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On Periodic Solutions of Certain *n*th Order Differential Equations

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We present two theorems on periodic solutions of *n*th-order ordinary differential equations with the right-hand side T-periodic with respect to the time variable. We apply a topological method based on the Lefschetz Fixed Point Theorem and the Ważewski Principle. © 1995 Academic Press, Inc.

The problem of the existence of periodic solutions of an *n*th-order nonlinear differential equation has been considered by numerous authors, [OZ, P, R, RSC, S1, W], for example. [OZ] and [RSC] contain extensive bibliographies of the subject. The techniques of [P, S1] are topological, while the approach of [OZ, R, W] is based on an application of functional analysis methods to the study of solutions of the considered differential equations. The purpose of this note is to present an application of an alternative approach to the study of periodic solutions of differential equations based on the Lefschetz Fixed Point Theorem and ideas coming from the Ważewski retract principle. This approach, introduced in [Sr1, Sr2] (cf. also [Sr3]), permits one not only to obtain different proofs of known results (compare Theorem 1), but also to improve these (see Theorem 2).

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The basis for the paper will be the following Theorem 0 (See [Sr2, Theorem 7.1 and Corollary 7.4; Sr3, Theorems 3.1 and 3.2]). It is a consequence of a version of the Lefschetz Fixed Point Theorem stated in [D]. The first assertion of the theorem was established in [Sr1]:

THEOREM 0. *Assume that the Cauchy problem*

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{0}$$

where $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and T-periodic in t (for some* $T > 0$*), has the uniqueness property. Denote by* $t \to \Phi_{(t_0,t-t_0)}(x_0)$ *its solution (i.e.,* Φ *is the (local) process generated by* (0); *cf.* [H]). *Let* (E, E^{exit}) *be a pair of compact Euclidean neighborhood retracts (ENR—see* [D] *for the definition) contained in* $\mathbb{R}^n$ *and assume that*

$$\mathbb{R} \times E^{\text{exit}} = \{(t,x) \in \mathbb{R} \times E : \exists\{\varepsilon_k\}, \varepsilon_k > 0, \lim_{k\to\infty} \varepsilon_k = 0, \Phi_{(t,\varepsilon_k)}(x) \notin E\}.$$

If

$$\chi(E) - \chi(E^{\text{exit}}) \neq 0$$

where χ *denotes the Euler chracteristic, then there exists a T-periodic solution of* (0) *staying in the set E. If for every* $x \in \partial E$ *and* $t_0 \in \mathbb{R}$ *there exists a* $t \in \mathbb{R}$ *such that* $\Phi_{(t_0,t)}(x) \notin E$ *then the set*

$$K_E = \{x \in \mathbb{R}^n : \Phi_{(0,T)}(x) = x, \forall t \in [0,T] : \Phi_{(0,t)}(x) \in E\}$$

is compact and isolated in the set of fixed points of the Poincaré operator $\Phi_{(0,T)}$, *and*

$$\operatorname{ind}(\Phi_{(0,T)}, K_E) = \chi(E) - \chi(E^{\text{exit}})$$

where ind *denotes the fixed point index.*

The equation considered here has the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' = b(t, y, y', \ldots, y^{(n-1)}), \tag{1}$$

where $a_1, \ldots, a_{n-1}$ denote real numbers and $b\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous map. $\mathbb{R}^k$ is a Euclidean space with the scalar product $x \cdot y$ and the norm $|x| = (x \cdot x)^{1/2}$. T is a positive number and let ε denote the number 1 or

-1. In the following we will impose the following conditions on $a_1, ..., a_{n-1}$, b, and T:

(H1) $b(\cdot, x_1, ..., x_n)$ is T-periodic for each $(x_1, ..., x_n) \in \mathbb{R}^n$.

(H2) The equation

$$\lambda^{n-1} + a_{n-1}\lambda^{n-2} + \cdots + a_2\lambda + a_1 = 0 \tag{2}$$

has no pure imaginary roots.

(H3) $b(t, x_1, ..., x_n)/|(x_2, ..., x_n)| \to 0$ as $|(x_2, ..., x_n)| \to \infty$ uniformly in $(t, x_1) \in \mathbb{R}^2$.

