Differential Geometry and its Applications 2 (1992) 99–121
North-Holland

Infinitesimal natural and gauge-natural lifts

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Communicated by I. Kolář
Received 9 April 1990

Janyška, J. and M. Modugno, Infinitesimal natural and gauge-natural lifts, Diff. Geom. Appl. 2 (1992) 99–121.

Abstract: Infinitesimal natural and gauge-natural lifts are defined as special "systems" of vector fields on a fibred manifold $p : E \to B$. (Infinitesimally) natural and gauge-natural operators are defined via the commutativity with Lie derivatives.

Keywords: Natural lift functor, natural differential operator, gauge-natural lift functor, gauge-natural differential operator, system of vector fields, Lie derivative, infinitesimal natural lift, infinitesimal gauge-natural lift.

MS classification: 53B99, 58A20.

Introduction

Starting from the famous papers by Nijenhuis, [15,16], geometrical objects and invariant operations with geometrical objects have been studied by using the concepts of natural bundles and natural differential operators.

In physics another sort of invariance plays an important role, the so called "gauge-invariance". Its geometrical description is the following, [2]. Let $\pi : P \to B$ be a G-principal bundle over a space-time manifold B and $E \to B$ be a bundle associated with P. An automorphism of P, over B, induces a fibred automorphism of E, over B, which is said to be a change of gauge. A physical theory is said to be gauge-invariant if it is invariant with respect to changes of gauge. Gauge-invariant theories can be described geometrically by using the concepts of gauge-natural bundle functors and natural or gauge-natural operators between gauge-natural bundles, Eck [1]. The aim of this paper

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is to express naturality and gauge-naturality at infinitesimal level by using the concept of "systems" introduced by the second author, [14]. The main idea is to replace the action of a Lie group on the standard fibre by the action of a Lie algebra on the space of vector fields on the standard fibre of the bundle. We consider a system of projectable vector fields on a fibred manifold which induces the structure of an infinitesimal natural lift on the fibred manifold. A similar result is obtained for gauge-natural lifts.

In the classical theory, if a differential operator is natural or gauge-natural, then it commutes with the Lie derivatives associated with any vector field. We use this fact for our definition of (infinitesimally) natural and gauge-natural operators and we express the naturality of differential operators by using the distinguished vector fields of the given "systems".

Throughout the paper we use the following notation from jet theory. If M, N are two differentiable manifolds then the space of k-jets from M to N with source $x \in M$ and target $y \in N$ will be denoted by $J^k_x(M,N)_y$. If $E \to B$ is a fibred manifold then the space of k-jets of local sections of E is $J^k E$ and $\pi^k_l : J^k E \to J^l E$, $k \geqslant l$, is the canonical projection. The k-jet prolongation of a fibred manifold morphism $\varphi : E \to \bar{E}$ (covering a diffeomorphism $f : B \to \bar{B}$ of base spaces) is $J^k\varphi : J^k E \to J^k \bar{E}$. If $\sigma : B \to E$ is a section then $j^k\sigma : B \to J^k E$ is its k-jet prolongation. If $\Xi : E \to TE$ is a projectable vector field of E, then its k-jet prolongation is the vector field $j^k\Xi : J^k E \to TJ^k E$ of $J^k E$ defined by $j^k\Xi \equiv r^k \circ J^k\Xi$, where $r^k : J^k TE \to TJ^k E$ is the canonical fibred morphism, [13]. The sheaf of local sections of E will be denoted by $C^\infty E$.

All manifolds and mappings are assumed to be in the category C^∞.

1. Natural lift functors

We recall here definitions and basic properties from the theory of natural lift functors, [10, 16, 19].

Let $\mathcal{M}$ be the category of smooth manifolds and smooth manifold mappings and $\mathcal{M}_n$ be the category of C^∞ n-dimensional manifolds and smooth embeddings. Let $\mathcal{FM}$ be the category of smooth fibred manifolds and smooth fibred manifold mappings and $\mathcal{FM}_n$ be the category of smooth fibred manifolds over n-dimensional base spaces and smooth fibred manifold morphisms over embeddings of base spaces.

Definition 1.1. A *natural lift functor* is a covariant functor F from $\mathcal{M}_n$ to $\mathcal{FM}_n$ satisfying

i) for each manifold $B \in \operatorname{Ob} \mathcal{M}_n$,

$$p_B : FB \to B$$

is a fibred manifold over B,

ii) for each embedding $f \in \operatorname{Mor} \mathcal{M}_n$, Ff is a fibred manifold morphism over f, which maps fibres diffeomorphically onto fibres.

A *natural bundle* is then a triplet (FB, p_B, B).

In the definition of natural lift functors further continuity condition is sometimes added, requiring that a smoothly parametrized family of diffeomorphisms is prolonged into a smoothly parametrized family of diffeomorphisms. But this condition turns out to be a consequence of i) and ii), [3].

The concept of natural lift functor was generalized, [4, 8, 10], to the concept of natural bundle functor.

Definition 1.2. A *natural bundle functor* (in literature it is currently denoted simply as "prolongation functor") on a subcategory $\mathcal{C}$ of $\mathcal{M}$ is a covariant functor F from $\mathcal{C}$ to the category $\mathcal{FM}$ satisfying

i′) for each manifold $B \in \operatorname{Ob}\mathcal{C}$, $p_B : FB \to B$ is a fibred manifold[1] over B,

ii′) for each $f \in \operatorname{Mor}\mathcal{C}$, Ff is a fibred manifold map covering f such that $F\iota(U) = \iota(FU)$ for any open subset $\iota : U \hookrightarrow B$.

A natural bundle functor on the subcategory $\mathcal{M}_n$ of $\mathcal{M}$, for a certain n, is a natural lift functor.

We say that a natural lift functor F is of order r if, for any $f \in \operatorname{Mor}\mathcal{M}_n$, the map Ff depends only on the r-jet of f.

Let F be an r-order natural lift functor and let $F_0 \equiv (F\mathbb{R}^n)_0$ be the standard fibre of F. On F_0 we have the canonical action of the Lie group

$$G^r_n = \operatorname{inv} J^r_0(\mathbb{R}^n, \mathbb{R}^n)_0$$

of invertible r-jets (with source and target 0) of diffeomorphisms of $\mathbb{R}^n$ which preserve 0. It is well known that any natural lift functor has finite order, [10, 17], and that there is, up to equivalence, a one-to-one correspondence between r-order natural lift functors and left smooth G^r_n-manifolds, [11, 19].

The continuity condition allows us to prolong a vector field ξ of B to the vector field $F\xi$ of FB by the rule

$$\exp(tF\xi) = F(\exp(t\xi)). \tag{1.1}$$

This flow prolongation defines for an r-order natural lift functor F the associated smooth fibred mapping

$$\mu : J^rTB \times_B FB \longrightarrow TFB \tag{1.2}$$

which is linear over FB, [8]. So, we obtain

$$F : C^\infty TB \longrightarrow C^\infty(TFB \to FB); \quad \xi \mapsto F\xi(u) \equiv \mu \circ (j^r\xi, u), \; u \in FB,$$

where we used the letter F again (by abuse of language). Later (Section 3) we shall meet a different abstract approach to formula (1.2) in terms of "systems".

Local coordinate charts on B and F_0 induce a fibred coordinate chart on FB, which is said to be *natural*.

Example 1.1. The tangent functor T is a natural bundle functor of order 1 on the category $\mathcal{M}$. In dimension n the corresponding standard fibre is $\mathbb{R}^n$ on which $G^1_n =$

$\mathrm{Gl}(n,\mathbb{R})$ acts in the standard way. The tangent prolongation of a vector field $\xi = \xi^\lambda \partial_\lambda$ of B is

$$T\xi = \xi^\lambda \frac{\partial}{\partial x^\lambda} + \dot{x}^\mu \partial_\mu \xi^\lambda \frac{\partial}{\partial \dot{x}^\lambda},$$

where $(x^\lambda, \dot{x}^\lambda)$ is the natural coordinate chart on TB.

Example 1.2. The cotangent functor T^* is a natural lift functor of order 1 with the standard fibre $\mathbb{R}^{n*}$ and the standard action of G^1_n. The cotangent prolongation of a vector field $\xi = \xi^\lambda \partial_\lambda$ of B is

$$T\xi = \xi^\lambda \frac{\partial}{\partial x^\lambda} - x_\mu \partial_\lambda \xi^\mu \frac{\partial}{\partial x_\lambda},$$

where (x^λ, x_λ) is the natural coordinate chart on T^*B.

