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Cellular automata and strongly irreducible shifts of finite type

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Abstract

If A is a finite alphabet and Γ is a finitely generated amenable group, Ceccherini-Silberstein, Machì and Scarabotti have proved that a local transition function defined on the full shift A^Γ is surjective if and only if it is pre-injective; this equivalence is the so-called *Garden of Eden* theorem. On the other hand, when Γ is the group of the integers, the theorem holds in the case of irreducible shifts of finite type as a consequence of a theorem of Lind and Marcus but it no longer holds in the two-dimensional case.

Recently, Gromov has proved a GOE-like theorem in the much more general framework of the spaces of bounded propagation. In this paper we apply Gromov's theorem to our class of spaces proving that all the properties required in the hypotheses of this theorem are satisfied.

We give a definition of strong irreducibility that, together with the finite-type condition, it allows us to prove the GOE theorem for the strongly irreducible shifts of finite type in A^Γ (provided that Γ is amenable). Finally, we prove that the bounded propagation property for a shift is strictly stronger than the union of strong irreducibility and finite-type condition.

1. Introduction

A *cellular automaton* (*CA*) is given by the set A^Γ of all functions defined on (the Cayley graph of) a finitely generated group Γ with values in a finite *alphabet* A and by a *transition function* $\tau : A^\Gamma \to A^\Gamma$ which is *local* (i.e. the value of $\tau(c)$, where $c \in A^\Gamma$ is a *configuration*, at a point $\gamma \in \Gamma$ only depends on the values of c at the points of a fixed

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neighborhood of γ). The *Garden of Eden* (*GOE*) theorem, as proved in [1] (see also [7]), states that if Γ is a finitely generated amenable group, then the local transition function of a CA has a *GOE pattern* (i.e. a configuration with finite support that has no pre-image under τ), if and only if it has two *mutually erasable patterns* (that is, a sort of non-injectivity of the transition function on the "finite" configurations). This theorem is a generalization of the theorem that Moore [8] and Myhill [9] proved in the case $\Gamma = \mathbb{Z}^2$.

Instead of the non-existence of mutually erasable patterns, we deal with the notion of *pre-injectivity* (a function $\tau : X \subseteq A^\Gamma \to A^\Gamma$ is *pre-injective* if whenever two configurations $c, \bar{c} \in X$ differ only on a finite non-empty subset of Γ, then $\tau(c) \neq \tau(\bar{c})$); this notion has been introduced by Gromov [5]. In fact, these two properties are equivalent for local functions defined on the full shift, but in the case of proper subshifts the former may be meaningless. On the other hand, the non-existence of GOE patterns is equivalent to the non-existence of GOE configurations, that is to the surjectivity of the transition function. Hence, in this language, the GOE theorem states that τ is surjective if and only if it is pre-injective. We call *Moore's property* the implication surjective $\Rightarrow$ pre-injective and *Myhill's property* the inverse one. We call *Moore–Myhill property* (*MM-property*) the union of these two properties and this last is an invariant of the shift.

As is well known, A^Γ is a compact metric space, and the local transition functions are those that are both continuous and commute with the natural action of Γ on A^Γ. We investigate the extent to which the MM-property holds for the closed and Γ-invariant subsets of A^Γ, the so-called *shifts*. As proved in [2] it is possible to prove, as a consequence of a theorem due to Lind and Marcus [6, Theorem 8.1.16], that the irreducible shifts of finite type in $A^{\mathbf{Z}}$ have the MM-property.

Recently, Gromov has proved a GOE-like theorem in a setting of graphs much more general than Cayley graphs, for alphabets not necessarily finite and for subset of the "universe" not necessarily invariant under translation. Because of the weakness of these hypotheses, in his theorem properties that are stronger than ours are needed, for example the *bounded propagation* of the spaces. In Section 3 we apply Gromov's theorem to our cellular automata proving that all the properties required in the hypotheses of this theorem are satisfied.

In Section 4, we generalize the previous result showing that the MM-property holds for *strongly irreducible* shifts of finite type of A^Γ (and we also show that strong irreducibility together with the finite type condition is strictly weaker that the bounded propagation property).

In this paper the notation $A \subseteq B$ means that the set A is contained in the set B and $A \subset B$ means that $A \subseteq B$ and $A \neq B$.

2. Shift spaces and cellular automata

In this section, we recall some definitions and we state some preliminary results about our class of CA.

If Γ is a finitely generated group and $\mathscr{X}$ is a fixed finite set of generators for Γ, then each $\gamma \in \Gamma$ can be written as

$$\gamma = x_{i_1}^{\delta_1} x_{i_2}^{\delta_2} \ldots x_{i_n}^{\delta_n}, \tag{1}$$

where the x_{i_j}'s are generators and $\delta_j \in \mathbb{Z}$. The *length of* γ (*with respect to* $\mathscr{X}$) is defined as the natural number

$$\|\gamma\|_{\mathscr{X}} := \min\{|\delta_1| + |\delta_2| + \cdots + |\delta_n| \mid \gamma \text{ is written as in } (1)\},$$

so that Γ is naturally endowed with a metric space structure, with the distance given by

$$\mathrm{dist}_{\mathscr{X}}(\alpha, \beta) := \|\alpha^{-1}\beta\|_{\mathscr{X}} \tag{2}$$

and

$$D_n^{\mathscr{X}} := \{\gamma \in \Gamma \mid \|\gamma\|_{\mathscr{X}} \leqslant n\}$$

is the disk of radius n centered at 1. Notice that $D_1^{\mathscr{X}}$ is the set $\mathscr{X} \cup \mathscr{X}^{-1} \cup \{1\}$. The asymptotic properties of the group being independent on the choice of the set of generators $\mathscr{X}$, from now on we fix a set $\mathscr{X}$ which is also symmetric (i.e. $\mathscr{X}^{-1} = \mathscr{X}$) and we omit the index $\mathscr{X}$ in all the above definitions.

For each $\gamma \in \Gamma$, the set D_n provides, by left translation, a *neighborhood of* γ, that is the set $\gamma D_n = D(\gamma, n)$, where $D(\gamma, n)$ is the disk of radius n centered at γ. Indeed, if $\alpha \in \gamma D_n$ then $\alpha = \gamma\beta$ with $\|\beta\| \leqslant n$. Hence $\mathrm{dist}(\alpha, \gamma) = \|\alpha^{-1}\gamma\| = \|\beta^{-1}\| \leqslant n$. Conversely, if $\alpha \in D(\gamma, n)$ then $\|\gamma^{-1}\alpha\| \leqslant n$ (that is $\gamma^{-1}\alpha \in D_n$), and $\alpha = \gamma\gamma^{-1}\alpha$.

