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On an Error Term Related to the Jordan Totient Function $J_k(n)$

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We investigate the error terms

$$E_k(x) = \sum_{n \leqslant x} J_k(n) - \frac{x^{k+1}}{(k+1)\,\zeta(k+1)} \qquad \text{for} \quad k \geqslant 2,$$

where $J_k(n) = n^k \prod_{p \mid n}(1 - 1/p^k)$ for $k \geqslant 1$. For $k \geqslant 2$, we prove

$$\sum_{n \leqslant x} E_k(n) \sim \frac{x^{k+1}}{2(k+1)\,\zeta(k+1)}.$$

Also,

$$\limsup_{x \to \infty} \frac{E_k(x)}{x^k} \leqslant \frac{D}{\zeta(k+1)},$$

where $D = .7159$ when $k = 2$, $.6063$ when $k \geqslant 3$. On the other hand, even though

$$\liminf_{x \to \infty} \frac{E_k(x)}{x^k} \leqslant -\frac{1}{2\zeta(k+1)},$$

$E_k(n) > 0$ for integers n sufficiently large. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let

$$E_k(x) = \sum_{n \leqslant x} J_k(n) - \frac{x^{k+1}}{(k+1)\,\zeta(k+1)}, \tag{1.1}$$

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where $J_k(n)$, the Jordan totient function, is defined by

$$J_k(n) = n^k \prod_{p \mid n} \left(1 - \frac{1}{p^k}\right). \tag{1.2}$$

For $k=1$, $J_1(n)$ is the Euler's totient function $\phi(n)$ and the investigation of the error term $E_1(x)$ has a long history. Sylvester conjectured in 1883 [6, 7] that $E_1(n)>0$ for all positive integers n. The conjecture was wrong ($n=820$ is a counter example, as noted by Sarma [5]). In 1950, Erdős and Shapiro [2] showed that $E_1(n)$ changes sign infinitely often. In fact, they proved the stronger result $E_1(n) = \Omega_{\pm}(n \log\log\log\log n)$.

(Before this, Pillai and Chowla [4] had proved that $E_1(x) = \Omega(x \log\log\log x)$ and

$$\sum_{n \leqslant x} E_1(n) \sim \frac{3}{2\pi^2} x^2).$$

For $k \geqslant 2$, we prove the following theorems.

THEOREM 1. *We have*

$$\sum_{n \leqslant x} E_k(n) \sim \frac{x^{k+1}}{2(k+1)\,\zeta(k+1)}.$$

COROLLARY 1.1. *We have*

$$\limsup_{n \to \infty} \frac{E_k(n)}{n^k} \geqslant \frac{1}{2\zeta(k+1)}.$$

THEOREM 2. *For real* x,

$$\liminf_{x \to \infty} \frac{E_k(x)}{x^k} \leqslant -\frac{1}{2\zeta(k+1)}.$$

As regards the sign change for the error term $E_k(n)$ at integer points, the situation in the cases $k \geqslant 2$ is quite different from that of $k=1$, as can be seen from

THEOREM 3. *There is* $n_k > 0$ *such that* $E_k(n) > 0$ *for all integers* $n \geqslant n_k$.

THEOREM 4. *We have*

$$\limsup_{x \to \infty} \frac{E_k(x)}{x^k} \leqslant \frac{D}{\zeta(k+1)}$$

where $D = .7159$ *when* $k=2$, $.6063$ *when* $k \geqslant 3$.

We prove Theorems 1, 2, and 3 in Section 4. Theorem 4 is proved in Section 5. We should mention here that the technique of averaging over arithmetic progressions, which was developed by Erdős and Shapiro [2] to deal with the case $k=1$ and which was later used successfully for proving Ω-results for error terms related to different arithmetic functions by Petermann and others (see [3] and [1]), does not seem to give the best possible results here (see Remark 2, Section 5). In our proof of Theorem 4 we use some ad hoc arguments instead.