Example 1.3. The functor $T^{(r,s)}$ of (r,s) tensors is a natural lift functor of order 1. The standard fibre is $(\bigotimes^r \mathbb{R}^n) \otimes (\bigotimes^s \mathbb{R}^{n*})$ on which G^1_n acts in the standard tensor way. The tensor prolongation of a vector field ξ of B is

$$T^{(r,s)}\xi = \xi^\lambda \frac{\partial}{\partial x^\lambda} + (\; t^{\rho\lambda_2\ldots\lambda_r}_{\mu_1\ldots\mu_s} \partial_\rho \xi^{\lambda_1} + \cdots + t^{\lambda_1\ldots\lambda_{r-1}\rho}_{\mu_1\ldots\mu_s} \partial_\rho \xi^{\lambda_r} -$$
$$- \, t^{\lambda_1\ldots\lambda_r}_{\rho\mu_2\ldots\mu_s} \partial_{\mu_1} \xi^\rho - \cdots - t^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_{s-1}\rho} \partial_{\mu_s} \xi^\rho) \, \frac{\partial}{\partial t^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s}},$$

where $(x^\lambda, t^{\lambda_1\ldots\lambda_r}_{\mu_1\ldots\mu_s})$ is the natural coordinate chart on $T^{(r,s)}B$.

Example 1.4. The functor of metrics Met is a natural lift functor of order 1. Its standard fibre $(\mathrm{Met})_0$ is the subspace in $\bigodot^2 \mathbb{R}^{n*}$ of non-degenerate symmetric matrices with the coordinate chart $(g_{\lambda\mu})$ and the tensor action of G^1_n.

Example 1.5. The functor of k^r-velocities T^r_k is a natural bundle functor of order r on the category $\mathcal{M}$. For any $B \in \mathrm{Ob}\,\mathcal{M}$, we define $T^r_k B = J^r_0(\mathbb{R}^k, B)$ and, for any $f \in \mathrm{Mor}\,\mathcal{M}$, $f : B \to \bar{B}$, we define $T^r_k f(J^r_0 \alpha) = J^r_0(f \circ \alpha)$, where $J^r_0 \alpha \in T^r_k B$. The standard fibre of T^r_k in dimension n is $J^r_0(\mathbb{R}^k, \mathbb{R}^n)_0$ and the action of G^r_n on the standard fibre is given by the composition of jets.

Example 1.6. The functor of r-order frames F^r is a natural lift functor of order r. For any $B \in \mathrm{Ob}\,\mathcal{M}_n$, we define $F^r B = \mathrm{inv} J^r_0(\mathbb{R}^n, B)$ and, for any $f \in \mathrm{Mor}\,\mathcal{M}_n$, $F^r f$ is defined as in Example 1.5. The values of the functor F^r are in the category $\mathcal{PB}_n(G^r_n)$ of smooth principal bundles with n-dimensional base spaces, the structure group G^r_n and smooth principal bundle mappings.

Example 1.7. The bundle of linear connections C on a given manifold is a natural lift functor of order 2. Its standard fibre is $\mathbb{R}^n \otimes (\bigotimes^2 \mathbb{R}^{n*})$ on which G^2_n acts via the well known transformation relations of the Christoffel symbols. The flow prolongation of a vector field $\xi = \xi^\lambda \partial_\lambda$ of B is

$$C\xi = \xi^\lambda \frac{\partial}{\partial x^\lambda} + \left(\Gamma^\rho_{\mu\nu}\partial_\rho \xi^\lambda - \Gamma^\lambda_{\rho\nu}\partial_\mu \xi^\rho - \Gamma^\lambda_{\mu\rho}\partial_\nu \xi^\rho + \partial_{\mu\nu}\xi^\lambda\right) \frac{\partial}{\partial \Gamma^\lambda_{\mu\nu}},$$

where $(x^\lambda, \Gamma^\lambda_{\mu\nu})$ is the natural coordinate chart on CB.

Remark 1.1. Let F be a natural lift functor of order r. For any $f \in \mathrm{Mor}\,\mathcal{M}_n$, $f : B \to \bar{B}$, by using the standard jet prolongation, we get the commutative diagram

$$\begin{array}{ccc} J^s FB & \xrightarrow{J^s Ff} & J^s F\bar{B} \\ {\scriptstyle \pi^s_o}\downarrow & & \downarrow{\scriptstyle \pi^s_o} \\ FB & \xrightarrow{Ff} & F\bar{B} \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle p_B} \\ B & \xrightarrow[f]{} & \bar{B} \end{array}$$

which implies that $J^s F \equiv J^r \circ F$ is a natural lift functor of order $(r+s)$. If F_0 is the standard fibre of F then the standard fibre of $J^s F$ is $J^s F_0 = T^s_n F_0$ and the action of G^{r+s}_n on $J^s F_0$ is obtained by the jet prolongation of the action of G^r_n on F_0.

Remark 1.2. In the theory of natural lift functors, the functor of r-order frames defined in Example 1.6 plays a fundamental role. Namely, any natural lift functor F of order r, with standard fibre F_0, is canonically represented by

$$FB = [F^r B, F_0], \quad Ff = [F^r f, \mathrm{id}], \tag{1.3}$$

where $B \in \mathrm{Ob}\,\mathcal{M}_n$, $f \in \mathrm{Mor}\,\mathcal{M}_n$, and $[F^r B, F_0] = (F^r B, F_0)/G^r_n$ is the bundle associated with $F^r B$, [10,11,19].

Example 1.8. With respect to the adjoint action of G^r_n on its Lie algebra $\mathcal{G}^r_n$, we can define the r-order natural lift functor

$$\mathcal{G}^r_n(B) = [F^r B, \mathcal{G}^r_n], \quad \mathcal{G}^r_n(f) = [F^r f, \mathrm{id}]. \tag{1.4}$$

So $\mathcal{G}^r_n$ can be viewed as a natural r-order lift functor and we shall call it the *adjoint r-order natural lift functor.*

Let F be a natural lift functor, $f : B \to \bar{B}$ be a mapping in $\mathrm{Mor}\,\mathcal{M}_n$ and $\sigma : B \to FB$ be a section. Then we define the section $f^*\sigma : \bar{B} \to F\bar{B}$ by $f^*\sigma = Ff \circ \sigma \circ f^{-1}$.

Definition 1.3. A *natural differential operator* D from a natural lift functor F_1 to a natural lift functor F_2 is a family of differential operators

$$\{D(B) : C^\infty F_1 B \to C^\infty F_2 \bar{B}\}_{B \in \mathrm{Ob}\,\mathcal{M}_n}$$

such that $D(\bar{B})(f^*\sigma) = f^*D(B)(\sigma)$ for all sections $\sigma : B \to F_1 B$ and all $f : B \to \bar{B}$ in $\operatorname{Mor} \mathcal{M}_n$.

A natural differential operator is of a finite order k if all $D(B)$, $B \in \operatorname{Ob} \mathcal{M}_n$, depend on k-order jets of sections. Thus, a k-order natural differential operator D from F_1 to F_2 is characterized by the associated fibred manifold morphisms $\mathcal{D}(B) : J^k F_1 B \to F_2 B$, over B, according to the formula $\mathcal{D}(B)(j^k_x \sigma) = D(B)(\sigma)(x)$. The family $\mathcal{D} = \{\mathcal{D}(B)\}_{B \in \operatorname{Ob} \mathcal{M}_n}$ defines a natural transformation of the functors $J^k F_1$ and F_2.

Let D be a natural operator of order k from F_1 to F_2 and $\mathcal{D}$ be its associated natural transformation. Then we can define the tangent prolongation TD of D as the k-order operator TD from TF_1 to TF_2 defined by the associated transformation $\mathcal{TD} : J^k T F_1 \to T F_2$, where $\mathcal{TD} = T\mathcal{D} \circ r^k$ and $r^k : J^k T F_1 \to T J^k F_1$ is the natural transformation defined in [13]. It is easy to see that $\mathcal{TD}$ is a natural transformation of covariant functors and it implies that TD defined by $TD(B)(\Sigma) = \mathcal{TD}(B) \circ (j^k \Sigma)$, for any section $\Sigma : B \to TF_1 B$, is the natural operator such that $q_{F_2 B}(TD(\Sigma)) = D(q_{F_1 B}(\Sigma))$, where $q_E : TE \to E$ is the canonical projection.

