On an Error Term Related to the Jordan Totient Function $J_k(n)$

SUKUMAR DAS ADHIKARI

The Institute of Mathematical Sciences, Madras 600 113, India

AND

A. SANKARANARAYANAN

School of Mathematics, Tata Institute of Fundamental Research, Colaba, Bombay 400 005, India

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We investigate the error terms

$$E_k(x) = \sum_{n \leq x} J_k(n) - \frac{x^{k+1}}{(k+1) \zeta(k+1)}$$

for $k \geq 2$, where $J_k(n) = n^k \prod_{p \mid n} (1 - 1/p^k)$ for $k \geq 1$. For $k \geq 2$, we prove

$$\sum_{n \leq x} E_k(n) \sim \frac{x^{k+1}}{2(k+1) \zeta(k+1)}.$$

Also,

$$\limsup_{x \to \infty} \frac{E_k(x)}{x^k} \leq \frac{D}{\zeta(k+1)},$$

where $D = .7159$ when $k = 2, .6063$ when $k \geq 3$. On the other hand, even though

$$\liminf_{x \to \infty} \frac{E_k(x)}{x^k} \leq -\frac{1}{2 \zeta(k+1)},$$

$E_k(n) > 0$ for integers $n$ sufficiently large. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let

$$E_k(x) = \sum_{n \leq x} J_k(n) - \frac{x^{k+1}}{(k+1) \zeta(k+1)},$$

(1.1)
where $J_k(n)$, the Jordan totient function, is defined by

$$J_k(n) = n^k \prod_{p \mid n} \left(1 - \frac{1}{p^k}\right). \quad (1.2)$$

For $k = 1$, $J_1(n)$ is the Euler's totient function $\phi(n)$ and the investigation of the error term $E_1(x)$ has a long history. Sylvester conjectured in 1883 [6, 7] that $E_1(n) > 0$ for all positive integers $n$. The conjecture was wrong ($n = 820$ is a counter example, as noted by Sarma [S]). In 1950, Erdős and Shapiro [2] showed that $E_1(n)$ changes sign infinitely often. In fact, they proved the stronger result $E_1(n) = \Omega_+(n \log \log \log \log n)$.

(Before this, Pillai and Chowla [4] had proved that $E_1(x) = \Omega(x \log \log \log x)$ and

$$\sum_{n \leq x} E_1(n) \sim \frac{3}{2\pi^2} x^2.$$  

For $k \geq 2$, we prove the following theorems.

**Theorem 1.** We have

$$\sum_{n \leq x} E_k(n) \sim \frac{x^k + 1}{2(k + 1) \zeta(k + 1)}.$$

**Corollary 1.1.** We have

$$\limsup_{n \to \infty} \frac{E_k(n)}{n^k} \geq \frac{1}{2\zeta(k + 1)}.$$

**Theorem 2.** For real $x$,

$$\liminf_{x \to \infty} \frac{E_k(x)}{x^k} \leq -\frac{1}{2\zeta(k + 1)}.$$

As regards the sign change for the error term $E_k(n)$ at integer points, the situation in the cases $k \geq 2$ is quite different from that of $k = 1$, as can be seen from

**Theorem 3.** There is $n_k > 0$ such that $E_k(n) > 0$ for all integers $n \geq n_k$.

**Theorem 4.** We have

$$\limsup_{x \to \infty} \frac{E_k(x)}{x^k} \leq \frac{D}{\zeta(k + 1)}$$

where $D = .7159$ when $k = 2$, $.6063$ when $k \geq 3$. 

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We prove Theorems 1, 2, and 3 in Section 4. Theorem 4 is proved in Section 5. We should mention here that the technique of averaging over arithmetic progressions, which was developed by Erdős and Shapiro [2] to deal with the case $k = 1$ and which was later used successfully for proving $\Omega$-results for error terms related to different arithmetic functions by Petermann and others (see [3] and [1]), does not seem to give the best possible results here (see Remark 2, Section 5). In our proof of Theorem 4 we use some ad hoc arguments instead.

2. Preliminaries

From Definition (1.2), it is clear that $J_k(n)$ is multiplicative and

$$\sum_{n=1}^{\infty} \frac{J_k(n)}{n^s} = \frac{\zeta(s-k)}{\zeta(s)}$$

for $\sigma > k + 1$ ($s = \sigma + it$).

From (2.1) it follows that

$$J_k(n) = \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^k$$

and

$$n^k = \sum_{d|n} J_k(d).$$

Notation

Symbols $x$ and $n$ will represent real and integer variables, respectively, and $k$ will be an integer $\geq 2$. $[x]$ and $\{x\}$ will respectively stand for the integral part and the fractional part of $x$. For any two integers $m$ and $n$, $(m, n)$ denotes the g.c.d. of $m$ and $n$.

