Gabriel–Popescu type theorems and applications

Un analogue du théorème de Gabriel–Popescu et applications

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Abstract

In this paper we obtain a general version of Gabriel–Popescu theorem representing any Grothendieck category $\mathcal{A}$ as a quotient category of the category of modules over a ring (not necessarily with unit) with enough idempotents to right using a family of generators $(U_i)_{i \in I}$ of $\mathcal{A}$ where $U_i$ are not supposed to be small. Applications to locally finite categories are obtained. In particular, for a coalgebra $C$ (over a field) we prove that $C$ is right semiperfect if and only if the category $\mathcal{M}^C$ has the AB4* condition.

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Résumé

Dans ce travail nous établissons une version généralisée du théorème de Gabriel–Popescu en représentant toute catégorie de Grothendieck $\mathcal{A}$ comme un quotient d’une catégorie de modules sur un anneau (non nécessairement unitaire) ayant assez d’idempotents à droite, en utilisant une famille de générateurs $(U_i)_{i \in I}$ de $\mathcal{A}$; où les $U_i$ ne sont pas supposés superflus (small). Des applications aux catégories localement finies sont obtenues. En particulier, pour une coalgèbre $C$ (sur un corps) on montre que $C$ est semiparfaite à droite si et seulement si la catégorie $\mathcal{M}^C$ possède la condition AB4*.

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Introduction

Throughout this paper $R$ will denote an associative ring (not necessarily with unit) and $\text{Mod-R}$ will denote the category of all right $R$-modules and morphisms being given by $R$-modules maps. We say that $R$ has enough idempotents to right, if there exists a family $\{e_i\}_{i \in I}$ of pairwise orthogonal non-zero idempotents $e_i \in R$ with $R = \bigoplus_{i \in I} Re_i$. If $X$ is a finite subset of $R$, then there is an idempotent $e \in R$ such that $x = xe$ for every $x \in X$. Indeed, for $x = \sum_{i=1}^n r_i^j e_i^j$ we put $A_x = \bigcup_{i \leq n} \{e_i^j, 1 \leq i \leq n\}$ and $e = \sum_{f \in A_x} f$. Clearly $e$ is an idempotent. Since $x e_i^j = r_i^j e_i^j$ and $x f = 0$ when $f \notin \{e_i^j, 1 \leq i \leq n\}$ we have $x = xe$ for any $x \in X$.

For a ring $R$ with enough idempotents to right we denote by $\text{MOD-R}$ the subcategory the all right $R$-modules with the property $M R = M$. Recall that if $C$ is a non-empty class of objects of a Grothendieck category $A$, then $C$ is called a localizing subcategory [2] of $A$ if is closed under subobjects, quotient objects, extensions and arbitrary direct sums. If moreover, $C$ is closed under direct products, then the localizing subcategory $C$ is called $TTF$-class.

**Proposition 0.1.** Let $R$ be a ring with enough idempotents to right. Then

(i) The class $\text{MOD-R}$ is a localizing subcategory of $\text{Mod-R}$. In particular, $\text{MOD-R}$ is a Grothendieck category. Moreover, if $(M_i)_{i \in I}$ is a family of modules $M_i \in \text{MOD-R}$, then the direct product in $\text{MOD-R}$ is $\prod_{i \in I} M_i = (\prod_{i \in I} M_i)_R$, where $\prod_{i \in I} M_i$ denoting the direct product in $\text{Mod-R}$.

(ii) For every $M \in \text{MOD-R}$, the canonical morphism $\alpha : M \otimes_R R \to M$ defined by $\sum_{i=1}^n m_i \otimes r_i \mapsto \sum_{i=1}^n m_i r_i$, is an isomorphism.

**Proof.** (i) Let

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

an exact sequence in $\text{Mod-R}$. Assume that $M \in \text{MOD-R}$. Clearly $M'' R = M''$, so $M'' \in \text{MOD-R}$. Likewise, for $m' \in M'$, $m' = m' e$ which implies that $M' \in \text{MOD-R}$. Conversely, let $m \in M$. Then $v(m) = v(m)e = v(me)$. So $m - me \in M'$. Therefore, there exist an idempotent $e'$ such that $m - me = (m - me)e'$. This implies that $m = me + me' - mee' = m(e + e' - ee')$. Then exist an idempotent $e''$ such that $e + e' - ee' = (e + e' - ee')e''$. So $m = me''$ and $M \in \text{MOD-R}$.