Definition 1.4. Let F be a natural lift functor, ξ be a vector field of B and $\exp(t\xi)$ its flow. Then the *Lie derivative* of a section $\sigma : B \to FB$ with respect to the vector field ξ is defined by

$$\mathcal{L}_\xi \sigma = \frac{d}{dt}\Big|_0 \{\exp(-t\xi)^* \sigma\}. \tag{1.5}$$

In [5,9,18] it was proved that the Lie derivative can be expressed geometrically as the mapping

$$\mathcal{L}_\xi \sigma = T\sigma \circ \xi - F\xi \circ \sigma. \tag{1.6}$$

Hence $\mathcal{L}_\xi \sigma$ turns out to be a section $\mathcal{L}_\xi \sigma : B \to VFB$ projectable onto the section σ.

Lemma 1.1. *If a k-order differential operator D from a natural lift functor F_1 to a natural lift functor F_2 is natural, then*

$$\mathcal{L}_\xi D(B)(\sigma) = TD(B)(\mathcal{L}_\xi \sigma), \tag{1.7}$$

for $B \in \operatorname{Ob} \mathcal{M}_n$, any section $\sigma : B \to F_1 B$ and any vector field ξ of B.

Proof. From the infinitesimal expression (1.5) for $\mathcal{L}_\xi \sigma$ we get

$$\begin{aligned} TD(B)(\mathcal{L}_\xi \sigma) &= T\mathcal{D}(B) \circ r^k \circ j^k \frac{d}{dt}\Big|_0 \{\exp(-t\xi)^* \sigma\} = \\ &= T\mathcal{D}(B) \circ \left[\frac{d}{dt}\Big|_0 \{ j^k (\exp(-t\xi)^* \sigma)\} \right] = \frac{d}{dt}\Big|_0 \{\mathcal{D}(B) \circ (j^k (\exp(-t\xi)^* \sigma))\} = \\ &= \frac{d}{dt}\Big|_0 \{\exp(-t\xi)^* (\mathcal{D}(B) \circ (j^k \sigma))\} = \mathcal{L}_\xi D(B)(\sigma). \end{aligned}$$

which proves our Lemma 1.1. □

For the case of linear operators on natural vector bundles Lemma 1.1 was used in [6].

Remark 1.3. Since the values of the Lie derivative are in VFB it is sufficient to consider in (1.7) only the vertical prolongation VD of the operator D instead of the tangent prolongation TD. The vertical prolongation VD can be defined as the restriction of TD on VF_1 or equivalently by

$$VD(B)\left(\frac{d}{dt}\Big|_0 \sigma_t\right) = \frac{d}{dt}\Big|_0 D(B)(\sigma_t),$$

where $\sigma_t : B \to F_1 B$ is a smoothly parametrized family of sections.

Definition 1.5. A differential operator D from a natural lift functor F_1 to a natural lift functor F_2 is said to be *infinitesimally natural* if (1.7) holds for any section $\sigma : B \to F_1 B$ and any vector field ξ of B.

Many geometrical constructions are in fact natural differential operators between natural lift functors. The study of natural differential operators is based on relations between natural differential operators and equivariant mappings. The basic tool is the following theorem, [10, 12, 19].

Theorem 1.1. *There is a bijective correspondence between the set of k-order natural differential operators from a natural lift functor F_1 to a natural lift functor F_2 and equivariant mappings from the standard fibre of $J^k F_1$ to the standard fibre of F_2.*

Example 1.9. The exterior derivative d is a first order natural operator from $\bigwedge^p T^*$, $p \geqslant 1$, to $\bigwedge^{p+1} T^*$. The corresponding G^2_n-equivariant mapping from $J^1(\bigwedge^p T^*)_0 = T^1_n(\bigwedge^p \mathbb{R}^{n*})$ to $(\bigwedge^{p+1} T^*)_0 = \bigwedge^{p+1} \mathbb{R}^{n*}$ is given in the canonical coordinate chart $(\omega_{i_1 \dots i_p})$ $1 \leqslant i_1 < \dots < i_p \leqslant n$ on $(\bigwedge^p \mathbb{R}^{n*})$ by

$$\omega_{i_1 \dots i_{p+1}} \circ \mathrm{d} = \omega_{[i_1 \dots i_p, i_{p+1}]},$$

where [...] denotes the antisymmetrization. It can be proved that the naturality determines d up to a constant.

Example 1.10. The Levi-Civita connection is a first order natural differential operator from Met to C. The corresponding G^2_n-equivariant mapping from $J^1(\mathrm{Met})_0$ to C_0 is given by

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2} g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}),$$

where $(g^{\lambda\mu})$ is the inverse matrix of $(g_{\lambda\mu})$.

Example 1.11. The curvature tensor is a first order natural differential operator from C to $T \otimes (\bigotimes^3 T^*)$. The corresponding G^3_n-equivariant mapping from $J^1 C_0$ to $(T \otimes (\bigotimes^3 T^*))_0 = \mathbb{R}^n \otimes (\bigotimes^3 \mathbb{R}^{n*})$ is given by

$$t^\lambda_{\mu\nu\kappa} = \Gamma^\lambda_{\mu\nu,\kappa} - \Gamma^\lambda_{\mu\kappa,\nu} + \Gamma^\lambda_{\rho\kappa}\Gamma^\rho_{\mu\nu} - \Gamma^\lambda_{\rho\nu}\Gamma^\rho_{\mu\kappa}.$$

Example 1.12. The Nijenhuis tensor is a first order natural differential operator from $T \otimes T^*$ to $T \otimes (\bigwedge^2 T^*)$. The corresponding G^2_n-equivariant mapping from $J^1(T^{(1,1)})_0$ to $(T \otimes (\bigwedge^2 T^*))_0$ is given by

$$t^\lambda_{\mu\nu} = t^\lambda_\rho t^\rho_{\mu,\nu} - t^\lambda_\rho t^\rho_{\nu,\mu} + t^\lambda_{\nu,\rho} t^\rho_\mu - t^\lambda_{\mu,\rho} t^\rho_\nu .$$

2. Gauge-natural bundle functors

In this section we recall some basic definitions and properties of gauge-natural bundle functors, [1, 7].

Let $\mathcal{PB}_n(G)$ be the category of smooth principal G-bundles, whose base manifolds are n-dimensional, and smooth G-bundle morphisms (φ, f), where $f \in \operatorname{Mor} \mathcal{M}_n$.

Definition 2.1. A *gauge-natural bundle functor* is a covariant functor F from the category $\mathcal{PB}_n(G)$ to the category $\mathcal{FM}_n$ satisfying

i) for each $\pi : P \to B$ in $\mathcal{PB}_n(G)$, $\pi_P : FP \to B$ is a fibred manifold over B,

ii) for each embedding (φ, f) in $\mathcal{PB}_n(G)$, $F\varphi = F(\varphi, f)$ is a fibred manifold morphism covering f,

iii) for any open subset $U \subset B$, the immersion $\iota : \pi^{-1}(U) \hookrightarrow P$ is transformed into the immersion $F\iota : \pi_P^{-1}(U) \hookrightarrow FP$.

A *gauge-natural bundle* is then a quadruple $(FP, \pi_P, B, \pi : P \to B)$.

In the original definition, [1], there is one more continuity condition which says that a smoothly parametrized family of diffeomorphisms of P is "prolonged" into a smoothly parametrized family of isomorphisms of FP. But this condition is a consequence of i), ii) and iii), [10].

Example 2.1. Let $(\pi : P \to B) \in \operatorname{Ob} \mathcal{PB}_n(G)$, let $W^r P$ be the space of all r-jets $J^r_{(0,e)}\varphi$, where $\varphi : \mathbb{R}^n \times G \to P$ is in $\operatorname{Mor} \mathcal{PB}_n(G)$, $0 \in \mathbb{R}^n$ and e is the unity in G. The space $W^r P$ is a principal fibre bundle over B with structure group $W^r_n G = J^r_{(0,e)}(\mathbb{R}^n \times G, \mathbb{R}^n \times G)$ of all r-jets of principal fibre bundle isomorphisms $\Psi : \mathbb{R}^n \times G \to \mathbb{R}^n \times G$ covering the diffeomorphisms $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi(0) = 0$. The group $W^r_n G$ is the semidirect product of G^r_n and $T^r_n G$ with respect to the action of G^r_n on $T^r_n G$ given by jet composition. Let $(\varphi : P \to \bar{P}) \in \operatorname{Mor} \mathcal{PB}_n(G)$, then we can define the principal bundle morphism $W^r\varphi : W^r P \to W^r \bar{P}$ by jet composition. The rule transforming any $P \in \operatorname{Ob} \mathcal{PB}_n(G)$ into $W^r P \in \operatorname{Ob} \mathcal{PB}_n(W^r_n G)$ and any $\varphi \in \operatorname{Mor} \mathcal{PB}_n(G)$ into $W^r\varphi \in \operatorname{Mor} \mathcal{PB}_n(W^r_n G)$ is a gauge-natural bundle functor, [7].