2. Preliminaries

From Definition (1.2), it is clear that $J_k(n)$ is multiplicative and

$$\sum_{n=1}^{\infty} \frac{J_k(n)}{n^s} = \frac{\zeta(s-k)}{\zeta(s)} \tag{2.1}$$

for $\sigma > k+1$ $(s=\sigma+it)$.

From (2.1) it follows that

$$J_k(n) = \sum_{d \mid n} \mu(d) \left(\frac{n}{d}\right)^k \tag{2.2}$$

and

$$n^k = \sum_{d \mid n} J_k(d). \tag{2.3}$$

Notation

Symbols x and n will represent real and integer variables, respectively, and k will be an integer $\geqslant 2$. $[x]$ and $\{x\}$ will respectively stand for the integral part and the fractional part of x. For any two integers m and n, (m, n) denotes the g.c.d. of m and n.

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In this section, we first prove some lemmas.

Lemma 3.1. *Let $\lambda \geqslant 1$ be an integer. Then*

(a) *We have*

$$\sum_{n \leqslant x} n^{\lambda} = \frac{x^{\lambda+1}}{\lambda+1} + (\tfrac{1}{2} - \{x\})\, x^{\lambda} + O(x^{\lambda-1}).$$

(b) *For integers j and d* ($0 \leqslant j < d$), *we have*

$$\sum_{\substack{n \leqslant x \\ n \equiv j(d)}} n^{\lambda} = \frac{x^{\lambda+1}}{(\lambda+1)d} + O(x^{\lambda}).$$

(c) *For integers j, d, and A* ($0 \leqslant j < d$, $A \geqslant 1$), *we have*

$$\sum_{\substack{n \leqslant x \\ An \equiv j(d)}} n^{\lambda} = \begin{cases} \dfrac{x^{\lambda+1}}{\lambda+1}\dfrac{(A,d)}{d} + O(x^{\lambda}) & \text{if } (A,d) \mid j \\ 0 & \text{otherwise.} \end{cases}$$

(d) *For integers j, d, A, and B* ($0 \leqslant j < d$, $0 \leqslant B < A$) *we have*

$$\sum_{\substack{n \leqslant x \\ An \equiv j(d)}} (An-B)^{\lambda} = \begin{cases} \dfrac{A^{\lambda}x^{\lambda+1}}{\lambda+1} \cdot \dfrac{(A,d)}{d} + O(A^{\lambda}x^{\lambda}) & \text{if } (A,d) \mid j \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (a) and (b) are standard results. To prove (c), we note that $An \equiv j \pmod{d}$ has no solution if $(A,d) \nmid j$. If $(A,d) \mid j$, then $An \equiv j \pmod{d}$ is equivalent to $n \equiv \beta \pmod{d/(A,d)}$ for some β and we now apply (b). (d) follows from (c), since

$$(An-B)^{\lambda} = A^{\lambda}n^{\lambda} + O(A^{\lambda}n^{\lambda-1}).$$

LEMMA 3.2. *For given positive integers A and d, we have*

$$\sum_{\substack{j=0 \\ (A,d) \mid j}}^{d-1} \left(\frac{1}{2} - \frac{j}{d}\right) = \frac{1}{2}.$$

Proof. It is clear that

$$\sum_{j=0}^{\lambda-1} \left(\frac{1}{2} - \frac{j}{\lambda}\right) = \frac{1}{2}. \tag{3.1}$$

If $(A,d) \mid j$ and $0 \leqslant j \leqslant d-1$, then j looks like $j = r(A,d)$, where $0 \leqslant r \leqslant d/(A,d) - 1$.

