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A new analytical and numerical treatment for singular two-point boundary value problems via the Adomian decomposition method

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ABSTRACT

Based on the Adomian decomposition method, a new analytical and numerical treatment is introduced in this research to investigate linear and non-linear singular two-point BVPs. The effectiveness of the proposed approach is verified by several linear and non-linear examples.

1. Introduction

Many problems in applied mathematics, such as gas dynamics, nuclear physics, chemical reaction, studies of atomic structures and atomic calculations lead to singular boundary value problems of the form:

$$u''(x) + p(x)u'(x) + q(x)f(u(x)) = r(x), \quad x \in (a, b), \tag{1}$$

subject to the boundary conditions

$$u(a) = \alpha \quad \text{and} \quad u(b) = \beta, \tag{2}$$

where at least one of the functions $p(x)$, $q(x)$ and $r(x)$ has a singular point and a, b, α and β are finite constants. When $p(x) = 0$ and $f(u(x)) = [u(x)]^{-\sigma}$, Eq. (1) is known as the generalized Emden–Fowler equation with a negative exponent and arises frequently in applied mathematics (see [1,2] and the references cited therein). Also when $p(x) = r(x) = 0$, $q(x) = -x^{-1/2}$ and $f(u(x)) = [u(x)]^{\frac{3}{2}}$, Eq. (1) is known as the Thomas–Fermi equation [3], given by the singular equation

$$u'' = x^{-1/2}u^{3/2}, \tag{3}$$

which arises in the study of the electrical potential in an atom. Another example is given by the singular equation:

$$u'' + \frac{p}{x}u' + g(u) = 0, \tag{4}$$

which results from an analysis of heat conduction through a solid with heat generation. The function $g(u)$ represents the heat generation within the solid, u is the temperature and the constant p is equal to 0, 1 or 2 depending on whether the solid is a plate, a cylinder or a sphere [4]. In recent years, an increasing interest has been observed in investigating singular two-point boundary value problems and a number of methods have been proposed, see [4–20]. Although these numerical methods have many advantages, a huge amount of computational work is required for getting accurate approximations, especially for nonlinear problems. So, we hope in this research to introduce a direct and a unified approach to deal with these singular two-point BVPs.

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Over the last 25 years, the Adomian decomposition method (ADM) and its modification (MADM) [21–36] have been used to solve effectively and easily a large class of linear and nonlinear ordinary and partial differential equations. However, little attention was devoted to their applications in solving the singular two-point boundary value problems. Recently, an attempt has been done in [37]. Very recently, approximate solutions of linear singular two-point BVPs were obtained in [38] using the modified homotopy analysis method. Also in [39] Ravi and Aruna used another analytical method, the differential transformation method to obtain the exact solutions for some linear singular two-point BVPs. Although these analytical methods [38,39] are effective in the linear case, their applicability for non-linear problems was not examined. In this work, the ADM is improved to deal with linear and non-linear singular two-point BVPs. This improvement is based on the ADM and a modification of Lesnic's work [40]. It might seem reasonable before launching into the main idea of this paper to present a brief outline of Lesnic's work.

2. Lesnic's work

In [40] Lesnic proposed the operators

$$L_{xx}^{-1}(.) = \int_{x_0}^{x} dx' \int_{x_0}^{x''} (.)dx'' - \frac{x - x_0}{1 - x_0} \int_{x_0}^{1} dx' \int_{x_0}^{x''} (.)dx'', L_t^{-1}(.) = \int_0^t (.)dt', \tag{5}$$

to solve the Dirichlet problem for the heat equation $u_t = u_{xx}, x_0 < x < 1, t > 0$ under the boundary conditions

$$u(x_0, t) = f_0(t), \qquad u(1, t) = f_1(t), \tag{6}$$

and the initial condition

$$u(x, 0) = p(x). \tag{7}$$

Using the definition in (5) we note that

$$L_{xx}^{-1}(u_{xx}) = u(x, t) - u(x_0, t) - \frac{x - x_0}{1 - x_0}[u(1, t) - u(x_0, t)], \tag{8}$$

i.e., the boundary conditions can be used directly. However, from (5) we note that the lower bound of all integrations is restricted to the initial point x_0. In fact we can avoid this restriction by using a new definition of L_{xx}^{-1} which gives the same result as in Eq. (8) and given by

