Automorphisms of a linear Lie algebra over a commutative ring

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Abstract

Suppose that \( m \geq 5 \) and that \( R \) is a commutative ring with identity in which 2 is invertible. This paper determines all automorphisms of the standard Borel subalgebra of the orthogonal Lie algebra \( o(2m, R) \).

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1. Introduction

Let \( m, n \) be positive integers, \( R \) a commutative ring with identity, \( R^* \) the group consisting of all invertible elements in \( R \), \( E^{(n)} \) the \( n \times n \) identity matrix (\( E^{(m)} \) is abbreviated to \( E \)), \( R^{m \times n} \) the set of all \( m \times n \) matrices over \( R \), \( gl(m, R) \) the general linear Lie algebra consisting of all \( m \times m \) matrices over \( R \) with bracket: \( [X, Y] = XY - YX \). Let \( t(m, R) \) be the subalgebra of \( gl(m, R) \) consisting of all upper triangular matrices. Set \( I = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \). The orthogonal Lie algebra \( o(2m, R) \) over \( R \) is defined to be the subalgebra of \( gl(2m, R) \), consisting of all \( X \in gl(2m, R) \) satisfying \( X'I = -IX \). The condition for \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} (A, B, C, D \in R^{m \times m}) \) to be orthogonal is that \( B' = -B \), \( C' = -C \) and \( D' = -A \).

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Let
\[ l = \left\{ \begin{pmatrix} A & B \\ 0 & -A' \end{pmatrix} \mid A \in t(m, R), \ B \in R^{m \times m} \text{satisfies } B' = -B \right\}. \]

It is called the standard Borel subalgebra of \( o(2m, R) \).

The problem to determine automorphisms of the Borel subalgebras of classical Lie algebras was initiated by Doković in [1]. Recently, some further progress has been made on relative problems (see [2–7]). In this paper, using the main theorem in [2], we determine all automorphisms of the standard Borel subalgebra \( l \) of \( o(2m, R) \), when \( m \geq 5 \) and \( R \) is a commutative ring with identity in which 2 is invertible. The main idea of this paper is to reduce the problem on \( l \) to that on \( t(m, R) \).

2. Preliminaries

In the following, we always suppose that \( m \geq 5 \) and \( 2 \in R^* \).

If all diagonal entries of \( T \in t(m, R) \) are 0, we call \( T \) strictly upper triangular. Set
\[ h = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \mid A \text{ is a diagonal matrix in } gl(m, R) \right\}; \]
\[ t = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \mid A \in t(m, R) \right\}; \]
\[ v = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A' \end{pmatrix} \mid A \in t(m, R) \text{ is strictly upper triangular} \right\}; \]
\[ w = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \mid B \in R^{m \times m}, \ B' = -B \right\}; \]
\[ s = \{ \text{diag}(A, 0, -A', 0) \mid A \in t(m - 1, R) \}; \]

and let \( u = v + w \), then \( t = h + v \), \( l = h + u = t + w \).

For \( 1 \leq i < j \leq m \), let \( E_{i,j} \) denote the \((2m) \times (2m)\) matrix, whose \((i, j)\)-entry is 1, all other entries are 0; \( E_{i,-j} \) the \((2m) \times (2m)\) matrix, whose \((i, m + j)\)-entry is 1, all other entries are 0; \( E_{-j,-i} \) the \((2m) \times (2m)\) matrix, whose \((j + m, i + m)\)-entry is 1, all other entries are 0.

For \( a \in R \), \( 1 \leq i < j \leq m \), set
\[ T_{i,j}(a) = a(E_{i,j} - E_{-j,-i}); \]
\[ T_{i,-j}(a) = a(E_{i,-j} - E_{j,-i}); \]
\[ T_{i,j} = \{ T_{i,j}(a) \mid a \in R \}; \]
\[ T_{i,-j} = \{ T_{i,-j}(a) \mid a \in R \}. \]

For \( 1 \leq i \leq m \), \( a \in R \), set
\[ H_i(a) = aE_{i,i} - aE_{-i,-i}; \]
\[ H(a) = \text{diag}(aE, -aE); \]
\[ h_i = \{ H_i(a) \mid a \in R \}; \]
\[ d_i = h_1 + h_2 + \cdots + h_i. \]
Definition 2.1. An ideal \( L \) of \( l \) is called invariant in \( l \) if it is stable under each automorphism \( \phi \) of \( l \), i.e., \( \phi(L) = L \).

