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Discreteness of space from GUP II: Relativistic wave equations

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ABSTRACT

Various theories of Quantum Gravity predict modifications of the Heisenberg Uncertainty Principle near the Planck scale to a so-called Generalized Uncertainty Principle (GUP). In some recent papers, we showed that the GUP gives rise to corrections to the Schrödinger equation, which in turn affect all quantum mechanical Hamiltonians. In particular, by applying it to a particle in a one-dimensional box, we showed that the box length must be quantized in terms of a fundamental length (which could be the Planck length), which we interpreted as a signal of fundamental discreteness of space itself. In this Letter, we extend the above results to a relativistic particle in a rectangular as well as a spherical box, by solving the GUP-corrected Klein–Gordon and Dirac equations, and for the latter, to two and three dimensions. We again arrive at quantization of box length, area and volume and an indication of the fundamentally grainy nature of space. We discuss possible implications.

Various approaches to quantum gravity (such as String Theory and Doubly Special Relativity (or DSR) Theories), as well as black hole physics, predict a minimum measurable length, and a modification of the Heisenberg Uncertainty Principle to a so-called Generalized Uncertainty Principle, or GUP, and a corresponding modification of the commutation relations between position coordinates and momenta. The following GUP which we proposed in [1] is (and as far as we know the only one) consistent with DSR theories, String Theory and Black Holes Physics *and* which ensure $[x_i, x_j] = 0 = [p_i, p_j]$ (via the Jacobi identity)[1]

$$[x_i, p_j] = i\hbar\left[\delta_{ij} - a\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + a^2\left(p^2\delta_{ij} + 3p_i p_j\right)\right], \tag{1}$$

$$\begin{aligned}\Delta x\Delta p &\geqslant \frac{\hbar}{2}\left[1 - 2a\langle p\rangle + 4a^2\langle p^2\rangle\right] \\ &\geqslant \frac{\hbar}{2}\left[1 + \left(\frac{a}{\sqrt{\langle p^2\rangle}} + 4a^2\right)\Delta p^2 + 4a^2\langle p\rangle^2 - 2a\sqrt{\langle p^2\rangle}\right]\end{aligned} \tag{2}$$

where $a = a_0/M_{Pl}c = a_0\ell_{Pl}/\hbar$, M_{Pl} = Planck mass, $\ell_{Pl} \approx 10^{-35}$ m = Planck length, and $M_{Pl}c^2$ = Planck energy $\approx 10^{19}$ GeV. It should be stressed that the GUP-induced terms become important near the Planck scale. It is normally assumed that $a_0 \approx 1$. (For earlier versions of GUP, motivated by String Theory, Black Hole Physics, DSR, etc., see e.g. [2–7], and for some phenomenological implications see [8,9,1].) Note that although Eqs. (1) and (2) are not Lorentz covariant, they are at least approximately covariant under DSR transformations [7]. We expect the results of our Letter to have similar covariance as well. In addition, since DSR transformations preserve not only the speed of light, but also the Planck momentum and the Planck length, it is not surprising that Eqs. (1) and (2) imply the following minimum measurable length *and* maximum measurable momentum

$$\Delta x \geqslant (\Delta x)_{\min} \approx a_0\ell_{Pl}, \tag{3}$$

$$\Delta p \leqslant (\Delta p)_{\max} \approx \frac{M_{Pl}c}{a_0}. \tag{4}$$

It can be shown that the following definitions

$$x_i = x_{0i}, \qquad p_i = p_{0i}\left(1 - ap_0 + 2a^2p_0^2\right) \tag{5}$$

(with x_{0i}, p_{0j} satisfying the canonical commutation relations $[x_{0i}, p_{0j}] = i\hbar\delta_{ij}$, such that $p_{0i} = -i\hbar\partial/\partial x_{0i}$) satisfy Eq. (1). In [1] we showed that any non-relativistic Hamiltonian of the form $H = p^2/2m + V(\vec{r})$ can be written as $H = p_0^2/2m - (a/m)p_0^3 + V(r) + \mathcal{O}(a^2)$ using Eq. (5), where the second term can be treated as a perturbation. Now, the third order Schrödinger equation has a new *non-perturbative* solution of the form $\psi \sim e^{ix/2a\hbar}$, which when superposed with the regular solutions perturbed by terms $\mathcal{O}(a)$,

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[1] (a) In [1,8,9] we had used α in place of a.
(b) The results of this article do not depend on this particular form of GUP chosen, and continue to hold for a large class of variants, so long as an $\mathcal{O}(a)$ term is present in the right-hand side of Eq. (1).

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implies not only the usual quantization of energy, but also that the box length L is quantized according to

$$\frac{L}{a\hbar} = \frac{L}{a_0 \ell_{Pl}} = 2p\pi + \theta, \quad p \in \mathbb{N}, \tag{6}$$

where $\theta = \mathcal{O}(1)$. We interpreted this as the quantization of measurable lengths, and effectively that of space itself, near the Planck scale. In this Letter, we re-examine the above problem, but now assuming that the particle is relativistic. This we believe is important for several reasons, among which are that extreme high energy (ultra-)relativistic particles are natural candidates for probing the nature of spacetime near the Planck scale, and that most elementary particles in nature are fermions, obeying some form of the Dirac equation. Furthermore, as seen from below, attempts to extend our results to 2 and 3 dimensions seem to necessitate the use of matrices. However, we first start by examining the simpler Klein–Gordon equation.