$$L_{xx}^{-1}(.) = \int_{x_0}^{x} dx' \int_{c}^{x''} (.)dx'' - \frac{x - x_0}{1 - x_0} \int_{x_0}^{1} dx' \int_{c}^{x''} (.)dx'', \tag{9}$$

where c is free lower point. This free point plays an important role if the equation being solved has a singular point. Consequently, we propose the following operator to solve the singular two-point BVPs (1) and (2):

$$L_{xx}^{-1}(.) = \int_{a}^{x} dx' \int_{c}^{x'} (.)dx'' - \frac{x - a}{b - a} \int_{a}^{b} dx' \int_{c}^{x'} (.)dx'', \quad a \neq b, c \text{ is arbitrary.} \tag{10}$$

In the next section, we introduce a theoretical derivation of the operator given by Eq. (10).

3. Derivation of the proposed operator

First, we define L_{xx}^{-1} as

$$L_{xx}^{-1}(.) = \int_{a}^{x} dx' \int_{c}^{x'} (.)dx'' - z(x) \int_{d}^{b} dx' \int_{e}^{x'} (.)dx'', \tag{11}$$

where $z(x)$ is to be determined such that $L_{xx}^{-1}(u''(x))$ can be expressed only in terms of the boundary conditions given in (2). With this definition we can easily get

$$\begin{aligned} L_{xx}^{-1}(u''(x)) &= u(x) - u(a) - (x - a)u'(c) - z(x)[u(b) - u(d) - (b - d)u'(e)] \\ &= u(x) - u(a) - z(x)[u(b) - u(d)] - (x - a)u'(c) + z(x)[(b - d)u'(e)]. \end{aligned} \tag{12}$$

Setting $d = a$ and $e = c$ we obtain

$$L_{xx}^{-1}(u''(x)) = u(x) - u(a) - z(x)[u(b) - u(a)] - (x - a)u'(c) + z(x)[(b - a)u'(c)]. \tag{13}$$

In order to express $L_{xx}^{-1}(u''(x))$ in terms of the two boundary conditions only, we have to eliminate the coefficient multiplying $u'(c)$ by setting

$$-(x - a)u'(c) + z(x)(b - a)u'(c) = 0. \tag{14}$$

Solving this equation for $z(x)$ assuming that $u'(c) \neq 0$, we get

$$z(x) = \frac{x-a}{b-a}. \tag{15}$$

Substituting (15) into (11) and (13) respectively, we obtain

$$L_{xx}^{-1}(u''(x)) = u(x) - u(a) - \frac{x-a}{b-a}[u(b) - u(a)], \tag{16}$$

and Eq. (10). From Eq. (16), we note that $L_{xx}^{-1}(u''(x))$ is already expressed in terms of the given boundary conditions without any restrictions on c. So, the choice of the value that c can be take depends properly on the singular point of the equation under consideration. For example, if the equation has a singular point at $x = x_0$, say, we will choose c to be any real value except the value of x_0. Moreover, if the equation has two singular points at $x = x_1$ and $x = x_2$, then we choose c to be any real value except these values of x_1 and x_2. In general, if the equation has n-singular points $x_1, x_2, \ldots, x_n$, then c takes any real value except the values of these singular points. In the next section, we use Eq. (10) with the standard ADM to establish the improved ADM (IADM) for solving linear and non-linear singular two-point boundary value problems.

4. Analysis of the improved ADM (IADM) for solving singular two-point BVPs

In this section, the Adomian decomposition method with the modification of Lesnic's work developed in the previous section are used to construct algorithms for solving Eq. (1) under the Dirichlet boundary conditions (2). First, we rewrite Eq. (1) in the form:

$$u''(x) = r(x) - p(x)u'(x) - q(x)f(u(x)). \tag{17}$$

Now, applying the operator $L_{xx}^{-1}(.)$ presented in the previous section and given by Eq. (10) on both sides of Eq. (17), we obtain

$$u(x) = u(a) + \frac{x-a}{b-a}[u(b) - u(a)] + L_{xx}^{-1}[r(x)] - L_{xx}^{-1}[p(x)u'(x)] - L_{xx}^{-1}[q(x)f(u(x))]. \tag{18}$$

