Iterative methods for nonlinear matrix equations \( X + A^*X^{-\alpha}A = I \)

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Abstract

The present paper treats iterative methods for a class of nonlinear matrix equations \( X + A^*X^{-\alpha}A = I \), with two cases: \( \alpha \geq 1 \) and \( 0 < \alpha < 1 \). Here, the matrix \( A \) is nonsingular. We will derive the necessary and sufficient conditions for the existence of positive definite solutions.

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1. Introduction

This paper considers a class of nonlinear matrix equations

\[ X + A^*X^{-\alpha}A = I, \quad (1.1) \]

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where \( A \) is nonsingular matrix and with two cases: \( \alpha \geq 1 \) and \( 0 < \alpha < 1 \). We can see the matrix equation (1.1) from another point of view as a particular case of the nonlinear matrix equation

\[
X^\alpha + A^*X^{-\beta}A = I,
\]

where \( \alpha, \beta > 0 \) are real. Eq. (1.1) with \( \alpha \geq 1 \) which can arise from Eq. (1.2) when \( 0 < \alpha \leq \beta \). Also Eq. (1.1) with \( 0 < \alpha < 1 \) arises from Eq. (1.2) when \( 0 < \beta < \alpha \).

Some special cases of matrix equation (1.1) have appeared. The existence of positive definite solutions is studied in several articles, see [1,3–12]. Liu and Gao [9] studied Eq. (1.2), when \( \alpha \) and \( \beta \) are natural numbers. They obtained the necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation. They introduce three iterative methods for calculating the positive definite solutions. Du and Hou [3] discussed Eq. (1.2), when \( \alpha \) and \( \beta \) are positive integers. They derived the necessary and sufficient conditions for the existence of a positive semidefinite solution. They suggested two effective iterative methods for finding a positive definite solution of the equation.

A natural question is to consider the positive definite solution of a more general equations \( X^\alpha + A^*X^{-\beta}A = I \), where \( \alpha, \beta > 0 \) are real numbers. For that, the aim of this paper is to discuss Eq. (1.1), with two cases: \( \alpha \geq 1 \) and \( 0 < \alpha < 1 \). We suggest two algorithms for solving the two cases of Eq. (1.1). We derive the necessary and sufficient conditions for the existence of positive definite solutions.

This paper is organized as follows: Section 2 deals with the properties and the existence of positive definite solutions of the problem (1.1) with \( \alpha \geq 1 \), while Section 3 deals with problem (1.1) with \( 0 < \alpha < 1 \). Finally, some concluding remarks are reported.

The following notations are used throughout the rest of the paper. The notation \( A > 0 \) (\( A \geq 0 \)) means that \( A \) is positive definite (semidefinite). \( A^* \) denotes the complex conjugate transpose of \( A \), and \( I \) is the identity matrix. The notation \( A > B \) (\( A \geq B \)) indicates that \( A - B \) is positive definite (semidefinite). We denote by \( \rho(A) \) the spectral radius of \( A \) (i.e., \( \max_{\lambda_i} |\lambda_i| \) and \( \lambda_i \) are the eigenvalues of \( A \)). In what the following we denote by \( \| \cdot \| \) the spectral norm (i.e., \( \| A \| = \sqrt{\rho(A^*A)} \)) unless otherwise noted.

### 2. The matrix equation \( X + A^*X^{-\alpha}A = I \), with \( \alpha \geq 1 \)

In this section, we will discuss in detail the matrix equation (1.1)

\[
X + A^*X^{-\alpha}A = I,
\]

with \( \alpha \geq 1 \). We will start with a review of some elementary useful facts, see [2]. If \( r \in (0, 1] \) and \( 0 \leq P \leq Q \) then \( P^r \leq Q^r \), and if \( 0 \leq P < Q \) then \( P^r < Q^r \). Also if \( r \in [-1, 0) \) and \( 0 \leq P \leq Q \) then \( P^r \geq Q^r \). Finally, if \( r \in (-1, 0) \) and \( 0 \leq P < Q \) then \( P^r > Q^r \).
We will investigate the iterative solution of this equation. For this purpose, let us consider the iterative process

\[ X_{k+1} = \sqrt{A(I - X_k)^{-1}A^*}, \quad k = 0, 1, \ldots, \quad X_0 = 0, \quad (2.2) \]

such that \( I - X \) is invertible matrix.