$$\therefore \sum_{\substack{j=0 \\ (A,d) \mid j}}^{d-1} \left(\frac{1}{2} - \frac{j}{d}\right) = \sum_{r=0}^{d/(A,d)-1} \left(\frac{1}{2} - \frac{r(A,d)}{d}\right)$$

$$= \frac{1}{2} \qquad \text{(by 3.1)}.$$

LEMMA 3.3. *Let $f(d)$ be any multiplicative arithmetic function such that*

$$\sum_{d=1}^{\infty} \frac{f(d)}{d^k} < \infty.$$

Let $F_{k,f}(n) = \sum_{d \mid n} f(d)(n/d)^k$ and

$$E_{k,f}(x) = \sum_{n \leqslant x} F_{k,f}(n) - \frac{x^{k+1}}{k+1} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}. \tag{3.2}$$

Then we have

$$E_{k,f}(x) = x^k \sum_{d=1}^{\infty} \frac{f(d)}{d^k} \left(\frac{1}{2} - \left\{ \frac{x}{d} \right\} \right) + o(x^k).$$

Proof. We have

$$\begin{aligned}
\sum_{n \leqslant x} F_{k,f}(n) &= \sum_{n \leqslant x} \sum_{d \mid n} f(d) \left(\frac{n}{d} \right)^k \\
&= \sum_{md \leqslant x} f(d)\, m^k \\
&= \sum_{d \leqslant x} f(d) \sum_{m \leqslant x/d} m^k \\
&= \sum_{d \leqslant x} f(d) \left(\frac{x^{k+1}}{(k+1)\, d^{k+1}} + \left(\frac{1}{2} - \left\{ \frac{x}{d} \right\} \right) \left(\frac{x}{d} \right)^k + O\left(\left(\frac{x}{d} \right)^{k-1} \right) \right) \\
&= \frac{x^{k+1}}{k+1} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} + x^k \sum_{d=1}^{\infty} \frac{f(d)}{d^k} \left(\frac{1}{2} - \left\{ \frac{x}{d} \right\} \right) + o(x^k).
\end{aligned}$$

which proves the lemma, since

$$\begin{aligned}
x^{k-1} \sum_{d \leqslant x} \frac{f(d)}{d^{k-1}} &= x^{k-1} \left\{ \sum_{d \leqslant \sqrt{x}} \frac{f(d)}{d^{k-1}} + \sum_{x \geqslant d \geqslant \sqrt{x}} \frac{f(d)}{d^{k-1}} \right\} \\
&= O(x^{k-1/2}) + \varepsilon x^k \left(\because \frac{x}{d} > 1 \right)
\end{aligned}$$

and $\varepsilon > 0$ small.

LEMMA 3.4. *If*

$$H_{k,f}(x) = \sum_{n \leqslant x} \frac{F_{k,f}(n)}{n} - \frac{x^k}{k} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}$$

then

$$H_{k,f}(x)=x^{k-1}\sum_{d=1}^{\infty}\frac{f(d)}{d^k}\left(\frac{1}{2}-\left\{\frac{x}{d}\right\}\right)+o(x^{k-1}).$$

Proof. Proof follows by a similar argument as in Lemma 3.3.

Remark. From Lemmas 3.3 and 3.4 we have

$$E_{k,f}(x)=xH_{k,f}(x)+o(x^k). \tag{3.3}$$

LEMMA 3.5. *For integers A and B $(0\leqslant B<A)$ and f as in Lemma* 3.3, *if*

$$S=\sum_{n\leqslant x}\sum_{d=1}^{\infty}\frac{f(d)}{d^k}\left(\frac{1}{2}-\left\{\frac{An}{d}\right\}\right)(An-B)^{k-1},$$

then we have

$$S=\frac{A^{k-1}x^k}{2k}\sum_{d=1}^{\infty}\frac{f(d)}{d^{k+1}}(A,d)+O(A^{k-1}x^{k-1}).$$