The gauge-natural bundle functor described in Example 2.1 plays a fundamental role in the theory of gauge-natural bundle functors. We have, [1, 7],

Theorem 2.1. *Every gauge-natural bundle FP is a fibred bundle associated with the gauge-natural bundle $W^r P$ for a certain order r.*

The number r from Theorem 2.1 is called the *order* of the gauge-natural bundle functor F. So if F is an r-order gauge-natural bundle functor then

$$FP = [W^r P, F_0], \quad F\varphi = [W^r\varphi, \mathrm{id}_{F_0}], \tag{2.1}$$

where F_0 is a $W^r_n G$-manifold called the *standard fibre* of F.

A local fibred coordinate chart on P and a coordinate chart on F_0 induce a fibred coordinate chart on FP, which is said to be *gauge-natural.*

Let $s \leqslant r$ be the minimum number such that the action of $W^r_n G = G^r_n \times_S T^r_n G$ on F_0 can be factorized through the canonical projection $\pi^r_s : T^r_n G \to T^s_n G$, $r \geqslant s$, via the commutative diagram

$$\begin{array}{ccc} (G^r_n \times_S T^r_n G) \times F_0 & \longrightarrow & F_0 \\ \downarrow & \nearrow & \\ (G^r_n \times_S T^s_n G) \times F_0 & & \end{array}$$

Then s is called the *gauge-order* of F and we say that F is of order (r, s).

The regularity condition allows us to "prolong" any G-invariant vector field Ξ of P to the vector field $F\Xi$ of FP. Namely, $\exp(tF\Xi) = F(\exp(t\Xi))$. The vector fields Ξ and $F\Xi$ are projected on the same vector field of B. The flow prolongation of a G-invariant vector field of P defines the linear mapping

$$J^r(TP/G) \times_B FP \longrightarrow TFP \tag{2.2}$$

over FP, where TP/G is the space of G-invariant vector fields of P. Later (Section 3) we shall meet a different abstract approach to formula (2.2) in terms of "systems".

Example 2.2. Any r-order natural lift functor in the sense of Definition 1.1 is the $(r, 0)$-order gauge-natural bundle functor with the trivial gauge action, i.e. the action

$$(G^r_n \times G) \times F_0 \longrightarrow F_0$$

does not depend on G.

Example 2.3. Let $(\pi : P \to B) \in \mathrm{Ob}\,\mathcal{PB}_n(G)$ and let us denote by $CP \to B$ the bundle of principal connections on P. Then C is a $(1, 1)$-order gauge-natural bundle functor with the standard fibre $\mathcal{G} \otimes \mathbb{R}^{n*}$. In particular, let $G = G^r_m$, then CP can be viewed as the bundle of linear connections on an associated vector bundle with m-dimensional fibres. The standard fibre of C is $C_0 = \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{n*}$ with coordinates $(\Gamma^i_{j\lambda})$, $i, j = 1, \ldots, m$, $\lambda = 1, \ldots, n$, and the action of $W^1_n = G^1_n \times_S T^1_n G^1_m$ on C_0 is given, in the canonical coordinates $(a^\lambda_\mu, a^i_j, a^i_{j\lambda})$ on $G^1_n \times_S T^1_n G^1_m$, by

$$\bar{\Gamma}^i_{j\lambda} = a^i_p \Gamma^p_{q\rho} \tilde{a}^q_j \tilde{a}^\rho_\lambda + a^i_{p\rho} \tilde{a}^p_j \tilde{a}^\rho_\lambda,$$

where the tilde denotes the inverse element.

Example 2.4. With respect to the adjoint action of $W^r_n G$ on its Lie algebra $\mathcal{W}^r_n\mathcal{G}$ we can define the r-order gauge-natural bundle functor

$$\mathcal{W}^r_n\mathcal{G}(P) = [W^r P, \mathcal{W}^r_n\mathcal{G}], \quad \mathcal{W}^r_n\mathcal{G}(\varphi) = [W^r\varphi, \mathrm{id}]. \tag{2.3}$$

So $\mathcal{W}^r_n\mathcal{G}$ can be viewed as a gauge-natural bundle functor which will be called the *adjoint r-order gauge-natural bundle functor.* In particular, $\mathcal{G}$ is the adjoint 0-order gauge-natural bundle functor.

Example 2.5. If F is a gauge-natural bundle functor of order (r, s) then J^kF is a gauge-natural bundle functor of order at most $(r + k, s + k)$. The number $(r + k)$ is exact, but $(s + k)$ may be too big, for instance if F is an r-order natural lift functor, i.e. an $(r, 0)$-order gauge-natural bundle functor, then J^kF is an $(r + k)$-order natural lift functor, i.e. an $(r + k, 0)$-order gauge-natural bundle functor.

Example 2.6. $\mathcal{G} \otimes (\bigwedge^p T^*)$ is a $(1, 0)$-order gauge-natural bundle functor.

Let $(\varphi, f) \in \operatorname{Mor}\mathcal{PB}_n(G)$, $\varphi : P \to \bar{P}$, $f : B \to \bar{B}$, F be a gauge-natural bundle functor and $\sigma : B \to FP$ be a section. Then we define the section $\varphi^*\sigma : \bar{B} \to F\bar{P}$ by $\varphi^*\sigma = F\varphi \circ \sigma \circ f^{-1}$.

Definition 2.4. A *natural differential operator* D from a gauge-natural bundle functor F_1 to a gauge-natural bundle functor F_2 is a family of differential operators

$$\{D(P) \mid C^\infty F_1 P \to C^\infty F_2 P\}_{P \in \operatorname{Ob}\mathcal{PB}_n(G)}$$

such that $D(\bar{P})(\varphi^*\sigma) = \varphi^* D(P)(\sigma)$ for all sections $\sigma : B \to F_1P$ and all $(\varphi, f) \in \operatorname{Mor}\mathcal{PB}_n(G)$, $\varphi : P \to \bar{P}$ over $f : B \to \bar{B}$.

Definition 2.5. A differential operator D from a gauge-natural bundle functor F_1 to a gauge-natural bundle functor F_2 is said to be *gauge-natural* if

$$D(\bar{P})(F_1\varphi \circ \sigma) = F_2\varphi \circ D(P)(\sigma)$$

for any $\varphi \in \operatorname{Mor}\mathcal{PB}_n(G)$, over the identity, and any section $\sigma : B \to F_1P$.

A natural differential operator D from F_1 to F_2 is of a finite order k if all $D(P)$, $(\pi : P \to B) \in \operatorname{Ob}\mathcal{PB}_n(G)$, depend on k-order jets of sections of F_1P. Thus, a k-order natural operator from F_1 to F_2 is characterized by the associated fibred manifold morphism $\mathcal{D}(P) : J^kF_1P \to F_2P$, over B, such that the family $\mathcal{D} = \{\mathcal{D}(P)\}_{P \in \operatorname{Ob}\mathcal{PB}_n(G)}$ is a natural transformation of J^kF_1 to F_2. The following fundamental theorem is due to Eck, [1].

Theorem 2.2. *Let F_1 and F_2 be gauge-natural bundle functors of order $\leqslant r$. Then we have a one-to-one correspondence between natural differential operators of order k from F_1 to F_2 and $W^{r+k}_n G$-equivariant mappings from $(J^kF_1)_0$ to $(F_2)_0$.*

For the case of gauge-natural operators of order k we obtain that the corresponding equivariant mappings are equivariant with respect to the actions of the group $T_n^{r+k}G \approx \{J_0^{r+k}\mathrm{id}\} \times T_n^{r+k}G$.

Example 2.7. The curvature operator is a 1-order natural operator from C to $\mathcal{G} \otimes (\bigwedge^2 T^*)$.

Definition 2.6. Let Ξ be a G-invariant vector field of P over a vector field ξ of B and $\sigma : B \to FP$ be a section. The *Lie derivative* of σ with respect to Ξ is defined by

$$\mathcal{L}_\Xi \sigma = \frac{d}{dt}\Big|_0 \{\exp(-t\Xi)^* \sigma\}. \tag{2.4}$$

This derivation can be expressed by

$$\mathcal{L}_\Xi \sigma = T\sigma \circ \xi - F\Xi \circ \sigma.$$

Thus $\mathcal{L}_\Xi \sigma : B \to VFP$ is a section projectable onto the section σ.