1. Klein–Gordon equation in one dimension

The Klein–Gordon (KG) equation in 1-spatial dimension[2]

$$p^2 \Phi(t,x) = \left(\frac{E^2}{c^2} - m^2c^2\right)\Phi(t,x). \tag{7}$$

We see that this is identical to the Schrödinger equation, when one makes the identification: $2mE/\hbar^2 \equiv k^2 \to E^2/\hbar^2c^2 - m^2c^2/\hbar^2$. As a result, the quantization of length, which does not depend on k, continues to hold [1].

However, in addition to fermions being the most fundamental entities, the 3-dimensional version of KG equation (7), when combined with Eq. (5), suffers from the drawback that the p^2 term translates to $p^2 = p_0^2 - 2ap_0^3 + \mathcal{O}(a^2) = -\hbar^2\nabla^2 + i2a\hbar^3\nabla^{3/2} + \mathcal{O}(a^2)$, of which the second term is evidently non-local. As we shall see in the next section, the Dirac equation can address both issues at once.

2. Dirac equation in one dimension

First we linearize $p_0 = \sqrt{p_{0x}^2 + p_{0y}^2 + p_{0z}^2}$ using the Dirac prescription, i.e. replace $p_0 \to \vec{\alpha}\cdot\vec{p}$, where α_i $(i = 1,2,3)$ and β are the Dirac matrices, for which we use the following representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{8}$$

The GUP-corrected Dirac equation can thus be written to $\mathcal{O}(a)$ as[3]

$$\begin{aligned} H\psi &= \left(c\vec{\alpha}\cdot\vec{p} + \beta mc^2\right)\psi(\vec{r}) \\ &= \left(c\vec{\alpha}\cdot\vec{p}_0 - ca(\vec{\alpha}\cdot\vec{p}_0)(\vec{\alpha}\cdot\vec{p}_0) + \beta mc^2\right)\psi(\vec{r}) \\ &= E\psi(\vec{r}) \end{aligned} \tag{9}$$

which for 1-spatial dimension, say z, is in the position representation

$$\left(-i\hbar c\alpha_z\frac{d}{dz} + ca\hbar^2\frac{d^2}{dz^2} + \beta mc^2\right)\psi(z) = E\psi(z). \tag{10}$$

Note that this is a second order differential equation instead of the usual first order Dirac equation (we have used $\alpha_z^2 = 1$). Thus, it has two linearly independent, positive energy solutions, which to $\mathcal{O}(a)$ are

$$\psi_1 = N_1 e^{ikz}\begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix}, \tag{11}$$

$$\psi_2 = N_2 e^{i\frac{z}{a\hbar}}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix} \tag{12}$$

where m is the mass of the Dirac particle, $k = k_0 + a\hbar k_0^2$, k_0 satisfies the usual dispersion relation $E^2 = (\hbar k_0 c)^2 + (mc^2)^2$, $r \equiv \frac{\hbar k_0 c}{E+mc^2}$ and $\chi^\dagger\chi = I$. Note that r runs from 0 (non-relativistic) to 1 (ultra-relativistic). k, k_0 could be positive (right moving) or negative (left moving). N_1, N_2 are suitable normalization constants. As in the case of Schrödinger equation, here too a new *non-perturbative* solution ψ_2 appears, which should drop out in the $a \to 0$ (i.e. no GUP) limit. This has a characteristic wavelength $2\pi a\hbar$.

As noted in [11], to confine a relativistic particle in a box of length L in a consistent way avoiding the Klein paradox (in which an increasing number of negative energy particles are excited), one may take its mass to be z-dependent as was done in the MIT bag model of quark confinement

$$\begin{aligned} m(z) &= M, \quad z < 0 \text{ (Region I)}, \\ &= m, \quad 0 \leqslant z \leqslant L \text{ (Region II)}, \\ &= M, \quad z > L \text{ (Region III)}, \end{aligned} \tag{13}$$

where m and M are constants and we will eventually take the limit $M \to \infty$. Thus, we can write the general wavefunctions in the three regions

$$\psi_I = Ae^{-iKz}\begin{pmatrix} \chi \\ -R\sigma_z\chi \end{pmatrix} + Ge^{i\frac{z}{a\hbar}}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}, \tag{14}$$

$$\psi_{II} = Be^{ikz}\begin{pmatrix} \chi \\ r\sigma_z\chi \end{pmatrix} + Ce^{-ikz}\begin{pmatrix} \chi \\ -r\sigma_z\chi \end{pmatrix} + Fe^{i\frac{z}{a\hbar}}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}, \tag{15}$$

$$\psi_{III} = De^{iKz}\begin{pmatrix} \chi \\ R\sigma_z\chi \end{pmatrix} + H\,e^{i\frac{z}{a\hbar}}\begin{pmatrix} \chi \\ \sigma_z\chi \end{pmatrix}, \tag{16}$$

where $E^2 = (\hbar K_0 c)^2 + (Mc^2)^2$, $K = K_0 + a\hbar K_0^2$ and $R = \hbar K_0 c/(E + Mc^2)$. Thus, in the limit $M \to \infty$, $K \to +i\infty$, the terms associated with A and D go to zero. However, those with G and H do not. Moreover, it can be shown that the fluxes due to these terms do not vanish. Thus, we must set $G = 0 = H$. In addition, without loss of generality we choose $B = 1$ and $C = e^{i\delta}$ where δ is a real number. It can be shown that if one chooses $|C| \neq 1$ then the energy of the relativistic particle is complex. Finally, we must have $F \sim a^s$, $s > 0$, such that this term goes to zero in the $a \to 0$ limit. Now, boundary conditions akin to that for the Schrödinger equation, namely $\psi_{II} = 0$ at $z = 0$ and $z = L$ will require ψ_{II} to vanish identically. Thus, they are disallowed. Instead, we require the outward component of the Dirac current to be zero at the boundaries (the MIT bag model). This ensures that the particle is indeed confined within the box [12].