**Lemma 2.1.** If \( X \) is a positive definite solution of Eq. (2.1) and \( A \) is an invertible matrix, then

\[ \sqrt{AA^*} < X < I. \quad (2.3) \]

**Proof.** The second inequality is evident. To prove the first inequality, we have

\[ A^*X^{-\alpha}A < I, \]

\[ X^\alpha > AA^*, \]

i.e.,

\[ X > \sqrt{AA^*}. \quad \Box \quad (2.4) \]

**Lemma 2.2.** If Eq. (2.1) has a positive definite solution \( X \), therefore

\[ AA^* + \sqrt{AA^*} < I. \quad (2.5) \]

**Proof.** Using the inequality (2.4), \( X^{-\alpha} > I \) and from (2.1), then (2.5) holds. \( \Box \)

**Theorem 2.1.** Let \( A \) be an invertible matrix. Eq. (2.1) has a positive definite solution in the interval \((0, I)\) if and only if there is a number \( \chi \in (0, 1) \) such that \( X_k < \chi I \) for all \( k \). Moreover, in this case, the iteration (2.2) converges to the smallest positive definite solution.

**Proof.** Assume that there exists a number \( \chi \) as in the theorem. Since \( X_0 = 0 \) one has

\[ X_1 = \sqrt{AA^*} > X_0 = 0. \]

Further, we get

\[ X_2 = \sqrt{A(I - X_1)^{-1}A^*}, \]

\[ > \sqrt{A(I - X_0)^{-1}A^*} = X_1. \]

Now, if \( X_k > X_{k-1} \), we have

\[ X_{k+1} = \sqrt{A(I - X_k)^{-1}A^*}, \]

\[ > \sqrt{A(I - X_{k-1})^{-1}A^*} = X_k. \]
Then, the sequence \( \{X_k\} \) is a monotonically increasing sequence and bounded above by some positive definite matrix \( \chi I \). Consequently the sequence \( \{X_k\} \) convergences to a positive definite matrix \( X \), which is a solution Eq. (2.1), i.e.,

\[
X = \sqrt[\alpha]{A(I - X)^{-1}A^*}.
\]

Conversely, let Eq. (2.1) have a positive definite solution \( X \in (0, I) \) and \( \chi \in (0, 1) \) be the largest eigenvalue of \( X \). In order to prove \( X_k < \chi I \) for every \( X_k \) generated from (2.2), we shall prove that

\[
X_k < X.
\]

Really \( X_0 = 0 < \chi I \) and

\[
X_1 = \alpha \sqrt{A(I - X)} \frac{1}{H_2} < X.
\]

Now, let \( X_k < X \), for some fixed \( k \). Hence, we have

\[
X_{k+1} = \alpha \sqrt{A(I - X_k)} \frac{1}{H_2} < \sqrt{A(I - X)} \frac{1}{H_2} = X,
\]

i.e., \( X_{k+1} \in (0, X) \). Then, the theorem is proved.

**Theorem 2.2.** If the matrix \( A \) is normal and \( \rho(A) \leq \left( \frac{\alpha^{\alpha}}{(\alpha + 1)^{\alpha + 1}} \right)^{1/2} \) is fulfilled, then the matrix equation (2.1) has a positive definite solution.