Analogously to the case of natural lift functors, we can define the tangent prolongation TD of a k-order natural operator D from a gauge-natural bundle functor F_1 to a gauge-natural bundle functor F_2 and the following lemma can be proved in the same way as Lemma 1.1.

Lemma 2.1. *If a k-order differential operator D from a gauge-natural bundle functor F_1 to a gauge-natural bundle functor F_2 is natural, then*

$$\mathcal{L}_\Xi D(P)(\sigma) = TD(P)(\mathcal{L}_\Xi \sigma) \tag{2.5}$$

for any section $\sigma : B \to F_1 P$ and any G-invariant vector field Ξ of P.

The infinitesimal version of the naturality and gauge-naturality of a differential operator is given by the following definition.

Definition 2.7. A differential operator D from a gauge-natural bundle functor F_1 to a gauge-natural bundle functor F_2 is said to be *infinitesimally natural* (resp. *infinitesimally gauge-natural*) if (2.5) holds for any section $\sigma : B \to F_1 P$ and any G-invariant vector field (resp. G-invariant vertical vector field) Ξ of P.

3. Systems of projectable vector fields and connections

In this section we shall recall basic properties of systems introduced by the second author, [14], and add the new definition of Lie derivative in the context of systems and a result on induced systems.

Let $p \colon E \to B$ be a fibred manifold. A *projectable, linear, regular system of vector fields* on a fibred manifold E is a pair (H, η), where

$$q_H \colon H \to B \tag{3.1}$$

is a vector bundle, called the *space* of the system, and

$$\eta \colon H \times_B E \to TE \tag{3.2}$$

is a linear fibred morphism over E, called the *evaluation morphism* of the system, which is projectable over a linear fibred morphism over B, of maximum rank,

$$\bar{\eta} \colon H \to TB \colon h \mapsto \bar{h} \tag{3.3}$$

by means of the following commutative diagram

$$\begin{array}{ccc} H \times_B E & \xrightarrow{\eta} & TE \\ \downarrow & & \downarrow \\ H & \xrightarrow[\bar{\eta}]{} & TB \end{array}$$

Let us set $q_A \colon A \equiv \ker \bar{\eta} \subset H \to B$. The dimension of the fibre of A is called the *rank* of the system. Then we have the exact sequence of vector bundles, over B,

$$0 \longrightarrow A \xrightarrow{j} H \xrightarrow{\bar{\eta}} TB \longrightarrow 0 \tag{3.4}$$

and the following diagram commutes

$$\begin{array}{ccc} A \times_B E & \xrightarrow{\eta_A} & VE \\ {\scriptstyle j \times \mathrm{id}} \downarrow & & \downarrow \\ H \times_B E & \xrightarrow[\eta]{} & TE \end{array}$$

Any (local) section $h \colon B \to H$ induces the vector field $\tilde{\eta}(h)$ on E by

$$\tilde{\eta}(h)(y) = \eta(h(p(y)), y), \quad y \in E. \tag{3.5}$$

These (local) vector fields $\tilde{\eta}(h)$ are the *distinguished* vector fields of the system.

We say that the system is *canonical* if there exists a fibred atlas, constituted by linear fibred charts $(x^\lambda, z^\lambda, z^a)$, with $1 \leqslant a \leqslant r$, of H and fibred charts (x^λ, y^i) of E, such that the coordinate expression of η is

$$\eta = z^\lambda \frac{\partial}{\partial x^\lambda} + \eta^i_a z^a \frac{\partial}{\partial y^i},$$

where $\eta^i_a \in C^\infty(E, \mathbb{R})$, with $\partial_\lambda \eta^i_a = 0$.

Moreover, we say that the system is *monic* if the construction of the distinguished vector fields $h \mapsto \tilde{\eta}(h)$ is injective. The monicity is expressed by the equivalence, for any $x \in B$,

$$\eta^i_a z^a = 0 \iff z^a = 0, \text{ for any } y \in E_x.$$

Let (H, η) be a projectable, linear, regular, canonical and monic system of vector fields on E. We say that the system is *involutive* if, for any two local sections $h, k \colon B \to$

H, the vector field $[\tilde{\eta}(h), \tilde{\eta}(k)]$ is associated with a section of H, which turns out to be unique, and will be also denoted by $[h, k]$. Hence

$$\tilde{\eta}([h,k]) = [\tilde{\eta}(h), \tilde{\eta}(k)].$$

So $[\cdot\,,\cdot]$ is a sheaf (bilinear) mapping (operator) from $C^\infty H \times C^\infty H$ to $C^\infty H$.

A projectable, linear, regular, canonical, monic and involutive system (H, η) is briefly called *strong*. Let $h = (h^\lambda(x), h^a(x))$, $k = (k^\lambda(x), k^a(x))$ be the coordinate expressions of the sections h, k. Then we obtain the following coordinate expression

$$[h,k] = (h^\mu \partial_\mu k^\lambda - k^\mu \partial_\mu h^\lambda, h^\mu \partial_\mu k^a - k^\mu \partial_\mu h^a + C^a_{bc} h^b k^c),$$

where $C^a_{bc} \in \mathbb{R}$. Hence, we have a unique associated (bilinear) fibred morphism over B

$$[\cdot\,,\cdot]\colon J^1 H \times_B J^1 H \to H, \tag{3.6}$$

such that, for any sections $h, k\colon B \to H$,

$$[h,k] = [\cdot\,,\cdot] \circ (j^1 h, j^1 k).$$

Moreover, it restricts to a bilinear fibred morphism over B

$$\beta\colon A \times_B A \to A,$$

which endows the bundle $q_A\colon A \to B$ with a Lie algebra bundle structure. Furthermore, in the canonical fibred chart on A we obtain

$$\beta^a_{bc} = C^a_{bc} \in \mathbb{R}.$$

In conclusion, a strong system (H, η) determines a subalgebra of the Lie algebra of infinitesimal generators of local fibred automorphisms of $p : E \to B$. The assumption of a strong system is essentially a generalized version of the hypothesis that E is locally associated with a principal bundle.

Example 3.1. Let $p : E \to B$ be a right principal bundle with structure group G. Then we have the quotient vector bundles $q_H : H = TE/G \to B$ and $q_A : A = VE/G \to B$ and the exact sequence

$$0 \to VE/G \longrightarrow TE/G \longrightarrow TB \to 0.$$

Moreover, we have a canonical linear fibred isomorphism $\eta : TE/G \times_B E \to TE$, over E, which restricts to $\eta_A : VE/G \times_B E \to VE$. Then the system (H, η) of G-invariant vector fields on E is strong.

Remark 3.1. If $\eta : H \times_B E \to TE$ is a linear, projectable and regular system such that $\eta_A = 0$, then η factorizes through a fibred morphism over E

$$\gamma : TB \times_B E \to TE$$

which turns out to be linear and projectable over $\mathrm{id} : TB \to TB$. Thus, γ is a general connection on $E \to B$. Additionally, in this case, canonicity of η and integrability of γ are equivalent.

Remark 3.2. A projectable, linear, regular and canonical system (H, η) of vector fields of E induces the projectable, linear, regular and canonical system $(H^{\oplus}, \eta^{\oplus})$ of tangent valued forms on E, where

$$q^r : H^r = \bigwedge^r T^*B \otimes H \to B, \tag{3.7}$$

and the linear fibred morphism

$$\eta^r : H^r \times_B E \to \bigwedge^r T^*B \otimes TE : (\alpha \otimes z, y) \mapsto \alpha \otimes \eta(z, y) \tag{3.8}$$

is projectable over the linear fibred morphism over B

$$\overline{\eta}^r : H^r \to \bigwedge^r T^*B \otimes TB : \alpha \otimes z \mapsto \alpha \otimes \overline{\eta}(z). \tag{3.9}$$

If the system (H, η) is involutive (with respect to the Lie bracket), then the system $(H^{\oplus}, \eta^{\oplus})$ turns out to be involutive with respect to the Frölicher-Nijenhuis bracket of tangent valued forms on E, [14].