Given a subset $E \subseteq \Gamma$ and for each $n \in \mathbb{N}$ we denote by

$$E^{+n} := \bigcup_{\alpha \in E} D(\alpha, n), \quad E^{-n} := \{\alpha \in E \mid D(\alpha, n) \subseteq E\} \quad \text{and} \quad \partial_n E := E^{+n} \backslash E^{-n}$$

the *n-closure of E*, the *n-interior of E* and the *n-boundary of E*, respectively; by

$$\partial_n^+ E := E^{+n} \backslash E \quad \text{and} \quad \partial_n^- E := E, \backslash E^{-n},$$

the *n-external boundary of E* and the *n-internal boundary of E*, respectively. For all these sets, we will omit the index n if $n = 1$.

The *Cayley graph* of Γ, is the graph in which Γ is the set of vertices and there is an edge from γ to $\bar{\gamma}$ if there exists a generator $x \in \mathscr{X}$ such that $\bar{\gamma} = \gamma x$. Obviously this graph depends on the presentation of Γ. For example, we may look at the classical cellular decomposition of Euclidean space $\mathbf{R}^n$ as the Cayley graph of the group $\mathbb{Z}^n$ with the presentation $\langle a_1, \ldots, a_n | a_i a_j = a_j a_i \rangle$.

If $\mathbf{G} = (\mathscr{V}, \mathscr{E})$ is a graph with set of vertices $\mathscr{V}$ and set of edges $\mathscr{E}$, the *graph distance* (or *geodetic distance*) between two vertices $v_1, v_2 \in \mathscr{V}$ is the minimal length of a path from v_1 to v_2. Hence the distance defined in (2) coincides with the graph distance on the Cayley graph of Γ.

Let A be a finite *alphabet*; in the classical theory of cellular automata, the "universe" is the Cayley graph of the free abelian group $\mathbb{Z}^n$ and a *configuration* is an element of

$A^{\mathbb{Z}^n}$, that is a function $c:\mathbb{Z}^n \to A$ assigning to each point of the graph a letter of A. We generalize this notion taking as universe a Cayley graph of a generic finitely generated group Γ and taking suitable subsets of configurations in A^Γ. On this set we have a natural metric and hence a topology. This latter is equivalent to the usual product topology, where the topology in A is the discrete one. An element of A^Γ is called a *configuration*.

If $c_1, c_2 \in A^\Gamma$ are two configurations, we define the distance

$$\mathrm{dist}(c_1, c_2) := \frac{1}{n+1},$$

where n is the least natural number such that $c_1 \neq c_2$ in D_n. If such an n does not exist, that is if $c_1 = c_2$, we set their distance equal to zero.

Observe that the group Γ acts on A^Γ on the right as follows:

$$(c^\gamma)_{|\alpha} := c_{|\gamma\alpha}$$

for each $c \in A^\Gamma$ and each $\gamma, \alpha \in \Gamma$ (where $c_{|\alpha}$ is the value of c at α).

Now we give a topological definition of a *shift space* (briefly *shift*); as stated in Theorem 2.5, this definition is equivalent (in the Euclidean case) to the classical combinatorial one.

Definition 2.1. A subset X of A^Γ is called a *shift* if it is topologically closed and Γ-invariant (i.e. $X^\Gamma = X$).

For every $X \subseteq A^\Gamma$ and $E \subseteq \Gamma$, we set

$$X_E := \{c_{|E} \mid c \in X\};$$

a *pattern of* X is an element of X_E where E is a non-empty finite subset of Γ. The set E is called the *support of* the pattern; a *block of* X is a pattern of X with support a disk. The *language of* X is the set $L(X)$ of all the blocks of X. If X is a subshift of $A^{\mathbf{Z}}$, a configuration is a bi-infinite word and a block of X is a finite word appearing in some configuration of X.

Hence a pattern with support E is a function $p:E \to A$. If $\gamma \in \Gamma$, we have that the function $\bar{p}:\gamma E \to A$ defined as $\bar{p}_{|\gamma\alpha} = p_{|\alpha}$ (for each $\alpha \in E$), is the pattern obtained copying p on the translated support γE. Moreover, if X is a shift, we have that $\bar{p} \in X_{\gamma E}$ if and only if $p \in X_E$. For this reason, in the sequel we do not make distinction between p and $\bar{p}$ (when the context makes it possible). For example, a word $a_1 \ldots a_n$ is simply a finite sequence of symbols for which we do not specify (if it is not necessary), if the support is the interval $[1, n]$ or the interval $[-n, -1]$.

Definition 2.2. Let X be a subshift of A^Γ; a function $\tau: X \to A^\Gamma$ is *M-local* if there exists $\delta: X_{D_M} \to A$ such that for every $c \in X$ and $\gamma \in \Gamma$

$$(\tau(c))_{|\gamma} = \delta((c^\gamma)_{|D_M}) = \delta(c_{|\gamma\alpha_1}, c_{|\gamma\alpha_2}, \ldots, c_{|\gamma\alpha_m}),$$

where $D_M = \{\alpha_1, \ldots, \alpha_m\}$.

In this definition, we have assumed that the alphabet of the shift X is the same as the alphabet of its image $\tau(X)$. In this assumption there is no loss of generality because if $\tau : X \subseteq A^\Gamma \to B^\Gamma$, one can always consider X as a shift over the alphabet $A \cup B$.

Definition 2.3. Let Γ be a finitely generated group with a fixed symmetric set of generators $\mathscr{X}$, let A be a finite alphabet with at least two elements and let D_M the disk in Γ centered at 1 and with radius M. A *cellular automaton* is a triple (X, D_M, τ) where X is a subshift of the compact space A^Γ, D_M is the neighborhood of 1 and $\tau : X \to X$ is an M-local function.

Let $\tau : X \to A^\Gamma$ be an M-local function; if c is a configuration of X and E is a subset of Γ, $\tau(c)_{|E}$ only depends on $c_{|E^{+M}}$. Thus we have a family of functions $(\tau_{E^{+M}} : X_{E^{+M}} \to \tau(X)_E)_{E \subseteq \Gamma}$.

There is a characterization of local functions that, in the one-dimensional case, is known as the Curtis–Lyndon–Hedlund theorem. A shift being compact, it holds for a general local function and states that *a function $\tau : X \to A^\Gamma$ is local if and only if it is continuous and commutes with the Γ-action* (i.e. for each $c \in X$ and each $\gamma \in \Gamma$, one has $\tau(c^\gamma) = \tau(c)^\gamma$). From this result, it is clear that *the composition of two local functions is still local*. In any case, this can be easily seen by a direct proof that follows Definition 2.2.

Now, fix $\gamma \in \Gamma$ and consider the function $X \to A^\Gamma$ that associates with each $c \in X$ its translated configuration c^γ. In general, this function does not commute with the Γ-action (and therefore it is not local). Indeed, if Γ is not abelian and $\gamma\alpha \neq \alpha\gamma$, then $(c^\gamma)^\alpha \neq (c^\alpha)^\gamma$. However this function is continuous. In order to see this, if $n \geqslant 0$, fix a number $m \geqslant 0$ such that $\gamma D_n \subseteq D_m$; if $\mathrm{dist}(c, \bar{c}) \leqslant 1/(m+1)$, then c and $\bar{c}$ agree on D_m and hence on γD_n. If $\alpha \in D_n$, we have $c_{|\gamma\alpha} = \bar{c}_{|\gamma\alpha}$ and then $c^\gamma_{|\alpha} = \bar{c}^\gamma_{|\alpha}$ that is c^γ and $\bar{c}^\gamma$ agree on D_n so that $\mathrm{dist}(c^\gamma, \bar{c}^\gamma) < 1/(n+1)$.