**Proof.** As \( A \) is normal, it can be presented in the form \( A = U^*DU \), where \( U \) is unitary and \( D = \text{diag}(d_1, d_2, \ldots, d_n) \), with \( \{d_s\} \) the eigenvalues of \( A \). According to the condition in the theorem on \( \rho(A) \), we have

\[
|d_s|^2 \leq \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha + 1}}. \tag{2.6}
\]

Now we shall try to find the positive definite solution of Eq. (2.1) of the form

\[
X = U^*\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)U,
\]

where \( \lambda_s \in (0, 1) \) for every \( s \). But it is easy to see that such \( X \) will be a solution of Eq. (2.1) if

\[
\lambda_s + |d_s|^2 \lambda_s^{-\alpha} = 1 \quad \text{or} \quad \lambda_s^\alpha (1 - \lambda_s) = |d_s|^2. \tag{2.7}
\]

From

\[
|d_s|^2 \leq \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha + 1}} = \max_{\lambda \in [0, 1]} \left( (1 - \lambda)\lambda^{-\alpha} \right),
\]

it follows that for every \( s \), there exist \( \lambda_s \in (0, 1) \), such that (2.7) holds. So the matrix \( X = U^*\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)U \) will be the positive definite solution of (2.1).

The next theorem will give us another sufficient condition.

**Theorem 2.3.** Let \( A \) be an invertible matrix. If there exists a \( \chi \in (0, 1) \) such that \( AA^* \leq \chi^\alpha (1 - \chi)I \), then, Eq. (2.1) has a positive definite solution.
**Proof.** Consider the iteration process (2.2). We have

\[ X_1 = \sqrt[\alpha]{AA^*} > X_0 = 0. \]

Let now \( X_k > X_{k-1} \), for fixed \( k \geq 1 \), therefore

\[ X_{k+1} = \sqrt[\alpha]{A(I - X_k)^{-1}A^*}, \]

\[ > \sqrt[\alpha]{A(I - X_{k-1})^{-1}A^*} = X_k. \]

So, we proved that the sequence \( \{X_k\} \) is increasing. Now we prove that for every \( k \), we have \( X_k < \chi I \). Indeed \( X_0 = 0 < \chi I \).

\[ X_1 = \sqrt[\alpha]{AA^*} < \sqrt[\alpha]{\chi^\alpha (1 - \chi)} I \]

\[ = \chi \sqrt[\alpha]{(1 - \chi)} I < \chi I. \]

Now, let \( X_k < \chi I \), for some \( k \). Then,

\[ X_{k+1} = \sqrt[\alpha]{A(I - X_k)^{-1}A^*}, \]

\[ < \sqrt[\alpha]{A(I - \chi I)^{-1}A^*}, \]

\[ < \sqrt[\alpha]{AA^*(I - \chi)^{-1}}, \]

\[ \leq \sqrt[\alpha]{\chi^\alpha (1 - \chi)(I - \chi)^{-1} I}, \]

\[ = \chi I. \]

Therefore, we have \( 0 < X_k < X_{k+1} < \chi I \). Then, we get

\[ \lim_{k \to \infty} X_k = X = \sqrt[\alpha]{A(I - X)^{-1}A^*} > 0, \]

i.e., \( X \) is a positive definite solution of Eq. (2.1). So, we complete the proof. □

3. **The matrix equation** \( X + A^*X^{-\alpha}A = I \) **for** \( \alpha \in (0, 1) \)

In this section, we turn our attention to the equation

\[ X + A^*X^{-\alpha}A = I, \tag{3.1} \]

with \( 0 < \alpha < 1 \). By setting \( X = Y^{1/\alpha} \) in Eq. (3.1), we get

\[ Y^{1/\alpha} + A^*Y^{-1}A = I. \tag{3.2} \]

To solve Eq. (3.1), we will apply Eq. (3.2) to construct the following iterative process:

\[ Y_{k+1} = (I - A^*Y_k^{-1}A)^\alpha, \quad k = 0, 1, \ldots, \quad Y_0 = I. \tag{3.3} \]

For Eq. (3.2), we can formulate some theorems similar to the theorems from Section 2.
Lemma 3.3. If Eq. (3.2) has a positive definite solution \( Y \), then \( Y \in (AA^\ast, I) \).