Clearly, $\text{MOD-R}$ is closed under direct sums.

(ii) $\alpha$ is surjective so $MR = M$. Assume that $\alpha(\sum_{i=1}^n m_i \otimes r_i) = \sum_{i=1}^n m_i r_i = 0$. For $1 \leq i \leq n$ there exist a right idempotent $e \in R$ such that $r_i = re_i$. Then $\sum_{i=1}^n m_i \otimes r_i = \sum_{i=1}^n m_i \otimes r_i e = \sum_{i=1}^n m_i r_i \otimes e = 0$. Hence, $\alpha$ is injective. \[\square\]
The Gabriel–Popescu theorem [3] states essentially that any Grothendieck category $A$ is equivalent to a quotient category of the category of modules over the endomorphism ring of a generator of $A$. Recall that a quick proof of this theorem was found by Mitchell [9]. For a systems of small generators $(U_i)_{i \in I}$ of $A$, Menini [8] showed that if $U = \bigoplus_{i \in I} U_i$ and $\text{Hom}_A(U, X)$ consists of the morphisms which vanish on almost all the $U_i$, then $S = \text{Hom}_A(U, -): A \rightarrow \text{Mod-} \overline{A}$ has an exact left adjoint $T$ such that the adjunction $TS \rightarrow 1_C$ is an isomorphism. Therefore, $S$ induces an equivalence between $A$ and the quotient category $\text{Mod-} \overline{A}$ by $\ker T$, where $\overline{A} = \text{Hom}_A(U, U)$.

The aim of our paper is to obtain a Gabriel–Popescu type theorem when the $U_i$ are not small. In Section 2, using the classical Gabriel–Popescu theorem, a very elegant and simple proof of our version is obtained (Theorem 2.2). We also construct projective objects in quotient categories. These results are applied to locally finite Grothendieck categories. Next, for a coalgebra $C$ (over a field) we prove that $C$ is right semiperfect if and only if the category $M^C$ has the AB4* condition.

1. The ring associated to a family of objects

Let $(U_i)_{i \in I}$ be a family of objects of a Grothendieck category $A$. Put $U = \bigoplus_{i \in I} U_i$ and let $A = \text{End}_A(U)$ be its endomorphism ring with multiplication given by the composition. Following [8], we can consider a family $(\eta_i)_{i \in I}$ of orthogonal idempotents of $A$ where for every $i \in I$, $\eta_i: U \rightarrow U$ is defined by $\eta_i = \epsilon_i \circ \pi_i$, with $\epsilon_i$ and $\pi_i$ the canonical injection and projection respectively. For every object $X \in A$ and for every $f \in \text{Hom}_A(U, X)$, we can consider the set

$$
\overline{\text{Hom}}_A(U, X) = \{ f \in \text{Hom}_A(U, X) \mid \text{Supp}(f) < \infty \},
$$

where Supp$(f) = \{ i \in I \mid f \circ \eta_i \neq 0 \}$. Let $R = \overline{\text{Hom}}_A(U, U)$ denote the endomorphism ring of $U$. By [8, Proposition 2.2], $R$ is a left idempotent ideal of $A$ and $R = \bigoplus_{i \in I} R \eta_i$. So $R$ is a ring with enough idempotents to right. Also $\overline{\text{Hom}}_A(U, X) R = \text{Hom}_A(U, X) R$. Moreover (to see [8]) if each $U_i$ is small, then $R = \bigoplus_{i \in I} R \eta_i = \bigoplus_{i \in I} \eta_i R$, i.e. $R$ is a ring with enough idempotents.

Between the categories $\text{MOD-R}$ and $\text{Mod-A}$ we have the pair of functors

$$
- \overline{\otimes}_R A: \text{MOD-R} \rightleftarrows \text{Mod-A}: t
$$

where $t$ is defined by $t(N) = NR$ for every $N \in \text{Mod-A}$.

**Proposition 1.1.** With the above notations, the functor $- \overline{\otimes}_R A$ is a left adjoint to $t$ functor. Moreover $t \circ (- \overline{\otimes}_R A) \simeq 1_{\text{MOD-R}}$ and $t$ is an exact functor.