The Adomian decomposition method (ADM) is based on decomposing u and the non-linear term $f(u)$ as

$$u = \sum_{n=0}^{\infty} u_n \quad \text{and} \quad f(u) = \sum_{n=0}^{\infty} A_n, \tag{19}$$

where A_n are Adomian's polynomials for the non-linear term $f(u(x))$ and can be found from the formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} f\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{20}$$

Substituting (19) into (18) and according to the ADM, the solution $u(x)$ can be elegantly computed by using the recurrence relation

$$\begin{aligned} u_0(x) &= u(a) + \frac{x-a}{b-a}[u(b) - u(a)] + L_{xx}^{-1}[r(x)], \\ u_{n+1}(x) &= -L_{xx}^{-1}[p(x)u_n'(x) + q(x)A_n(x)], \quad n \geq 0. \end{aligned} \tag{21}$$

If $f(u) = u$, i.e., linear function, then the solution $u(x)$ can be computed by using the recurrence relation

$$\begin{aligned} u_0(x) &= u(a) + \frac{x-a}{b-a}[u(b) - u(a)] + L_{xx}^{-1}[r(x)], \\ u_{n+1}(x) &= -L_{xx}^{-1}[p(x)u_n'(x) + q(x)u_n(x)], \quad n \geq 0, \end{aligned} \tag{22}$$

where algorithms (21) and (10), (22) and (10) improve the standard ADM, (IADM) and can be used to solve linear and non-linear singular two-point BVPs as shown in the next section.

5. Numerical examples

Example 5.1. Consider the inhomogeneous Bessel equation [15]:

$$u''(x) + \frac{1}{x}u'(x) + u(x) = 4 - 9x + x^2 - x^3, \quad x \in (0, 1) \tag{23}$$

with the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 0. \tag{24}$$

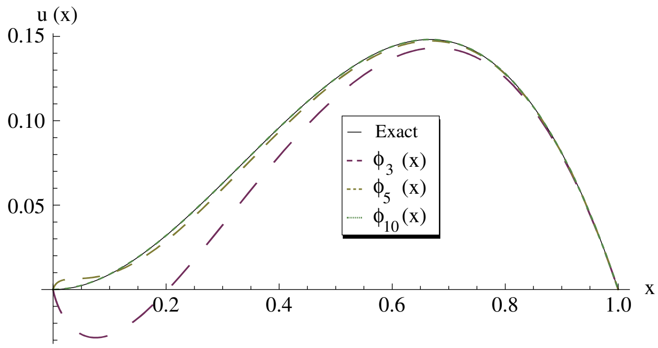

Fig. 1. The truncated ADM series solutions and the exact solution of Example 5.1.

Table 1
Numerical results for Example 5.1.

x	$\lvert u(x) - u_{26}\rvert$ [15]	$\lvert u(x) - \Phi_{10}(x)\rvert$
0.0	0	0
0.1	2.3E−05	2.2418E−05
0.2	1.1E−05	2.5259E−05
0.3	5.5E−05	3.0855E−05
0.4	2.3E−04	2.4171E−05
0.5	1.1E−04	1.6781E−05
0.6	1.2E−04	1.1127E−05
0.7	1.6E−04	7.0442E−06
0.8	1.5E−04	4.0377E−06
0.9	4.1E−05	1.7590E−06
1.0	0	0

In this problem, Eq. (23) has a singular point at $x = 0$. So, we choose c to be any real value except zero. Using the IADM for this linear boundary value problem, we obtain

$$\begin{aligned} u_0(x) &= -\frac{8}{15}x + 2x^2 - \frac{3}{2}x^3 + \frac{1}{12}x^4 - \frac{1}{20}x^5, \\ u_{n+1}(x) &= -L_{xx}^{-1}\left[\frac{1}{x}u_n'(x) + u_n(x)\right], \quad n \geq 0, \\ L_{xx}^{-1}(.) &= \int_0^x \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'' - x\int_0^1 \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'', \quad c \neq 0. \end{aligned} \tag{25}$$

Using this recurrence relation, we obtain the approximate solution as

$$\Phi_n(x) = \sum_{i=0}^{n-1} u_i(x). \tag{26}$$

In order to verify numerically whether the proposed approach (IADM) leads to accurate solutions, we use MATHEMATICA to evaluate the decomposition series solutions using the n-terms approximation (26). Then we compare the approximate solutions $\Phi_3(x)$, $\Phi_5(x)$ and $\Phi_{10}(x)$ with the exact solution $u = x^2 - x^3$ in Fig. 1. The numerical results show that a good approximation is achieved using small values of n-terms of the decomposition series solution. It is also important to note that the approach proposed by Lesnic in [40] fails to overcome the singularity at $x = 0$, for this singular problem. Moreover, a comparison of the numerical results for the absolute errors $|u(x) - \Phi_{10}(x)|$ with those of Ref. [15] are shown in Table 1. This shows that our approach not only more accurate but also used in more easier way.