The conserved current corresponding to Eq. (10) can be shown to be

$$\begin{aligned} J_z &= \bar{\psi}\gamma^z\psi + ic\hbar a\left(\psi^\dagger\frac{d\psi}{dz} - \frac{d\psi^\dagger}{dz}\psi\right) \\ &\equiv J_{0z} + J_{1z} \end{aligned} \tag{17}$$

where $J_{0z} + J_{1z}$ are the usual and new GUP-induced currents, respectively. We will comment on J_{1z} shortly. First, the vanishing of the Dirac current $J^\mu = \bar{\psi}\gamma^\mu\psi$ at a boundary is equivalent to the condition $i\gamma\cdot n\psi = \psi$ there, where n is the outward normal to

[2] In this and in subsequent sections, we start with the usual forms of the KG and Dirac equations, as is indeed the case for massless particles, as well as for massive particles *and* in their stationary states, with the re-definition $m \to m(1 - \ell_{Pl}E/\hbar c)$, see e.g. Eq. (11) of [6] or Eq. (14) of [10].

[3] In this section, we closely follow the formulation of [11].

the boundary [12]. Applying this to J_{0z} for the wavefunction ψ_{II} at $z=0$ and $z=L$ gives [11]

$$i\beta\alpha_z\psi_{II}|_{z=0} = \psi_{II}|_{z=0}, \tag{18}$$

and

$$-i\beta\alpha_z\psi_{II}|_{z=L} = \psi_{II}|_{z=L}, \tag{19}$$

respectively. Using the expression for ψ_{II} from (15), we get from (18) and (19), respectively,

$$\frac{B+C+F'e^{-i\pi/4}}{B-C} = ir, \tag{20}$$

$$\frac{Be^{ikL}+Ce^{-ikL}+F'e^{i(L/a\hbar+\pi/4)}}{Be^{ikL}-Ce^{-ikL}} = -ir \tag{21}$$

(where $F' = \sqrt{2}F$), which in turn yield

$$(ir-1) - F'e^{-i\pi/4} = (ir+1)e^{i\delta}, \tag{22}$$

$$(ir-1) - F'e^{i(L/a\hbar+\pi/4)}e^{ikL}e^{-i\delta} = (ir+1)e^{i(2kL-\delta)}. \tag{23}$$

Note that conditions (22) and (23) imply

$$|B| = |C| + \mathcal{O}(a), \tag{24}$$

which guarantees that

$$J_{1z} = -2ca\hbar k\left(1+r^2\right)\left[|B|^2 - |C|^2\right] = 0. \tag{25}$$

Furthermore, from (22) and (23) it follows that

$$kL = \delta = \arctan\left(-\frac{\hbar k}{mc}\right) + \mathcal{O}(a), \tag{26}$$

and

$$\frac{L}{a\hbar} = \frac{L}{a_0\ell_{Pl}} = 2p\pi - \frac{\pi}{2}, \quad p \in \mathbb{N}. \tag{27}$$

The transcendental equation (26) gives the quantized energy levels for a relativistic particle in a box. Its $a \to 0$ limit gives $k_0 L = \arctan(-\frac{\hbar k_0}{mc})$ which is Eq. (17) of Ref. [11], its non-relativistic limit gives $(k_0 + a\hbar k_0^2)L = n\pi$, while its non-relativistic *and* $a \to 0$ limit yields the Schrödinger equation result $k_0 L = n\pi$. Eq. (27) on the other hand shows that such a particle cannot be confined in a box, unless the box length is quantized according to this condition. Note that this is *identical* to the quantization condition (6), which was derived using the Schrödinger equation (with the identification $\theta \equiv -\pi/2$). This indicates the robustness of the result. As measuring spatial dimensions requires the existence and observation of at least one particle, the above result once again seems to indicate that effectively all measurable lengths are quantized in units of $a_0\ell_{Pl}$.