**Proof.** Let $M \in \text{MOD-R}$ and $N \in \text{Mod-A}$. We define the applications

$$
\alpha: \text{Hom}_A(M \overline{\otimes}_R A, N) \cong \text{Hom}_R(M, NR): \beta
$$

$\alpha(f)(m) = f(m \otimes 1)$ and $\beta(g)(m \overline{\otimes} a) = g(m)a$, where $m \in M$, $a \in A$, $f \in \text{Hom}_A(M \overline{\otimes}_R A, N)$ and $g \in \text{Hom}_R(M, NR)$. Clearly $\alpha(f)$ is a $R$-linear map and $\beta(g)$ is an $A$-linear map. Moreover, $(\beta \circ \alpha)(f)(m \overline{\otimes} a) = \alpha(f)(m)a = f(m \otimes 1)a = f((m \otimes 1)a) = \beta(g)(m \overline{\otimes} a)$.
Thus \( \beta \circ \alpha = 1_{\text{Hom}(M \otimes_R A, N)} \). Analogously, \((\alpha \circ \beta)(g)(m) = \alpha(\beta(g))(m) = \beta(g)(m \otimes 1) = g(m)1 = g(m)\). So \( \alpha \circ \beta = 1_{\text{Hom}(N, R)} \).

Now, \( t(M \otimes_R A) = (M \otimes_R A)R = M \otimes_R AR = M \otimes_R R \cong M \). Hence, \( t \circ (\otimes_R A) \cong 1_{\text{MOD}-R} \). That \( t \) is an exact functor result immediately. □

2. A version of Gabriel–Popescu theorem

Let \( A \) be a Grothendieck category and \((U_i)_{i \in I}\) a family of generators of \( A \). Then \( U = \bigoplus_{i \in I} U_i \) is a generator of \( A \). If we consider the canonical functor

\[
S = \text{Hom}_A(U, -) : A \to \text{Mod}-A,
\]

where \( A = \text{End}_A(U) \) denote the endomorphism ring with \( T \) its left adjoint functor, then the classical Gabriel–Popescu theorem states that \( T \) is an exact functor and \( T \circ S \cong 1_A \) (to see [10]). This implies that \( S \) is a faithful functor and \( T \) induces an equivalence between \( A \) and the quotient category \( \text{Mod}-A/\ker T \), where \( \ker T = \{ M \in \text{Mod}-A \mid T(M) = 0 \} \) is a localizing subcategory.

We now denote by \( F_T \) the Gabriel topology associated to \( \ker T \), i.e.

\[
F_T = \{ I \text{ right ideal of } A \mid A/I \in \ker T \}.
\]

If \( I \leq A \) is a right ideal of \( A \), then we have the canonical morphism \( \theta : U(I) \to U \), where \( \text{Im}(\theta) = \sum_{f \in I} \text{Im}(f) \). If we denote by \( IU = \text{Im}(\theta) \), then it is well known that

\[
F_T = \{ I \text{ right ideal of } A \mid IU = U \}.
\]

We observe the following simple, but useful, fact.

**Lemma 2.1.** The right ideal \( RA \in F_T \). In particular, if \( X(RA) = 0 \) for \( X \in \text{Mod}-A \), then \( X \in \ker T \).

**Proof.** Clearly \( RA \) is a right ideal of \( A \). In fact, is a two-sided ideal. Since \( \eta_i \in RA \) and \( \text{Im}(\eta_i) = U_i \) for any \( i \in I \), this implies that \( (RA)U = U \). □

Consider now the sequence of categories and adjoint functors

\[
A \xrightarrow{S} \text{Mod}-A \xleftarrow{t} \text{MOD}-R.
\]

In this situation, we give a version of Gabriel–Popescu theorem between the category \( A \) and the category \( \text{MOD}-R \). It is the main result from this section.

**Theorem 2.2.** Let \( A \) be a Grothendieck category and \((U_i)_{i \in I}\) a family of generators of \( A \). If \( R \) is the ring associated to family \((U_i)_{i \in I}\), then the following assertions hold.

(a) \( T' = T \circ (\otimes_R A) \) is a left adjoint of the functor \( S' = t \circ S \).
(b) \( T' \) is an exact functor.
(c) \( T' \circ S' \cong 1_A \).
(d) $\mathcal{A}$ is equivalent to the quotient category $\text{MOD-R}/\ker T'$ via the functor $T'$.