Example 5.2. In this example we discuss the advantage of the IADM over other analytical methods [39] for solving the following linear singular BVP of Cauchy–Euler type:

$$\begin{aligned} &u''(x) + \frac{2}{x}u'(x) - \frac{2}{x^2}u(x) = \frac{\sin[\ln(x)]}{x^2}, \quad 1 \leq x \leq 2, \\ &y(1) = 1, \qquad y(2) = 2. \end{aligned} \tag{27}$$

Table 2
Numerical results for Example 5.3.

x	$\lvert u(x) - \Phi_7(x)\rvert$
0.0	0
0.1	3.0047E−05
0.2	1.7524E−04
0.3	9.3554E−06
0.4	4.3842E−05
0.5	9.3002E−05
0.6	1.9170E−04
0.7	3.1501E−04
0.8	3.9364E−04
0.9	3.1680E−04
1.0	0

First, we apply the IADM for this problem to get

$$\begin{aligned} u_0 &= x + \frac{1}{2}[x - 2 + \cos(\ln x) - \sin(\ln x) + (1-x)(\cos(\ln 2) -)\sin(\ln 2)], \\ u_{n+1} &= -L_{xx}^{-1}\left[\frac{2}{x}y_n'(x) - \frac{2}{x^2}y_n(x)\right], \quad n \geq 0, \\ L_{xx}^{-1}(.) &= \int_1^x \mathrm{d}x \int_c^x (.)\mathrm{d}x - (x-1)\int_1^2 \mathrm{d}x \int_c^x (.)\mathrm{d}x, \quad c \neq 0. \end{aligned} \tag{28}$$

Using only two terms of the decomposition series, a good agreement with the exact solution:

$$u = \frac{1}{70x^2}[8 + 69x^3 + 4(x^3 - 1)\cos(\ln 2) + 12(x^3 - 1)\sin(\ln 2) - 7x^2[\cos(\ln 2) + 3\sin(\ln 2)]], \tag{29}$$

is shown in Fig. 2. In this example we showed the effectiveness of the IADM for achieving good numerical results using only two terms. On other hand, it may be difficult to handle Eq. (27) by the differential transformation method [39]. We may indicate this point as follows. As in [39], we first write the equation as $x^2u''(x) + 2xu'(x) - 2u(x) = \sin(\ln x)$. Here, we note that the differential transformation for the left-hand side can be easily obtained, however, it is not so for the right-hand side. This because the differential transformation of $\sin(\ln x)$ is not known; it is only available for the elementary functions, i.e., x^m, e^x, $\sin(\alpha x + \beta)$, ..., which may be one of the disadvantages of the differential transformation method.

Example 5.3. Consider the linear singular equation [6,12]:

$$u''(x) + \frac{\alpha}{x}u'(x) = \beta x^{\beta-2}[\alpha + \beta - 1 + \beta x^\beta]u(x), \quad x \in (0, 1) \tag{30}$$

with the boundary conditions

$$u(0) = 1 \quad \text{and} \quad u(1) = e. \tag{31}$$

Here, the equation has a singular point at $x = 0$. So, as indicated in the previous examples, we choose c to be any real value except zero. Consequently, the solution for this singular boundary value problem can be obtained as

$$\begin{aligned} u_0(x) &= 1 + (e-1)x, \\ u_{n+1}(x) &= L_{xx}^{-1}\left[-\frac{\alpha}{x}u_n'(x) + \beta x^{\beta-2}(\alpha + \beta - 1 + \beta x^\beta)u_n(x)\right], \quad n \geq 0, \\ L_{xx}^{-1}(.) &= \int_0^x \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'' - x\int_0^1 \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'', \quad c \neq 0. \end{aligned} \tag{32}$$

To verify how close the approximate solution is to the exact one: $u(x) = e^{x^\beta}$, we plot different approximate solutions and the exact solution in Figs. 3a–3d for various values of α and β. It is shown from these figures that the approximate solution obtained through the IADM is very close to the exact one using a few terms. Furthermore, the numerical results for the absolute errors $|u(x) - \Phi_7(x)|$ are shown in Table 2, at $\alpha = 1$ and $\beta = 3$. In [12] the author used a three-point finite difference method in which a huge amount of computational work is needed to obtain the numerical solution. However, the absolute error obtained in [12] was $\|E\| = 2.8 \times 10^{-4}$ using $N = 64$.

