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A Remark on the Global Solvability of the Cauchy Problem for Quasilinear Parabolic Equations

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The present paper is concerned with the global solvability of the Cauchy problem for the quasilinear parabolic equations with two independent variables: $u_t = a(t,x,u,u_x)u_{xx} + f(t,x,u,u_x)$. We investigate the case of the arbitrary order of growth of the function $f(t,x,u,p)$ with respect to p when $|p| \to +\infty$. Conditions which guarantee the global classical solvability of the problem are given.
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INTRODUCTION AND MAIN RESULT

In the present paper we consider the following problem

$$(0.1)\qquad u_t = a(t,x,u,u_x)u_{xx} + f(t,x,u,u_x) \quad \text{in } \Pi_T = (0,T) \times \mathbf{R},$$

$$(0.2)\qquad u(0,x) = u_0(x) \quad \text{for } x \in \mathbf{R},$$

where the functions $a(t,x,u,p) > 0$, $f(t,x,u,p)$ take finite values for $(t,x) \in \Pi_T$ and any finite u, p.

It is well known (see [1]) that the global solvability (i.e., solvability in domains of arbitrary size without assumptions of the smallness of the initial conditions and their derivatives) of the Cauchy problem as well as of the boundary value problems for quasilinear parabolic equations is not merely a consequence of sufficient smoothness of the coefficients. The essential role played here is the character of nonlinearities of the coefficients. Fulfillment of the condition

$$(0.3)\qquad uf(t,x,u,0) \le b_1 u^2 + b_2,$$

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where $b_1 \geq 0$, $b_2 \geq 0$ are some constants, guarantees the global a priori estimate of $|u|$ (see [1, 2]). The following condition (Bernstein's type condition [3])

$$|f(t,x,u,p)| \leq a(t,x,u,p)\psi(|p|), \tag{0.4}$$

where $\psi(\rho) \in C^1(0,+\infty)$ is a nondecreasing positive function of $\rho \geq 0$ satisfying the relation

$$\int_1^{+\infty} \frac{\rho\, d\rho}{\psi(\rho)} = +\infty, \tag{0.5}$$

guarantees the boundedness of $|u_x|$ (see [1, 2]).

In [2] it was shown that the global solvability of problem (0.1), (0.2) (and of the boundary value problems) takes place when conditions (0.3)–(0.5) are fulfilled and functions a, b are (only) Hölder continuous functions. For the first initial boundary value problem conditions (0.4), (0.5) are generally speaking necessary for the global solvability. Examples show that in the case of violation of these assumptions the gradient of the bounded solution may blow-up (on the boundary of the domain, see [4–6], as well as in the interior of the domain, see [7–9]), i.e., there exists t^* such that $|u_x(t,x)| \to +\infty$ when $t \to t^*$ at least for some x. As it was mentioned in [10] for the Cauchy problem even if conditions (0.4), (0.5) are violated, it is possible to obtain the global solvability for a certain class of equations, such as

$$u_t = u_{xx} + f(u_x)$$

with f an arbitrary function from C^1. Let us mention here that in [11, 12] it was shown that conditions (0.4), (0.5) are unnecessary for a certain class of second and third initial boundary value problems. We should also mention the following result: in [13] the solvability of problem (0.1), (0.2) was proved for any $0 < T < +\infty$ under assumption (0.4) and the following assumption (which replaces (0.5)),

$$\sup_{\Pi_T} |u(t,x) - u(t,y)| < \int_K^{+\infty} \frac{\rho\, d\rho}{\psi(\rho)},$$

where $|u_0(x) - u_0(y)| \leq K|x-y|$.

The goal of this paper is to present a new (weaker) sufficient condition guaranteeing the global solvability of the Cauchy problem.

Let us formulate the main result. Suppose that the function $f(t,x,u,p)$ can be represented in the form

$$f(t,x,u,p) = f_1(t,x,u,p) + f_2(t,x,u,p), \tag{0.6}$$

where

$$|f_1(t,x,u,p)| \le a(t,x,u,p)\psi(|p|) \tag{0.7}$$

($\psi(\rho)$ is the same as in (0.4), (0.5)) and f_2 satisfies the condition

$$\begin{gathered} f_2(t,y,v,p) - f_2(t,x,u,p) \ge 0, \\ f_2(t,x,v,-p) - f_2(t,y,u,-p) \ge 0, \end{gathered} \tag{0.8}$$

when $x \ge y$, $u \ge v$, $p \ge 0$. The function f_2 satisfies condition (0.8), for example, in the following cases:

(1) f_2 is an arbitrary function of two variables t and p ($f_2 = f_2(t,u_x)$);

(2) $f_2 = g(x)f_3(t,p)$ or $f_2 = g(u)f_4(t,p)$ where g is an arbitrary nondecreasing function and $pf_3(t,p) \le 0$, $f_4(t,p) \le 0$ for $t \in (0,T)$ and all p.