Proof. From Eq. (3.2), we have \( Y^{1/\alpha} < I \). Then, the second inequality is evident. To prove the first inequality, we have
\[
A^\ast Y^{-1} A < I, \\
Y^{-1} < A^{-\ast} A^{-1},
\]
i.e.,
\[
Y > AA^\ast.
\] (3.4)

Theorem 3.4. Let \( A \) be an invertible matrix. Eq. (3.2) has a positive definite solution in the interval \( (0, I) \) if and only if there exists a \( \chi \in (0, 1) \) such that for the matrices \( Y_k \) generated by the iteration (3.3) we have \( Y_k > \chi I \) for all \( k \geq 0 \). Moreover, in that case the matrices \( Y_k \) converge to a positive definite solution.

Proof. Suppose that there exists a number \( \chi \in (0, 1) \) such that for the matrices \( Y_k \) as in the theorem we have \( Y_k > \chi I \). Since \( Y_0 = I \), we have
\[
Y_1 = (I - A^\ast A)^{\alpha} < I = Y_0.
\]
Moreover,
\[
Y_2 = (I - A^\ast Y_1^{-1} A)^{\alpha} < (I - A^\ast Y_0^{-1} A)^{\alpha} = Y_1.
\]
Suppose that \( Y_k < Y_{k-1} \), we get
\[
Y_{k+1} = (I - A^\ast Y_k^{-1} A)^{\alpha} < (I - A^\ast Y_{k-1}^{-1} A)^{\alpha} = Y_k.
\] (3.5)

Hence, the sequence \( \{Y_k\} \) is monotonically decreasing and bounded below by some positive definite matrix \( \chi I \). Consequently, the sequence \( \{Y_k\} \) convergences to a positive definite matrix \( Y \), which is a solution Eq. (3.2).

Conversely, let Eq. (3.2) have a positive definite solution \( Y \in (0, I) \) and \( \chi \in (0, 1) \) be the smallest eigenvalue of \( Y \). In order to prove \( Y_k > \chi I \) for every \( Y_k \) generated from (3.3), we shall prove that \( Y_k > Y \). Indeed \( Y_0 = I > Y \) and
\[
Y_1 = (I - A^\ast A)^{\alpha} > (I - A^\ast Y^{-1} A)^{\alpha} = Y.
\]
Now, let \( Y_k > Y \), for some fixed \( k \geq 1 \). Hence, we have
\[
Y_{k+1} = (I - A^\ast Y_k^{-1} A)^{\alpha} > (I - A^\ast Y^{-1} A)^{\alpha} = Y,
\]
i.e., \( Y_{k+1} \in (Y, I) \). Then, the theorem is proved. \( \Box \)

The next theorem will give us another sufficient condition.

Theorem 3.5. Let \( A \) be an invertible matrix. If there exists a \( \chi \in (0, 1) \) such that \( A^\ast A \leq (1 - \chi^{1/\alpha}) I \), then Eq. (3.2) has a positive definite solution \( Y \). Consequently, Eq. (3.1) has a positive definite solution \( X = Y^{1/\alpha} \).
Proof. First, we shall prove that $Y_k > \chi I$ for every $k$. For $k = 0$, we have $Y_0 = I > \chi I$ then this condition is fulfilled. Suppose that for some fixed $k$, we have $Y_k > \chi I$. Therefore,

$$Y_{k+1} = (I - A^*Y_k^{-1}A)^\alpha,$$

$$> (I - A^*A)^\alpha,$n

$$\geq (I - \chi (1 - \chi^{1/\alpha})^\alpha I,$n

$$= \chi I.$$

Further, from the iteration process (3.3), we get $Y_1 = (I - A^*A)^\alpha$. By the assumptions on $A$ we have $0 < A^*A < I$, and hence $Y_1 < I = Y_0$. Also if $Y_k < Y_{k-1}$, for fixed $k \geq 1$ and from (3.5), then $Y_{k+1} < Y_k$.

From the above, we proved the sequence $\{Y_k\}$ is monotonically decreasing and bounded from below, i.e., $\chi I < Y_{k+1} < Y_k$. Hence, the sequence has a limit, which we denote by $Y$. Obviously, $Y = (I - A^*A)^\alpha > 0$. That is, $Y$ is a positive definite solution of Eq. (3.2). Therefore, $X = Y^{1/\alpha}$ is a positive definite solution of Eq. (3.1). This completes the proof. □

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References