3. Dirac equation in two and three dimensions

We now generalize to a box in two or three dimensions defined by $0 \leqslant x_i \leqslant L_i$, $i = 1, \ldots, d$ with $d = 1, 2, 3$. We start with the following ansatz for the wavefunction

$$\psi = e^{i\vec{t}\cdot\vec{r}}\begin{pmatrix}\chi\\ \vec{\rho}\cdot\vec{\sigma}\,\chi\end{pmatrix} \tag{28}$$

where $\vec{t}$ and $\vec{\rho}$ are d-dimensional (spatial) vectors, and $\chi^\dagger\chi = I$ as before. In this case, Eq. (9) translates to

$$\begin{aligned} H\psi &= e^{i\vec{t}\cdot\vec{r}}\begin{pmatrix}((mc^2 - ca\hbar^2t^2) + c\hbar(\vec{t}\cdot\vec{\rho} + i\vec{\sigma}\cdot(\vec{t}\times\vec{\rho})))\chi\\ (c\hbar\vec{t} - (mc^2 + ca\hbar^2t^2)\vec{\rho})\cdot\vec{\sigma}\,\chi\end{pmatrix}\\ &= E\psi, \end{aligned} \tag{29}$$

where we have used the identity $(\vec{t}\cdot\vec{\sigma})(\vec{\rho}\cdot\vec{\sigma}) = \vec{t}\cdot\vec{\rho} + i\vec{\sigma}\cdot(\vec{t}\times\vec{\rho})$. Eq. (29) implies $\vec{t}\times\vec{\rho} = 0$, i.e. $\vec{\rho}$ is parallel to $\vec{t}$, and two solutions for t, namely $t = k$ and $t = 1/a\hbar$, and correspondingly $\rho = r$ and $\rho = 1$. The latter solutions for t and ρ are the (new) non-perturbative ones, which as we shall see, will give rise to quantization of space. Thus the vector $\vec{t}$ for the two cases are $\vec{t} = \vec{k}$ and $\vec{t} = \frac{\hat{q}}{a\hbar}$ and $\vec{\rho} = r\hat{k}$ and $\vec{\rho} = \hat{q}$ respectively, where $\hat{q}$ is an arbitrary unit vector.[4] Thus, putting in the normalizations, the two independent positive energy solutions are

$$\psi_1 = N_1 e^{i\vec{k}\cdot\vec{r}}\begin{pmatrix}\chi\\ r\hat{k}\cdot\vec{\sigma}\,\chi\end{pmatrix}, \tag{30}$$

$$\psi_2 = N_2 e^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}\begin{pmatrix}\chi\\ \hat{q}\cdot\vec{\sigma}\,\chi\end{pmatrix} \tag{31}$$

with ψ_2 being the new GUP-induced eigenfunction.

Next, we consider the following wavefunction

$$\psi = \begin{pmatrix}[\prod_{i=1}^d (e^{ik_ix_i} + e^{-i(k_ix_i-\delta_i)}) + Fe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}]\chi\\ \sum_{j=1}^d [\prod_{i=1}^d (e^{ik_ix_i} + (-1)^{\delta_{ij}}e^{-i(k_ix_i-\delta_i)})r\hat{k}_j\\ + Fe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}\hat{q}_j]\sigma_j\chi\end{pmatrix} \tag{32}$$

where $d = 1, 2, 3$, depending on the number of spatial dimensions and an overall normalization has been set to unity. The number of terms in row I and row II are $2^d + 1$ and $(2^d + 1)\times d$ respectively, i.e. $(3, 3)$, $(5, 10)$ and $(9, 27)$ in 1, 2 and 3 dimensions, respectively. It can be easily shown that the above is a superposition of $F\psi_2$ and the following 2^d eigenfunctions, for all possible combinations with ϵ_i $(i = 1, \ldots, d)$, with $\epsilon_i = \pm 1$

$$\Psi = e^{i(\sum_{i=1}^d \epsilon_i k_i x_i + \frac{(1-\epsilon_i)}{2}\delta_i)}\begin{pmatrix}\chi\\ r\sum_{i=1}^d \epsilon_i\hat{k}_i\sigma_i\chi\end{pmatrix} \tag{33}$$

where δ_i $(i = 1, \ldots, d)$ are phases to be determined shortly using boundary conditions.

Again, we impose the MIT bag boundary conditions $\pm i\beta\alpha_k\psi = \psi$, $k = 1, \ldots, d$, with the $+$ and $-$ signs corresponding to $x_k = 0$ and $x_k = L_k$ respectively, ensuring vanishing flux through all six boundaries. First, we write the above boundary condition for any x_k, for the wavefunction given in Eq. (32). This yields the following 2-component equation

$$\pm\begin{pmatrix}i\sum_{j=1}^d [\prod_{i=1}^d (e^{ik_ix_i} + (-1)^{\delta_{ij}}e^{-i(k_ix_i-\delta_i)})\sigma_k r_j\sigma_j\\ + Fe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}\sigma_k\,\hat{q}_j\sigma_j]\chi\\ -i[\prod_{i=1}^d (e^{ik_ix_i} + e^{-i(k_ix_i-\delta_i)}) + Fe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}]\sigma_k\chi\end{pmatrix} = \psi. \tag{34}$$

Employing the MIT bag model boundary conditions and thus equating rows I and II of Eq. (32) with the corresponding ones of Eq. (34) yields, respectively

$$\begin{aligned} &\prod_{i=1}^d (e^{ik_ix_i} + e^{-i(k_ix_i-\delta_i)}) + Fe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}\\ &= \pm\Bigg[i\prod_{i=1}^d (e^{ik_ix_i} + (-1)^{\delta_{ik}}e^{-i(k_ix_i-\delta_i)})r\hat{k}_k + iFe^{i\frac{\hat{q}\cdot\vec{r}}{a}}\,\hat{q}_k\\ &\quad + i\sum_{j=1\neq k}^d \Bigg[\prod_{i=1}^d (e^{ik_ix_i} + (-1)^{\delta_{ij}}e^{-i(k_ix_i-\delta_i)})r\hat{k}_j\sigma_k\sigma_j\\ &\quad + Fe^{i\frac{\hat{q}\cdot\vec{r}}{a}}\,\hat{q}_j\sigma_k\sigma_j\Bigg]\Bigg] \end{aligned} \tag{35}$$

