal{K}^{-1} \) is a closed, unbounded, connected region in the open cone \( \mathcal{P}_d \) with smooth boundary \( \mathcal{B}^{-1} \). \( \mathcal{K}^{-1} \) is disjoint from a suitable neighborhood of \( O \).

It remains to show that

(76) \( \mathcal{K}^{-1} \) is strictly convex and \( N \) is an interior normal vector of \( \mathcal{K}^{-1} \) at its boundary point \( I \).

By (75) and the result (74) of Tietze it is sufficient for the proof of (76) to show the following:

(77) Let \( A^{-1} \in \text{bd} \mathcal{K}^{-1} = \mathcal{B}^{-1} \). Then \( \mathcal{K}^{-1} \) is locally strictly supported at \( A^{-1} \). The vector \( N_I = N \) is an interior normal vector of the local support hyperplane of \( \mathcal{K}^{-1} \) at \( I \).

Since each \( X \in \mathbb{E}^{\frac{1}{2}(d+1)} \) can be represented in the form \( X = A^{\frac{1}{2}} (I - Y) A^{-\frac{1}{2}} \) with unique \( Y \in \mathbb{E}^{\frac{1}{2}(d+1)} \), the set

\[
\mathcal{N} = \left\{ X = A^{\frac{1}{2}} (I - Y) A^{-\frac{1}{2}} \in \mathcal{P}_d : \|Y\| < \frac{1}{2} \right\}
\]

is a neighborhood of \( A \) in \( \mathcal{P}_d \). By (47), the mapping \( X \rightarrow X^{-1} \) of \( \mathcal{P}_d \) onto itself, is a diffeomorphism of \( \mathcal{P}_d \). Thus

\[
\mathcal{N}^{-1} = \left\{ X^{-1} = A^{-\frac{1}{2}} (I - Y)^{-1} A^{\frac{1}{2}} \in \mathcal{P}_d : \|Y\| < \frac{1}{2} \right\}
\]

is a neighborhood of \( A^{-1} \) in \( \mathcal{P}_d \). Clearly,
\[\begin{align*}
N^{-1} \cap K^{-1} &= (N \cap K)^{-1} \\
&= \left\{ X^{-1} = A^{-\frac{1}{2}}(I - Y)^{-1}A^{-\frac{1}{2}} \in \mathbb{P}^d : \|Y\| < \frac{1}{2}, \ A^\frac{1}{2}YA^{-\frac{1}{2}} \cdot N \geq 0 \right\}.
\end{align*}\]

By (3),
\[\begin{align*}
N &= \sum_{i=1}^{d} \lambda_i u_i \otimes u_i, \quad \text{with l.i. } u_1, \ldots, u_d \in \mathbb{R}^d \text{ and } \lambda_1, \ldots, \lambda_d > 0.
\end{align*}\]

Let
\[\begin{align*}
N_A &= \sum_{i=1}^{d} \lambda_i Au_i \otimes Au_i.
\end{align*}\]

If we can show that
\[\begin{align*}
(78) \ X^{-1} \cdot N_A > A^{-1} \cdot N_A \text{ for } X^{-1} \in N^{-1} \cap K^{-1} \setminus \{A^{-1}\},
\end{align*}\]
the proof of (77) is complete. Since
\[\begin{align*}
X^{-1} &= A^{-\frac{1}{2}}(I - Y)^{-1}A^{-\frac{1}{2}} = A^{-\frac{1}{2}}(I + Y + Y^2 + Y^3 + \cdots)A^{-\frac{1}{2}} \\
&= A^{-1} + A^{-\frac{1}{2}}YA^{-\frac{1}{2}} + A^{-\frac{1}{2}}YYA^{-\frac{1}{2}} + \sum_{k=1}^{\infty} A^{-\frac{1}{2}}YY^kYA^{-\frac{1}{2}}
\end{align*}\]