In this section, we first prove some lemmas.

Lemma 3.1. Let $\lambda \geq 1$ be an integer. Then

(a) We have

$$\sum_{n \leq x} n^\lambda = \frac{x^{\lambda+1}}{\lambda+1} + \left(\frac{1}{2} - \{x\}\right)x^\lambda + O(x^{\lambda-1}).$$
(b) For integers \( j \) and \( d \) (\( 0 \leq j < d \)), we have
\[
\sum_{\substack{n \leq x \\mod d\equiv j(d)\}} n^j = \frac{x^{j+1}}{(j+1)d} + O(x^j).
\]

(c) For integers \( j, d, \) and \( A \) (\( 0 \leq j < d, A \geq 1 \)), we have
\[
\sum_{\substack{n \leq x \\mod d\equiv j(d)\}} n^j = \begin{cases} 
\frac{x^{j+1}}{(j+1)d} + O(x^j) & \text{if } (A, d) \mid j \\
0 & \text{otherwise}.
\end{cases}
\]

(d) For integers \( j, d, A, \) and \( B \) (\( 0 \leq j < d, 0 \leq B < A \)) we have
\[
\sum_{\substack{n \leq x \\mod d\equiv j(d)\}} (An-B)^j = \begin{cases} 
\frac{A^j x^{j+1}}{(j+1)d} + O(A^j x^j) & \text{if } (A, d) \mid j \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. (a) and (b) are standard results. To prove (c), we note that \( An \equiv j \mod d \) has no solution if \( (A, d) \nmid j \). If \( (A, d) \mid j \), then \( An \equiv j \mod d \) is equivalent to \( n \equiv \beta \mod d/(A, d) \) for some \( \beta \) and we now apply (b). (d) follows from (c), since
\[
(An-B)^j = A^j n^j + O(A^j n^{j-1}).
\]

Lemma 3.2. For given positive integers \( A \) and \( d \), we have
\[
\sum_{\substack{j=0 \\\mod d/(A, d)\mid j}} \left( \frac{1}{2} - \frac{j}{d} \right) = \frac{1}{2}.
\]

Proof. It is clear that
\[
\sum_{\substack{j=0 \\\mod d/(A, d)\mid j}} \left( \frac{1}{2} - \frac{j}{\lambda} \right) = \frac{1}{2}. \tag{3.1}
\]

If \( (A, d) \mid j \) and \( 0 \leq j \leq d-1 \), then \( j \) looks like \( j = r(A, d) \), where \( 0 \leq r \leq d/(A, d) - 1 \).
\[
\sum_{\substack{j=0 \\\mod d/(A, d)\mid j}} \left( \frac{1}{2} - \frac{j}{d} \right) = \sum_{r=0}^{d/(A, d)-1} \left( \frac{1}{2} - \frac{r(A, d)}{d} \right) = \frac{1}{2} \quad \text{(by 3.1).}
\]
Lemma 3.3. Let \( f(d) \) be any multiplicative arithmetic function such that
\[
\sum_{d=1}^{\infty} \frac{f(d)}{d^k} < \infty.
\]

Let \( F_{k,f}(n) = \sum_{d|n} f(d)(n/d)^k \) and
\[
E_{k,f}(x) = \sum_{n \leq x} F_{k,f}(n) - \frac{x^{k+1}}{k+1} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}.
\] (3.2)

Then we have
\[
E_{k,f}(x) = x^k \sum_{d=1}^{\infty} \frac{f(d)}{d^k} \left( \frac{1}{2} - \left( \frac{x}{d} \right) \right) + o(x^k).
\]

Proof. We have
\[
\sum_{n \leq x} F_{k,f}(n) = \sum_{n \leq x} \sum_{d|n} f(d) \left( \frac{n}{d} \right)^k
= \sum_{m \leq x/d} f(d) \sum_{m \leq x/d} m^k
= \sum_{d \leq x} f(d) \sum_{m \leq x/d} m^k
= \sum_{d \leq x} f(d) \left( \frac{x^{k+1}}{(k+1) d^{k+1}} + \left( \frac{1}{2} - \left( \frac{x}{d} \right) \right) \left( \frac{x}{d} \right)^k + O\left( \left( \frac{x}{d} \right)^{k-1} \right) \right)
= \frac{x^{k+1}}{k+1} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} + x^k \sum_{d=1}^{\infty} \frac{f(d)}{d^k} \left( \frac{1}{2} - \left( \frac{x}{d} \right) \right) + o(x^k).
\]

which proves the lemma, since
\[
x^{k-1} \sum_{d \leq x} \frac{f(d)}{d^k} = x^{k-1} \left\{ \sum_{d \leq \sqrt{x}} \frac{f(d)}{d^k} + \sum_{x/d \geq \sqrt{x}} \frac{f(d)}{d^{k-1}} \right\}
= O(x^{k-1/2}) + \varepsilon x^k \left( \frac{x}{d} > 1 \right)
\]
and \( \varepsilon > 0 \) small.