**Proof.** (a) This is a direct consequence of [7, Theorem 1, p. 101].

(b) Let
$$0 \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$$
an exact sequence in $\text{MOD-R}$. Apply the functor $- \otimes_R A$ to obtain the exact sequence
$$M' \otimes_R A \xrightarrow{u \otimes A} M \otimes_R A \xrightarrow{v \otimes A} M'' \otimes_A R \to 0.$$If we put $X = \ker (u \otimes A)$, then using Proposition 1.1,
$$t(X) = t(\ker (u \otimes A)) \cong t(\ker (u)) = 0.$$Thus $XR = 0$ or $X(RA) = 0$. Since $RA \in \ker T$ we have $X \in \ker T$, so $T(X) = 0$. Therefore, the functor $T'$ is exact.

(c) Let $X$ be a object of $\mathcal{A}$. We consider the canonical morphism of $A$-modules
$$\gamma : S(X)R \otimes_R A \to S(X), \quad \gamma(x \otimes a) \mapsto xa,$$where $x \in S(X)R$ and $a \in A$. If we apply the functor $t$ to exact sequence
$$0 \to \ker (\gamma) \to S(X)R \otimes_R A \xrightarrow{\gamma} S(X) \to \coker (\gamma) \to 0$$we obtain that $t(\ker (\gamma)) = t(\coker (\gamma)) = 0$. This implies that $\ker (\gamma)R = \coker (\gamma)R = 0$ and hence $\ker (\gamma)(RA) = \coker (\gamma)(RA) = 0$. Therefore, $\ker (\gamma), \coker (\gamma) \in \ker T$. Finally, $T'(X) \cong T(\ker (\gamma)) \cong T(S(X)) \cong X$.

(d) It follows of Proposition 5 in [10, p. 374]. ☐

**Corollary 2.3** (Menini [8]). If $(U_i)_{i \in I}$ is a family of small generators of $\mathcal{A}$, then the functor $S'$ commutes with coproducts. If in addition every $U_i$ is projective, the functor $S'$ induces an equivalence between the categories $\mathcal{A}$ and $\text{MOD-R}$ with functor inverse $T'$ where $R$ has enough idempotents.

**Lemma 2.4.** Let $M$ be a simple $A$-module. Then either $t(M) = 0$ or $t(M)$ is a simple object in $\text{MOD-R}$. Moreover $M \in \ker T$ whenever $t(M) = 0$ and if $t(M) \neq 0$, then $t(M) \notin \ker T'$.

**Proof.** If $t(M) = MR = 0$, then $M(RA) = 0$ and $M \in \ker T$ by Lemma 2.1. Assume now that $t(M) \neq 0$ and put $X = MR$. If $Y$ is a non-zero subobject of $X$, then $0 \neq YA \subset MRA \subset M$ and so $M = YA$. Therefore, $MR = YAR = YR = Y$, i.e., $Y = X$. Suppose that $MR \in \ker T'$. Then $0 = T'(MR) = T(MR \otimes R A)$. So $MR \otimes_R A \in \ker T$. Since there exists a non-zero canonical epimorphism $\theta : MR \otimes_R A \to M$ defined by $\theta (mr \otimes a) = (mr)a$, this implies that $M \in \ker T$ which yields a contradiction. ☐

Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{C}$ be a localizing subcategory. From [2] it can be defined the quotient category $\mathcal{A}/\mathcal{C}$ and canonical functors $T : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ and $S : \mathcal{A}/\mathcal{C} \to \mathcal{A}$.
such that $T$ is an exact functor and $S$ is a right adjoint functor to $T$. If $\psi : 1_A \to S \circ T$ is the unit of the adjunction, then $\psi_M : M \to (S \circ T)(M)$ has $C$-torsion kernel and cokernel for every object $M$ of $A$. Moreover, the counit $\phi : T \circ S \to 1_{A/C}$ is a natural isomorphism. We also recall that a projective cover of an object $M \in A$ is an epimorphism $\xi : P \to M$, where $P$ is a object projective in $A$ and $\ker(\alpha)$ is small in $P$ (see [5]). An object $X$ of $A$ is called \textit{finitely generated} if given $X = \bigcup_{i \in I} X_i$ for some subobjects $X_i$ of $X$, then $X = \bigcup_{j \in J} X_j$ for a finite subset $J$ of $I$.