It is easy to see that \( u \), being exactly \([l, l]\), is invariant in \( l \). Let \( u^{(1)} = [u, u], \ u^{(2)} = [u, u^{(1)}], \ldots, u^{(k)} = [u, u^{(k-1)}], \ldots \).

By calculation, we see that
\[
u^{(m-2)} = T_{1,m} + \left( \sum_{1 \leq i < j \leq m} T_{i,j} \right);
\]
\[
u^{(m-1)} = \sum_{1 \leq i < j \leq m} T_{i,j};
\]
\[
u^{(2m-4)} = T_{1,2};
\]
\[
u^{(2m-5)} = T_{1,2} + T_{1,3}.
\]

They are naturally all invariant in \( l \). Let \( x = \sum_{1 \leq i < j \leq m-1} T_{i,j}; \)
\[
y = x + T_{1,m} + T_{1,-m};
\]
\[
z = \left( \sum_{i=1}^{m-1} T_{i,m} \right) + u;
\]
\[
p = T_{1,m-1} + z;
\]
\[
q = h_m + z.
\]

Lemma 2.2. The subalgebras \( x, y, z, p \) and \( q \), defined as above, are all invariant in \( l \).

Proof. The centralizer of \( u^{(m-2)} \) (resp., \( u^{(m-1)} \)) in \( u \) is \( y \) (resp., \( p \)), so \( y \) and \( p \) both are invariant in \( l \). If we can prove that \( z \) is invariant in \( l \), then \( x \), being the center of \( z \), is invariant in \( l \). Furthermore, \( q \) being the centralizer of \( x \) in \( l \), is also invariant in \( l \). So for our goal, it suffices to prove that \( z \) is invariant in \( l \). Let \( \phi \) be an arbitrary automorphism of \( l \). Since \( \phi(y) = y \subseteq z \), it suffices to prove that \( \phi(T_{i,m}), \phi(T_{i,-m}) \) are all contained in \( z \) for \( i = 2, 3, \ldots, m - 1 \). Note that \( [T_{2,m}, y] = T_{1,-2} \). This shows that
\[
\phi([T_{2,m}, y]) = \phi(T_{1,-2}) = T_{1,-2}.
\]
If \( \phi(T_{2,m}) \) is not contained in \( z \), then there exists \( a_0 \in R \) such that \( \phi(T_{2,m}(a_0)) = T_{1,m-1}(r_0) + Z_0 \), where \( 0 \neq r_0 \in R, Z_0 \in z \) (note that \( \phi(p) = p \)). Since \( \phi(y) = y \), we may choose \( Y_0 \in y \) such that \( \phi(Y_0) = T_{m-2, -(m-1)}(1) \). Thus
\[
\phi([T_{2,m}(a_0), Y_0]) = [T_{1,m-1}(r_0) + Z_0, T_{m-2, -(m-1)}(1)] = T_{1,-2}(r_0).
\]
This is absurd (note that \( \phi([T_{2,m}, y] = T_{1,-2}) \). So \( \phi(T_{2,m}) \subseteq z \).

If \( 3 \leq i \leq m - 1 \), then \( [T_{i,m}, y] = T_{1,-i} \). If \( \phi(T_{i,m}) \) is not contained in \( z \), then exists \( 0 \neq a_i \in R \) such that \( \phi(T_{i,m}(a_i)) = T_{1,m-1}(r_i) + Z_i \), where \( r_i \neq 0, Z_i \in z \). Choose \( Y_i \in y \) such that \( \phi(Y_i) = T_{2, -(m-1)}(1) \), then
\[
\phi([T_{i,m}(a_i), Y_i]) = [T_{1,m-1}(r_i) + Z_i, T_{2, -(m-1)}(1)] = T_{1,-2}(r_i),
\]
which leads to $\phi^{-1}(T_{1,-2}(r_i)) \in T_{1,-t}$, contradicting the fact that $T_{1,-2}$ is invariant in $l$. So $\phi(T_{i,m}) \subseteq z$ for all $2 \leq i \leq m - 1$. We can also prove that $\phi(T_{i,-m}) \subseteq z$ for all $2 \leq i \leq m - 1$. The process, being similar to above, is omitted. So $\phi(z) = z$. □

3. Standard automorphisms of $t$

It is obvious that $t$ is isomorphic to $(m, R)$. Cao [2] has described the automorphisms of $(m, R)$, we now transfer them to $t$ for later use. $t$ has the following standard automorphisms.