(H4) $\varepsilon b(t, x_1, ..., x_n)x_1 \to \infty$ as $|x_1| \to \infty$ uniformly in $(t, x_2, ..., x_n)$ on compact subsets of $\mathbb{R}^n$.

Our aim is to prove the following results:

THEOREM 1. *The equation* (1) *has a T-periodic solution provided the conditions* (H1)–(H4) *hold.*

THEOREM 2. *Assume* (H1)–(H4). *Assume in addition that*

$$b(t, x_1, ..., x_n) = b_1x_1 + \cdots + b_nx_n + c(t, x_1, ..., x_n)$$

with c continuous satisfying the condition

$$\frac{c(t, x_1, ..., x_n)}{|(x_1, ..., x_n)|} \to 0 \text{ as } |(x_1, ..., x_n)| \to 0 \text{ uniformly in } t \in \mathbb{R},$$

and constants $b_1, ..., b_n$ such that the equation

$$\lambda^n + (a_{n-1} - b_n)\lambda^{n-1} + \cdots + (a_1 - b_2)\lambda - b_1 = 0 \tag{3}$$

has no roots on $i\mathbb{R}$. Denote by k and l the number of roots (counted with their multiplicities) of, respectively, (2) *and* (3) *which have real positive parts. If*

$$(-1)^k(\operatorname{sgn} a_1)\varepsilon = (-1)^l$$

then (1) *has a nontrivial T-periodic solution. If, moreover, b is odd in x, i.e.,*

$$b(t, -x_1, ..., -x_n) = -b(t, x_1, ..., x_n)$$

then there are at least two distinct such solutions.

Let us mention that Theorem 1 is known; it is a corollary from [W]. We present it here, because our proof differs from the proof in [W] and can be given simultaneously with the proof of Theorem 2. Our proof does not use functional analysis and is based on Theorem 0. It seems that in the published literature there are not many results similar to Theorem 2 on the existence of nontrivial periodic solutions for the considered equations, even in the case $n = 2$. In the latter case [Sr2, Corollary 11.2] represents another such result, but it is slightly different from the result in Theorem 2.

Proof of Theorems 1 *and* 2. Assume additionally that the Cauchy problem for (1) has the unique solution. Rewrite the equation (1) in an equivalent form

$$x' = Ax + B(t,x), \tag{4}$$

where $x = \mathrm{col}(x_1, ..., x_n)$, A is the $(n \times n)$-matrix defined by

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix},$$

and $B(t, x) = \mathrm{col}(0, ..., 0, b(t, x))$. Consider an auxiliary equation

$$u' = Pu \tag{5}$$

where P is an $(n - 1) \times (n - 1)$-matrix,

$$P = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}.$$

By (H2),

$$\mathbb{R}^{n-1} = U_+ \oplus U_- \tag{6}$$

where U_+ (and U_-) is the generalized eigenspace of the eigenvalues with positive real parts (negative real parts, respectively). In order to simplify notation, suppose that both U_+ and U_- are different from $\{0\}$. (The case

U_+ or U_- equal to $\mathbb{R}^n$ follows by a similar argument; it has been also considered in [S1, S2].) Let dim $U_+ = k$ and let

$$P_+ = P|_{U_+}, \qquad P_- = P|_{U_-},$$

hence

$$P = \operatorname{diag}(P_+, P_-).$$

For an element $u \in \mathbb{R}^{n-1}$ denote by u_+ and u_- its components in the decomposition (6). For a $\lambda > 0$ define

$$\psi_+^\lambda(u) = \phi_+(\lambda^{-1}u_+) - 1,$$
$$\psi_-^\lambda(u) = \phi_-(\lambda^{-1}u_-) - 1,$$

where ϕ_+, ϕ_- denote quadratic forms being Lyapunov functions associated with the equations

$$v' = -P_+v$$
$$w' = P_-w,$$

respectively. Since the right-hand side of (5) is homogeneous of degree 1, by (H3) and the argument in the proof of [Sr2, Proposition 9.1] there exists $\mu > 0$ such that for any $(t, y) \in \mathbb{R}^2$,

$$\operatorname{grad} \psi_+^\mu(u) \cdot (P(u) + \operatorname{col}(0, ..., 0, b(t, y, u))) > 0, \tag{7}$$