Example 5.4. Consider the linear singular equation [41]:

$$x^2u''(x) - xu'(x) + u(x) = 0, \quad 1 \leq x \leq 2 \tag{33}$$

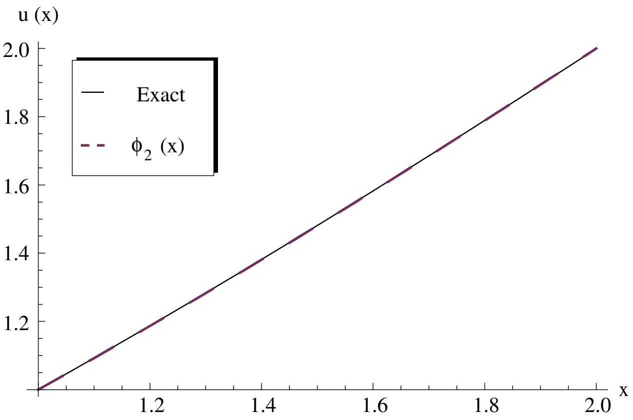

Fig. 2. The truncated ADM series solutions and the exact solution of Example 5.2.

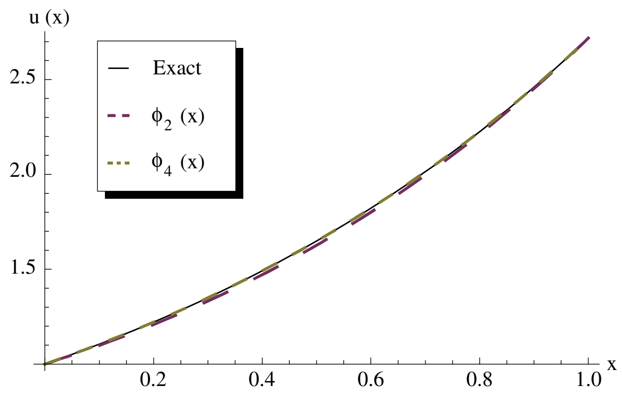

Fig. 3a. The truncated ADM series solutions and the exact solution of Example 5.3 for $\alpha = 0$, $\beta = 1$.

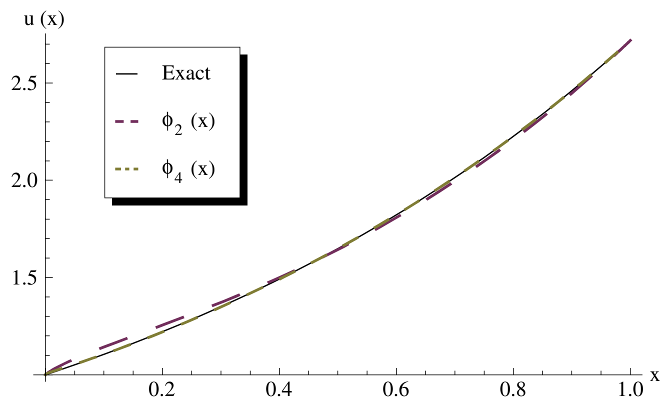

Fig. 3b. The truncated ADM series solutions and the exact solution of Example 5.3 for $\alpha = 0.5$, $\beta = 1$.