(a) Inner automorphisms

For invertible $A \in t(m, R)$, set $T = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$, and define $Int_t T : t \to t$, sending $X \in t$ to $TX^{-1}$. Then $Int_t T$ is an automorphism of $t$, called the inner automorphism of $t$ induced by $T$.

(b) Central automorphisms

The map $\phi_t,\eta : X \mapsto X + \left( \begin{array}{cc} \eta(X)E & 0 \\ 0 & -\eta(X)E \end{array} \right)$, for all $X \in t$, where $\eta : t \to R$ is a homomorphism of Lie algebras with $1 + \eta(0) - \eta(0) = 1 \in R^*$ and $R$ is regarded as an abelian Lie algebra, is an automorphism of $t$, called the central automorphism of $t$ induced by $\eta$.

(c) Graph automorphisms

Let $\epsilon = e^2$ be an idempotent in $R$, $J = E_{1,m}^{(m)} + E_{2,m-1}^{(m)} + \cdots + E_{m-1,2}^{(m)} + E_{m,1}^{(m)}$. Define $\phi_{t,\epsilon} : t \to t$ by sending any $\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$ to $\begin{pmatrix} \epsilon A - (1-\epsilon)JA'J & 0 \\ 0 & -\epsilon A' + (1-\epsilon)JAJ \end{pmatrix}$. Then $\phi_{t,\epsilon}$ is an automorphism of $t$, called the graph automorphism of $t$ induced by $\epsilon$.

The main theorem in [2] is as follows.

**Theorem 3.1** [2]. If $R$ is a commutative ring with identity, $m \geq 3$. Then every automorphism $\phi_t$ of $t$ can be written uniquely in the form

$$\phi_t = \phi_t,\eta \cdot \phi_{t,\epsilon} \cdot Int_t T,$$

where $\phi_t,\eta$, $\phi_{t,\epsilon}$, $Int_t T$ are the central, graph and inner automorphisms of $t$ defined above.

4. Standard automorphisms of $l$

We now define some standard automorphisms for the standard Borel subalgebra $l$ of $\mathfrak{o}(2m, R)$.

(a) Inner automorphisms

For invertible $A \in t(m, R)$ and $B' = -B \in R^{m \times m}$, set $X = \begin{pmatrix} A & AB \\ 0 & A^{-1} \end{pmatrix}$, and define $Int_l X : l \to l$, sending $Y \in l$ to $XYX^{-1}$. Then $Int_l X$ is an automorphism of $l$, called the inner automorphism of $l$ induced by $X$.

(b) Graph automorphisms

Let $\omega = E^{(2m)} - E_{m,m} - E_{m,-m} - E_{-m,-m} + E_{-m,m}$, $\pi = \pi^2$ be an idempotent in $R$, we define $\phi_{l,\pi} : l \to l$, sending any $X \in l$ to $\pi X + (1-\pi)\omega X \omega$. Then $\phi_{l,\pi}$ is an automorphism of $l$ (note that $\phi_{l,\pi}$ is the identity), called the graph automorphism of $l$ induced by $\pi$.

(c) Extremal automorphisms

Let $c \in R^*$, and define $\phi_{l,c} : l \to l$, sending $\begin{pmatrix} A & B \\ 0 & -A \end{pmatrix} \in l$ to $\begin{pmatrix} A & cB \\ 0 & -A \end{pmatrix} \in l$. Then $\phi_{l,c}$ is an automorphism of $l$, called the extremal automorphism of $l$ induced by $c \in R^*$. Note that if $c = r^2$
for certain $r \in R^*$, then $\phi_{l,c}$ is exactly the inner automorphism of $l$ induced by $\begin{pmatrix} rE & 0 \\ 0 & r^{-1}E \end{pmatrix}$. If $c \notin (R^*)^2$, $\phi_{l,c}$ is not an inner automorphism.