and
\[\begin{align*}
A^{-\frac{1}{2}}YA^{-\frac{1}{2}} \cdot N_A &= \sum_{i=1}^{d} \lambda_i \frac{1}{2} A^{-\frac{1}{2}}YA^{-\frac{1}{2}} \cdot Au_i \otimes Au_i \\
&= \sum_i \lambda_i u_i^T A^T A^{-\frac{1}{2}}YA^{-\frac{1}{2}}Au_i = \sum_i \lambda_i u_i^T A^\frac{1}{2} Y A^\frac{1}{2} u_i \\
&= A^\frac{1}{2} YA^\frac{1}{2} \cdot \sum \lambda_i u_i \otimes u_i = A^\frac{1}{2} YA^\frac{1}{2} \cdot N = (A - X) \cdot N \geq 0
\end{align*}\]
(since \(X \cdot N \leq A \cdot N\)),
\[\begin{align*}
A^{-\frac{1}{2}}YY^kYA^{-\frac{1}{2}} \cdot N_A &= \cdots = \sum_i \lambda_i u_i^T A^\frac{1}{2} Y Y A^\frac{1}{2} u_i = \sum_i \lambda_i \left( A^\frac{1}{2} u_i \right)^2 > 0
\end{align*}\]
(since \(Y \neq O\) (note that \(X^{-1} \neq A^{-1}\) and \(A^\frac{1}{2} u_1, \ldots, A^\frac{1}{2} u_d\) are l.i.),
\[\begin{align*}
A^{-\frac{1}{2}}YY^kYA^{-\frac{1}{2}} \cdot N_A &= \cdots = \sum_i \lambda_i u_i^T A^\frac{1}{2} Y Y^k Y A^\frac{1}{2} u \\
&= \sum_i \lambda_i Y^k \cdot (YA^\frac{1}{2} u_i \otimes YA^\frac{1}{2} u_i) \geq - \sum_i \lambda_i \|Y\|^k (YA^\frac{1}{2} u_i)^2.
\end{align*}\]
we have
\[ X^{-1} \cdot N_A \geq A^{-1} \cdot N_A + \sum_{i} \lambda_i (YA^{rac{1}{2}}u_i)^2 (1 - \|Y\| - \|Y\|^2 - \cdots) \]
\[ = A^{-1} \cdot N_A + \sum_{i} \lambda_i (YA^{rac{1}{2}}u_i)^2 \frac{1 - 2\|Y\|}{1 - \|Y\|} > A^{-1} \cdot N_A \]
(since \(Y \neq 0\) and \(A^{rac{1}{2}}u_1, \ldots, A^{rac{1}{2}}u_d\) are l.i.)

This concludes the proof of (78) and thus of (77) which, in turn, implies (76). Propositions (75) and (76) together yield the lemma in case when \(N \in P^d\).

Assume now that \(N \in Q^d \setminus P^d\). Then we may choose vectors \(N_1, N_2, \ldots \in P^d\) and numbers \(\alpha_1, \alpha_2, \ldots > 0\) such that
\[ \{A: A \cdot N_1 \leq \alpha_1\} \subseteq \{A: A \cdot N_2 \leq \alpha_2\} \subseteq \ldots \to K \]
and the result follows from the first part of the proof. \(\square\)

A simple proof yields the following well-known formula:

\[ (79) \quad (AC)^* = A^{-1} C^* \quad \text{for} \quad A \in P^d \quad \text{and} \quad C \in C \quad \text{where} \quad o \in \text{int} \ C. \]

**Proof of Theorem 14.** By (65) it is sufficient to consider minimization with respect to \(A \in P^d\) and thus uniqueness in \(P^d\). Remember the definition of \(F\):

\[ F(AC) = \frac{M(AC, t^2)}{\det A} = \int_C \|Ax\|^2 \, dx \quad \text{for} \quad A \in P^d. \]