Proof. We have

$$\begin{aligned}S&=\sum_{d=1}^{\infty}\frac{f(d)}{d^k}\sum_{j=0}^{d-1}\left(\frac{1}{2}-\frac{j}{d}\right)\sum_{\substack{n\leqslant x\\ An\equiv j\ (\mathrm{mod}\ d)}}(An-B)^{k-1}\\ &=\sum_{d=1}^{\infty}\frac{f(d)}{d^k}\sum_{\substack{j=0\\ (A,d)\mid j}}^{d-1}\left(\frac{1}{2}-\frac{j}{d}\right)\left(\frac{A^{k-1}x^k}{k}\frac{(A,d)}{d}+O(A^{k-1}x^{k-1})\right)\\ &\qquad\text{(by Lemma 3.1(d))}\\ &=\frac{A^{k-1}x^k}{2k}\sum_{d=1}^{\infty}\frac{f(d)}{d^{k+1}}(A,d)+O(A^{k-1}x^{k-1})\qquad\text{(by Lemma 3.2).}\end{aligned}$$

Note. Henceforth the symbols $F_{k,f}(n)$, $E_{k,f}(x)$, $H_{k,f}(x)$ will be used with the assumptions on f as in Lemma 3.3.

LEMMA 3.6. *We have*

$$\sum_{n\leqslant x}H_{k,f}(n)=\frac{x^k}{2k}\sum_{d=1}^{\infty}\frac{f(d)}{d^{k+1}}+o(x^k).$$

Proof. By Lemma 3.4, we have

$$\sum_{n\leqslant x}H_{k,f}(n)=\sum_{n\leqslant x}\left(n^{k-1}\sum_{d=1}^{\infty}\frac{f(d)}{d^k}\left(\frac{1}{2}-\left\{\frac{n}{d}\right\}\right)+o(n^{k-1})\right)$$

and now the result follows by application of Lemma 3.5 with $A=1$ and $B=0$.

LEMMA 3.7. *We have*

$$\sum_{n\leqslant x} E_{k,f}(n) \sim \frac{x^{k+1}}{2(k+1)} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}.$$

Proof. Proof follows on using Lemmas 3.3 and 3.5 with $A=1$ and $B=0$.

LEMMA 3.8. *If* $\sum_{d=1}^{\infty} f(d)/d^{k+1} > 0$ *(respectively* <0*), then we have*

$$\limsup_{x\to\infty} \frac{E_{k,f}(x)}{x^k} \geqslant \frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} \qquad \left(\textit{respectively} \geqslant -\frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}\right)$$

and

$$\liminf_{x\to\infty} \frac{E_{k,f}(x)}{x^k} \leqslant -\frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} \qquad \left(\textit{respectively} \leqslant \frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}\right).$$

Proof. Suppose $\sum_{d=1}^{\infty} f(d)/d^{k+1} > 0$. Then Lemma 3.7 implies that

$$\limsup_{n\to\infty} \frac{E_{k,f}(n)}{n^k} \geqslant \frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}.$$

It also follows from Lemma 3.7 that, for infinitely many positive integers n, we have

$$E_{k,f}(n) \leqslant \frac{n^k}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} + o(n^k). \tag{3.4}$$

If x lies in the open interval $(n, n+1)$, then from Eq. (3.2) we have

$$\begin{aligned}\lim_{\theta\to 1^-} (E_{k,f}(n+\theta) - E_{k,f}(n)) &= \left(-\sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}\right)\left(\frac{(n+1)^{k+1} - n^{k+1}}{k+1}\right) \\ &= -n^k \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} + o(n^k).\end{aligned}$$

From the inequality (3.4), it follows that $E_{k,f}(n)$ becomes negative between n and $n+1$. More precisely,

$$\liminf_{x\to\infty} \frac{E_k(x)}{x^k} \leqslant -\frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}.$$

The case $\sum_{d=1}^{\infty} f(d)/d^{k+1} < 0$ can be dealt with in the same way.

4. Proof of Theorems 1, 2, and 3

If we take $\mu(d)=f(d)$, all the conditions for $f(d)$ in Lemma 3.3 are satisfied with $k\geqslant 2$. Also we have $F_{k,\mu}(n)=J_k(n)$ and $E_{k,\mu}(x)=E_k(x)$.