Remark 3.3. A projectable, linear, regular and canonical system (H, η) of vector fields of E yields also a "system" (C, ξ) of connections on E. Namely, we have the bundle

$$p_C : C \to B, \tag{3.10}$$

which is defined as the subbundle in $T^*B \otimes H$, which projects onto $\mathbf{1}_B \subset T^*B \otimes TB$. Hence, $p_C : C \to B$ is an affine bundle whose vector bundle is $T^*B \otimes A \to B$. Moreover, ξ is the restriction of η^1 on C and we obtain the affine fibred morphism over E

$$\xi : C \times_B E \to J^1E \subset T^*B \otimes_E TE. \tag{3.11}$$

A coordinate chart $(x^\lambda, z^\lambda, z^a)$ on H induces the coordinate chart (x^λ, v^a_λ) on C and the coordinate expression of ξ is

$$\xi = dx^\lambda \otimes \frac{\partial}{\partial x^\lambda} + \eta^i_a v^a_\lambda dx^\lambda \otimes \frac{\partial}{\partial y^i}. \tag{3.12}$$

Any (local) section $c : B \to C$ induces the connection $\tilde{\xi}(c)$ on E by

$$\tilde{\xi}(c)(y) = \xi(c(p(y)), y), \quad y \in E.$$

These (local) connections are the *distinguished* connections of the system.

If (H, η) is strong, then we say that (C, ξ) is *strong*.

The bracket on $H^{\oplus}$ given by the involutivity of the system $(H^{\oplus}, \eta^{\oplus})$ with respect to the Frölicher-Nijenhuis bracket of tangent valued forms on E allows us to define the differential calculus connected with a given connection $c : B \to C$, [14]. Namely, the *strong covariant differential* $d_c : C^\infty H^r \to C^\infty H^{r+1}$ is defined by

$$d_c \Phi = [c, \Phi]. \tag{3.13}$$

Moreover, the *strong curvature form* of a given connection c is

$$\omega = d_c c = [c, c]. \tag{3.14}$$

The values of the strong curvature form are in $C^\infty A^2$, where $A^2 = A \otimes \bigwedge^2 T^*B$, and the coordinate expression is

$$d_c c = (\partial_\mu v^a_\lambda + \tfrac{1}{2} C^a_{bc} v^b_\lambda v^c_\mu) dx^\lambda \wedge dx^\mu \otimes \frac{\partial}{\partial z^a}. \tag{3.15}$$

Definition 3.1. Let (H, η) be a projectable, linear and regular system of vector fields of E. Let h be a section of $H \to B$ and σ be a section of $E \to B$. By using the geometrical interpretation of Lie derivative we can define the *Lie derivative* of σ with respect to infinitesimal fibred transformation h of $E \to B$ by

$$\mathcal{L}_h \sigma = T\sigma \circ \underline{\eta}(h) - \tilde{\eta}(h) \circ \sigma. \tag{3.16}$$

Thus

$$\mathcal{L}_h \sigma : B \to VE$$

and is projectable on the section σ. For studying infinitesimal gauge-natural lifts we shall need the following result.

Theorem 3.1. *Let (H, η) be a strong system on a fibred manifold $E \to B$. Then we obtain in a natural way a linear, projectable, regular and canonical system*

$$\zeta_A : H \times_B A \to TA. \tag{3.17}$$

Its coordinate expression is

$$\zeta_A = z^\lambda \frac{\partial}{\partial x^\lambda} - C^a_{bc} z^b \bar{z}^c \frac{\partial}{\partial z^a}. \tag{3.18}$$

Proof. The fibred morphism (3.6) over B restricts to

$$[\cdot, \cdot] : H \times_B J^1 A \to A,$$

and can be viewed as a fibred morphism over A

$$H \times_B J^1 A \to VA$$

with coordinate expression

$$[\cdot, \cdot] = \bar{z}^a_\lambda z^\lambda \frac{\partial}{\partial z^a} + C^a_{bc} z^b \bar{z}^c \frac{\partial}{\partial z^a}.$$

On the other hand, we have the canonical fibred morphism over A

$$\Pi : TB \times_B J^1 A \to TA$$

which extends to the fibred morphism over A

$$\Pi : H \times_B J^1 A \to TA$$

with coordinate expression

$$\Pi = z^\lambda \frac{\partial}{\partial x^\lambda} + \bar{z}^a_\lambda z^\lambda \frac{\partial}{\partial z^a}.$$

Then the fibred difference of above fibred morphisms yields our map ζ_A. □

We remark that if a system (H, η) is non-monic then it does not make sense to check involutivity, unless we have an extra bracket on the sections of H. Later we shall use the following

Definition 3.2. Let (H, η) be a linear, regular, projectable and canonical system of $E \to B$. Suppose we have an additional bracket $[\cdot,\cdot]$ which makes $C^\infty H$ into a sheaf of Lie algebras. Then we say that the system is *almost involutive* if

$$\tilde{\eta}([h,k]) = [\tilde{\eta}(h), \tilde{\eta}(k)], \tag{3.19}$$

where the left bracket is the additional one and the right is the Lie bracket.

Sometimes the additional bracket is not given on the whole $C^\infty H$ but on a certain subsheaf; in such a case we shall say that the system is almost involutive with respect to this subsheaf.

4. Infinitesimal natural lifts

In this section we shall define infinitesimal natural lifts by using the concept of systems of vector fields. Our approach will be motivated by the following remark and lemma.

Remark 4.1. Let $\mathcal{J}^rTB$ be the sheaf of local integrable sections $j^rh : B \to J^rTB$, where $h : B \to TB$ is a local section. Then $\mathcal{J}^rTB$ becomes a sheaf of Lie algebras by means of the bracket given by

$$[j^rh, j^rk] \equiv j^r[h,k]. \tag{4.1}$$

In general the bracket on $\mathcal{J}^rTB$ will involve the $(r+1)$-jet prolongation of vector fields of B. Namely, we obtain a well defined fibred morphism

$$[\cdot,\cdot] : J^{r+1}TB \times_B J^{r+1}TB \to J^rTB. \tag{4.2}$$

The restriction of (4.2) to the subbundle $J^{r+1}TB_0 \equiv \operatorname{Ker} \pi^{r+1}_0$ factorizes through the canonical projection $J^{r+1}TB_0 \to J^rTB_0$ and defines a structure of Lie algebra bundle on J^rTB_0. This Lie algebra bundle is isomorphic to the r-order adjoint natural bundle $\mathcal{G}^r_n(B)$ defined in Example 1.8.

Lemma 4.1. *Let F be an r-order natural lift functor. The construction of the flow prolongation of vector fields defines a projectable, linear, regular and canonical system of FB, $B \in \mathcal{M}_n$,*

$$\mu : J^rTB \times_B FB \to TFB,$$

with $\eta \equiv \mu, H \equiv J^rTB, E \equiv FB$. This system is almost involutive with respect to the subsheaf of integrable sections of J^rTB.

Proof. The evaluation morphism μ of the system is given by the mapping (1.2). It is easy to see that this system is projectable, linear, regular and canonical, [8]. From the property of natural bundles, [18],

$$\tilde{\mu}([j^r h, j^r k]) \equiv F[h,k] = [Fh, Fk] \equiv [\tilde{\mu}(h), \tilde{\mu}(k)] \tag{4.3}$$

for any local sections h, k of TB, we get that it is involutive with respect to the subsheaf of integrable sections of J^rTB. □

By generalization of Lemma 4.1 we introduce the following new notion.

Definition 4.1. An *infinitesimal natural lift* of order r is a fibred manifold $p : E \to B$ together with a system (J^rTB, μ) of vector fields of E which is linear, regular, canonical, projectable over $\pi^r_0 : J^rTB \to TB$ and almost involutive with respect to the subsheaf of integrable sections of $J^rTB \to TB$.

We shall say that a structure of *r-order infinitesimal natural lift* is given on E. The system (J^rTB, μ) will be said to be a *natural r-order system.*

We remark that in this definition we do not need that the fibred manifold $p : E \to B$ be a bundle.

Remark 4.2. The system $(H, \eta) \equiv (J^rTB, \mu)$ in the above Definition 4.1 is not assumed to be monic. For this reason we refer to the extra bracket in the subsheaf $\mathcal{J}^rTB \subset C^\infty(J^rTB)$ defined in Remark 4.1.

Example 4.1. If F is a natural lift functor of order r in the sense of Definition 1.1, then from Lemma 4.1 it follows that a structure of infinitesimal r-order lift bundle is induced on every FB, $B \in \mathrm{Ob}\,\mathcal{M}_n$.