Since
\[ F(AC) F((AC)^*) = F(AC) F(A^{-1} C^*) = M(AC, t^2) M((AC)^*, t^2) \quad \text{for} \quad A \in P^d, \]
the following remark holds:

\[ (80) \quad C \quad \text{is in minimum} \quad MM^*\text{-position if and only if it is in minimum} \quad FF^*\text{-position with respect to} \quad A \in P^d. \]

Note also that
\[ (81) \quad F(AC) F(A^{-1} C^*) = \int_{AC} \|x\|^2 \, dx \int_{A^{-1} C^*} \|x\|^2 \, dx \quad \text{for} \quad A \in P^d. \]

Define
\( F = F(C) = \{ A \in \mathbb{P}^d : F(AC) \leq F(C) \}, \)
\( F^* = F^*(C) = \{ A \in \mathbb{P}^d : F((AC)^*) \leq F(C^*) \} \)
\[= \{ A \in \mathbb{P}^d : F(A^{-1}C^*) \leq F(C^*) \} = \{ A \in \mathbb{P}^d : F(AC^*) \leq F(C^*) \}^{-1}. \]

By Lemma 4, the sets

\( F \) and \( F^*-1 \) are smooth convex regions in \( \mathbb{P}^d \) which contain a neighborhood of \( O \) in \( \mathbb{P}^d \) and
\[
\int_C x \otimes x \, dx, \int_{C^*} x \otimes x \, dx \in \mathbb{P}^d
\]
are exterior normal vectors of the regions \( F \) and \( F^*-1 \) at their common boundary point \( I \).

(i): Existence: Let \( A \in \text{bd} \mathcal{D} \). The eigenvalues of \( A \) are positive and their product is 1. The eigenvalues of \( A^{-1} \) are the inverses of those of \( A \). Thus, if \( A \) has at least one large eigenvalue, then also \( A^{-1} \) has at least one large eigenvalue. The equality (81) shows that in this case \( F(AC)F(A^{-1}C^*) \) is large. Thus, if \( F(AC)F(A^{-1}C^*) \) is close to its infimum, the eigenvalues of \( A \) cannot be very large and a simple compactness argument yields the existence of a minimum \( FF^* \)-position of \( C \). Now apply (80).

Uniqueness: We may assume that \( C \) is in minimum \( FF^* \)-position. Then

\( F \) and \( F^*-1 \) have the same exterior normal at their common boundary point \( I \), say \( N \), where \( N \in \mathbb{P}^d \).

If the first assertion on (84) does not hold, there is \( A \in \text{bd} \mathcal{F} \cap \text{int} \mathcal{F}^*-1 \), not a multiple of \( I \), and we obtain
\[
F(AC)F(A^{-1}C^*) < F(C)F(C^*).\]

a contradiction to the assumption that \( C \) is in minimum \( FF^* \)-position. That \( N \in \mathbb{P}^d \) now follows from (83). As a consequence of (84) we have

\( F, F^*-1 \subseteq \{ A \in \mathbb{P}^d : A \cdot N \leq I \cdot N \} \) where \( N \in \mathbb{P}^d \).

Lemma 5 then shows that

\( F^* \subseteq \{ A \in \mathbb{P}^d : A \cdot N > I \cdot N \} \cup \{ I \}. \)

Since \( F \) is convex by (83), the inclusions (85) and (86) show that

\( F \cap F^* = \{ I \}. \)

We now show that the following holds:

\( \text{Let} \ A \in \mathbb{P}^d \). Then \( AC \) is in minimum \( FF^* \)-position if and only if \( A = t I \) where \( t > 0 \).
Since $C$ is in minimum $FF^*$-position, also $tC$ is in minimum $FF^*$-position for all $t > 0$. If $A$ is not a multiple of $I$, we may choose $t > 0$ by (87), such that $tA \notin \mathcal{F}, \mathcal{F}^*$. Then

$$F(AC)F(A^{-1}C^*) = F(tAC)F(t^{-1}A^{-1}C^*) > F(C)F(C^*)$$

by (82). Hence $AC$ is not a minimum $FF^*$-position of $C_i$, concluding the proof of (88).