[4] Although one can choose $\hat{q} = \hat{k}$, *per se* our analysis does not require this to be the case. We will comment on this towards the end of the Letter.

and

$$\prod_{i=1}^{d}\left(e^{ik_ix_i}+e^{-i(k_ix_i-\delta_i)}\right)+Fe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}$$
$$=\pm\Bigg[i\prod_{i=1}^{d}\left(e^{ik_ix_i}+(-1)^{\delta_{ik}}e^{-i(k_ix_i-\delta_i)}\right)r\hat{k}_k+iFe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}\hat{q}_k$$
$$+i\sum_{j=1\neq k}^{d}\Bigg[\prod_{i=1}^{d}\left(e^{ik_ix_i}+(-1)^{\delta_{ij}}e^{-i(k_ix_i-\delta_i)}\right)r\hat{k}_j\sigma_j\sigma_k$$
$$+Fe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}\hat{q}_j\sigma_j\sigma_k\Bigg]\Bigg]. \tag{36}$$

Note that the only difference between Eqs. (35) and (36) is in the order of σ_k and σ_j in the last two terms in the RHS. Thus, adding the two equations and using $\{\sigma_k,\sigma_j\}=0$, these terms simply drop out. Next, dividing the rest by $f_{\bar{k}}(x_i,k_i,\delta_i)\equiv\prod_{i=1\neq k}^{d}(e^{ik_ix_i}+e^{-i(k_ix_i-\delta_i)})$, where the subscript $\bar{k}$ of $f_{\bar{k}}$ signifies the lack of dependence on (x_k,k_k,δ_k), we get

$$e^{ik_kx_k}+e^{-i(k_kx_k-\delta_k)}+f_{\bar{k}}^{-1}Fe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}$$
$$=\pm i\left(e^{ik_kx_k}-e^{-i(k_kx_k-\delta_k)}\right)r\hat{k}_k\pm if_{\bar{k}}^{-1}Fe^{i\frac{\hat{q}\cdot\vec{r}}{a\hbar}}\hat{q}_k. \tag{37}$$

Note that for all practical purposes the boundary condition has factorized into its Cartesian components, at least in the a independent terms, which contain (x_k,k_k,δ_k) alone, i.e. no other index i. Eq. (37) yields, at $x_k=0$ and $x_k=L_k$, respectively,

$$e^{i\delta_k}(1+ir\hat{k}_k)=(ir\hat{k}_k-1)+f_{\bar{k}}^{-1}F'_ke^{-i\theta_k} \tag{38}$$

and

$$e^{i(2k_kL_k-\delta_k)}(1+ir\hat{k}_k)=(ir\hat{k}_k-1)+f_{\bar{k}}^{-1}F'_ke^{i\theta_k}e^{i(k_kL_k-\delta_k)} \tag{39}$$

where $F'_k\equiv\sqrt{1+|\hat{q}_k|^2}F$, $\theta_k\equiv\arctan\hat{q}_k$ and we have assumed that $f_{\bar{k}}$ is evaluated at the same x_i $(i\neq k)$ at both boundaries of x_k. Comparing Eqs. (38) and (39), which are the d-dimensional generalizations of Eqs. (22) and (23), we see that the following relations must hold

$$k_kL_k=\delta_k=\arctan\left(-\frac{\hbar k_k}{mc}\right)+\mathcal{O}(a), \tag{40}$$

$$\frac{|\hat{q}_k|L_k}{a\hbar}=\frac{|\hat{q}_k|L_k}{a_0\ell_{Pl}}=2p_k\pi-2\theta_k. \tag{41}$$

While Eq. (40) yields quantization of energy levels in d dimensions ($k_kL_k=n\pi$ in the non-relativistic limit), Eq. (41) shows that lengths in all directions are quantized. Further, one may choose the symmetric case $|\hat{q}_k|=1/\sqrt{d}$,[5] in which case, it follows from Eq. (41) above

$$\frac{L_k}{a_0\ell_{Pl}}=(2p_k\pi-2\theta_k)\sqrt{d},\quad p\in\mathbb{N}, \tag{42}$$

which reduces to Eq. (27) for $d=1$. Note that the above also gives rise to quantization of measured areas ($N=2$) and volumes ($N=3$), as follows

$$A_N\equiv\prod_{k=1}^{N}\frac{L_k}{a_0\ell_{Pl}}=d^{N/2}\prod_{k=1}^{N}(2p_k\pi-2\theta_k),\quad p_k\in\mathbb{N}. \tag{43}$$