Observe that if X is a subshift of A^Γ and $\tau : X \to A^\Gamma$ is a local function, then, by the (generalized) Curtis–Lyndon–Hedlund theorem, the image $Y := \tau(X)$ is still a subshift of A^Γ. Indeed Y is closed (or, equivalently, compact) and Γ-invariant:

$$Y^\Gamma = (\tau(X))^\Gamma = \tau(X^\Gamma) = \tau(X) = Y.$$

Moreover, if τ is injective then $\tau : X \to Y$ is a homeomorphism; if $c \in Y$ then $c = \tau(\bar{c})$ for a unique $\bar{c} \in X$ and we have

$$\tau^{-1}(c^\gamma) = \tau^{-1}(\tau(\bar{c})^\gamma) = \tau^{-1}(\tau(\bar{c}^\gamma)) = \bar{c}^\gamma = (\tau^{-1}(c))^\gamma,$$

that is, τ^{-1} commutes with the Γ-action. By the Curtis–Lyndon–Hedlund theorem, τ^{-1} is local. Hence the well-known theorem (see [11]), stating that *the inverse of an invertible Euclidean cellular automaton is a cellular automaton*, holds also in this more general setting. In the one-dimensional case, Lind and Marcus [6, Theorem 1.5.14] give a direct proof of this fact.

This result leads us to give the following definition.

Definition 2.4. Two subshifts $X, Y \subseteq A^\Gamma$ are *conjugate* if there exists a local bijective function between them (namely a *conjugacy*). The *invariants* are the properties of a shift invariant under conjugacy.

It is easy to prove that the topological definition of a shift space is equivalent to the following combinatorial one involving the avoidance of certain *forbidden blocks*, this fact is well known in the Euclidean case.

Theorem 2.5. *A subset $X \subseteq A^\Gamma$ is a shift if and only if there exists a subset $\mathscr{F} \subseteq \bigcup_{n\in\mathbb{N}} A^{D_n}$ such that $X = X_{\mathscr{F}}$, where*

$$X_{\mathscr{F}} := \{c \in A^\Gamma \mid c^\alpha_{|D_n} \notin \mathscr{F} \text{ for every } \alpha \in \Gamma,\ n \in \mathbb{N}\}.$$

In this case, $\mathscr{F}$ is a set of forbidden blocks *of X.*

Now we give the first, fundamental notion of irreducibility for a one-dimensional shift and we see how to generalize this notion to a generic shift.

Definition 2.6. A shift $X \subseteq A^{\mathbf{Z}}$ is *irreducible* if for each pair of words $u, v \in L(X)$, there exists a word $w \in L(X)$ such that the concatenated word $uwv \in L(X)$.

The natural generalization of this property to any group Γ is the following.

Definition 2.7. A shift $X \subseteq A^\Gamma$ is *irreducible* if for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_F$, there exists an element $\gamma \in \Gamma$ such that $E \cap \gamma F = \emptyset$ and a configuration $c \in X$ such that $c_{|E} = p_1$ and $c_{|\gamma F} = p_2$.

In other words, a shift is irreducible if whenever we have $p_1, p_2 \in L(X)$, there exists a configuration $c \in X$ in which these two blocks appear simultaneously on disjoint supports. This definition could seem weaker than Definition 2.6, in fact in the latter one we establish that each word $u \in L(X)$ must always appear in a configuration on the left of each other word of the language. In order to prove that the two definitions agree, suppose that $X \subseteq A^{\mathbf{Z}}$ is an irreducible shift in accordance with Definition 2.7. If u, v are words in $L(X)$, there exists a configuration $c \in X$ such that $c_{|E} = u$ and $c_{|F} = v$ where E and F are finite and disjoint intervals. If $\max E < \min F$ then there exists a word w such that $uwv \in L(X)$ (where $w = c_{|I}$ and I is the interval $[\max E + 1, \min F - 1]$). If, otherwise, $\max F < \min E$ there exists a word w such that $vwu \in L(X)$; consider the word vwu two times, there exists another word x such that $vwu\ x\ vwu \in L(X)$ and hence $uxv \in L(X)$.

Now we give the fundamental notion of shift of finite type. The basic definition is in terms of forbidden blocks; in a sense we may say that a shift is of finite type if we can decide whether or not a configuration belongs to the shift only by checking its blocks of a fixed (and only depending on the shift) size. This fact implies the useful characterization of one-dimensional shifts of finite type, a sort of "overlapping" property for the words of the language. As stated below, this overlapping property still holds for a generic shift of finite type.

Definition 2.8. A shift is *of finite type* if it admits a finite set of forbidden blocks.

If X is a shift of finite type, since a finite set $\mathscr{F}$ of forbidden blocks of X has a maximal support, we can always suppose that each block of $\mathscr{F}$ has the disk D_M as support (indeed each block that contains a forbidden block is forbidden). In this case the shift X is called *M-step* and the number M is called the *memory of X*. If X is a subshift of $A^{\mathbf{Z}}$, we define the memory of X as the number M, where $M+1$ is the maximal length of a forbidden word.

For the shifts of finite type in $A^{\mathbf{Z}}$, we have the following useful property.

Proposition 2.9 ([6, Theorem 2.1.8]). *A shift $X \subseteq A^{\mathbf{Z}}$ is an M-step shift of finite type if and only if whenever $uv, vw \in L(X)$ and $|v| \geqslant M$, then $uvw \in L(X)$.*

It is easy to prove that this "overlapping" property holds more generally for subshifts of finite type of A^{Γ}, as stated in the following proposition.

Proposition 2.10. *Let X be an M-step shift of finite type and let E be a subset of Γ. If $c_1, c_2 \in X$ are two configurations that agree on $\partial^+_{2M}E$, then the configuration $c \in A^{\Gamma}$ that agrees with c_1 on E and with c_2 on $\complement E$ is still in X.*

Corollary 2.11. *Let X be an M-step shift of finite type and let E be a finite subset of Γ; if $p_1, p_2 \in X_{E+2M}$ are two patterns that agree on $\partial^+_{2M}E$, than there exist two extensions $c_1, c_2 \in X$ of p_1 and p_2, respectively, that agree on $\complement E$.*

Now we give the definition of entropy for a generic shift. This definition involves the existence of a suitable sequence of sets that, as one can see, in the case of non-exponential growth of the group can be taken as balls centered at 1 and with increasing radius.