As a consequence of (88) we show the following:

(89) Let $M$ be a non-singular $d \times d$ matrix. Then $MC$ is in minimum $FF^*$-position if and only if $M$ is a similarity.

Since $C$ is in minimum $FF^*$-position, also $RC$ is in minimum $FF^*$-position for any rotation $R$.

To see (89), represent $M$ in the form $M = AR$ where $A \in \mathcal{P}_d$ and $R$ is a rotation; this is possible by (65). With $C$ also $RC$ is in minimum $FF^*$-position. Hence (88) implies that $MC = ARC$ is in minimum $FF^*$-position if and only if $A = tI$, i.e. $M = AR = tR$ for a suitable $t > 0$, concluding the proof of (89) and thus of the uniqueness part of statement (i).

(ii)$\Rightarrow$(iii): If $C$ is in (the unique) minimum $FF^*$-position (up to similarities), then (84) holds as shown above. An application of (83) then yields (iii).

(iii)$\Rightarrow$(ii). If (iii) holds then $\mathcal{F}$ and $\mathcal{F}^* - \mathcal{I}$ have the same exterior normal at their common boundary point $I$. By Lemma 4 this normal is contained in $\mathcal{P}_d$. Using this, it was shown in the uniqueness part of the proof of (i) that then $C$ is in the unique minimum $FF^*$-position (up to similarities), i.e., (ii) holds. $\Box$

4.3. The non-symmetric case

The first result for minimum moments proved above, can easily be extended to the non-symmetric case, where for a convex body $C \in \mathcal{C}$, we consider the following polar $f$-moment (with center $a \in \mathbb{E}^d$):

$$M(AC - a, f) = \int_{AC - a} f(\|x\|) \, dx \quad \text{for} \quad (A, a) \in \mathcal{P}_d \times \mathbb{E}^d.$$ 

**Theorem 15.** Let $C \in \mathcal{C}$ and assume that $f : [0, +\infty) \to [0, +\infty)$ is convex and $f(t) = 0$ precisely for $t = 0$. Then statement (i) holds and statements (ii) and (iii) are equivalent:

(i) Up to rigid motions of $\mathbb{E}^d$, the body $C$ has a unique minimum polar $f$-moment position with respect to volume-preserving affine transformations.

(ii) $C$ is in minimum polar $f$-moment position.

(iii) $I = \lambda \int_C \frac{f'(\|x\|)}{\|x\|} x \otimes x \, dx$ for suitable $\lambda > 0$ and $\int_C \frac{f'(\|x\|)}{\|x\|} x \, dx = o$.

Let $f(t) = t^2$ for $t \geq 0$. In the following result we minimize the expression $M(AC - a, t^2) \times M((AC - a)^*, t^2)$ where $A$ is a non-singular $d \times d$ matrix and $a \in \text{int} \, AC$.

**Theorem 16.** Let $C \in \mathcal{C}$ and assume that $f(t) = t^2$. Then statement (i) holds and statements (ii) and (iii) are equivalent:
(i) Up to similarities of $\mathbb{R}^d$ the convex body $C$ has a unique minimum $MM^*$-position with respect to non-singular affinities.

(ii) $C$ is in minimum $MM^*$-position.

(iii) $\int_C x \otimes x \, dx = \lambda \int_{C^*} x \otimes x \, dx$ and $\int_C x \, dx = \int_{C^*} x \, dx$ for suitable $\lambda > 0$.

Statement (iii) means that $C$ and $C^*$ both have their centroids at the origin $o$ and their Legendre ellipsoids coincide.