5. Automorphisms of $l$

In this paper, we obtain the main theorem as follows.

**Theorem 5.1.** Let $R$ be a commutative ring with identity, $2 \in R^*$ and $m \geq 5$. Then every automorphism $\phi$ of $l$ can be written in the form

$$\phi = \phi_{l,\pi} \cdot \phi_{l,c} \cdot \text{Int}_l X,$$

where $\phi_{l,\pi}$, $\phi_{l,c}$, and $\text{Int}_l X$ are the graph, extremal and inner automorphisms of $l$ defined above.

**Proof.** Let $\phi$ be an automorphism of $l$. We shall give the proof by steps.

**Step 1:** There exists $S = \text{diag}(A, 1, A^{-1}, 1)$ with $A \in t(m - 1, R)$ invertible, such that $(\phi \cdot \text{Int}_l S^{-1})(T_i, j(a)) = T_i, j(a)(\text{mod} z)$ for all $a \in R$ and $1 \leq i < j \leq m - 1$.

Because $q$ is stable under $\phi$, then $\phi$ induces an automorphism $\overline{\phi}$ of $l/\phi$ by $\overline{\phi}(X) = \phi(X)$, $X \in l$. Since $l/\phi$ is isomorphic to $s$, we now directly view $l/\phi$ as $s$. By Section 3, we know that

$$\overline{\phi} = \phi_{s,\eta} \cdot \phi_{s,\epsilon} \cdot \text{Int}_s S,$$

where $\phi_{s,\eta}, \phi_{s,\epsilon}$ and $\text{Int}_s S$ are the central, graph and inner automorphisms of $s$ respectively (defined in Section 3). It is easy to see that $\text{Int}_s S = \text{Int}_l S$. So $\overline{\phi} \cdot \text{Int}_l S^{-1} = \phi_{s,\eta} \cdot \phi_{s,\epsilon}$. Replace $\phi$ with $\phi \cdot \text{Int}_l S^{-1}$, then $\overline{\phi} = \phi_{s,\eta} \cdot \phi_{s,\epsilon}$. If we can prove that $\epsilon = 1$, then for any $a \in R$ and $1 \leq i < j \leq m - 1$, $\phi(T_i, j(a)) \equiv T_i, j(a)(\text{mod} z)$ (note that $u$ is stable under $\phi$). We know that $\phi(T_{2,3}(1)) = T_{2,3}(\epsilon) + T_{m-2,m-1}(\epsilon - 1) + Z$ for some $Z \in z$, $\phi(T_{1,-2}) = T_{1,-2}$, and $\phi(T_{1,-3}) \subseteq T_{1,-3} + T_{1,-2}$. Suppose that $\phi(T_{1,-2}(a)) = T_{1,-2}(1)$, $\phi(T_{1,-3}(a)) = T_{1,-3}(b) + T_{1,-2}(c)$, where $a, b, c \in R$. It is obvious that $a, b$ are invertible. By applying $\phi$ on $T_{2,3}(1), T_{1,-3}(a) = T_{1,-2}(a)$, we have that

$$[T_{2,3}(\epsilon) + T_{m-2,m-1}(\epsilon - 1) + Z, T_{1,-3}(b) + T_{1,-2}(c)] = T_{1,-2}(1),$$

which shows that $\epsilon b = 1$, thus $\epsilon \in R^*$, leading to $\epsilon = 1$. Hence $\phi_{s,\epsilon}$ is the identity.

**Step 2:** There exists a graph automorphism $\phi_{l,\pi}$ of $l$ induced by an idempotent $\pi \in R$ such that $w$ is stable under $\phi_{l,\pi} \cdot \phi$.