**Proposition 2.5.** Let $C$ be a localizing subcategory of a Grothendieck category $A$.

(a) If $X$ is a non-zero simple object of $A/C$ and $C$ is a TTF class, then there exists a non-zero simple object $M \in A$ such that $T(M) \simeq X$.

(b) Let $M$ be a non-zero simple object of $A$ with $M \notin C$. If $P \xrightarrow{\xi} M$ is a projective cover of $M$, then $T(P)$ is a non-zero projective object in $A/C$. In this case, $P$ and $T(P)$ are finitely generated objects in $A$ and $A/C$ respectively.

**Proof.** (a) Because $S$ is a faithful functor, $S(X) \neq 0$. Since $C$ is a TTF-class, there exists a non-zero subobject $M \subseteq S(X)$, where $M = \bigcap_{Y \subseteq S(X)} Y$ with the property $S(X)/Y \in C$. If $Y \neq 0$, $Y \subseteq S(X)$, then $T(Y) \neq 0$ and $T(Y) \subseteq T(S(X)) \simeq X$. Because $X$ is a simple object result that $T(Y) = TS(X)$ and $S(X)/Y \in C$. Therefore, $M$ is a simple object of $A$. Finally, $T(M) \simeq T(S(X)) \simeq X$.

(b) We begin by showing that for every $Y \in C$, $\text{Hom}_A(P, Y) = 0$. Suppose that the statement is false. Then there exists a morphism $f : P \to Y$, ($f \neq 0$) with $\ker(f) \subseteq P$. If $\ker(f) \subseteq \ker(\xi)$, because $\ker(\xi)$ is a maximal subobject of $P$, then $\ker(f) + \ker(\xi) = P$. Hence, $\ker(f) = P$, as contradiction. This implies that $\ker(f) \subset \ker(\xi)$ and we have the epimorphism $P/\ker(f) \to P/\ker(\xi) = M$. But $P/\ker(f) \simeq \text{Im}(f) \subseteq Y$. Then $P/\ker(f) \in C$ and hence, $M \in C$. This is a contradiction.

We now prove that $T(P)$ is a projective object. Consider the diagram in $A/C$

$$
\begin{array}{c}
T(P) \\
\downarrow g \\
X \xrightarrow{u} X' \xrightarrow{0}
\end{array}
$$

Applying the functor $S$ we obtain

$$
P \xrightarrow{\psi_P} ST(P) \xrightarrow{S(g)} S(X) \xrightarrow{S(u)} S(X') \xrightarrow{\pi} \text{coker}(S(u)) \quad (1)
$$

where $\text{coker}(S(u)) \in C$. Since $\text{Hom}_A(P, Y) = 0$ for every $Y \in C$, $\pi \circ S(g) \circ \psi_P = 0$. So $(S(g) \circ \psi_P)(P) \subseteq \text{Im}(S(u))$. As $P$ is projective, there exists a morphism $f : P \to S(X)$ such that $S(g) \circ \psi_P = S(u) \circ f$. 

Applying now the functor \( T \) to commutative diagram (1) we obtain the commutative diagram

\[
\begin{array}{ccccc}
T(P) & \xrightarrow{T\phi_P} & TST(P) & \xrightarrow{\phi T(P)} & T(P) \\
\downarrow{T(f)} & & \downarrow{T\phi} & & \downarrow{g} \\
TST(X) & \xrightarrow{TST(u)} & TST(X') & \xrightarrow{\phi X'} & X'
\end{array}
\]

\[
\begin{array}{ccccc}
X & \xrightarrow{u} & X' & \xrightarrow{1_{X'}} & X'
\end{array}
\]

Since \( \phi T(P) \circ T\phi_P = 1_{T(P)} \), using the commutativity of the two upper square, \( g = \phi_X' \circ TS(u) \circ T(f) \). But \( \phi_X' \circ TS(u) = u \circ \phi_X \). Then \( g = u \circ \phi_X \circ T(f) \) and \( T(P) \) is projective (because \( T(\psi_P) \) is an isomorphism).