In the proof we use the function $F : \mathcal{F}^d \times \mathbb{E}^d \to \mathbb{R}$ defined by

$$F(AC - a) = \frac{M(AC - a)}{\det A}.$$

The sets $\mathcal{F} = \mathcal{F}(C) = \{(A, a) : F(AC - a) \leq F(C)\}$ and $\mathcal{F}^* = \mathcal{F}^*(C) = \{(A, a) : F((AC - a)^*) \leq F(C^*)\}^{-1}$ are convex regions, each smooth in a neighborhood of their common boundary point $(I, o)$. If $C$ is in minimum $MM^*$-position, they have a common normal at this point.

5. Minimum mean width and surface area position

The mean width $W(C)$ of a convex body $C \in \mathcal{C}$ is defined by

$$W(C) = \frac{2}{S(B^d)} \int_{S^{d-1}} h_C(u) \, d\sigma(u).$$

This section contains characterizations of the minimum mean width, resp. surface area position of a convex body $C$ and a characterization of the minimum positions of the product of the mean widths of $C$ and $C^*$ and, similarly, for the surface area.

Using the tools for minimum moment problems developed in the last section, the proofs are quite easy.

5.1. Characterization of minimum mean width and surface area position

The surface area measure $\sigma_C$ of a convex body $C \in \mathcal{C}$ is a Borel measure on $S^{d-1}$ which is defined as follows: given a Borel set $B \subseteq S^{d-1}$, consider the set of all boundary points of $C$ at which there is a support hyperplane of $C$ with exterior normal vector in $B$. Then $\sigma_C(B)$ is the $(d - 1)$-dimensional Hausdorff measure of this set.

A first characterization of the minimum surface area position of a convex body $C$ is due to Petty [32], see also Giannopoulos and Papadimitrakis [13]: $C \in \mathcal{C}$ is in minimum surface area position with respect to volume preserving affinities, if and only if the surface area measure $\sigma_C$ of $C$ on $S^{d-1}$ is isotropic, that is,

$$\int_{S^{d-1}} u \otimes u \, d\sigma_C(u) = \lambda I \quad \text{for suitable } \lambda > 0.$$

Equivalently, the Legendre ellipsoid corresponding to the surface area measure $\sigma_C$ on $S^{d-1}$ is a ball with center $o$. An application of the minimum surface area position to hyperplane projections of convex bodies was given by Giannopoulos and Papadimitrakis [13].
Giannopoulos and Milman [10] and Giannopoulos, Milman and Rudelson [12] show that a convex body \( C \) is in minimum mean width position, if and only if the measure \( h_C \sigma \) on \( S^{d-1} \) is isotropic. For the minimum positions with respect to the other intrinsic volumes, these authors give similar necessary conditions. Bastero and Romance [4] proved analogous results for dual quermassintegrals of starshaped bodies, as introduced by Lutwak [27].

In this section we characterize minimum mean width and minimum surface area positions.

**Minimum mean width and surface area position.** We show the following result:

**Theorem 17.** Let \( C \in \mathbb{C} \). Then statement (i) holds and statements (ii) and (iii) are equivalent:

(i) Up to rigid motions of \( \mathbb{E}^d \), the body \( C \) has a unique minimum mean width position with respect to volume preserving affinities.

(ii) \( C \) is in minimum mean width position.

(iii) \( I = \lambda \int_{S^{d-1}} \left( \nabla h_C(u) \otimes u + u \otimes \nabla h_C(u) \right) d\sigma(u) \) for suitable \( \lambda > 0 \).

As a consequence of Theorem 17 we have the following result:

**Corollary 4.** Let \( C \in \mathbb{C} \). Then statement (i) holds and the statements (ii) and (iii) are equivalent:

(i) Up to rigid motions of \( \mathbb{E}^d \), the body \( C \) has a unique minimum surface area position with respect to volume preserving affinities.

(ii) \( C \) is in minimum surface area position.