We know that $w = x + (\sum_{i=1}^{m-1} T_{i,-m})$, and $\phi(x) = x \subseteq w$. For our goal, we need to choose an idempotent $\pi \in R$ such that $(\phi_{l,\pi} \cdot \phi)(T_{i,-m}) \subseteq w$ for all $1 \leq i \leq m - 1$. Since $y$ is stable under $\phi$, we assume that

$$\phi(T_{1,-m}(1)) = T_{1,-m}(a) + T_{1,m}(b) + X,$$

with $a, b \in R$ and $X \in x$. Since $T_{1,-m} + X$ is an ideal of $l$, then so does $\phi(T_{1,-m} + x)$. Thus

$$[H_m(1), \phi(T_{1,-m}(1))] = T_{1,-m}(a) + T_{1,m}(-b) \in \phi(T_{1,-m} + x).$$

This shows that $T_{1,m}(b)$ and $T_{1,-m}(a)$ both lie in $\phi(T_{1,-m} + x)$. Suppose that

$$\phi(T_{1,-m}(\pi)) = T_{1,-m}(a)(\text{mod} x);$$

$$\phi(T_{1,-m}(\rho)) = T_{1,m}(b)(\text{mod} x).$$

Then $\pi + \rho = 1$ and $\pi \cdot \rho = 0$ and $\phi(T_{1,-m}(\pi \cdot \rho)) \equiv 0(\text{mod} x)$, we see that $\pi \cdot \rho = 0$. Now we see that $\pi$ and $\rho$ both are idempotents in $R$. We now construct the graph automorphism
\(\phi_{l,\pi}\) of \(l\), and replace \(\phi_{l,\pi} \cdot \phi\) with \(\phi\), then we see that \(\phi(T_1, -m(1)) \in w\), leading to \(\phi(T_1, -m) \subseteq w\). Since \(\phi(z) = z\), we may assume that

\[
\phi(T_{m-1, -m}(1)) = \sum_{i=1}^{m-1} T_{i,m}(a_i) (\text{mod } w).
\]

For \(1 \leq k \leq m - 3\), assume that \(\phi(T_{k,k+1}(1)) = T_{k,k+1}(1) + Z_k\), and \(\phi(T_{m-1, -m}(1)) = \sum_{i=1}^{m-1} T_{i,m}(a_i) + W_0\) with \(Z_k \in w, W_0 \in w\). By applying \(\phi\) on \([T_{k,k+1}(1), T_{m-1, -m}(1)] = 0\), we see that

\[
\left[ T_{k,k+1}(1) + Z_k, \sum_{i=1}^{m-1} T_{i,m}(a_i) + W_0 \right] = 0.
\]

This shows that \(a_{k+1} = 0\) for \(k = 1, 2, \ldots, m - 3\). Hence

\[
\phi(T_{m-1, -m}(1)) = T_{1,m}(a_1) + T_{m-1, -m}(a_{m-1}) (\text{mod } w).
\]

By applying \(\phi\) on \([T_{1,m}(1), T_{m-1, -m}(1)] = T_{1,-m}(1)\), we see that \([T_{1,m}(1), \phi(T_{m-1, -m}(1))] \subseteq w\), which leads to \(a_{m-1} = 0\). So \(\phi(T_{m-1, -m}(1)) \equiv T_{1,m}(a_1) (\text{mod } w)\). Furthermore, for any \(2 \leq i \leq m - 2\),

\[
\phi(T_{i,-m}(1)) = \phi([T_{i,m}(1), T_{m-1, -m}(1)]) = [\phi(T_{i,m}(1)), \phi(T_{m-1, -m}(1))] \subseteq w.
\]

Hence \(\phi(T_{i,-m}) \subseteq w\) for any \(2 \leq i \leq m - 2\). Now only \(\phi(T_{m-1, -m}) \subseteq w\) is left, for this we only need to prove that \(a_1 = 0\). By above, we see that \(x + \sum_{i=1}^{m-2} T_{i,-m}\) is stable under \(\phi\). We may choose \(W_1 \in x + \sum_{i=1}^{m-2} T_{i,-m}\) such that \(\phi(W_1) = T_{m-2,-m}(1)\). By applying \(\phi\) on \([T_{m-1, -m}(1), W_1] = 0\), we have that

\[
[T_{1,m}(a_1) + W_0, T_{m-2,-m}(1)] = T_{1,-(m-2)}(a_1) = 0.
\]

Thus \(a_1 = 0\), as desired.