Concerning the initial function $u_0(x)$ we suppose that

$$\begin{gathered} |u_0(x) - u_0(y)| \le K|x-y| \text{ for } |x|,|y| \le N; \\ u_0(x) \equiv 0 \text{ for } |x| \ge N, \end{gathered} \tag{0.9}$$

where $K > 0$, $N > 0$ are arbitrary constants. Note that condition (0.9) can be weakened (see the Remark in Section 2).

THEOREM. *Suppose that* $a(t,x,u,p)$, $f(t,x,u,p) \in C^{\alpha}(\Pi_T \times R^2)$ *for some* $\alpha \in (0,1)$ *and suppose conditions* (0.3), (0.6)–(0.9) *hold. Then for any* $0 < T < \infty$ *there exists a bounded solution of problem* (0.1), (0.2) *which belongs to* $C^{1+\beta/2,2+\beta}_{t,x}(\Pi_T) \cap C^0(\overline{\Pi}_T)$ *for some* $\beta \in (0,1)$.

If the functions $a(t,x,u,p)$, $f(t,x,u,p)$ *and their partial derivatives with respect to* u *and* p *are bounded for* $(t,x) \in \Pi_T$ *and finite* u, p *then such a solution is unique.*

Remark. Usually the solution of the Cauchy problem is obtained as the limit of a sequence of solutions of the first initial boundary value problem in cylinders $(0,T) \times \Omega$ under an unlimited dilation of the domain Ω (see, for example, [1]). In our case the use of the first initial boundary value problem does not work, because, as it was mentioned above, for the first initial boundary value problem the conditions of the theorem do not guarantee the boundedness of $|u_x|$. To approximate the Cauchy problem we use here a certain third initial boundary value problem.

1. AUXILIARY LEMMA

In this section we consider the problem

$$u_t^\varepsilon = a(t,x,u^\varepsilon,u_x^\varepsilon)u_{xx}^\varepsilon + f(t,x,u^\varepsilon,u_x^\varepsilon) \quad \text{in } \Pi_T^\varepsilon, \tag{1.1}$$

$$u^\varepsilon(0,x) = u_0(x) \quad \text{for } |x| \le 1/\varepsilon, \tag{1.2}$$

$$u_x^\varepsilon - u^\varepsilon|_{x=-1/\varepsilon} = 0, \qquad u_x^\varepsilon + u^\varepsilon|_{x=1/\varepsilon} = 0, \tag{1.3}$$

where $\Pi_T^\varepsilon = \{(t,x): t \in (0,T), |x| < 1/\varepsilon\}$ with $\varepsilon < \varepsilon_0$ and $1/\varepsilon_0 \ge N$. Due to (0.9) the following compatibility condition is fulfilled

$$u_0'(-1/\varepsilon) - u_0(-1/\varepsilon) = 0, \qquad u_0'(1/\varepsilon) + u_0(1/\varepsilon) = 0.$$

For the sake of simplicity we drop the index ε in $u^\varepsilon(t,x)$.

In order to estimate the $|u_x|$ we use Kruzhkov's idea of introducing a new spatial variable [2].

LEMMA. *Let $u(t,x)$ be a classical solution of problem* (1.1)–(1.3) (*i.e.*, $u(t,x) \in C_{t,x}^{1,2}(\Pi_T^\varepsilon) \cap C_{t,x}^{0,1}(\overline{\Pi}_T^\varepsilon)$) *and suppose that conditions* (0.3), (0.6)–(0.9) *are fulfilled. Then in $\overline{\Pi}_T^\varepsilon$ the inequalities*

$$|u(t,x)| \le M, \qquad |u_x(t,x)| \le C_1$$

hold, where the constant M depends only on b_1, b_2, $\sup|u_0(x)|$ *and the constant C_1 only on K, $1/\varepsilon$, ψ, and M.*

Proof. Due to boundary conditions (1.3) the function $u(t,x)$ cannot attain neither positive maximum nor negative minimum when $x = \pm 1/\varepsilon$. Taking this into account we can easily obtain the required estimate of $|u|$ on the basis of the maximum principle by means of standard arguments (see, for example, [1]).