Lemma 3.4. If
\[
H_{k,f}(x) = \sum_{n \leq x} \frac{F_{k,f}(n)}{n} = \frac{x^k}{k} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}
\]
then
\[ H_{k,f}(x) = x^{k-1} \sum_{d=1}^{\infty} \frac{f(d)}{d^k} \left( \frac{1}{2} - \left\lfloor \frac{x}{d} \right\rfloor \right) + o(x^{k-1}). \]

Proof. Proof follows by a similar argument as in Lemma 3.3.

Remark. From Lemmas 3.3 and 3.4 we have
\[ E_{k,f}(x) = xH_{k,f}(x) + o(x^k). \] (3.3)

Lemma 3.5. For integers \( A \) and \( B \) \((0 \leq B < A)\) and \( f \) as in Lemma 3.3, if
\[ S = \sum_{n \leq x} \sum_{d=1}^{\infty} \frac{f(d)}{d^k} \left( \frac{1}{2} - \left\lfloor \frac{An}{d} \right\rfloor \right) (An - B)^{k-1}, \]
then we have
\[ S = \frac{A^{k-1}x^k}{2k} \sum_{d=1}^{\infty} \frac{f(d)}{d^k+1} (A, d) + O(A^{k-1}x^{k-1}). \]

Proof. We have
\[ S = \sum_{d=1}^{\infty} \frac{f(d)}{d^k} \sum_{j=0}^{d-1} \left( \frac{1}{2} - \frac{j}{d} \right) \sum_{n \leq x} \frac{1}{\left( An - j \right)^{k-1}} \]
\[ = \sum_{d=1}^{\infty} \frac{f(d)}{d^k} \sum_{j=0}^{d-1} \left( \frac{1}{2} - \frac{j}{d} \right) \left( \frac{A^{k-1}x^k}{d} \frac{(A, d)}{d} + O(A^{k-1}x^{k-1}) \right) \]
\[ = \frac{A^{k-1}x^k}{2k} \sum_{d=1}^{\infty} \frac{f(d)}{d^k+1} (A, d) + O(A^{k-1}x^{k-1}) \quad \text{(by Lemma 3.2).} \]

Note. Henceforth the symbols \( F_{k,f}(n) \), \( E_{k,f}(x) \), \( H_{k,f}(x) \) will be used with the assumptions on \( f \) as in Lemma 3.3.

Lemma 3.6. We have
\[ \sum_{n \leq x} H_{k,f}(n) = \frac{x^k}{2k} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} + o(x^k). \]

Proof. By Lemma 3.4, we have
\[ \sum_{n \leq x} H_{k,f}(n) = \sum_{n \leq x} \left( n^{k-1} \sum_{d=1}^{\infty} \frac{f(d)}{d^k} \left( \frac{1}{2} - \left\lfloor \frac{n}{d} \right\rfloor \right) + o(n^{k-1}) \right) \]
and now the result follows by application of Lemma 3.5 with \( A = 1 \) and \( B = 0 \).

**Lemma 3.7.** We have

\[
\sum_{n \leq x} E_{k,f}(n) \sim \frac{x^{k+1}}{2(k+1)} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}.
\]

**Proof.** Proof follows on using Lemmas 3.3 and 3.5 with \( A = 1 \) and \( B = 0 \).

**Lemma 3.8.** If \( \sum_{d=1}^{\infty} f(d)/d^{k+1} > 0 \) (respectively \( < 0 \)), then we have

\[
\lim \sup_{x \to \infty} \frac{E_{k,f}(x)}{x^k} \geq \frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} \quad \text{(respectively } \geq -\frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} \text{)}
\]

and

\[
\lim \inf_{x \to \infty} \frac{E_{k,f}(x)}{x^k} \leq -\frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} \quad \text{(respectively } \leq \frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} \text{)}
\]

**Proof.** Suppose \( \sum_{d=1}^{\infty} f(d)/d^{k+1} > 0 \). Then Lemma 3.7 implies that

\[
\lim \sup_{n \to \infty} \frac{E_{k,f}(n)}{n^k} \geq \frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}.
\]