with the conditions

$$u(1) = 1 \quad \text{and} \quad u(2) = 1. \tag{34}$$

In this example, we rewrite Eq. (33) in the form:

$$u''(x) - \frac{1}{x}u'(x) + \frac{1}{x^2}u(x) = 0. \tag{35}$$

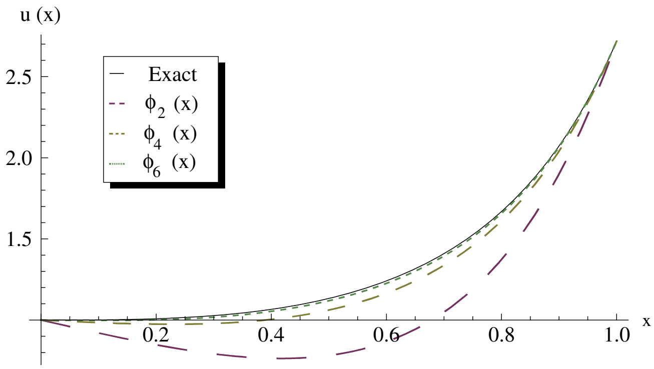

Fig. 3c. The truncated ADM series solutions and the exact solution of Example 5.3 for $\alpha = 0$, $\beta = 3$.

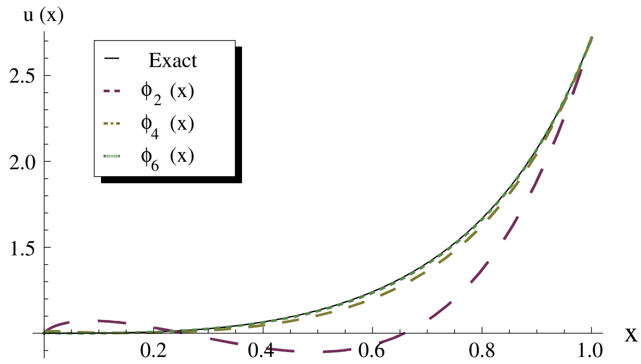

Fig. 3d. The truncated ADM series solutions and the exact solution of Example 5.3 for $\alpha = 0.5$, $\beta = 3$.

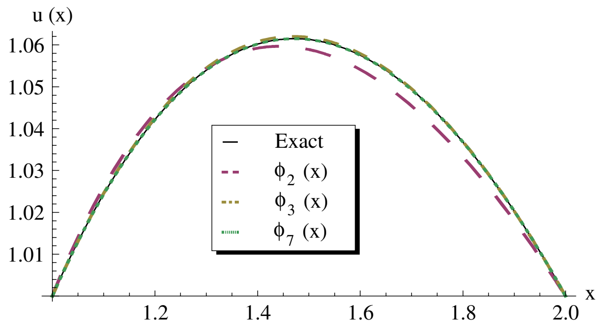

Fig. 4. The truncated ADM series solutions and the exact solution of Example 5.4.

Using the IADM we obtain

$$
\begin{aligned}
&u_0(x) = 1, \\
&u_{n+1}(x) = L_{xx}^{-1}\left[\frac{1}{x}u_n'(x) - \frac{1}{x^2}u_n(x)\right], \quad n \geq 0, \\
&L_{xx}^{-1}(.) = \int_1^x \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'' - (x-1)\int_1^2 \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'', \quad c \neq 0.
\end{aligned}
\tag{36}
$$

In Fig. 4, we plot the approximate solutions Φ_3, Φ_5 and Φ_7 and the exact solution $u(x) = \frac{2x\ln(2) - x\ln(x)}{2\ln(2)}$. It can be concluded from this figure that an accurate numerical solution is obtained through the proposed IADM. Also, in order to compare our approach with another modification of the ADM [41] we present the numerical results for the absolute errors $|u(x) - \Phi_7(x)|$ in Table 3. In [41] the authors used the 7-terms of the ADM in indirect way in which a huge amount of computational work is needed to obtain the numerical solution with absolute errors $\leq 6 \times 10^{-4}$, see example 2 in [41]. By this, not only is our approach more accurate but also more simple and direct.

Table 3
Numerical results for Example 5.4.

x	$\lvert u(x) - \Phi_7(x)\rvert$
1.0	0
1.1	2.6258E−08
1.2	5.3793E−08
1.3	6.7857E−08
1.4	6.3608E−08
1.5	4.4892E−08
1.6	2.0090E−08
1.7	1.8739E−09
1.8	1.4216E−08
1.9	1.3555E−08
2.0	0

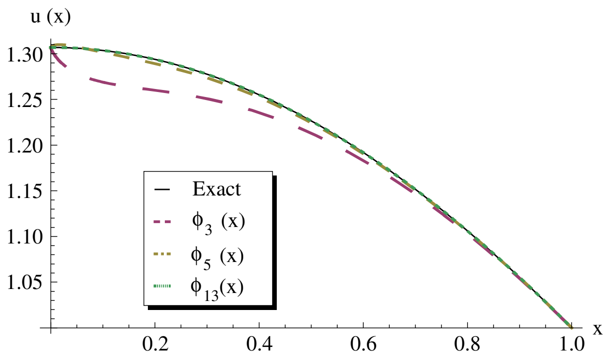

Fig. 5. The truncated ADM series solutions and the exact solution of Example 5.5.