(iii) \( I = \lambda \int_{S^{d-1}} \left( \nabla h_{\Pi C}(u) \otimes u + u \otimes \nabla h_{\Pi C}(u) \right) d\sigma(u) \) for suitable \( \lambda > 0 \).

**Preliminaries.** The main step of the proof of Theorem 17, and thus of Corollary 4, is to show the following result:

**Lemma 6.** Let \( C \in \mathbb{C} \). The set \( W = W(C) = \{ A \in \mathbb{Q}^d : W(AC) \leq W(C) \} \) is a bounded convex region in \( \mathbb{Q}^d \) which contains a neighborhood of \( O \) in \( \mathbb{Q}^d \) and the surface \( \mathbb{P}^d \cap \text{bd} W \) is smooth. Each ray in \( \mathbb{P}^d \) with endpoint \( O \) meets this surface in precisely one point. Further,

\[
\nabla W(AC) = \int_{S^{d-1}} \left( \nabla h_C(Au) \otimes u + u \otimes \nabla h_C(Au) \right) d\sigma(u) \neq 0 \quad \text{for } A \in \mathbb{P}^d.
\]

We give only outlines of the proofs of Theorem 17 and Lemma 6.

**Outline of the proof of Lemma 6.** The proof that \( W \) is a bounded region which contains a neighborhood of \( O \) in \( \mathbb{P}^d \) is almost identical to the proof that \( F \) is a region in Lemma 4. The proof of the convexity of \( W(\cdot) \) makes use of the definition of \( W(\cdot) \), proposition (44) and the convexity of \( h_C \). The proof that \( W(\cdot) \) is of class \( C^1 \) in \( \mathbb{P}^d \) and of the expression for \( \nabla W(AC) \) for \( A \in \mathbb{P}^d \) follows the proof of the corresponding properties of \( F \) in Lemma 4. The argument is simplified by the fact that \( h_C(A\cdot) \) is Lipschitz and thus almost everywhere differentiable by Rademacher’s theorem. □

In the proofs of Theorem 17 and Corollary 4 it is, by (65), sufficient to prove uniqueness of the minimum position with respect to \( A \in \text{bd} \mathbb{P} \).
Outline of the proof of Theorem 17. It is easy to show that $C$ has a minimum position. If $C$ is in minimum position, then the same proof as above shows that the regions $D$ and $W$ touch precisely at their common boundary point $I$.

The remaining parts of the proof are very similar to the corresponding parts of the proof of Theorem 13.

Proof of Corollary 4. Since $h_{\Pi C}(u) = v(C|u^{-1})$ for $u \in S^{d-1}$, Cauchy’s formula (45) shows that

$$S(C) = \frac{1}{v(B^{d-1})} \int_{S^{d-1}} h_{\Pi C}(u) d\sigma(u) = \beta W(\Pi C) \text{ for } C \in \mathcal{C},$$

where $\beta = \frac{S(B^d)}{2v(B^{d-1})}$.

The definition of minimum position together with the identities (43), (44) and Cauchy’s formula then yield the following equivalences, where minimum position is with respect to $A \in \partial D$.

Up to rigid motions, $C$ is in unique minimum surface area position

$$\Leftrightarrow \int_{S^{d-1}} h_{\Pi C}(u) d\sigma(u) \leq \int_{S^{d-1}} h_{\Pi AC}(u) d\sigma(u)$$

for $A \in \partial D$, where equality holds precisely in case $A = I$,

$$\Leftrightarrow \int_{S^{d-1}} h_{\Pi C}(u) d\sigma(u) \leq \det A \int_{S^{d-1}} h_{A^{-1} \Pi C}(u) d\sigma(u) = \int_{S^{d-1}} h_{\Pi C}(A^{-1} u) d\sigma(u)$$

for $A \in \partial D$, where equality holds precisely in case $A = I$,

$$\Leftrightarrow \int_{S^{d-1}} h_{\Pi C}(u) d\sigma(u) \leq \int_{S^{d-1}} h_{\Pi C}(Bu) d\sigma(u)$$

for $B \in \partial D$, where equality holds precisely in case $B = I$,

$$\Leftrightarrow \text{ up to rigid motions, } \Pi C \text{ is in unique minimum mean width position.}$$

Taking this into account, the corollary is an immediate consequence of Theorem 17, applied to $\Pi C$ instead of $C$.