Let $(E_n)_{n\geqslant 1}$ be a sequence of subsets of Γ such that $\bigcup_{n\in\mathbb{N}} E_n = \Gamma$ and

$$\lim_{n\to\infty} \frac{|\partial_M E_n|}{|E_n|} = 0; \tag{3}$$

if $X \subseteq A^{\Gamma}$ is a shift, the *entropy of X respect to* $(E_n)_{n\geqslant 1}$ is defined as

$$\mathrm{ent}(X) := \limsup_{n\to\infty} \frac{\log |X_{E_n}|}{|E_n|}.$$

Condition (3) is necessary to prove the next theorem and hence that *the entropy is invariant under conjugacy*; other aspects of its importance will be clarified in Section 4.

Theorem 2.12. *Let X be a shift and $\tau : X \to A^{\Gamma}$ a local function. Then* $\mathrm{ent}(\tau(X)) \leqslant \mathrm{ent}(X)$ (*that is, the entropy is invariant under conjugacy*).

Proof. Let τ be M-local and let $Y := \tau(X)$; we have that the function $\tau_{E_n^{+M}} : X_{E_n^{+M}} \to Y_{E_n}$ is surjective and hence

$$|Y_{E_n}| \leqslant |X_{E_n^{+M}}| \leqslant |X_{E_n}||X_{\partial_M^+ E_n}| \leqslant |X_{E_n}||A^{\partial_M^+ E_n}|.$$

From the previous inequalities we have

$$\frac{\log |Y_{E_n}|}{|E_n|} \leqslant \frac{\log |X_{E_n}|}{|E_n|} + \frac{|\partial_M^+ E_n| \log |A|}{|E_n|}$$

and hence, taking the maximum limit, $\mathrm{ent}(Y) \leqslant \mathrm{ent}(X)$. □

Now we give the definition of a *pre-injective* function. This notion is equivalent to the notion of non-existence of *mutually erasable patterns* used in the original works of Moore [8] and Myhill [9]. Indeed they prove that a transition function τ of a Euclidean cellular automaton on a full shift admits two mutually erasable patterns if and only if it admits a *GOE pattern*, that is a pattern without pre-image. Recall that two different patterns with the same support are called *τ-mutually erasable* if each pair of configurations extending them and coinciding out of the support, have the same image under τ.

In order to consider GOE-like theorems not in the whole of A^Γ but in a subshift $X \subseteq A^\Gamma$, notice first that two patterns of X are not necessarily extendible by the same configuration of X. Therefore, it could happen that two patterns with support F for which there does not exist a common extension $c_{|\complement F}$, are τ-mutually erasable although the function τ is bijective. The notion that seems to be a good generalization of the non-existence of mutually erasable patterns, is that of *pre-injectivity*; it can be seen that if $X = A^\Gamma$ then the non-existence of τ-mutually erasable patterns is equivalent to the pre-injectivity of τ.

Definition 2.13. A function $\tau : X \subseteq A^\Gamma \to A^\Gamma$ is called *pre-injective* if whenever $c_1, c_2 \in X$ and $c_1 \neq c_2$ only on a finite non-empty subset of Γ, then $\tau(c_1) \neq \tau(c_2)$.

One can prove (see [7, Theorem 5]) that a transition function on A^Γ is surjective if and only if there are no GOE patterns. It is easy to prove that this property holds also for the local functions between shifts. Hence we can state the GOE theorem as follows.

Theorem 2.14. *If $\tau : A^{\mathbb{Z}^2} \to A^{\mathbb{Z}^2}$ is a transition function, then τ is pre-injective if and only if it is surjective.*

Definition 2.15. A shift $X \subseteq A^\Gamma$ has the *Moore–Myhill property* (briefly *MM-property*), if for every cellular automaton (X, D_M, τ) the transition function τ is pre-injective if and only if it is surjective. The *Moore-property* is surjective ⇒ pre-injective and the *Myhill-property* is pre-injective ⇒ surjective.

In the sequel we will distinguish between these properties and the GOE theorems for a local function. Indeed the former are properties of a single shift but, on the other hand, we will speak of GOE theorem whenever we have a GOE-like theorem for a local function between two possibly different shifts.

As can be easily seen, the composition of two local pre-injective function is still a (local) pre-injective function. Hence we have that *the MM-property is invariant under conjugacy*.

A group Γ is called *amenable* if it admits a Γ-invariant probability measure; using the following characterization of it due to Følner (see [3,4,10]), the GOE theorem holds in the case of local transition functions $\tau : A^\Gamma \to A^\Gamma$.

Theorem 2.16 (Følner). *A group Γ is amenable if and only if for each finite subset $F \subseteq \Gamma$ and each $\varepsilon > 0$ there exists a finite subset $K \subseteq \Gamma$ such that*

$$\frac{|KF \backslash K|}{|K|} < \varepsilon.$$

Using this characterization we can prove the existence of a (nested) sequence $(E_n)_{n\geqslant 1}$ satisfying condition (3). Such a sequence is called *amenable* (or *Følner sequence*); from now on we fix the amenable sequence $(E_n)_{n\geqslant 1}$ found above and the entropy of a shift will be defined with respect to $(E_n)_{n\geqslant 1}$. Notice that condition (3) implies $\lim_{n\to\infty} |\partial_M^+ E_n|/|E_n| = 0$ and $\lim_{n\to\infty} |\partial_M^- E_n|/|E_n| = 0$.

Using the existence of an amenable sequence in the amenable group Γ, Ceccherini-Silberstein, Machì and Scarabotti have proved, in our language, the *the full shift A^Γ has the MM-property*.

3. Gromov's theorem

In [5], Gromov has proved a GOE-like theorem in a setting of graphs much more general than Cayley graphs, for alphabets not necessarily finite and for subset of the "universe" not necessarily invariant under translation. Because of the weakness of these hypotheses, in his theorem properties that are stronger than ours are needed (as we will see in the next section), for example the *bounded propagation* of the spaces. In this section we apply Gromov's theorem to our cellular automata proving that all the properties required in the hypotheses of this theorem are satisfied.

Definition 3.1. A closed subset $X \subseteq A^\Gamma$ is of *bounded propagation* $\leqslant M$ if for each pattern $p \in A^F$ with support F one has

$$p_{|F\cap D(\alpha,M)} \in X_{F\cap D(\alpha,M)} \quad \text{for each } \alpha \in F \Rightarrow p \in X_F.$$

If $\gamma \in \Gamma$, the left translation $i_\gamma : \Gamma \to \Gamma$ defined by $i_\gamma(\alpha) = \gamma\alpha$ is an isometry.