Example 5.5. Consider the Bessel equation of order zero [14,16]:

$$u''(x) + \frac{1}{x}u'(x) + u(x) = 0, \quad x \in (0, 1) \tag{37}$$

with the boundary conditions

$$u(0) = \frac{1}{J_0(1)} \quad \text{and} \quad u(1) = 1. \tag{38}$$

Proceeding as above, we obtain

$$\begin{aligned} &u_0(x) = 1, \\ &u_{n+1}(x) = -L_{xx}^{-1}\left[\frac{1}{x}u_n'(x) + u_n(x)\right], \quad n \geq 0, \\ &L_{xx}^{-1}(.) = \int_0^x dx' \int_c^{x'} (.)dx'' - x\int_0^1 dx' \int_c^{x'} (.)dx'', \quad c \neq 0. \end{aligned} \tag{39}$$

It can be concluded from Fig. 5 that the approximate solutions converge rapidly to the exact solution $u(x) = \frac{J_0(x)}{J_0(1)}$. The numerical results for the absolute errors $|u(x) - \Phi_{13}(x)|$ and those of Ref. [16] by using the cubic spline are compared in Table 4. It is also clear that the overall errors can be made smaller by adding new terms in the decomposition series (19).

Example 5.6. Consider the Thomas–Fermi equation [3]:

$$u''(x) = x^{-1/2}u^{3/2}(x), \quad x \in (0, 1) \tag{40}$$

subject to the boundary conditions

$$u(0) = 1 \quad \text{and} \quad u(1) = 0. \tag{41}$$

Table 4
Numerical results for Example 5.5.

x	Ref. [16], $h = \frac{1}{40}$	$\lvert u(x) - \Phi_{13}(x)\rvert$
0.0	1.16E−04	0
0.1	1.16E−04	1.6729E−06
0.2	1.15E−04	1.8402E−06
0.3	1.13E−04	2.2581E−06
0.4	1.10E−04	1.7706E−06
0.5	1.07E−04	1.2296E−06
0.6	1.02E−04	8.1541E−07
0.7	9.60E−05	5.1621E−07
0.8	9.00E−05	2.9587E−07
0.9	7.20E−05	1.2888E−07
1.0	7.10E−05	0

On applying the IADM with the modified decomposition method, the solution of this non-linear boundary value problem is obtained by using the recurrence relation:

$$\begin{aligned}
&u_0(x) = 1,\\
&u_1(x) = -x + L_{xx}^{-1}[x^{-1/2}A_0(x)],\\
&u_{n+1}(x) = L_{xx}^{-1}(x^{-1/2}A_n(x)), \quad n \geq 1,\\
&L_{xx}^{-1}(.) = \int_0^x \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'' - x\int_0^1 \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'', \quad c \neq 0.
\end{aligned} \tag{42}$$

The first few terms of Adomian's polynomials can be evaluated from formula (20) as

$$A_0 = u_0^{3/2}, \qquad A_1 = \frac{3}{2}u_0^{1/2}u_1, \qquad A_2 = \frac{3}{8}u_0^{-1/2}(u_1^2 + 4u_0u_2), \qquad A_3 = \frac{1}{16}u_0^{-3/2}(-u_1^3 + 12u_0u_1u_2 + 24u_0^2u_3). \tag{43}$$

The approximate solutions $\Phi_2(x)$, $\Phi_4(x)$, $\Phi_6(x)$, $\Phi_8(x)$ and $\Phi_9(x)$ are plotted in Fig. 6. It is clear from this figure that the numerical solutions converge rapidly to a certain function as the number of terms of the decomposition series solution increases. Also, it should be noted that the differential transformation method [38] and the homotopy analysis method [39] cannot be applied directly to solve this BVP due to the existence of the nonlinear term $u^{3/2}$. So, it may be concluded that the current approach has many advantages over the others, mainly because it can be applied directly to singular two-point BVPs with complex nonlinearities.