Finally, we prove that \( P \) and \( T(P) \) are finitely generated objects. Let \( P \xrightarrow{\xi} M \) a projective cover of \( M \) with \( P = \bigcup_{i \in I} P_i \) where \( (P_i)_{i \in I} \) is a direct family of subobjects of \( P \). Then \( M = \xi(P) = \xi(\bigcup_{i \in I} P_i) = \bigcup_{i \in I} \xi(P_i) \). Since \( M \) is simple, there exists \( i_0 \in I \) such that \( M = \xi(P_{i_0}) \). Then \( \ker(\xi) + P_{i_0} = P \). Because \( \ker(\xi) \) is small in \( P \), then \( P = P_{i_0} \) and hence, \( P \) is finitely generated in \( A \). Let now \( T(P) = \bigcup_{i \in I} X_i \) where \( (X_i)_{i \in I} \) is a direct family. Then \( ST(P) = S(\bigcup_{i \in I} X_i) \). We consider the canonical morphism

\[
\bigcup_{i \in I} S(X_i) \xrightarrow{\alpha} S\left( \bigcup_{i \in I} X_i \right) = ST(P).
\]

Since \( T \) commute with direct limits, \( T(\alpha) \) is an isomorphism. Thus, \( \operatorname{coker}(\alpha) \in C \). Therefore the composition morphism

\[
P \xrightarrow{\psi_P} ST(P) \xrightarrow{\pi} \operatorname{coker}(\alpha)
\]

is zero. Hence, \( \operatorname{Im}(\psi_P) \subset \bigcup_{i \in I} S(X_i) \) and \( P = \bigcup_{i \in I} \psi_P^{-1}(S(X_i)) \). On the other hand, since \( P \) is finitely generated, there exists \( X_{i_0} \) such that \( P = \psi_P^{-1}(S(X_{i_0})) \). We now consider the exact sequence

\[
0 \to \ker(\psi_P|_{S(X_{i_0})}) \to \psi_P^{-1}(S(X_{i_0})) \xrightarrow{\psi_P|_{S(X_{i_0})}} S(X_{i_0}) \to \operatorname{coker}(\psi_P|_{S(X_{i_0})}) \to 0
\]

where \( \ker(\psi_P|_{S(X_{i_0})}), \operatorname{coker}(\psi_P|_{S(X_{i_0})}) \in C \). Then \( T(\psi_P|_{S(X_{i_0})}) \) is an isomorphism. Hence, \( T(P) \simeq T(S(X_{i_0})) \simeq X_{i_0} \). So \( T(P) \) is finitely generated. \( \square \)

**Remark 2.6.** In general, if \( M \) is a finitely generated and projective object in \( A \) does not true that \( T(M) \) is finitely generated in \( A/C \). This follows immediately from Gabriel–Popescu theorem.

3. Applications to locally finite categories

Recall that a Grothendieck category \( A \) is said to have *enough projective* if it has a family of projective generators. \( A \) is called *semiperfect* if every finitely generated object in \( A \) has
a projective cover and the category $\mathcal{A}$ has the AB4* condition if the direct products in $\mathcal{A}$ are exacts. Finally, $\mathcal{A}$ is locally finite if $\mathcal{A}$ has a generating family $(U_i)_{i \in I}$ of objects of finite length. Clearly each $U_i$ is a small object in $\mathcal{A}$. In this case, we can consider the generator $U = \bigoplus_{i \in I} U_i$ of $\mathcal{A}$ and its endomorphism ring $A = \text{End}_\mathcal{A}(U)$. From Section 2, the ring $R = \bigoplus_{i \in I} R_{\eta_i} = \bigoplus_{i \in I} \eta_i R$, i.e., $R$ has enough idempotents. On the other hand, for every $i \in I$, $U_i$ is of finite length. Then the ring $\text{End}_\mathcal{A}(U_i) \simeq \eta_i R_{\eta_i}$ is semi-primary. Therefore, $R$ is a semiperfect ring (left and right) and hence, $\text{MOD-R}$ is a semiperfect category [13, 49.10].

**Theorem 3.1.** The following conditions are equivalent for a locally finite Grothendieck category $\mathcal{A}$.

(a) $\mathcal{A}$ has enough projective objects.

(b) $\mathcal{A}$ has the AB4* condition.

(c) $\mathcal{A}$ is a semiperfect category.