Consider Eq. (0.1) at two different points of Π_T, (t,x), (t,y) where $x \ne y$,

$$u_t(t,x) = a(t,x,u(t,x),u_x(t,x))u_{xx} + f(t,x,u(t,x),u_x(t,x)),$$
$$u_t(t,y) = a(t,y,u(t,y),u_y(t,y))u_{yy} + f(t,y,u(t,y),u_y(t,y)).$$

Introduce the function $v(t,x,y) \equiv u(t,x) - u(t,y)$. One can easily see that the function v satisfies the equation

$$\begin{aligned} v_t &= a(t,x,u(t,x),v_x)v_{xx} + a(t,y,u(t,y),-v_y)v_{yy} \\ &\quad + f_1(t,x,u(t,x),v_x) - f_1(t,y,u(t,y),-v_y) \\ &\quad + f_2(t,x,u(t,x),v_x) - f_2(t,y,u(t,y),-v_y) \end{aligned}$$

in the domain $\{(t, x, y) : t \in (0, T), |x| < 1/\varepsilon, |y| < 1/\varepsilon\}$. Define the operator

$$Lv \equiv -v_t + A(t,x)\left[v_{xx} + \psi(|v_x|)\right] + A(t,y)\left[v_{yy} + \psi\left(|v_y|\right)\right],$$

where $A(t, z) \equiv a(t, z, u(t, z), u_z(t, z))$. From (0.7) it follows that

$$Lv \geq f_2\big(t, y, u(t,y), u_y(t,y)\big) - f_2\big(t, x, u(t,x), u_x(t,x)\big).$$

Let the function $h(\tau)$ be a solution of the ordinary differential equation

$$h''(\tau) + \psi(|h'(\tau)|) = 0 \tag{1.4}$$

on the interval $(0, \tau_0)$, where $\tau_0 > 0$ will be selected below, and

$$\begin{gathered} h(0) = 0, \qquad h(\tau_0) = \mu \equiv \sup u - \inf u, \\ h'(\tau) > 0 \text{ for } \tau \in [0, \tau_0]. \end{gathered} \tag{1.5}$$

In order to select τ_0 represent the solution of (1.4) in parametrical form (using the substitution $h' = q$, $d/d\tau = q d/dh$),

$$h(q) = \int_q^{q_1} \frac{\rho\, d\rho}{\psi(\rho)}, \qquad \tau(q) = \int_q^{q_1} \frac{d\rho}{\psi(\rho)},$$

where the parameter q varies in the interval $[q_0, q_1]$ and q_0, q_1 are selected so that $q_1 > q_0 > \max\{K, M, \mu\varepsilon\}$ and

$$h(q_0) = \int_{q_0}^{q_1} \frac{\rho\, d\rho}{\psi(\rho)} = \mu.$$

This is possible due to (0.5). We put $\tau_0 = \tau(q_0)$. It is clear that

$$\tau_0 = \int_{q_0}^{q_1} \frac{d\rho}{\psi(\rho)} < \frac{1}{q_0} \int_{q_0}^{q_1} \frac{\rho\, d\rho}{\psi(\rho)} = \frac{\mu}{q_0} < \frac{1}{\varepsilon}.$$

Consider the function $w \equiv v(t, x, y) - h(x - y)$ in the prism

$$P = \{(t,x,y) : 0 < t \leq T, |x| < 1/\varepsilon, |y| < 1/\varepsilon, 0 < x - y < \tau_0\}.$$

By virtue of (1.4), $Lh = 0$ and

$$\begin{aligned} Lv - Lh \equiv \tilde{L}w \equiv & -w_t + A(t,x)(w_{xx} + \beta_1 w_x) + A(t,y)(w_{yy} + \beta_2 w_y) \\ & \geq f_2\big(t, y, u(t,y), u_y(t,y)\big) - f_2\big(t, x, u(t,x), u_x(t,x)\big), \end{aligned}$$

where $|\beta_i| < +\infty$, $i = 1, 2$, due to the smoothness of ψ and to the fact that $u(t, x)$ is a classical solution. Let $\tilde{w} = we^{-t}$. Then

$$(1.6) \quad \tilde{L}\tilde{w} \equiv -\tilde{w}_t + A(t,x)\big(\tilde{w}_{xx} + \beta_1\tilde{w}_x\big) + A(t,y)\big(\tilde{w}_{yy} + \beta_2\tilde{w}_y\big) - \tilde{w}$$
$$\geq e^{-t}\big[f_2\big(t,y,u(t,y),u_y(t,y)\big) - f_2\big(t,x,u(t,x),u_x(t,x)\big)\big].$$