**Step 3:** There exists an inner automorphism \(\text{Int}_{t}T_0\) induced by certain \(T_0 = \text{diag}(A, A^{-1})\) with \(A \in t(m, R)\) invertible, such that \(\phi \cdot \text{Int}_{t}T_0^{-1}\) fixes each \(T + w\) for \(T \in t\).

Since \(w\) is stable under \(\phi\), thus \(\phi\) induces an automorphism \(\tilde{\phi}\) of \(l/w\) by \(\tilde{\phi}(X) = \phi(X)/X \in L\). Since \(l/w\) is isomorphic to \(t\), we may directly view \(l/w\) as \(t\). Thus by Theorem 3.1, \(\tilde{\phi}\) can be written in the form

\[
\tilde{\phi} = \phi_{t,\eta_1} \cdot \phi_{t,\epsilon_1} \cdot \text{Int}_{t}T_0,
\]

where \(T_0 = \text{diag}(A, A^{-1})\) with \(A \in t(m, R)\) invertible, \(\epsilon_1\) is an idempotent in \(R\), \(\eta_1 : l \rightarrow R\) is an homomorphism of Lie algebras such that \(1 + \eta_1(\text{diag}(E, -E)) \in R^*\). The fact that \(\phi(T_{1,2}(1)) \equiv T_{1,2}(1) (\text{mod } w)\) shows that \(\epsilon_1 = 1\). It is easy to see that \(\text{Int}_{t}T_0 = \text{Int}_{t}T_0\). Replace \(\phi\) with \(\phi \cdot \text{Int}_{t}T_0^{-1}\). Then \(\tilde{\phi} = \phi_{t,\eta_1}\), thus \(\phi(T_{i,j}(a) + w) = T_{i,j}(a) + w\) for all \(a \in R\) and all \(1 \leq i < j \leq m\).

By \(\phi(T_{1,-2}) = T_{1,-2}\), we may suppose that \(\phi(T_{1,-2}(1)) = T_{1,-2}(c)\). Then \(c \in R^*\), and \(\phi(T_{1,-2}(a)) = T_{1,-2}(ac)\) for any \(a \in R\). For any \(H \in h\), by applying \(\phi\) on \([H, T_{1,-2}(1)] = T_{1,-2}(\chi_1(H) + \chi_2(H)), \chi_i\) denote the map \(h \rightarrow R\), sending \(H \in h\) to its \((i, i)\)-entry, we know that

\[
\left[ H + \eta_1(1)( \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, T_{1,-2}(1)) = T_{1,-2}(c\chi_1(H) + c\chi_2(H)).
\right.
\]

It follows that \(c\chi_1(H) + c\chi_2(H) = c\chi_1(H) + c\chi_2(H) + 2c\eta_1(H)\), leading to \(\eta_1(H) = 0\) for all \(H \in h\). It is easy to see that \(\eta_1(v) = 0\). Thus \(\eta_1(T) = 0\) for all \(T \in t\). Hence \(\phi_{t,\eta_1} = 1\) and \(\phi\) fixes each \(T + w\) for \(T \in t\).
Step 4: There exists certain $W = \begin{pmatrix} E & B \\ 0 & E \end{pmatrix}$, where $B' = -B \in R^{m \times m}$, such that $(Int_l W \cdot \phi)(H) = H$ for any $H \in h$.

Suppose that $\phi$ sends $H(2)$ to $\begin{pmatrix} 2E & B \\ 0 & -2E \end{pmatrix}$, and suppose that $\phi$ sends any $H = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \in h$ to $\begin{pmatrix} A & D \\ 0 & -A \end{pmatrix}$, where $B, D \in R^{m \times m}$ satisfy $B' + B = 0$, $D' + D = 0$ and $A \in t(m, R)$ is diagonal.