Consider a subgroup $\bar{\Gamma} \subseteq \Gamma$ and the set $\mathscr{I}(\bar{\Gamma})$ consisting of all restriction to each finite subset F of Γ of the left translations by an element of $\bar{\Gamma}$; a generic element of $\mathscr{I}(\bar{\Gamma})$ is $i_{\gamma_{|F}} : F \to \gamma F$. The set $\mathscr{I}(\bar{\Gamma})$ is, following Gromov's definition, a *pseudogroup of partial isometries*. Now consider a *stable* (i.e. closed) and $\bar{\Gamma}$-invariant space $X \subseteq A^{\Gamma}$; if we consider the finite subsets of Γ and the elements of $\bar{\Gamma}$, a family of functions

$$H_{F,\gamma}: X_F \to X_{\gamma F} = X_{i_\gamma(F)}$$

which commute with the restriction (i.e. $(H_{F,\gamma}(c_{|F}))_{|\gamma E} = H_{E,\gamma}(c_{|E})$), gives rise to a set of *holonomy maps*. In particular, we have a set of holonomy maps $\mathscr{H}(\bar{\Gamma})$ defining

$$H_{F,\gamma}(c_{|F}) := c^{\gamma^{-1}}_{|\gamma F}.$$

Following Gromov's definition, the set $\mathscr{H}(\bar{\Gamma})$ is a *pseudogroup of holonomies* and if $\mathscr{I}(\bar{\Gamma})$ is *dense* (that is, if $\bar{\Gamma}$ has finite index), we have defined a *dense pseudogroup of holonomies.*

If $Y \subseteq A^{\Gamma}$ is another stable and $\bar{\Gamma}$-invariant space, a function $\tau: X \to Y$ is *of bounded propagation* $\leqslant M$ if it is the limit of a family of functions $\tau_F: X_F \to Y_{F^{-M}}$ that commute with the restrictions; then a function of bounded propagation is such that $\tau(c)_{|\alpha} = \tau_{D(\alpha,M)}(c_{|D(\alpha,M)})_{|\alpha}$ and, in general, $\tau_F(c_{|F}) = \tau(c)_{|F^{-M}}$.

If τ is a function of bounded propagation, one can see that the holonomies in $\mathscr{H}(\bar{\Gamma})$ commute with τ if τ commutes with the $\bar{\Gamma}$-action and in this case, provided that $\mathscr{I}(\bar{\Gamma})$ is dense, we say that the function τ *admits a dense holonomy*.

Under these hypotheses and supposing that Γ is amenable, we have the following theorem.

Theorem 3.2 (Gromov [5]). *Let $X, Y \subseteq A^{\Gamma}$ be stable spaces of bounded propagation and $\tau: X \to Y$ a map of bounded propagation admitting a dense holonomy, then* $\text{ent}(X) = \text{ent}(Y)$ *implies that τ is surjective if and only if it is pre-injective.*

Suppose that τ is a bounded propagation $\leqslant M$ function between two $\bar{\Gamma}$-invariant stable spaces and τ commutes with the $\bar{\Gamma}$-action, if the pseudogroup $\mathscr{I}(\bar{\Gamma})$ is dense, we can write each $\alpha \in \Gamma$ as $\alpha = \gamma d$ ($\gamma \in \bar{\Gamma}$, $d \in D_R$) and

$$\tau(c)_{|\alpha} = \tau(c)_{|\gamma d} = (\tau(c^{\gamma}))_{|d} = \tau_{D(d,M)}(c^{\gamma}_{|D(d,M)})_{|d} = \tau_{D_{M+R}}(c^{\gamma}_{|D_{M+R}})_{|d}.$$

This means that in order to know the function τ it is sufficient to know how the image under τ of a configuration in X acts on D_R. In other words, it is sufficient to know the function $\tau_{D_{M+R}}: X_{D_{M+R}} \to \tau(X)_{D_R}$.

On the other hand, if τ is M-local between two shift spaces, we have

$$\tau(c)_{|\alpha} = \tau(c^{\alpha})_{|1} = \tau_{D_M}(c^{\alpha}_{|D_M})_{|1},$$

that is it suffices to know how the image under τ of a configuration in X acts on the identity of Γ, i.e. the local rule δ.

For these reasons, the notion of bounded propagation is a generalization of the notion of local function as far as stable spaces, not necessarily the Γ-invariant, are concerned. Hence, if Γ is amenable, the next theorem follows from Theorem 3.2.

Corollary 3.3 (GOE theorem for shifts of bounded propagation). *Let $X, Y \subseteq A^{\Gamma}$ be shift spaces of bounded propagation and $\tau : X \to Y$ a local function, then* $\text{ent}(X) = \text{ent}(Y)$ *implies τ surjective $\Leftrightarrow$ τ pre-injective.*

As a consequence of this fact, we have that *a shift of bounded propagation has the MM-property*.

4. Strongly irreducible shifts

In this section we give the definition of *strong irreducibility* for a shift. In general, as we have seen, it is possible to give a definition of irreducibility that generalizes the one-dimensional one. But although we can prove the MM-property for irreducible shifts of finite type of $A^{\mathbf{Z}}$ (see [2]), there are simple counterexamples showing that this irreducibility is too weak in the general case of subshifts of finite type of $A^{\mathbb{Z}^2}$. We prove the MM-property for the strongly irreducible shifts of finite type of A^{Γ}. On the other hand, we will see that a shift of bounded propagation (that has, by Gromov's theorem, the MM-property), is strongly irreducible and of finite type, but the converse does not hold.

Definition 4.1. A shift X is called *M-irreducible* if for each pair of finite sets $E, F \subseteq \Gamma$ such that $\text{dist}(E, F) > M$ and for each pair of patterns $p_1 \in X_E$ and $p_2 \in X_F$, there exists a configuration $c \in X$ that satisfies $c = p_1$ in E and $c = p_2$ in F. The shift X is called *strongly irreducible* if it is M-irreducible for some $M \in \mathbb{N}$.

In the particular case $\Gamma = \mathbb{Z}$, it can be easily seen that a shift $X \subseteq A^{\mathbf{Z}}$ is M-irreducible if for each $n \geqslant M$ and for each pair of words $u, v \in L(X)$, there exists a word $w \in L(X)$ with $|w| = n$, such that $uwv \in L(X)$.

Proposition 4.2. *Let Γ be an amenable group. Let X be a strongly irreducible shift of finite type and let $\tau : X \to A^{\Gamma}$ be a local and pre-injective function. Then* $\text{ent}(\tau(X)) = \text{ent}(X)$.

Proof. Suppose that the memory of X is M, that X is M-irreducible and that τ is M-local. Set $Y := \tau(X)$ and fix an amenable sequence $(E_n)_n$; we have

$$|Y_{E_n^{+2M}}| \leqslant |Y_{E_n}||A|^{|\partial_{2M}^{+} E_n|},$$

and then

$$\frac{\log |Y_{E_n^{+2M}}|}{|E_n|} \leqslant \frac{\log |Y_{E_n}|}{|E_n|} + \frac{|\partial_{2M}^{+} E_n| \log |A|}{|E_n|}.$$

Taking the maximum limit and being $\lim_{n\to\infty} |\partial_{2M}^{+} E_n| / |E_n| = 0$, we have

$$\limsup_{n\to\infty} \frac{\log |Y_{E_n^{+2M}}|}{|E_n|} \leqslant \text{ent}(Y).$$