5.2. Characterization of minimum $WW^*$- and $AA^*$-positions

Giannopoulos and Milman [10] proved, if a convex body $C \in \mathcal{C}$ is in minimum $WW^*$-position, then

$$\int_{S^{d-1}} (x \cdot u)^2 h_C(u) d\sigma(u) = \lambda \int_{S^{d-1}} (x \cdot u)^2 h_{C^*}(u) d\sigma(u) \text{ for } x \in S^{d-1},$$
or, equivalently,

\[ \int_{S^{d-1}} u \otimes u h_C(u) \, d\sigma(u) = \lambda \int_{S^{d-1}} u \otimes u h_{C^*}(u) \, d\sigma(u), \]

where \( \lambda > 0 \) is a suitable constant. That is, the measures \( h_C \sigma \) and \( h_{C^*} \sigma \) on \( S^{d-1} \) have homothetic Legendre ellipsoids. If \( h_C \) and \( h_{C^*} \) both are of class \( C^2 \), the necessary condition of Giannopoulos and Milman is also sufficient, as shown by Bastero and Romance [4]. Considering this result, the questions arise, first, to eliminate the differentiability assumption and, second, to characterize the minimum \( W_i W_i^* \)-positions of \( C \), where \( i = 0, \ldots, d - 1 \).

We prove the following version of the result Giannopoulos and Milman, resp. Bastero and Romance and, using the connection of the mean width and, as a corollary, characterize the minimum \( AA^* \)-position of \( C \).

**Minimum \( WW^* \)- and \( AA^* \)-position.** The aim of this section is to show the following result:

**Theorem 18.** Let \( C \in \mathcal{C} \) with \( o \in \text{int} \, C \). Then statement (i) holds and statements (ii) and (iii) are equivalent:

(i) Up to similarities of \( \mathbb{E}^d \), the convex body \( C \) has a unique minimum \( WW^* \)-position with respect to non-singular linear transformations.

(ii) \( C \) is in minimum \( WW^* \)-position.

(iii) \[ \int_{S^{d-1}} \left( \nabla h_C(u) \otimes u + u \otimes \nabla h_C(u) \right) \, d\sigma(u) = \lambda \int_{S^{d-1}} \left( \nabla h_{C^*}(u) \otimes u + u \otimes \nabla h_{C^*}(u) \right) \, d\sigma(u) \quad \text{for suitable } \lambda > 0. \]

Theorem 18 easily implies the following result for surface area:

**Corollary 5.** Let \( C \in \mathcal{C} \) with \( o \in \text{int} \, C \). Then statement (i) holds and statements (ii) and (iii) are equivalent:

(i) Up to similarities of \( \mathbb{E}^d \), the body \( C \) has a unique minimum \( AA^* \)-position with respect to non-singular linear transformations.

(ii) \( C \) is in minimum \( AA^* \)-position.

(iii) \[ \int_{S^{d-1}} \left( \nabla h_{\Pi_C}(u) \otimes u + u \otimes \nabla h_{\Pi_C}(u) \right) \, d\sigma(u) = \lambda \int_{S^{d-1}} \left( \nabla h_{\Pi_{C^*}}(u) \otimes u + u \otimes \nabla h_{\Pi_{C^*}}(u) \right) \, d\sigma(u) \quad \text{for suitable } \lambda > 0. \]

A result of Hadwiger [22, p. 260] yields the following plausible result. In view of the later application, we formulate it for \( C^* \).
The mean width of the orthogonal projection of $C^*$ onto a proper plane in $\mathbb{E}^d$ is less than the mean width of $C^*$.