4. Spherical cavity: Dirac equation in polar coordinates

Finally, we solve the Dirac equation with the GUP-induced terms in a spherical cavity, and show that only cavities of certain discrete dimensions can confine a relativistic particle. We follow the analysis of [13]. For related references, see [12,14]. A spherical cavity of radius R, defined by the potential

$$U(r)=0,\quad r\leqslant R,$$
$$=U_0\to\infty,\quad r>R, \tag{44}$$

yields the following Dirac equation in component form

$$c(\vec{\sigma}\cdot\vec{p}_0)\chi_2+\left(mc^2+U\right)\chi_1-cap_0^2\chi_1=E\chi_1, \tag{45}$$
$$c(\vec{\sigma}\cdot\vec{p}_0)\chi_1-\left(mc^2+U\right)\chi_2-cap_0^2\chi_2=E\chi_2 \tag{46}$$

where the Dirac spinor has the form $\psi=\binom{\chi_1}{\chi_2}$. It can be shown that the following operators commute with the GUP-corrected Hamiltonian: the total angular momentum operator (not to be confused with the Dirac current represented by the same letter) $\vec{J}=\vec{L}+\vec{\Sigma}/2$, $K=\beta(\vec{\Sigma}\cdot\vec{L}+I)$, where $\vec{L}$ is the orbital angular momentum, $\vec{\Sigma}=\begin{pmatrix}\vec{\sigma}&0\\0&\vec{\sigma}\end{pmatrix}$, and $K^2=J^2+1/4$. Thus, eigenvalues of J^2 and K, namely $j(j+1)$ and κ respectively, are related by $\kappa=\pm(j+1/2)$. Correspondingly, the Dirac spinor has the following form

$$\psi=\begin{pmatrix}\chi_1\\\chi_2\end{pmatrix}=\begin{pmatrix}g_\kappa(r)\mathcal{Y}^{j_3}_{j\ell}(\hat{r})\\if_\kappa(r)\mathcal{Y}^{j_3}_{j\ell'}(\hat{r})\end{pmatrix}, \tag{47}$$

$$\mathcal{Y}^{j_3}_{j\ell}=\left(l\ \ \frac{1}{2}\ \ j_3-\frac{1}{2}\ \ \frac{1}{2}\,\middle|\,j\ \ j_3\right)Y_\ell^{j_3-\frac{1}{2}}(\hat{r})\begin{pmatrix}1\\0\end{pmatrix}$$
$$+\left(l\ \ \frac{1}{2}\ \ j_3+\frac{1}{2}\ \ -\frac{1}{2}\,\middle|\,j\ \ j_3\right)Y_\ell^{j_3+\frac{1}{2}}(\hat{r})\begin{pmatrix}0\\1\end{pmatrix} \tag{48}$$

where $Y_l^{j\pm\frac{1}{2}}$ are spherical harmonics and $(j_1\ j_2\ m_1\ m_2\mid J\ M)$ are Clebsch–Gordon coefficients. χ_1 and χ_2 are eigenstates of L^2 with eigenvalues $\hbar^2\ell(\ell+1)$ and $\hbar^2\ell'(\ell'+1)$, respectively, such that the following hold

$$\text{if }\kappa=j+\frac{1}{2}>0,$$
$$\text{then }\ell=\kappa=j+\frac{1}{2},\quad \ell'=\kappa-1=j-\frac{1}{2}, \tag{49}$$
$$\text{and if }\kappa=-\left(j+\frac{1}{2}\right)<0,$$
$$\text{then }\ell=-(\kappa+1)=j-\frac{1}{2},\quad \ell'=-\kappa=j+\frac{1}{2}. \tag{50}$$

Next, we use the following identities

$$\vec{\sigma}\cdot\vec{p}_0=\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{p}_0)\right]$$
$$=\frac{\vec{\sigma}\cdot\vec{r}}{r^2}[\vec{r}\cdot\vec{p}_0+i\vec{\sigma}\cdot\vec{r}\times\vec{p}_0]$$
$$=\frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[-i\hbar r\frac{d}{dr}+i\vec{\sigma}\cdot\vec{L}\right], \tag{51}$$
$$(\vec{\sigma}\cdot\vec{L}+1)\chi_{1,2}=\mp\kappa\chi_{1,2}, \tag{52}$$
$$(\vec{\sigma}\cdot\hat{r})\mathcal{Y}^{j_3}_{j\ell}=-\mathcal{Y}^{j_3}_{jl'},\qquad(\vec{\sigma}\cdot\hat{r})\mathcal{Y}^{j_3}_{j\ell'}=-\mathcal{Y}^{j_3}_{jl} \tag{53}$$

[5] Alternatively, assuming no direction is intrinsically preferred in space and the only *special* direction is provided by the particle momentum $\vec{k}$, one can make the identification $\hat{q}=\hat{k}$, in which case $|\hat{q}_k|=n_k/\sqrt{\sum_{i=1}^{d}n_i^2}\approx1/\sqrt{d}$, assuming that the momentum quantum numbers $n_k\gg1$ and approximately equal, when space is probed at the fundamental level with ultra high energy super-Planckian particles.

where we have used $(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$, the related identity $(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{r}) = r^2$, and the relation $p_0^2 F(r) Y_\ell^m = \hbar^2[-\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}) + \frac{\ell(\ell+1)}{r^2}]F(r)Y_\ell^m$ for an arbitrary function $F(r)$, to obtain from Eqs. (45)–(46)

$$-c\hbar\frac{df_\kappa}{dr} + c\frac{(\kappa-1)}{r}f_\kappa + \big(mc^2+U\big)g_\kappa + ca\hbar^2\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg_\kappa}{dr}\right) - \frac{\ell(\ell+1)}{r^2}g_\kappa\right] = Eg_\kappa, \tag{54}$$

$$c\hbar\frac{dg_\kappa}{dr} + c\frac{(\kappa+1)}{r}g_\kappa - \big(mc^2+U\big)f_\kappa + ca\hbar^2\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_\kappa}{dr}\right) - \frac{\ell'(\ell'+1)}{r^2}f_\kappa\right] = Ef_\kappa. \tag{55}$$