Denote by Γ the parabolic boundary of P ($\Gamma = \partial P \setminus \{(t,x,y) : t = T, |x| < 1/\varepsilon, |y| < 1/\varepsilon, 0 < x - y < \tau_0\}$. If the function $\tilde{w}$ attains its positive maximum at the point $N^0 = (t^0, x^0, y^0) \in \bar{P} \setminus \Gamma$ then at this point $\tilde{w}_x = \tilde{w}_y = 0$, hence $u_x(t^0, x^0) = u_y(t^0, y^0) = h'(x^0 - y^0) > 0$ and $u(t^0, x^0) > u(t^0, y^0)$ $(x^0 > y^0)$. Thus from (1.6), taking into account the first inequality in (0.8), we obtain

$$\tilde{L}\tilde{w} \mid_{N^0} \geq 0.$$

On the other hand, the fact that at N^0 the function $\tilde{w}$ attains positive maximum implies the inequality

$$\tilde{L}\tilde{w} \mid_{N^0} < 0.$$

From this contradiction we conclude that $\tilde{w}$ cannot attain positive maximum in the interior of the domain P.

Consider Γ:

(1) For $x = y$ we have $\tilde{w} = 0$.

(2) For $x - y = \tau_0$ we have $\tilde{w} = e^{-t}(u(t,x) - u(t,y) - \mu) \leq e^{-t}(osc(u) - \mu) = 0$.

(3) For $t = 0$ we have $\tilde{w} = e^{-t}(u_0(x) - u_0(y) - h(x-y)) \leq e^{-t}(K(x-y) - (h(x-y) - h(0))) = e^{-t}(K(x-y) - h'(\tau^*)(x-y) \leq 0)$, where $0 < \tau^* < \tau_0$ (recall that $h' \geq q_0 > K$).

Now consider the following part of Γ

$$(4) \quad \big\{(t,x,y) : 0 < t < T, -1/\varepsilon < x < -1/\varepsilon + \tau_0, y = -1/\varepsilon\big\}.$$

Taking into account condition (1.3) we conclude that

$$(1.7) \quad \tilde{w}_y(t,x,-1/\varepsilon) = e^{-t}\big(-u_y(t,-1/\varepsilon) + h'(x + 1/\varepsilon)\big)$$
$$= e^{-t}(-u(t,-1/\varepsilon) + h'(x+1/\varepsilon))$$
$$\geq e^{-t}(-M + q_0) > 0$$

(recall that $q_0 > \max\{K, M, \mu\varepsilon\}$).

On the last part of Γ:

(5) $\{(t, x, y) : 0 < t < T, x = 1/\varepsilon, 1/\varepsilon - \tau_0 < y < 1/\varepsilon\}$

and (from (1.3)) we have

$$\tag{1.8} \begin{aligned} \tilde{w}_x(t, 1/\varepsilon, y) &= e^{-t}(-u(t, 1/\varepsilon) - h'(1/\varepsilon - y)) \\ &\le e^{-t}(M - q_0) < 0. \end{aligned}$$

From (1.7), (1.8) we conclude that $\tilde{w}$ cannot attain maximum on parts (4), (5). Hence $\tilde{w} \le 0$ on $\bar{P}$. Thus we have proved that

$$u(t, x) - u(t, y) \le h(x - y) \quad \text{in } \bar{P}.$$

By analogy, taking the function $v_1 \equiv u(t, y) - u(t, x)$ in the place of v, we obtain

$$u(t, x) - u(t, y) \ge -h(x - y) \quad \text{in } \bar{P}$$

(here, in order to prove that $\tilde{w}_1 = e^{-t}(v_1 - h(x - y))$ does not attain positive maximum in $\bar{P} \setminus \Gamma$, we use the second inequality (0.8)).