provided

$$\psi_+^\mu(u) = 0, \qquad \psi_-^\mu(u) \leq 0$$

and

$$\operatorname{grad} \psi_-^\mu(u) \cdot (P(u) + \operatorname{col}(0, ..., 0, b(t, y, u))) < 0, \tag{8}$$

provided

$$\psi_+^\mu(u) \leq 0, \qquad \psi_-^\mu(u) = 0.$$

Set

$$D = \{u \in \mathbb{R}^{n-1} : \psi_+^\mu(u) \le 0, \psi_-^\mu(u) \le 0\}.$$

Define a function $\omega : \mathbb{R}^n \to \mathbb{R}$ by

$$\omega(x_1, ..., x_n) = \left((a_2x_2 + \cdots + a_{n-1}x_{n-1} + x_n)x_1 + \frac{a_1}{2}x_1^2\right).$$

The direct calculation yields

$$\operatorname{grad} \omega(x) \cdot (Ax + B(t,x)) = a_2x_2^2 + a_3x_3x_2 + \cdots + a_{n-1}x_{n-1}x_2 + b(t,x)x_1. \tag{10}$$

Since D is compact, there exists an $M < \infty$ such that

$$M \ge \max\{|a_2x_2^2 + a_3x_3x_2 + \cdots + a_{n-1}x_{n-1}x_2| : (x_2, ..., x_n) \in D\}.$$

By (H1) and (H4), there exists an $R > 0$ such that if $|x_1| \ge R$ then

$$\varepsilon b(t,x)x_1 > M \tag{11}$$

for every $(t, x_2, ..., x_n) \in \mathbb{R} \times D$. Denote

$$N = \max\{|a_2x_2 + \cdots + a_{n-1}x_{n-1} + x_n| : (x_2, ..., x_n) \in D\}.$$

Let $r > 0$ be so chosen that

$$\frac{\sqrt{2|a_1|r} - N}{|a_1|} > R \tag{12}$$

and define as in [S1, S2] a function $L_2 : \mathbb{R}^n \to \mathbb{R}$ by the formula

$$L_2(x) = (\operatorname{sgn} a_1)\omega(x) - r.$$

From the formula for the roots of the equation $L_2(x) = 0$,

$$x_1^\pm = \frac{\pm\sqrt{\Delta} - (\operatorname{sgn} a_1)(a_2x_2 + \cdots + a_{n-1}x_{n-1} + x_n)}{|a_1|}, \tag{13}$$

where

$$\Delta = (a_2x_2 + \cdots + a_{n-1}x_{n-1} + x_n)^2 - 2|a_1|r,$$

and from (12) it is clear that for $(x_2, ..., x_n) \in D$ the roots $x_1^\pm$ are different from zero and of opposite signs. Moreover, $|x_1^\pm| > R$. Define also functions $L_1, L_3: \mathbb{R}^n \to \mathbb{R}$:

$$L_1(x) = \psi_+^\mu(x_2, ..., x_n),$$
$$L_3(x) = \psi_-^\mu(x_2, ..., x_n),$$

and let E be the set

$$E = \{x \in \mathbb{R}^n : L_i(x) \leq 0,\ i = 1, 2, 3\}.$$

Its boundary Γ is the union of sets

$$\Gamma_i = \{x \in E : L_i(x) = 0\},$$

$i = 1, 2, 3$. By (13), E is homeomorphic to $[0, 1] \times D$, and, as a consequence, it is also homeomorphic to the n-dimensional ball B^n. By (7) and (8),

$$\operatorname{grad} L_1(x) \cdot (Ax + B(t, x)) > 0, \qquad \text{for } x \in \Gamma_1, \tag{14}$$
$$\operatorname{grad} L_3(x) \cdot (Ax + B(t, x)) < 0, \qquad \text{for } x \in \Gamma_3, \tag{15}$$

and by (10), (11), and (14)

$$(\operatorname{sgn} a_1)\varepsilon \operatorname{grad} L_2(x) \cdot (Ax + B(t, x)) > 0, \qquad \text{for } x \in \Gamma_2. \tag{16}$$