Now Theorems 1 and 2 follow respectively from Lemmas 3.7 and 3.8. We have

$$E_k(n)=n^k\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k}\left(\frac{1}{2}-\left\{\frac{n}{d}\right\}\right)+o(n^k) \tag{4.1}$$

$$\frac{1}{2}-\left\{\frac{n}{d}\right\}\begin{cases}=\frac{1}{2} & \text{if } d=1\\ >-\frac{1}{2} & \text{if } d\geqslant 2.\end{cases}$$

Therefore,

$$\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k}\left(\frac{1}{2}-\left\{\frac{n}{d}\right\}\right)\geqslant\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots\right)$$
$$=\frac{1}{2}(1+1-\zeta(k))>1-\frac{\zeta(2)}{2}=1-\frac{\pi^2}{12}>0$$

and (4.1) implies Theorem 3.

5. Proof of Theorem 4

Writing $H_k(x)=H_{k,\mu}(x)$, we have from Lemma 3.4

$$H_k(n)=n^{k-1}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^k}\left(\frac{1}{2}-\left\{\frac{n}{d}\right\}\right)+o(n^{k-1}). \tag{5.1}$$

Also, from (3.2),

$$E_k(x)=xH_k(x)+o(x^k). \tag{5.2}$$

Now,

$$-\sum_{d=1}^{\infty}\left\{\frac{n}{d}\right\}\frac{\mu(d)}{d^k}\leqslant\sum_{p\leqslant 100}\frac{p-1}{p}\cdot\frac{1}{p^k}+\sum_{p_1,p_2,p_3\leqslant 100}\frac{p_1p_2p_3-1}{(p_1p_2p_3)^{k+1}}$$
$$+\frac{1}{10^{2k-4}}\sum_{d\geqslant 101}\frac{1}{d^2}$$
$$\leqslant\sum_{p\leqslant 100}\frac{1}{p^k}-\sum_{p\leqslant 100}\frac{1}{p^{k+1}}+\frac{5}{(30)^k}$$
$$+\frac{1}{10^{2k-4}}\sum_{d\geqslant 101}\left(\frac{1}{d-1}-\frac{1}{d}\right). \tag{5.3}$$

Case I. $k=2$. We have

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2}\left(\frac{1}{2}-\left\{\frac{n}{d}\right\}\right) \leqslant \frac{1}{2\zeta(2)} + \sum_{p\leqslant 100} \frac{1}{p^2} - \sum_{p\leqslant 100} \frac{1}{p^3} + \frac{5}{(30)^2} + .01 \qquad (\text{by } (5.3))$$

$$\leqslant \frac{.5\times\frac{\zeta(3)}{\zeta(2)} + .2915\times\zeta(3)}{\zeta(3)}$$

$$\leqslant \frac{.71582}{\zeta(3)} \qquad (\text{by numerical computations}). \qquad (5.4)$$

Case II. $k \geqslant 3$. Proceeding similarly,

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^k}\left(\frac{1}{2}-\left\{\frac{n}{d}\right\}\right) \leqslant \frac{1}{2\zeta(k)} + \sum_{p\leqslant 100} \frac{1}{p^3} - \sum_{p\leqslant 100} \frac{1}{p^4} + \frac{5}{(30)^3} + .0001$$

$$\leqslant \frac{.60628}{\zeta(k+1)}. \qquad (5.5)$$

Now, Theorem 4 follows from (5.1), (5.2), (5.4), (5.5), and the fact that $E_k(x)$ decreases between two consecutive integers.