In view of the symmetry of the variables x and y in the same way we examine the case $y > x$. As a result we have that for

$$0 \le t \le T, \qquad |x| \le 1/\varepsilon, \qquad |y| \le 1/\varepsilon, \qquad 0 < |x - y| \le \tau_0$$

the inequality

$$\frac{|u(t, x) - u(t, y)|}{|x - y|} \le \frac{h(|x - y|) - h(0)}{|x - y|}$$

holds, implying that $|u_x(t, x)| \le h'(0) = q_1$. The lemma is proved.

2. EXISTENCE AND UNIQUENESS THEOREM

Suppose that assumptions of the theorem are fulfilled. Consider the sequence of expanding domains $\Pi_T^\varepsilon = (0, T) \times (-1/\varepsilon, 1/\varepsilon)$, with $\varepsilon < 1/N$, that tend in the limit to Π_T. In each of the cylinders Π_T^ε there exists a solution $u^\varepsilon(t, x)$ of Eq. (0.1) from $C_{t,x}^{1+\beta/2, 2+\beta}(\Pi_T) \cap C_{t,x}^{0,1}(\bar{\Pi}_T)$ for some $\beta \in (0, 1)$ (see [2]). Due to (1.3), (1.4) and from the results of Section 1 it follows that the solutions and their first spatial derivatives are uniformly bounded

$$\sup_{\Pi_T^\varepsilon}|u^\varepsilon(t,x)| \le M, \ \sup_{\Pi_T^\varepsilon}|u_x^\varepsilon(t,x)| \le C_1.$$

Furthermore for any Π_T^δ and u^ε with $\varepsilon < \delta$ we have

$$\|u^\varepsilon\| \le C(\delta),$$

where $\|*\|$ is the norm in $C_{t,x}^{1+\beta/2,2+\beta}(\Pi_T^\delta)$ and the constant $C(\delta)$ depends on δ but not on ε. Employing the usual diagonal process we can extract from u^ε a subsequence u^{ε_n} that converges together with the derivatives $u_t^{\varepsilon_n}$, $u_x^{\varepsilon_n}$, $u_{xx}^{\varepsilon_n}$ at each point of Π_T to some function u and its corresponding derivatives. It is clear that $u(t,x) \in C_{t,x}^{1+\beta/2,2+\beta}(\Pi_T)$ is a solution of the Cauchy problem (0.1), (0.2) and

$$\sup_{\Pi_T}|u(t,x)| \le M, \qquad \sup_{\Pi_T}|u_x(t,x)| \le C_1.$$

The uniqueness of problem (0.1), (0.2) under the additional assumptions formulated in the theorem follows from [1].

EXAMPLES. Consider equations

$$u_t = a(t,x,u,u_x)u_{xx} + f_1(t,x,u,u_x) + f_2(t)e^{u_x}, \tag{2.1}$$

$$u_t = a(t,x,u,u_x)u_{xx} + f_1(t,x,u,u_x) - g_1(t,x)u_x^{2m+1}, \tag{2.2}$$

$$u_t = a(t,x,u,u_x)u_{xx} + f_1(t,x,u,u_x) - g_2(t,u)u_x^{2m}, \tag{2.3}$$

$$u_t = a(t,x,u,u_x)u_{xx} + f_1(t,x,u,u_x) - g_3(t,u)e^{u_x}, \tag{2.4}$$

where $m \in \mathbf{N}$; $a > 0$, f_1 satisfies conditions (0.3), (0.7), (0.5), function g_1 is nondecreasing with respect to x, functions g_2, g_3 are nondecreasing with respect to u, and $-ug_3(t,u) \le b_1u^2 + b_2$ (see (0.3)). If in addition the functions a, f_i, g_i, $i = 1, 2$, are Hölder continuous functions then there exists a global classical solution of the Cauchy problem (2.n), (0.2), $n = 1$, 2, 3, 4, with initial function $u_0(x)$ satisfying conditions (0.9). In (2.3) we can take $m \in (0, +\infty)$ if instead of $g_2(t,u)u_x^{2m}$ we take $g_2(t,u)|u_x|^{2m}$.

Remark. Instead of the initial function satisfying conditions (0.9) we can take the arbitrary Lipschitz continuous function $u_0(x)$ in $\overline{\mathbf{R}}$ vanishing when $|x| \to +\infty$ and approach it in the norm $|*|_L$ by functions $u_0^\varepsilon(x)$ satisfying conditions (0.9), where $|G(x)|_L = \sup|G(x)| + K$, $K = \sup(|G(x) - G(y)|/|x-y|)$.

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