In the proofs of Theorem 18 and Corollary 5 it is, by (65), sufficient to consider minimization and to prove uniqueness for $A \in \mathcal{P}^d$.

**Outline of the proof of Theorem 18.** We first make some preparations. Let

\[ W = W(C) = \{ A \in \mathcal{P}^d : W(AC) \leq W(C) \}, \]

\[ W^* = W^*(C) = \{ A \in \mathcal{P}^d : W((AC)^*) \leq W(C^*) \} = \{ A \in \mathcal{P}^d : W(A^{-1}C^*) \leq W(C^*) \} \]

= \{ A \in \mathcal{P}^d : W(AC^*) \leq W(C^*) \}^{-1}. \]

By Lemma 6,

\[ W, W^*-1 \text{ are bounded convex regions in } \mathcal{P}^d \text{ which contain a neighborhood of } O \text{ in } \mathcal{P}^d, \text{ the boundaries of which in } \mathcal{P}^d \text{ are smooth and} \]

\[ \int_{S^{d-1}} (\text{grad } h_C(u) \otimes u + u \otimes \text{grad } h_C(u)) \, d\sigma(u), \]

\[ \int_{S^{d-1}} (\text{grad } h_{C^*}(u) \otimes u + u \otimes \text{grad } h_{C^*}(u)) \, d\sigma(u) = N, \text{ say,} \]

are exterior normal vectors of $W$, resp. $W^*-1$ at their common boundary point $I$.

Next, the following will be shown:

\[ N \in \mathcal{P}^d. \]

The result (91) of Hadwiger together with proposition (4) implies that

for each proper face $F$ of $Q^d$ the orthogonal projection $I_F$ of $I$ into $F$ is contained in the relative interior of $bd Q^d \cap bd W^*-1$.

Hence $I_F \cdot N < I \cdot N$, or

\[ (I - I_E) \cdot N > 0 \text{ for each proper face } F \text{ of } Q^d. \]

If (93) does not hold, let $J = [I, N] \cap bd Q^d$. Then $J \in \text{relint } F$, where $F$ is a suitable proper face of $Q^d$. By (4) the hyperplane $\{ X : (I - I_F) \cdot X = (I - I_F) \cdot J = 0 \}$ through $J$ then is a support hyperplane of $Q^d$. Since it separates $N$ and $Q^d$, we have $(I - I_F) \cdot N \leq 0$, a contradiction to (94), concluding the proof of (93).

By (93) and (3), $N$ can be represented in the form
N = \lambda_1 u_1 \otimes u_1 + \cdots + \lambda_d u_d \otimes u_d, \text{ with suitable, l.i. } u_1, \ldots, u_d \in S^{d-1} \text{ and } \lambda_1, \ldots, \lambda_d > 0.

The proofs of statement (i) and of the implications (ii) \Rightarrow (iii) and (iii) \Rightarrow (ii) are almost verbatim the same as those of the corresponding assertions in Theorem 16.

**Proof of Corollary 5.** Since

\[ A(C) = \beta W(\Pi C) \quad \text{where } \beta = \frac{A(B^d)}{2v(B^{d-1})}, \]

the statements

- up to similarities, \( C \) is in unique minimum \( AA^* \)-position,
- up to similarities, \( \Pi C \) is in unique minimum \( WW^* \)-position,

are equivalent. Thus, Corollary 5 is an immediate consequence of Theorem 18.

**Acknowledgment**

For their help in the preparation of this article and valuable hints, I am grateful to Anthony Thompson, Apostolos Giannopoulos and the referee.

**References**


[31] A. Pelczyński, Remarks on John’s theorem on the ellipsoid of maximal volume inscribed into a convex symmetric region in \( \mathbb{R}^n \), Note Mat. 10 (Suppl. 2) (1990) 395–410.