Lemma 4.2. *If a natural system (J^rTB, μ) is given on a fibred manifold E, then the natural system $(J^{r+s}TB, j^s\mu)$ is induced on J^sE.*

Proof. This lemma follows from the properties of the jet prolongation and from the commutative diagram

$$\begin{array}{ccc} J^sJ^rTB \times_B J^sE & \xrightarrow{J^s\mu} & J^sTE \\ {\scriptstyle i^{r+s}\times \mathrm{id}}\uparrow & & \downarrow{\scriptstyle r^s} \\ J^{r+s}TB \times_B J^sE & \xrightarrow[j^s\mu]{} & TJ^sE \end{array}$$

where r^s is the fibred manifold mapping from J^sTE to TJ^sE defined in [13] and i^{r+s} is the canonical immersion of $J^{r+s}TB$ into J^sJ^rTB. □

Definition 4.2. Let E_1, E_2 be two fibred manifolds over B and assume that a structure of infinitesimal r-order natural lift is given on E_1 by a natural system (J^rTB, μ_1) and a structure of infinitesimal s-order natural lift is given on E_2 by a natural system (J^sTB, μ_2). A k-order operator D from $C^\infty E_1$ to $C^\infty E_2$ is said to be (*infinitesimally*) *natural* if

$$TD(\mathcal{L}_{j^r h}\sigma) = \mathcal{L}_{j^s h} D(\sigma),$$

for any section $\sigma : B \to E$ and any section $h : B \to TB$ (see Definition 3.1).

Lemma 4.3. *A k-order operator D from $C^\infty E_1$ to $C^\infty E_2$ is natural if and only if the distinguished vector fields $\widetilde{j^k\mu_1}(j^{r+k}h)$ of $J^{r+k}E_1$ and $\tilde{\mu}_2(j^s h)$ of E_2 are related by the associated fibred morphism $\mathcal{D} : J^k E_1 \to E_2$, i.e. if the following diagram*

$$\begin{array}{ccc} TJ^kE_1 & \xrightarrow{T\mathcal{D}} & TE_2 \\ {\scriptstyle \widetilde{j^k\mu_1}(j^{r+k}h)}\uparrow & & \uparrow{\scriptstyle \tilde{\mu}_2(j^s h)} \\ J^kE_1 & \xrightarrow[\mathcal{D}]{} & E_2 \end{array}$$

commutes for any section $h : B \to TB$.

Proof. From Lemma 4.2 we get that the distinguished vector field $\widetilde{j^k\mu_1}(j^{r+k}h)$ of $J^{r+k}E_1$ is given by $r^k \circ J^k\tilde{\mu}(j^r h)$. Then our Lemma 4.3 follows from the definition of the Lie derivative (3.16) and from the fact that $TD((T\sigma)\circ(h)) = T(D\sigma)\circ(h)$ for any section $h : B \to TB$. Indeed, we have

$$\begin{aligned} TD(\mathcal{L}_{j^r h}\sigma) &= TD(T\sigma \circ h - \tilde{\mu}_1(j^r h)\circ(\sigma)) \\ &= T(D\sigma)\circ(h) - TD(\tilde{\mu}_1(j^r h)\circ(\sigma)) \\ &= T(D\sigma)\circ(h) - T\mathcal{D}\circ(J^k(\tilde{\mu}_1(j^r h)\circ(\sigma))) \\ &= T(D\sigma)\circ(h) - T\mathcal{D}\circ(\widetilde{j^k\mu_1}(j^{r+k}h)\circ(j^k\sigma)) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{j^s h} D(\sigma) &= T(D\sigma)\circ(h) - \tilde{\mu}_2(j^s h)\circ(D\sigma) \\ &= T(D\sigma)\circ(h) - \tilde{\mu}_2(j^s h)\circ(\mathcal{D}\circ(j^k\sigma)). \end{aligned}$$ □

5. Infinitesimal gauge-natural lifts

Analogously to the case of infinitesimal natural lift we shall define infinitesimal gauge-natural lift by using the concept of systems. Our definition will be motivated by the following remark and lemma.

Remark 5.1. Let (H, η) be a strong system of vector fields on a fibred manifold $E \to B$. In Section 3 we have defined the bracket $[\cdot,\cdot]$ in $C^\infty H$ which makes $C^\infty H$ to be a sheaf of Lie algebras. This bracket can be prolonged to a bracket in the subsheaf $\mathcal{J}^r H \subset C^\infty J^r H$ of integrable sections of $J^r H \to B$ by

$$[j^r h, j^r k] \equiv j^r[h,k], \tag{5.1}$$

where h, k are local sections of H. From (3.6) we obtain the associated fibred morphism

$$[\cdot,\cdot] : J^{r+1}H \times J^{r+1}H \to J^r H. \tag{5.2}$$

The restriction of (5.2) to the subbundle $J^{r+1}H_0 \equiv \operatorname{Ker}(\pi_0^{r+1} \circ J^r\overline{\eta})$ factorizes through the canonical projection $J^{r+1}H_0 \to J^r H_0$ and defines a structure of Lie algebra bundle on $J^r H_0$. This Lie algebra bundle is locally a semidirect product of the Lie algebra bundles $J^r TB_0$ and $J^r A$.

Lemma 5.1. *Let F be an r-order gauge-natural bundle functor in the sense of Definition 2.1 defined on the category $\mathcal{PB}_n(G)$. The construction of the flow prolongation of G-invariant vector fields defines a projectable, linear, regular and canonical system*

$$\mu : J^r(TP/G) \times_B FP \to TFP$$

of $FP, (\pi : P \to B) \in \operatorname{Ob} \mathcal{PB}_n(G)$. This system is almost involutive with respect to the subsheaf of integrable sections of $J^r(TP/G) \to B$.

Proof. The evaluation morphism of the system $(J^r(TP/G), \mu)$ is given by the mapping (2.2). This system is projectable (over the mapping $T\pi \circ \pi_0^r$), linear, regular, canonical and almost involutive, [10]. □

The above results suggest the following new notion.

Definition 5.1. Let (H, η) be a strong system on $p : E \to B$. An *infinitesimal gauge-natural lift* of order r is a fibred manifold $p : E \to B$ together with a system $(J^r H, \mu)$ which is linear, regular, canonical, projectable over $(\pi_0^r \circ J^r\overline{\eta}) : J^r H \to TB$ and almost involutive with respect to the subsheaf of integrable sections of $J^r H \to B$.

We shall say that the system $(J^r H, \mu)$ defines a *structure of an infinitesimal gauge-natural lift of order r on E*. The system $(J^r H, \mu)$ will be called the *gauge-natural system.*

Let $s \leqslant r$ be the minimum number such that μ_A can be factorized through the canonical projection $\pi_s^r : J^r A \to J^s A$. Then we say that the infinitesimal gauge-natural lift is of order (r, s), and s is called the *gauge order* of the infinitesimal gauge-natural lift on E.

For any section $h : B \to H$ the distinguished vector field $\tilde{\mu}(j^r h) : E \to TE$ is induced and similarly for a section $h : B \to A$ the distinguished vertical vector field $\tilde{\mu}_A(j^r h) : E \to VE$ is induced. A structure of infinitesimal gauge-natural lift on E is said to be *gauge trivial* if its gauge order is 0 and $\tilde{\mu}_A(h)$ is the zero vector field for all sections $h : B \to A$.

Example 5.1. Let F be an r-order gauge-natural bundle functor in the sense of Definition 2.1 defined on the category $\mathcal{PB}_n(G)$, i.e. F can be represented by its standard fibre F_0 with the action of the group $W_n^r G = G_n^r \times_S T_n^r G$ on F_0. Then a structure of infinitesimal gauge-natural r-order lift is given on any $FP, P \in \mathrm{Ob}\,\mathcal{PB}_n(G)$, by means of the system $(J^r(TP/G), \mu)$ defined by Lemma 5.1. The adjoint bundle $\mathcal{W}_n^r\mathcal{G}(B) \to B$ is then isomorphic to $J^r(TP/G)_0 \to B$.

Example 5.2. Any infinitesimal natural lift of order r is an infinitesimal gauge-natural lift of the order $(r, 0)$ with the gauge trivial structure.

Example 5.3. Let a structure of (r, s)-order infinitesimal gauge-natural lift be given on E, then a structure of infinitesimal gauge-natural lift of order at most $(r + k, s + k)$ is induced on $J^k E$. The number $(r + k)$ is exact but the gauge order $(s + k)$ may be too big. For example, if a structure of infinitesimal natural r-order lift is given on E, i.e. a structure of infinitesimal gauge-natural lift of order $(r, 0)$ with the gauge trivial structure, then a structure of $(r + k)$-order infinitesimal natural lift, i.e. a structure of $(r + k, 0)$-order infinitesimal gauge-natural lift with the gauge trivial structure, is induced on $J^k E$.