Remarks. 1. Since

$$\left| \sum_{\substack{d\leqslant 100\\ d\neq p\\ d\neq p_1p_2p_3}} \left\{\frac{n}{d}\right\} \frac{\mu(d)}{d^2} \right| \leqslant .0711$$

(by numerical computations), it is clear from the calculations in Case I, that

$$-\sum_{d=1}^{\infty} \left\{\frac{n}{d}\right\} \frac{\mu(d)}{d^2} \geqslant .2$$

and therefore

$$\sum_{d=1}^{\infty} \left(\frac{1}{2}-\left\{\frac{n}{d}\right\}\right) \frac{\mu(d)}{d^2} \geqslant \frac{.35+.2\times 1.2}{\zeta(3)} = \frac{.59}{\zeta(3)}.$$

Thus, for $k=2$, we have

$$\limsup_{n\to\infty} \frac{E_k(n)}{n^2} \geqslant \frac{.59}{\zeta(3)} \qquad (\text{cf. Corollary 1.1}).$$

With more careful calculations this can be improved slightly and the lower bound for $\limsup_{n\to\infty} E_k(n)/n^k$ can be improved for other small k's in a similar way.

2. We give an outline of the technique of averaging over arithmetic progressions which yields a result weaker than Theorem 4.

One proves

LEMMA *. *For integers* $0 \leqslant \beta < A$,

$$\sum_{\substack{m \leqslant z \\ m \equiv \beta(A)}} \frac{J_k(m)}{m} = \frac{C(A)\, z^k}{kA} \sum_{d \mid (A,\beta)} \frac{\mu(d)}{d^k} + o(z^k),$$

where $C(A) = \prod_{p \nmid A}(1 - 1/p^{k+1}) > 0$.

Then we get

LEMMA **. *For integers* $0 < B < A$,

$$\sum_{n \leqslant x} H_k(An - B) = \frac{A^{k-1}x^k}{k}\left[\frac{B}{\zeta(k+1)} + \frac{C(A)}{2}\sum_{d \mid A}\frac{\mu(d)}{d^k} - C(A)\sum_{c=0}^{B-1}\sum_{d \mid (A,c)}\frac{\mu(d)}{d^k}\right] + O(A^{k-1}Bx^{k-1}) + o(A^{k-1}x^k).$$

Now, if we choose B to be a large positive integer, $A = \prod_{p<B} p^{[(\log B)/(\log 2)]}$ and $x = A^2$, for $1 \leqslant C < B$, $(A, C) = C$ and hence from Lemma **,

$$\sum_{n \leqslant x} H(An - B) = \frac{A^{k-1}x^k}{k}\left[\frac{B}{\zeta(k+1)} - \frac{C(A)}{2}\sum_{d \mid A}\frac{\mu(d)}{d^k} - C(A)\sum_{C=1}^{B-a}\sum_{d \mid c}\frac{\mu(d)}{d^k}\right] + o(A^{k-1}x^k) + O(A^{k-1}x^{k-1}B)$$

which leads to

$$\sum_{n \leqslant x} H(An - B) = \frac{C(A)\, A^{k-1}x^k}{k}\left[\frac{1}{\zeta(k+1)} - \frac{H(B-1)}{(B-1)^{k-1}} + O\left(\frac{1}{B^{k-1}}\right)\right] + o(A^{k-1}x^k) + O(A^{k-1}x^{k-1}B).$$

Since $C(A) \to 1/\zeta(k+1)$ as $B \to \infty$,

$$\frac{H(B-1)}{(B-1)^{k-1}} > \frac{1}{\zeta(k+1)} + \varepsilon \qquad \text{for} \quad \varepsilon > 0$$

and for infinitely many B's this would imply that $H(m)<0$ for infinitely many m's, which is false, as is apparent from the proof of Theorem 3.

Hence,

$$\frac{H_k(n)}{n^{k-1}} \leqslant \frac{1}{\zeta(k+1)} + \varepsilon$$

for $n \geqslant n_k$ for some $n_k > 0$, which gives

$$\limsup_{x \to \infty} \frac{E_k(x)}{x^k} \leqslant \frac{1}{\zeta(k+1)}.$$

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