**Proof.** (a) ⇒ (b) It is well-known. The proof that we give here it’s just routine. Consider the exact sequence in $\mathcal{A}$

$$M_i \xrightarrow{\eta_i} M_i' \rightarrow 0.$$

We will prove that the canonical map $\prod \eta_i : \prod_{i \in I} M_i \rightarrow \prod_{i \in I} M_i'$ is an epimorphism. We consider the commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{f} & \prod M_i' \\
\downarrow{g} & & \downarrow{\Pi'} \\
\Pi M_i & \xrightarrow{\Pi \eta_i} & \prod M_i'
\end{array}
$$

where $f$ is an epimorphism $P$ is a projective object of $\mathcal{A}$. Then there exists $g_i : P \rightarrow M_i$ such that $\Pi' \circ f = \eta_i \circ g_i$. Thus there exists a morphism $g : P \rightarrow \Pi M_i$ such that $g_i = \Pi \circ g$ for every $i \in I$. Clearly $f = (\Pi \eta_i) \circ g$ and hence, $\Pi \eta_i$ is an epimorphism.

(b) ⇒ (a) This implication is indicated in [12] but without doing proof. More exactly, he say that the implication result via homological considerations. From [12], any Grothendieck category locally noetherian has the AB6 condition. In particular, $\mathcal{A}$ verify the AB6 condition. With the notation of Theorem 2.2, $\mathcal{A}$ has a family of generators $(U_i)_{i \in I}$ of finite length. Then by [10, Theorem 21.6] or by [11] the localizing subcategory $\ker T$ is a TTF class. Let $X \in \mathcal{A}$ a simple object. By Proposition 2.5(a), there exists a simple object $N \in \text{Mod-A}$ with $N \notin \ker T$ and $X \simeq T(N)$. By Lemma 2.4, if $M = t(N) = NR$, then $M \notin \ker T'$ and $M$ is a simple object in $\text{MOD-R}$. From exact sequence $\gamma : M \otimes_R A \rightarrow N$...
defined by \( \gamma(m \otimes a) = ma \), since \( N \) is simple, \( \gamma \) is an epimorphism and \( \ker(\gamma) \in \ker T \). So \( T'(M) \simeq T(N) \simeq X \). Because MOD-R is a semiperfect category there exist a projective cover for \( M \). Let \( P \rightarrow M \rightarrow 0 \) this projective cover. By Proposition 2.5(b), we have \( T'(P) \rightarrow T'(M) \rightarrow X \rightarrow 0 \) where \( T'(P) \) is a finitely generated and projective object in \( A \). From this, if \( X \) has finite length, then there exists in \( A \) a projective object \( Y \) and an epimorphism \( Y \rightarrow X \rightarrow 0 \). Thus \( A \) has a family of projective generators. So \( A \) has enough projective objects.

(c) \( \Rightarrow \) (a) It is clear.

(a) \( \Rightarrow \) (c) Assume that \( A \) has a family of projective generators of finite length. Then by Corollary 2.3, \( A \) is equivalent to the semiperfect category MOD-R. So \( A \) is semiperfect (see also [5] and [6]). \( \square \)

3.1. Application to coalgebras

Let \( C \) be a coalgebra over a field \( k \) and \( M^C \) denote the Grothendieck category of all right \( C \)-comodules. The coalgebra \( C \) is called right semiperfect when \( M^C \) has enough projectives. The category \( M^C \) is locally finite since it is generated by finite dimensional comodules. The dual space \( C^* = \text{Hom}_k(C, k) \) is endowed naturally of a structure of a \( k \)-algebra such that every right \( C \)-comodules can be viewed as a left \( C^* \)-modules. These left \( C^* \)-modules are called rational left \( C^* \)-modules. In fact, we can define a functor \( \text{Rat}(-) : C^*-\text{Mod} \rightarrow M^C \) where for each \( M \in C^*-\text{Mod} \), \( \text{Rat}(M) \) is the unique maximal rational submodule of \( M \). We refere the reader to [1] for all results and notions on coalgebras.

Corollary 3.2. Let \( C \) be a \( k \)-coalgebra. The following assertion are equivalent.

(a) \( C \) is a right semiperfect coalgebra.
(b) The functor \( \text{Rat}(-) : C^*-\text{Mod} \rightarrow M^C \) is exact.
(c) \( M^C \) has the AB4* condition.

Proof. The equivalence (a) \( \Leftrightarrow \) (b) is given in [4, Proposition 2.2, Theorem 3.3].

(c) \( \Leftrightarrow \) (a) By Theorem 3.1. \( \square \)

References