Suppose that $\mathrm{ent}(Y) < \mathrm{ent}(X)$; then there exists $n \in \mathbb{N}$ such that

$$\frac{\log |Y_{E_n^{+2M}}|}{|E_n|} < \frac{\log |X_{E_n}|}{|E_n|},$$

that is $|Y_{E_n^{+2M}}| < |X_{E_n}|$. Fix $v \in X_{\partial^+_{2M} E_n^{+M}}$; since $\mathrm{dist}(\partial^+_{2M} E_n^{+M}, E_n) = M+1 > M$ for each $u \in X_{E_n}$ there exists a pattern $p \in X_{E_n^{+3M}}$ that coincides with u on E_n and with v on $\partial^+_{2M} E_n^{+M}$. Then

$$|\{ p \in X_{E_n^{+3M}} \mid p_{|\partial^+_{2M} E_n^{+M}} = v\}| \geqslant |X_{E_n}| > |Y_{E_n^{+2M}}|.$$

Since $\tau_{E_n^{+3M}} : X_{E_n^{+3M}} \to Y_{E_n^{+2M}}$ is surjective, there exist two patterns $p_1, p_2 \in X_{E_n^{+3M}}$ such that $p_1 \neq p_2$ but $p_1 = v = p_2$ on $\partial^+_{2M} E_n^{+M}$ and $\tau_{E_n^{+3M}}(p_1) = \tau_{E_n^{+3M}}(p_2)$. By Corollary 2.11, there exist two configurations $c_1, c_2 \in X$ which extend p_1 and p_2 and which coincide outside E_n^{+M}. We prove that $\tau(c_1) = \tau(c_2)$, and hence that τ is not pre-injective. If $\gamma \in E_n^{+2M}$ we have $\gamma D_M \subseteq E_n^{+3M}$ and hence, if $D_M = \{\alpha_1, \ldots, \alpha_m\}$, $\tau(c_1)_{|\gamma} = \delta(c_{1_{|\gamma\alpha_1}}, \ldots, c_{1_{|\gamma\alpha_m}}) = \delta(p_{1_{|\gamma\alpha_1}}, \ldots, p_{1_{|\gamma\alpha_m}}) = \tau_{E_n^{+3M}}(p_1)_{|\gamma} = \tau_{E_n^{+3M}}(p_2)_{|\gamma} = \delta(p_{2_{|\gamma\alpha_1}}, \ldots, p_{2_{|\gamma\alpha_m}}) = \delta(c_{2_{|\gamma\alpha_1}}, \ldots, c_{2_{|\gamma\alpha_m}}) = \tau(c_2)_{|\gamma}$. If $\gamma \notin E_n^{+2M}$, we have $\gamma D_M \subseteq \complement(E_n^{+M})$ and hence $\tau(c_1)_{|\gamma} = \tau(c_2)_{|\gamma}$, since c_1 coincide with c_2 on $\complement(E_n^{+M})$. $\square$

The proof of the following lemma only depends on the regularity of the Cayley graph of a finitely generated group. It is also implicit that the group is not finite and we do not treat the finite case because in the latter one the implications between surjectivity and injectivity are trivial.

Lemma 4.3. *If Γ is a finitely generated group, there exists a sequence of disks $(F_j)_{j\in\mathbb{N}}$ obtained by translation of a disk D and at distance $> M$ such that $\bigcup_{j\in\mathbb{N}} F_j^{+R} = \Gamma$ for a suitable $R > 0$. We call the above sequence a (D, M, R)*-net.

Proof. Let D be the disk centered at 1 and of radius ρ; define the following sequence of finite subsets of Γ:

$$\Gamma_0 := \{1\},$$

$$\Gamma_1 := \{\gamma \in \Gamma \mid \|\gamma\| = 2\rho + M + 1\}$$

and, in general,

$$\Gamma_n := \{\gamma \in \Gamma \mid \|\gamma\| = n(2\rho + M + 1)\}.$$

It is clear that for each n, $\mathrm{dist}(\Gamma_n, \Gamma_{n+1}) = 2\rho + M + 1$. Inside the set Γ_n, fix $\gamma_{n,1}$ and eliminate all the points in Γ_n whose distance from $\gamma_{n,1}$ is less than $2\rho + M + 1$.

Next, fix $\gamma_{n,2}$ among the remaining points and eliminate all the points whose distance from $\gamma_{n,2}$ is less than $2\rho + M + 1$. In this way, we will get a set $\bar{\Gamma}_n$ whose elements have mutual distance $\geqslant 2\rho + M + 1$ and such that for each element γ_n of Γ_n there exists an element of $\bar{\Gamma}_n$ whose distance from γ_n is less than $2\rho + M + 1$.

We now prove that, denoting by $(\beta_j)_{j\in\mathbb{N}}$ the sequence of the elements of $\bigcup_{n\in\mathbb{N}} \bar{\Gamma}_n$, the sequence $(\beta_j D)_{j\in\mathbb{N}}$ is a (D, M, R)-net with $R := 2\rho + 2M$; so that we can set $F_j := \beta_j D$.

Then let $\gamma \in \Gamma$; there exists $\gamma_n \in \Gamma_n$ such that $\mathrm{dist}(\gamma, \gamma_n) \leqslant \rho + M$. Since γ_n belongs to Γ_n, there is $\bar{\gamma}_n \in \bar{\Gamma}_n$ such that $\mathrm{dist}(\gamma_n, \bar{\gamma}_n) \leqslant 2\rho + M$ and hence $\mathrm{dist}(\gamma, \bar{\gamma}_n) \leqslant 3\rho + 2M$; then $\gamma \in (\bar{\gamma}_n D)^{+(2\rho+2M)}$. $\square$

Recall that a subshift of $A^{\mathbf{Z}}$ is *sofic* if it is the set of all labels of the bi-infinite paths in a finite labeled graph (or *finite-state automaton*). A fundamental result given in Section 4.4 of [6], is that *if X is an irreducible sofic shift and Y is a proper subshift of X, then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$. Now we prove a theorem of this kind in a much more general setting.

Lemma 4.4. *Let Γ be an amenable group and let $(E_n)_n$ be a fixed amenable sequence of Γ. Let $(F_j)_{j\in\mathbb{N}}$ be a $(D_r, 2M, R)$-net, let X be an M-irreducible shift and let Y be a subset of X such that $Y_{F_j} \subset X_{F_j}$ for every $j \in \mathbb{N}$. Then* $\mathrm{ent}(Y) < \mathrm{ent}(X)$.