It also follows from Lemma 3.7 that, for infinitely many positive integers \( n \), we have

\[
E_{k,f}(n) \leq \frac{n^k}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} + o(n^k). \quad (3.4)
\]

If \( x \) lies in the open interval \( (n, n+1) \), then from Eq. (3.2) we have

\[
\lim_{\theta \to 1^-} (E_{k,f}(n+\theta) - E_{k,f}(n)) = \left( - \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} \right) \frac{(n+1)^{k+1} - n^{k+1}}{k+1}
\]

\[
= -n^k \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}} + o(n^k).
\]

From the inequality (3.4), it follows that \( E_{k,f}(n) \) becomes negative between \( n \) and \( n+1 \). More precisely,

\[
\lim \inf_{x \to \infty} \frac{E_{k,f}(x)}{x^k} \leq -\frac{1}{2} \sum_{d=1}^{\infty} \frac{f(d)}{d^{k+1}}.
\]

The case \( \sum_{d=1}^{\infty} f(d)/d^{k+1} < 0 \) can be dealt with in the same way.
4. Proof of Theorems 1, 2, and 3

If we take $\mu(d) = f(d)$, all the conditions for $f(d)$ in Lemma 3.3 are satisfied with $k \geq 2$. Also we have $F_{k, \mu}(n) = J_k(n)$ and $E_{k, \mu}(x) = E_k(x)$.

Now Theorems 1 and 2 follow respectively from Lemmas 3.7 and 3.8. We have

$$E_k(n) = n^k \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \left( \frac{1}{2} - \frac{n}{d} \right) + o(n^k) \quad (4.1)$$

$$\frac{1}{2} - \frac{n}{d} \begin{cases} \frac{1}{2} & \text{if } d = 1 \\ > - \frac{1}{2} & \text{if } d \geq 2. \end{cases}$$

Therefore,

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \left( \frac{1}{2} - \frac{n}{d} \right) \geq \frac{1}{2} - \frac{1}{2} \left( \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots \right)$$

$$= \frac{1}{2} \left( 1 + 1 - \zeta(k) \right) > 1 - \frac{\zeta(2)}{2} = 1 - \frac{\pi^2}{12} > 0$$

and (4.1) implies Theorem 3.

5. Proof of Theorem 4

Writing $H_k(x) = H_{k, \mu}(x)$, we have from Lemma 3.4

$$H_k(n) = n^{k-1} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \left( \frac{1}{2} - \frac{n}{d} \right) + o(n^{k-1}). \quad (5.1)$$

Also, from (3.2),

$$E_k(x) = xH_k(x) + o(x^k). \quad (5.2)$$

Now,

$$- \sum_{d=1}^{\infty} \left\{ \frac{n}{d} \right\} \frac{\mu(d)}{d^k} \leq \sum_{p \leq 100} \frac{p-1}{p} \cdot \frac{1}{p^k} + \sum_{p_1, p_2, p_3 \leq 100} \frac{p_1 p_2 p_3 - 1}{(p_1 p_2 p_3)^{k+1}}$$

$$+ \frac{1}{10^{2k-4}} \sum_{d \geq 101} \frac{1}{d^2}$$

$$\leq \sum_{p \leq 100} \frac{1}{p^k} - \sum_{p \leq 100} \frac{1}{p^{k+1}} + \frac{5}{(30)^k}$$

$$+ \frac{1}{10^{2k-4}} \sum_{d \geq 101} \left( \frac{1}{d-1} - \frac{1}{d} \right). \quad (5.3)$$
Case I. \( k = 2 \). We have
\[
\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \left( \frac{1}{2} - \frac{n}{d} \right) \leq \frac{1}{2\zeta(2)} + \sum_{p \leq 100} \frac{1}{p^2} - \sum_{p \leq 100} \frac{1}{p^3} + \frac{5}{(30)^2} \cdot .01 \quad \text{(by (5.3))}
\]
\[
\leq \frac{.5 \times \zeta(3)}{\zeta(2)} + .2915 \times \zeta(3) \leq \frac{.71582}{\zeta(3)} \quad \text{(by numerical computations).} \quad (5.4)
\]

Case II. \( k \geq 3 \). Proceeding similarly,
\[
\sum_{d=1}^{\infty} \frac{\mu(d)}{d^k} \left( \frac{1}{2} - \frac{n}{d} \right) \leq \frac{1}{2\zeta(k)} + \sum_{p \leq 100} \frac{1}{p^3} - \sum_{p \leq 100} \frac{1}{p^4} + \frac{5}{(30)^3} + .0001
\]
\[
\leq \frac{.60628}{\zeta(k + 1)}. \quad (5.5)
\]

Now, Theorem 4 follows from (5.1), (5.2), (5.4), (5.5), and the fact that \( E_k(x) \) decreases between two consecutive integers.