As in the case of rectangular cavities, Eqs. (54)–(55) have the standard set of solutions, slightly perturbed by the GUP-induced term (represented by the $\mathcal{O}(a)$ terms below)

$$g_\kappa(r) = \tilde{N} j_\ell(p_0 r) + \mathcal{O}(a), \quad \text{where } \ell = \begin{cases} \kappa, & \text{if } \kappa > 0, \\ -(\kappa+1), & \text{if } \kappa < 0, \end{cases} \tag{56}$$

$$f_\kappa(r) = \tilde{N}\frac{\kappa}{|\kappa|}\sqrt{\frac{E-mc^2}{E+mc^2}}\, j_{\ell'}(p_0 r) + \mathcal{O}(a), \quad \text{where } \ell' = \begin{cases} (\kappa-1), & \text{if } \kappa > 0, \\ -\kappa, & \text{if } \kappa < 0, \end{cases} \tag{57}$$

where $j_l(x)$ are spherical Bessel functions. It can be shown that the MIT bag boundary condition (at $r = R$) is equivalent to [12,13]

$$\bar{\psi}_\kappa \psi_\kappa = 0 \tag{58}$$

which in the massless (high energy) limit yields

$$\big[g_\kappa^2(r) - f_\kappa^2(r)\big]\big(\mathcal{Y}_{jl}^{j_3}\big)^\dagger \mathcal{Y}_{jl}^{j_3} + \mathcal{O}(a) = 0 \tag{59}$$

which in turn gives the quantization of energy (for energy eigenvalues obtained numerically from Eq. (59), see Table 2.1, Chapter 2, Ref. [13]. These will also undergo tiny modifications $\mathcal{O}(a)$).

But from the analysis of previous sections, we expect new non-perturbative solutions of the form $f_\kappa = \mathcal{F}_\kappa(r)e^{i\epsilon r/a\hbar}$ and $g_\kappa = \mathcal{G}_\kappa(r)e^{i\epsilon r/a\hbar}$ (where $\epsilon = \mathcal{O}(1)$) for which Eqs. (54)–(55) simplify to

$$a\hbar\frac{d^2 g_\kappa}{dr^2} = \frac{df_\kappa}{dr}, \tag{60}$$

$$a\hbar\frac{d^2 f_\kappa}{dr^2} = -\frac{dg_\kappa}{dr} \tag{61}$$

where we have dropped terms which are ignorable for small a. These indeed have solutions

$$f_\kappa^{\mathcal{N}} = iN' e^{ir/a\hbar}, \tag{62}$$

$$g_\kappa^{\mathcal{N}} = N' e^{ir/a\hbar}, \tag{63}$$

where similar to the constant C in Ref. [1], here one must have $\lim_{a\to 0} N' = 0$, such that these new solutions drop out in the $a \to 0$ limit. The boundary condition (58) now gives

$$\big|g_\kappa(r) + g_\kappa^{\mathcal{N}}(r)\big|^2 = \big|f_\kappa(r) + f_\kappa^{\mathcal{N}}(r)\big|^2, \tag{64}$$

which to $\mathcal{O}(a)$ translates to

$$\big[j_\ell^2(p_0R) - j_{\ell'}^2(p_0R)\big] + 2N'\big[j_\ell(p_0R)\cos(R/a\hbar) - j_{\ell'}(p_0R)\sin(R/a\hbar)\big] = 0. \tag{65}$$

This again implies the following conditions

$$j_\ell(p_0R) = j_{\ell'}(p_0R), \tag{66}$$

$$\tan(R/a\hbar) = 1. \tag{67}$$

The first condition is identical to Eq. (59), and hence the energy quantization. The second implies

$$\frac{R}{a\hbar} = \frac{R}{a_0\ell_{pl}} = 2p\pi - \frac{\pi}{4}, \quad p \in \mathbb{N}. \tag{68}$$

This once again, the radius of the cavity, and hence the area and volume of spheres are seen to be quantized.

5. Conclusions

In this Letter, we have studied a relativistic particle in a box in one, two and three dimensions (including a spherical cavity in three dimensions), using the Klein–Gordon and Dirac equations with corrections that follow from the Generalized Uncertainty Principle. We have shown that to confine the particle in the box, the dimensions of the latter would have to be quantized in multiples of a fundamental length, which can be the Planck length. As measurements of lengths, areas and volumes require the existence and use of such particles, we interpret this as effective quantization of these quantities. Note that although existence of a fundamental length is apparently inconsistent with special relativity and Lorentz transformations (fundamental length in *whose frame*?), it is indeed consistent, and in perfect agreement with Doubly Special Relativity Theories. It is hoped that the essence of these results will continue to hold in curved spacetimes, and even if possible fluctuations of the metric can be take into account in a consistent way. In addition to exploring these issues, it would be interesting to study possible phenomenological implications of space quantization; e.g. if it has any measurable effects at distance scales far greater than the Planck length, such as at about 10^{-4} fm, the length scale to be probed by the Large Hadron Collider. We hope to make further studies in this direction and report elsewhere.