Since $H$ commutes with $H(2)$, by applying $\phi$, we see that

$$\begin{pmatrix} 2E & B \\ 0 & -2E \end{pmatrix} \begin{pmatrix} A & D \\ 0 & -A \end{pmatrix} = \begin{pmatrix} A & D \\ 0 & -A \end{pmatrix} \begin{pmatrix} 2E & B \\ 0 & -2E \end{pmatrix}.$$ 

Thus $D = 4^{-1}(BA + AB)$. Choose $W = \begin{pmatrix} E & 4^{-1}B \\ 0 & E \end{pmatrix} \in w$, we see that $(Int_l W \cdot \phi)(H) = H$.

Replace $\phi$ with $Int_l W \cdot \phi$. Now $\phi(H) = H$ for any $H \in h$ and $\phi(V + w) = V + w$ for any $V \in v$.

Step 5: $\phi(V) = V$ for all $V \in v$.

Since $v$ is generated by $T_{1,2}(1), T_{2,3}(1), \ldots, T_{m-1,m}(1)$, it suffices to prove that $\phi(T_{i,i+1}(1)) = T_{i,i+1}(1)$ for all $1 \leq i \leq m - 1$. Suppose that

$$\phi(T_{i,i+1}(1)) = T_{i,i+1}(1) + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

where $B \in R^{m \times m}$ satisfies $B + B' = 0$. By applying $\phi$ on $[H(2), T_{i,i+1}(1)] = 0$ we know that

$$\begin{pmatrix} H(2), T_{i,i+1}(1) + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \end{pmatrix} = 0,$$

which shows that $4B = 0$, thus $B = 0$. Hence $\phi$ fixes all $V \in v$.

Step 6: $\phi(T_{i,j}) = T_{i,j}$ for all $1 \leq i < j \leq m$.

Notice that $w$ is stable under $\phi$. For any $1 \leq i < j \leq m$, suppose that $\phi(T_{i,j}(1)) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, where $B = -B' = (b_{i,j})_{m \times m} \in R^{m \times m}$. By applying $\phi$ on $[H_i(-1), T_{i,j}(1)] = T_{i,j}(-1)$, we have that

$$\begin{pmatrix} H_i(-1), \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix},$$

which shows that all $b_{i,l} = 0$, except for the case that $k = i$ or $l = i$. By applying $\phi$ on $[H_j(-1), T_{i,j}(1)] = T_{i,j}(-1)$, we have that

$$\begin{pmatrix} H_j(-1), \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -B \\ 0 & 0 \end{pmatrix},$$

which shows that $b_{k,l} = 0$, except for the case that $k = j$ or $l = j$. So $\phi(T_{i,j}(1)) \in T_{i,j}$. It follows that $\phi(T_{i,j}) = T_{i,j}$ for all $1 \leq i < j \leq m$.

Step 7: $\phi$ is an extremal automorphism of $l$.

We have shown that there exists $c \in R^*$ such that $\phi(T_{1,2}(a)) = T_{1,2}(ac)$ for any $a \in R$. For any $2 < j \leq m$, by applying $\phi$ on $[T_{2,j}(1), T_{1,j}(a)] = T_{1,j}(a)$, we see that $\phi(T_{1,j}(a)) = T_{1,j}(ac)$ for all $a \in R$. For any $2 \leq i < j \leq m$, $a \in R$, by applying $\phi$ on $[T_{1,i}(1), T_{i,j}(a)] = T_{1,j}(a)$ we have that $\phi(T_{i,j}(a)) = T_{i,j}(ac)$. These show that $\phi$ is exactly the extremal automorphism $\phi_{l,c}$ induced by $c$. Now we see that

$$\phi = \phi_{l,\pi} \cdot Int_l W^{-1} \cdot \phi_{l,c} \cdot Int_l T_0 \cdot Int_l S.$$

By this one can easily obtain the desired expression for $\phi$. This completes the proof. $\square$
Remark. If 2 \notin R^*, Theorem 5.1 may be false. For example, suppose that the annihilator of 2 in R is R itself. Let \( \varphi : l \rightarrow l \), sending any \( X \in l \) to \( X + T_{1,-1}(\eta(X)) \), where \( \eta : l \rightarrow R \) is a homomorphism of Lie algebras such that \( 1 + \eta(T_{1,-1}(1)) \in R^* \). Then \( \varphi \) is an automorphism of \( l \) but can’t be written in the desired form.

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