Remarks. 1. Since
\[
\left| \sum_{d \leq 100} \frac{n}{d} \frac{\mu(d)}{d^2} \right| \leq .0711
\]
(by numerical computations), it is clear from the calculations in Case I, that
\[
- \sum_{d=1}^{\infty} \frac{n}{d} \frac{\mu(d)}{d^2} \geq .2
\]
and therefore
\[
\sum_{d=1}^{\infty} \left( \frac{1}{2} - \frac{n}{d} \right) \frac{\mu(d)}{d^2} \geq .35 + .2 \times 1.2 = .59 \frac{\zeta(3)}{\zeta(3)}
\]
Thus, for \( k = 2 \), we have
\[
\lim_{n \to \infty} \sup \frac{E_k(n)}{n^2} \geq \frac{.59}{\zeta(3)} \quad \text{(cf. Corollary 1.1)}. 
\]
With more careful calculations this can be improved slightly and the lower bound for \( \limsup_{n \to \infty} E_k(n)/n^k \) can be improved for other small \( k \)'s in a similar way.

2. We give an outline of the technique of averaging over arithmetic progressions which yields a result weaker than Theorem 4.

One proves

**Lemma **. For integers \( 0 \leq \beta < A \),

\[
\sum_{m \leq z \atop m = \beta(A)} \frac{J_k(m)}{m} = \frac{C(A)}{kA} \sum_{d \mid (A, \beta)} \frac{\mu(d)}{d^k} + o(z^k),
\]

where \( C(A) = \prod_{p \mid A} (1 - 1/p^{k+1}) > 0 \).

Then we get

**Lemma **. For integers \( 0 < B < A \),

\[
\sum_{n \leq x} H_k(An - B) = \frac{A^{k-1}x^k}{k} \left[ \frac{B}{\zeta(k+1)} + \frac{C(A)}{2} \sum_{d \mid A} \frac{\mu(d)}{d^k} \right]
\]

\[
- C(A) \sum_{c = 0}^{B-1} \sum_{d \mid (A, c)} \frac{\mu(d)}{d^k} \right]
\]

\[+ O(A^{k-1}Bx^{k-1}) + o(A^{k-1}x^k). \]

Now, if we choose \( B \) to be a large positive integer, \( A = \prod_{p < B} p^{[(\log B)/(\log 2)]} \) and \( x = A^2 \), for \( 1 \leq C < B \), \( (A, C) = C \) and hence from Lemma **,

\[
\sum_{n \leq x} H(An - B) = \frac{A^{k-1}x^k}{k} \left[ \frac{B}{\zeta(k+1)} - \frac{C(A)}{2} \sum_{d \mid A} \frac{\mu(d)}{d^k} \right]
\]

\[- C(A) \sum_{c = 1}^{B-a} \sum_{d \mid c} \frac{\mu(d)}{d^k} \right]
\]

\[+ o(A^{k-1}x^k) + O(A^{k-1}x^{k-1}B) \]

which leads to

\[
\sum_{n \leq x} H(An - B) = \frac{C(A)}{k} \frac{A^{k-1}x^k}{k} \left[ \frac{1}{\zeta(k+1)} - \frac{H(B-1)}{(B-1)^{k-1}} + O \left( \frac{1}{B^{k-1}} \right) \right]
\]

\[+ o(A^{k-1}x^k) + O(A^{k-1}x^{k-1}B). \]

Since \( C(A) \to 1/\zeta(k+1) \) as \( B \to \infty \),

\[
\frac{H(B-1)}{(B-1)^{k-1}} \geq \frac{1}{\zeta(k+1)} + \varepsilon \quad \text{for} \quad \varepsilon > 0
\]
and for infinitely many \( B \)'s this would imply that \( H(m) < 0 \) for infinitely many \( m \)'s, which is false, as is apparent from the proof of Theorem 3.

Hence,

\[
\frac{H_k(n)}{n^{k-1}} \leq \frac{1}{\zeta(k+1)} + \varepsilon
\]

for \( n \geq n_k \) for some \( n_k > 0 \), which gives

\[
\limsup_{x \to \infty} \frac{E_k(x)}{x^k} \leq \frac{1}{\zeta(k+1)}.
\]

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References