Note added

Following a suggestion of the anonymous referee to re-examine wave equations for a spinless particle with relativistic corrections, we explore two routes:

(i) First, we write the time-dependent version of the GUP-corrected Dirac equation (9)

$$\begin{aligned} H\psi &= \big(c\vec{\alpha}\cdot\vec{p} + \beta mc^2\big)\psi(\vec{r},t) \\ &= \big(c\vec{\alpha}\cdot\vec{p}_0 - ca(\vec{\alpha}\cdot\vec{p}_0)(\vec{\alpha}\cdot\vec{p}_0) + \beta mc^2\big)\psi(\vec{r},t) \\ &= i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}. \end{aligned} \tag{69}$$

To study the non-relativistic limit, we write the spinor ψ in a slightly different form as [15]

$$\psi = e^{-\frac{i}{\hbar}mc^2 t}\begin{pmatrix}\chi_1(\vec{r},t)\\ \chi_2(\vec{r},t)\end{pmatrix}, \tag{70}$$

and obtain the two component equations

$$i\hbar\frac{\partial\chi_1}{\partial t} = c(\vec{\sigma}\cdot\vec{p}_0)\chi_2 - ca(\vec{\sigma}\cdot\vec{p}_0)^2\chi_1, \tag{71}$$

$$i\hbar\frac{\partial\chi_2}{\partial t} = -2mc^2\chi_2 + c(\vec{\sigma}\cdot\vec{p}_0)\chi_1 - ca(\vec{\sigma}\cdot\vec{p}_0)^2\chi_2. \tag{72}$$

For $mc^2 \gg |\partial\chi_2/\partial t|$, Eq. (72) gives to $\mathcal{O}(a)$

$$\chi_2 = \frac{1}{2mc}\left[1 - \frac{a}{2mc}(\vec{\sigma}\cdot\vec{p}_0)^2\right](\vec{\sigma}\cdot\vec{p}_0)\chi_1, \tag{73}$$

which when substituted in Eq. (71) leads to

$$i\hbar\frac{\partial\chi_1}{\partial t} = \frac{1}{2m}(\vec{\sigma}\cdot\vec{p}_0)^2\chi_1 - \frac{a}{(2m)^2c}(\vec{\sigma}\cdot\vec{p}_0)^4\chi_1 - ca(\vec{\sigma}\cdot\vec{p}_0)^2\chi_1. \tag{74}$$

Using the identity $(\vec{\sigma}\cdot\vec{p}_0)^2 = p_0^2$, Eq. (74) becomes

$$i\hbar\frac{\partial\chi_1}{\partial t} = \left[\left(\frac{1}{2m} - ca\right)p_0^2 - \frac{a}{(2m)^2c}p_0^4\right]\chi_1. \tag{75}$$

Finally, substituting $\chi_1(\vec{r},t) = e^{-iEt/\hbar}\chi_1(\vec{r})$, we get

$$\left[\left(\frac{1}{2m} - ca\right)p_0^2 - \frac{a}{(2m)^2c}p_0^4\right]\chi_1 = E\chi_1. \tag{76}$$

Although the above GUP-corrected Pauli equation actually describes a 2-component, non-relativistic spinor, it is an interesting (and new) extension of the Schrödinger equation, and can have potential applications elsewhere. Note that the above holds in any spacetime dimension and is local. In particular, in one dimension, in addition to the usual plane wave solutions $\chi_1 = e^{\pm ik'z}$ (where $k' = \sqrt{2mE/\hbar^2} + \mathcal{O}(a)$), Eq. (76) also admits of the non-perturbative solutions $\chi_1 = e^{\pm\frac{i}{\hbar}\sqrt{2mc/a}z}$, which too are plane waves but with wavelength $\approx \hbar\sqrt{a/mc} = \sqrt{a_0\ell_{pl}\hbar/mc}$. Imposing standard boundary condition $\chi_1 = 0$ at $z = 0$ and $z = L$ and following the procedure outlined in [1], it is easy to show that $L/\sqrt{a_0\ell_{Pl}}$ is quantized.

(ii) Next, we square the operators on both sides of Eq. (9), use $\beta^2 = 1$ and the relation $(\vec{\alpha}\cdot\vec{p}_0)^2 = p_0^2$ to obtain

$$\begin{aligned} H^2 &= p_0^2c^2 + m^2c^4 - 2cap_0^2\left[c\vec{\alpha}\cdot\vec{p} + \beta mc^2\right] \\ &= p_0^2c^2 + m^2c^4 - 2cap_0^2H \\ &= p_0^2c^2(1 - 2aH/c) + m^2c^4 \end{aligned} \tag{77}$$

where in the last term of the intermediate step we have substituted $H = c\vec{\alpha}\cdot\vec{p}_0 + \beta mc^2 + \mathcal{O}(a)$. It is seen that the above too can be used in any dimension and is local. Furthermore, by construction, solutions of (9) are also solutions of (77) treated as a differential equation, resulting in identical space quantization results.

We expect similar results to hold for equations governing bosonic fields with higher spins, such as Maxwel's equations, including GUP corrections. It would be interesting to see the interplay of such a field with fermions, say via minimal coupling. We hope to report on it elsewhere.

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