Example 5.4. Let (H, η) be a strong system on $p : E \to B$. In Theorem 3.1 we have defined the canonical system (H, ζ_A) of vector fields on A. This system is linear, projectable, regular, canonical and almost involutive. So, we have the canonical structure of infinitesimal 0-order gauge-natural lift on A. The coordinate expression of ζ_A is

$$(x^\lambda, \mathfrak{z}^a, \dot{x}^\lambda, \dot{\mathfrak{z}}^a) \circ \zeta_A = (x^\lambda, \mathfrak{z}^a, z^\lambda, -C^a_{bc} z^b \mathfrak{z}^c).$$

Example 5.5. Let (H, η) be a strong system on $p : E \to B$. Let $A^r = A \otimes \bigwedge^r T^*B$. On A^r there is the $(1, 0)$-order structure of infinitesimal gauge-natural lift induced from the $(0, 0)$-order infinitesimal gauge-natural lift on A and $(1, 0)$-order infinitesimal gauge trivial lift on $\bigwedge^r T^*B$. The coordinate expression of the evaluation morphism μ^r is

$$\begin{aligned}(x^\lambda, \Phi^a_{\mu_1\dots\mu_r}, \dot{x}^\lambda, \dot{\Phi}^a_{\mu_1\dots\mu_r}) \circ \mu^r \\ = (x^\lambda, \Phi^a_{\mu_1\dots\mu_r}, z^\lambda, -r\Phi^a_{\rho[\mu_1\dots\mu_{r-1}} z^\rho_{\mu_r]} - C^a_{bc} z^b \Phi^c_{\mu_1\dots\mu_r}),\end{aligned} \tag{5.3}$$

where $(x^\lambda, \Phi^a_{\mu_1\dots\mu_r}, \dot{x}^\lambda, \dot{\Phi}^a_{\mu_1\dots\mu_r})$ is the induced fibred coordinate chart on TA^r.

Example 5.6. Let (H, η) be a strong system on $p : E \to B$. In Section 3 we have defined the strong system of connections (C, ξ) induced from a strong system of vector fields on a fibred manifold. In [14] a fibred morphism

$$\zeta_C : J^1 H \times_B C \to TC \tag{5.4}$$

is provided which turns out to be an infinitesimal gauge-natural lift on C. Let (x^λ, v^a_λ) be the induced fibred coordinate chart on C. Then the coordiante expression of ζ_C is

$$(x^\lambda, v^a_\lambda, \dot{x}^\lambda, \dot{v}^a_\lambda) \circ \zeta_C = (x^\lambda, v^a_\lambda, z^\lambda, z^a_\lambda - v^a_\rho z^\rho_\lambda - C^a_{bc} z^b v^c_\lambda). \tag{5.5}$$

Definition 5.2. Let E_1, E_2 be two fibred manifolds over B and let a structure of r-order infinitesimal gauge-natural lift is given on E_1 by a gauge-natural system $(J^r H, \mu_1)$ and a structure of s-order infinitesimal gauge-natural lift is given on E_2 by a gauge-natural system $(J^s H, \mu_2)$. A k-order operator D from $C^\infty E_1$ to $C^\infty E_2$ is said to be *natural* (respectively *gauge-natural*) if

$$TD(\mathcal{L}_{j^r h}\sigma) = \mathcal{L}_{j^s h} D\sigma, \tag{5.6}$$

for any section $\sigma : B \to E_1$ and any section $h : B \to H$ (respectively $h : B \to A$).

By using the same methods as in Lemma 4.3 we can prove

Lemma 5.2. *A k-order operator D from $C^\infty E_1$ to $C^\infty E_2$ is natural (resp. gauge-natural) if and only if the distinguished vector fields $\widetilde{j^k\mu_1}(j^{r+k}h)$ (resp. $\widetilde{j^k\mu_{1A}}(j^{r+k}h)$) of $J^k E_1$ and $\tilde{\mu}_2(j^s h)$ (resp. $\tilde{\mu}_{2A}(j^s h)$) of E_2 are related by the associated fibred morphism $\mathcal{D} : J^k E_1 \to E_2$, i.e. if the following diagrams*

$$\begin{array}{ccc}
TJ^k E_1 & \xrightarrow{T\mathcal{D}} & TE_2 \\
{\scriptstyle \widetilde{j^k\mu_1}(j^{k+r}h)}\uparrow & & \uparrow{\scriptstyle \tilde{\mu}_2(j^s h)} \\
J^k E_1 & \xrightarrow[\mathcal{D}]{} & E_2
\end{array}$$

resp.

$$\begin{array}{ccc}
VJ^k E_1 & \xrightarrow{V\mathcal{D}} & VE_2 \\
{\scriptstyle \widetilde{j^k\mu_{1A}}(j^{r+k}h)}\uparrow & & \uparrow{\scriptstyle \tilde{\mu}_{2A}(j^s h)} \\
J^k E_1 & \xrightarrow[\mathcal{D}]{} & E_2
\end{array}$$

commute for any section $h : B \to H$ (resp. $h : B \to A$).

Lemma 5.3. *The strong curvature form is a first order natural operator from $C^\infty C$ to $C^\infty A^2$.*

Proof. The $(1,1)$-order infinitesimal gauge-natural structure on C implies the $(2,2)$-order infinitesimal gauge-natural structure on $J^1 C$ which is given by the jet prolongation of the evaluation morphism (5.4). In the induced coordinate chart

$$(x^\lambda, v^a_\lambda, v^a_{\lambda,\mu}, \dot{x}^\lambda, \dot{v}^a_\lambda, \dot{v}^a_{\lambda,\mu})$$

on $TJ^1 C$ this prolonged evaluation morphism is given by

$$\begin{aligned}
&(x^\lambda, v^a_\lambda, v^a_{\lambda,\mu}, \dot{x}^\lambda, \dot{v}^a_\lambda, \dot{v}^a_{\lambda,\mu}) \circ j^1\zeta_C \\
&\quad = (x^\lambda, v^a_\lambda, v^a_{\lambda,\mu}, z^\lambda, z^a_\lambda - v^a_\rho z^\rho_\lambda - C^a_{bc} z^b v^c_\lambda, \\
&\qquad\quad z^a_{\lambda\mu} - v^a_{\rho,\mu} z^\rho_\lambda - v^a_{\lambda,\rho} z^\rho_\mu - v^a_\rho z^\rho_{\lambda\mu} - C^a_{bc} z^b v^c_{\lambda,\mu} - C^a_{bc} z^b_\mu v^c_\lambda).
\end{aligned} \tag{5.7}$$

For any local section h of H we get from (5.7) the vector field $\widetilde{j^1\zeta_C}(j^2h)$ of J^1C

$$\begin{aligned}\widetilde{j^1\zeta_C}(j^2h) = {} & h^\lambda \frac{\partial}{\partial x^\lambda} + (\partial_\lambda h^a - v^a_\rho \partial_\lambda h^\rho - C^a_{bc} h^b v^c_\lambda)\frac{\partial}{\partial v^a_\lambda} \\ & + (\partial_{\lambda\mu} h^a - v^a_{\rho,\mu}\partial_\lambda h^\rho - v^a_{\lambda,\rho}\partial_\mu h^\rho - v^a_\rho \partial_{\lambda\mu} h^\rho \\ & - C^a_{bc} h^b v^c_{\lambda,\mu} - C^a_{bc}\partial_\mu h^b v^c_\lambda)\frac{\partial}{\partial v^a_{\lambda,\mu}}\end{aligned}$$

and from (5.3) we get the vector field $\tilde{\mu}^2(j^1h)$ of A^2

$$\tilde{\mu}^2(j^1h) = h^\lambda \frac{\partial}{\partial x^\lambda} + (-\Phi^a_{\lambda\rho}\partial_\mu h^\rho - \Phi^a_{\rho\mu}\partial_\lambda h^\rho - C^a_{bc} h^b \Phi^c_{\lambda_1\lambda_2})\frac{\partial}{\partial \Phi^a_{\lambda_1\lambda_2}}.$$

Then the diagram

$$\begin{array}{ccc} TJ^1C & \xrightarrow{T\Omega} & TA^2 \\ {\scriptstyle \widetilde{j^1\zeta_C}(j^2h)}\uparrow & & \uparrow{\scriptstyle \tilde{\mu}^2(j^1h)} \\ J^1C & \xrightarrow[\Omega]{} & A^2 \end{array}$$

commutes for all sections $h : B \to H$, where $\Omega : J^1C \to A^2$ is the induced fibred manifold morphism corresponding to the strong curvature operator. From (3.15) we get its coordinate expression

$$(x^\lambda, \Phi^a_{\lambda\mu}) \circ \Omega = (x^\lambda, \tfrac{1}{2}(v^a_{\lambda,\mu} - v^a_{\mu,\lambda} + C^a_{bc} v^b_\lambda v^c_\mu)). \quad \square$$

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