Proof. Let $(p_j)_{j\in\mathbb{N}}$ be a sequence of patterns such that $p_j \in X_{F_j} \backslash Y_{F_j}$; let $N(n)$ be the number of F_j's such that $F_j^{+M} \subseteq E_n$ and denote by $F_{j_1}, \dots, F_{j_N}$ these disks. Set $\xi := |X_{D^{+M}}|$ (where $D = D_r$ and we omit the index r denoting the radius, if it is not necessary), and denote by $\pi_{j_i} : X_{E_n} \to X_{F_{j_i}}$ the restriction to F_{j_i} of the patterns of X_{E_n}. We prove that

$$\left| X_{E_n} \backslash \bigcup_{i=1}^{N} \pi_{j_i}^{-1}(p_{j_i}) \right| \leqslant (1 - \xi^{-1})^N |X_{E_n}| \tag{4}$$

by induction on $m \in \{1, \dots, N\}$. We have

$$|X_{E_n}| \leqslant |X_{F_{j_1}^{+M}}| |X_{E_n \backslash F_{j_1}^{+M}}|$$

then

$$|X_{E_n}| \leqslant \xi |X_{E_n \backslash F_{j_1}^{+M}}|.$$

Since X is an M-irreducible shift and since $\mathrm{dist}(F_{j_1}, E_n \backslash F_{j_1}^{+M}) > M$, given a pattern $p \in X_{E_n \backslash F_{j_1}^{+M}}$, there exists a pattern $\bar{p}$ defined on all E_n that coincides with p on $E_n \backslash F_{j_1}^{+M}$ and with p_{j_1} on F_{j_1}; then

$$|X_{E_n \backslash F_{j_1}^{+M}}| \leqslant |\pi_{j_1}^{-1}(p_{j_1})|.$$

Hence

$$\frac{1}{\xi} |X_{E_n}| \leqslant |\pi_{j_1}^{-1}(p_{j_1})|$$

and

$$|X_{E_n} \backslash \pi_{j_1}^{-1}(p_{j_1})| \leqslant |X_{E_n}| - \frac{1}{\xi} |X_{E_n}| = (1 - \xi^{-1}) |X_{E_n}|.$$

Suppose that (4) holds for $m - 1$; we have

$$\left| X_{E_n} \backslash \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i}) \right| \leqslant \xi \left| \left\{ p_{|E_n \backslash F_{j_m}^{+M}} \mid p \in X_{E_n} \backslash \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i}) \right\} \right|.$$

Moreover, since X is M-irreducible,

$$\left|\left\{ p_{|E_n\setminus F_{j_m}^{+M}} \mid p \in X_{E_n}\setminus \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})\right\}\right| \leqslant \left|\left\{ p \in X_{E_n}\setminus \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i}) \mid p_{|F_{j_m}} = p_{j_m}\right\}\right|.$$

Hence

$$\begin{aligned}\left|X_{E_n}\setminus \bigcup_{i=1}^{m} \pi_{j_i}^{-1}(p_{j_i})\right| &= \left|\left(X_{E_n}\setminus \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})\right)\setminus \pi_{j_m}^{-1}(p_{j_m})\right| \\ &\leqslant \left|\left(X_{E_n}\setminus \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})\right)\setminus \right. \\ &\quad \left.\left\{ p \in X_{E_n}\setminus \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i}) \mid p_{|F_{j_m}} = p_{j_m}\right\}\right| \\ &\leqslant \left|X_{E_n}\setminus \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})\right| - \frac{1}{\xi}\left|X_{E_n}\setminus \bigcup_{i=1}^{m-1} \pi_{j_i}^{-1}(p_{j_i})\right| \\ &\leqslant \left(1-\frac{1}{\xi}\right)(1-\xi^{-1})^{m-1}|X_{E_n}| = (1-\xi^{-1})^m|X_{E_n}|.\end{aligned}$$

Hence (4) holds, and since $|Y_{E_n}| \leqslant |X_{E_n}\setminus \bigcup_{i=1}^{N} \pi_{j_i}^{-1}(p_{j_i})|$, we have

$$\frac{\log|Y_{E_n}|}{|E_n|} \leqslant \frac{N(n)\log(1-\xi^{-1})}{|E_n|} + \frac{\log|X_{E_n}|}{|E_n|}. \tag{5}$$

Observe that

$$E_n \subseteq \bigcup_{i=1}^{N} F_{j_i}^{+R} \cup (E_n\setminus E_n^{-(R+2r+M)}). \tag{6}$$

Indeed suppose that $\gamma\in E_n$ and $\gamma\notin \bigcup_{i=1}^{N} F_{j_i}^{+R}$; $(F_j)_j$ being a $(D_r, 2M, R)$-net, we have that $\gamma\in F_k^{+R}$ for some k, that is $\gamma\in\beta D^{+R}$ with β such that $\beta D^{+M}\nsubseteq E_n$. Hence $\mathrm{dist}(\gamma,\beta) \leqslant r+R$ so that $\beta\in\gamma D^{+R}$. If $\gamma\in E_n^{-(R+2r+M)}$, then $\beta\in\gamma D^{+R}\subseteq E_n^{-(r+M)}$ so that $F_k^{+M} = \beta D^{+M}\subseteq E_n$ which is excluded.

From (6), we have

$$|E_n| \leqslant N(n)|D^{+R}| + |E_n\setminus E_n^{-(R+2r+M)}|,$$

so that

$$1 \leqslant \frac{N(n)}{|E_n|}|D^{+R}| + \frac{|\partial^-_{R+2r+M}E_n|}{|E_n|};$$

taking the minimum limit and being $\lim_{n\to\infty} |\partial^-_{R+2r+M} E_n|/|E_n| = 0$,

$$\zeta := \liminf_{n\to\infty} \frac{N(n)}{|E_n|} > 0$$

and then from (5) it follows

$$\mathrm{ent}(Y) \leqslant \zeta \log(1-\xi^{-1}) + \mathrm{ent}(X) < \mathrm{ent}(X). \qquad \square$$

Proposition 4.5. *Let X be a strongly irreducible shift of finite type, let $\tau : X \to A^\Gamma$ be a local function such that $\mathrm{ent}(\tau(X)) = \mathrm{ent}(X)$. Then τ is pre-injective.*

Proof. Suppose that X has memory M, that is X is M-irreducible and that τ is M-local. Moreover suppose that τ is not pre-injective; then there exist $c_1, c_2 \in X$ and a disk D contained in Γ, such that $c_1 \neq c_2$ on D, $c_1 = c_2$ out of D and $\tau(c_1) = \tau(c_2)$. Set $(F_j)_{j\in\mathbb{N}} = (\beta_j D^{+2M})_{j\in\mathbb{N}}$ a $(D^{+2M}, 2M, R)$-net and denote by Y the subset of X defined by

$$Y := \{c \in X \mid (c^{\beta_j})_{|D^{+2M}} \neq c_{2_{|D^{+2M}}} \text{ for every } j \in \mathbb{N}\},$$

that is the subset of X avoiding the pattern $c_{2|D^{+2M}}$ on the disk D^{+2M} and on the translated disks $F_j = \beta_j D^{+2M}$. The set Y is a subset of X such that $Y_{F_j} \subset X_{F_j}$; we prove that $\tau(Y) = \tau(X)$. Indeed if $c \in X \setminus Y$, there exists a subset $J \subseteq \mathbb{N}$ such that for every $j \in J$, we have $(c^{\beta_j})_{|D^{+2M}} = c_{2|D^{+2M}}$. Define $\bar{c} \in X$ in the following way:

- $\bar{c} = c_1^{\beta_j^{-1}}$ on F_j for every $j \in J$,
- $\bar{c} = c$ out of the union $\bigcup_{j\in J} F_j$.