Assume now that $(\operatorname{sgn} a_1)\varepsilon = 1$. Put

$$E^{\text{exit}} = \Gamma_1 \cup \Gamma_2.$$

By (14), (15), and (16) the pair (E, E^{exit}) fulfills the assumptions of Theorem 0. E^{exit} is homeomorphic to $\{0, 1\} \times D \cup [0, 1] \times S^{k-1} \times B^{n-k-1}$, hence it has the homotopy type of S^k. Following the notation used in Theorem 0, as its consequence we conclude that in our case the set K_E is compact and isolated, and

$$\operatorname{ind}(\Phi_{(0,T)}, K_E) = \chi(E) - \chi(E^{\text{exit}}) = (-1)^{k+1}.$$

If $(\operatorname{sgn} a_1)\varepsilon = -1$ then we put

$$E^{\text{exit}} = \Gamma_1,$$

hence it has the homotopy type of S^{k-1}. By the above argument we assert that in both of the cases

$$\operatorname{ind}(\Phi_{(0,T)}, K_E) = (-1)^{k+1}(\operatorname{sgn} a_1)\varepsilon. \tag{17}$$

We have already proved the conclusion of Theorem 1, because the difference of the Euler characteristics is always nonzero. Actually, the set K_E is nonempty and each of its points is an initial point of a T-periodic solution of (4) starting at the time 0.

We continue the proof under the assumptions of Theorem 2. By an argument similar to that above, using Lyapunov functions one can construct functions $\Lambda_1, \Lambda_2 : \mathbb{R}^n \to \mathbb{R}$ such that the set

$$E_0 = \{x \in \mathbb{R}^n : \Lambda_1(x) \leq 0, \Lambda_2(x) \leq 0\}$$

is a compact neighborhood of 0 homeomorphic to B^n satisfying conditions

$$\operatorname{grad} \Lambda_1(x) \cdot (Ax + B(t,x)) > 0 \qquad (x \in E_0, \Lambda_1(x) = 0), \tag{18}$$

$$\operatorname{grad} \Lambda_2(x) \cdot (Ax + B(t,x)) < 0 \qquad (x \in E_0, \Lambda_2(x) = 0), \tag{19}$$

and the set

$$E_0^{\text{exit}} = \{x \in E_0 : \Lambda_1(x) = 0\}$$

has the homotopy type of S^{l-1}. By the argument above we conclude that

$$K_{E_0} = \{x \in \mathbb{R}^n : \Phi_{(0,T)}(x) = x, \forall t \in [0, T] : \Phi_{(0,t)}(x) \in E_0\}$$

is compact and isolated, and

$$\operatorname{ind}(\Phi_{(0,T)}, K_{E_0}) = \chi(E_0) - \chi(E_0^{\text{exit}}) = (-1)^l. \tag{20}$$

If $x \equiv 0$ is the only periodic solution of (4), then

$$K_{E_0} = K_E = \{0\},$$

hence

$$\operatorname{ind}(\Phi_{(0,T)}, K_{E_0}) = \operatorname{ind}(\Phi_{(0,T)}, K_E).$$

This equation, together with (17) and (20), contradicts the assumptions on k and l.

The last conclusion concerning the existence of two distinct periodic solutions follows easily from the fact that, since b is odd with respect to x, the set of solutions of (1) is invariant under multiplication by -1. The proof with the additional hypothesis is finished.

To remove the uniqueness hypothesis, note that since inequalities (14)–(16), (18), and (19) are strict and hold in compact sets, they are also satisfied by any B_1 provided $|B(t, x) - B_1(t, x)|$ is small for $(t, x) \in K$, where $K \subset \mathbb{R}^{n+1}$ is a fixed compact set containing $[0, 1] \times D$ in its interior. Thus Theorems 1 and 2 remain valid for equations having the uniqueness property, approximating in K Eq. (1). By the standard limiting process, we conclude that theorems hold true also for the limit equation (1). The proof is finished.

EXAMPLE. The equation

$$y^{(3)} - y' = 2ye^{-y^2} \arctan y + \sin(y^3) \sin t$$

has two distinct nonzero 2π-periodic solutions.

Indeed, in this case $a_1 = \varepsilon = -1$ and $k = l = 1$.

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