That is, $\bar{c}$ is obtained from c substituting all the occurrences of $c_{2|D^{+2M}}$ with $c_{1|D^{+2M}}$.

By Proposition 2.10, we have $\bar{c} \in X$ and moreover $\bar{c} \in Y$; we prove that $\tau(\bar{c}) = \tau(c)$.

If $\gamma \in \beta_j D^{+M}$ for some $j \in J$, we have $\gamma D_M \subseteq F_j$ and then $\tau(\bar{c})_{|\gamma} = \tau(c_1^{\beta_j^{-1}})_{|\gamma} = \tau(c_1)_{|\beta_j^{-1}\gamma} = \tau(c_2)_{|\beta_j^{-1}\gamma} = \tau(c_2^{\beta_j^{-1}})_{|\gamma} = \tau(c)_{|\gamma}$.

Suppose that $\gamma \notin \beta_j D^{+M}$ for every $j \in J$; then $\gamma D_M \subseteq \complement(\beta_j D)$ and hence $\tau(\bar{c})_{|\gamma} = \tau(c)_{|\gamma}$. Indeed $\bar{c}$ and c coincide on $\bigcup_{j\in J} \complement(\beta_j D)$: if $j \in J$ and $\gamma \in \partial^+_{2M} \beta_j D = F_j \setminus \beta_j D$, we have $\bar{c}_{|\gamma} = (c_1^{\beta_j^{-1}})_{|\gamma} = c_{1_{|\beta_j^{-1}\gamma}}$. Since $\beta_j^{-1}\gamma \in \partial^+_{2M} D$ one has $c_{1_{|\beta_j^{-1}\gamma}} = c_{2_{|\beta_j^{-1}\gamma}} = (c_2^{\beta_j^{-1}})_{|\gamma} = c_{|\gamma}$.

Then, by Theorem 2.12 and Lemma 4.4,

$$\mathrm{ent}(\tau(X)) = \mathrm{ent}(\tau(Y)) \leqslant \mathrm{ent}(Y) < \mathrm{ent}(X). \qquad \square$$

Proposition 4.6. *Let Γ be an amenable group. Let X be a shift, let Y be a strongly irreducible shift and let $\tau : X \to Y$ be a local function such that $\mathrm{ent}(\tau(X)) = \mathrm{ent}(Y)$. Then τ is surjective.*

Proof. Let X and Y be as in the hypotheses and let $\tau : X \to Y$ be a local function. We prove that if $\tau(X) \subset Y$, then $\mathrm{ent}(\tau(X)) < \mathrm{ent}(Y)$. Indeed if $\tau(X) \subset Y$, there exists a configuration $c \in Y$ which does not belong to $\tau(X)$ and then there exists a disk D

such that $c_{|D} \in Y_D \setminus (\tau(X))_D$. Let $(F_j)_{j\in\mathbb{N}}$ be a $(D, 2M, R)$-net; then $(\tau(X))_{F_j} \subset Y_{F_j}$; by Lemma 4.4, $\text{ent}(\tau(X)) < \text{ent}(Y)$. □

Theorem 4.7. *Let Γ be an amenable group. Let X be a strongly irreducible shift of finite type, let Y be a strongly irreducible shift and let $\tau: X \to Y$ be a local function with $\text{ent}(X) = \text{ent}(Y)$. Then τ is pre-injective if and only if is surjective.*

Proof. If τ is pre-injective we have, by Proposition 4.2, that $\text{ent}(\tau(X)) = \text{ent}(X)$. Then $\text{ent}(\tau(X)) = \text{ent}(Y)$ so that, by Proposition 4.6, τ is surjective.

If, conversely, τ is surjective then $\text{ent}(\tau(X)) = \text{ent}(Y)$, that is $\text{ent}(\tau(X)) = \text{ent}(X)$. By Proposition 4.5, τ is pre-injective. □

Corollary 4.8 (MM-property for strongly irreducible shifts of finite type). *If Γ is an amenable group, a strongly irreducible subshift of finite type of A^Γ has the MM-property.*

We conclude this section proving that the property of bounded propagation for a shift is strictly stronger than the union of strong irreducibility and finite type condition. The following characterization of the finite type condition is an easy consequence of the definition.

Lemma 4.9. *A shift X is of finite type with memory M if and only if each configuration $c \in A^\Gamma$ such that $c_{|D(\alpha,M)} \in X_{D(\alpha,M)}$ for every $\alpha \in \Gamma$, belongs to X.*

Now we can prove the following statement.

Proposition 4.10. *If $X \subseteq A^\Gamma$ is a shift of bounded propagation, then X is strongly irreducible and of finite type.*

Proof. Suppose that X has bounded propagation $\leqslant M$; if $E, F \subseteq \Gamma$ are such that $\text{dist}(E, F) > M$ and $p_1 \in X_E$, $p_2 \in X_F$ are two patterns of X, consider the pattern p with support $E \cup F$ given by the union of the functions p_1 and p_2. Clearly $p \in X_{E\cup F}$ because if $\alpha \in E \cup F$ and, for instance $\alpha \in E$, we have $(E \cup F) \cap \alpha D_M \subseteq E$ and hence $p_{|(E\cup F)\cap \alpha D_M} \in X_{(E\cup F)\cap\alpha D_M}$. A configuration in X extending p is such that $c_{|E} = p_1$ and $c_{|F} = p_2$. Hence X is M-irreducible.

Now suppose that $c \in A^\Gamma$ is such that $c_{|D(\alpha,M)} \in X_{D(\alpha,M)}$ for every $\alpha \in \Gamma$. Then if $n \geqslant M$ and $\alpha \in D_n$ we have

$$c_{|D_n \cap D(\alpha,M)} = (c_{|D(\alpha,M)})_{|D_n\cap D(\alpha,M)} \in X_{D_n\cap D(\alpha,M)};$$

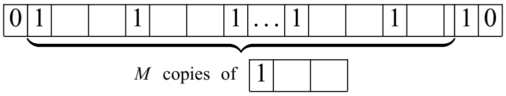
X being of bounded propagation we have $c_{|D_n} \in X_{D_n}$. X being closed we have $c \in X$. □

Recall that an *edge shift* is the set of all the bi-infinite paths in a finite graph. If $\Gamma = \mathbb{Z}$ and X is an edge shift, it can be seen that also the converse of the previous theorem holds.

Now we prove that in general strong irreducibility and finite type condition do not imply the bounded propagation property. Consider the subshift $X \subseteq \{0,1\}^{\mathbb{Z}}$ with a set of forbidden blocks:

$$\{010, 111\}.$$

Clearly X is a strongly irreducible (in fact 2-irreducible) shift of finite type; if $M \geqslant 1$ consider the following pattern p with $F := \mathrm{supp}(p)$



In this case we have $p_{|F \cap D(\alpha,M)} \in X_{F \cap D(\alpha,M)}$ but $p \notin X_F$; hence X is not of bounded propagation $\leqslant M$ for each $M \geqslant 1$.

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