Example 5.7. Consider the non-linear singular BVP [19]:

$$u''(x) + \frac{0.5}{x}u'(x) = e^{u(x)}(0.5 - e^{u(x)}), \quad x \in (0, 1) \tag{44}$$

subject to the boundary conditions

$$u(0) = \ln(2) \quad \text{and} \quad u(1) = 0. \tag{45}$$

Also, in this example we apply the modified decomposition method. On applying the IADM for this non-linear problem, we get

$$\begin{aligned}
&u_0(x) = 0,\\
&u_1(x) = (1 - x)\ln(2) + L_{xx}^{-1}\left[A_0(x) - \frac{0.5}{x}u_0'(x)\right],\\
&u_{n+1}(x) = L_{xx}^{-1}\left[A_n(x) - \frac{0.5}{x}u_n'(x)\right], \quad n \geq 1,\\
&L_{xx}^{-1}(.) = \int_0^x \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'' - x\int_0^1 \mathrm{d}x' \int_c^{x'} (.)\mathrm{d}x'', \quad c \neq 0.
\end{aligned} \tag{46}$$

The first few terms of Adomian's polynomials can be evaluated from the formula (20) as

$$\begin{aligned}
&A_0 = e^{u_0}\left(\frac{1}{2} - e^{u_0}\right), \qquad A_1 = \frac{1}{2}(1 - 4e^{u_0})u_1, \qquad A_2 = \frac{1}{4}e^{u_0}[u_1^2 + 2u_2 - 8e^{u_0}(u_1^2 + u_2)],\\
&A_3 = \frac{1}{12}e^{u_0}[u_1^3 + 6u_1u_2 + 6u_3 - 8e^{u_0}(2u_1^3 + 6u_1u_2 + 3u_3)].
\end{aligned} \tag{47}$$

Table 5
Numerical results for Example 5.7.

x	$\lvert y(x) - \text{Padé}[5/5](x)\rvert$	$\lvert y(x) - \Phi_{10}(x)\rvert$
10^{-1}	4.432E−04	1.734E−17
10^{-2}	1.368E−04	0.000E−00
10^{-3}	1.300E−05	1.602E−17
10^{-4}	1.219E−06	1.077E−17
10^{-5}	2.659E−06	8.279E−18
10^{-6}	2.803E−06	2.212E−17

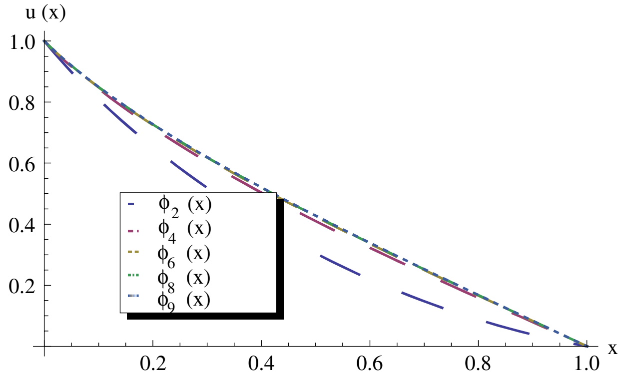

Fig. 6. The truncated ADM series solutions and the exact solution of Example 5.6.

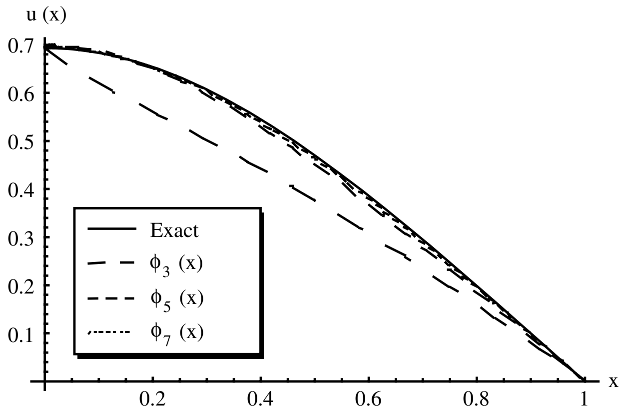

Fig. 7. The truncated ADM series solutions and the exact solution of Example 5.7.

The approximate solutions $\Phi_3(x)$, $\Phi_5(x)$ and $\Phi_7(x)$ are plotted in Fig. 7. It is shown from this figure that the approximate numerical solution $\Phi_7(x)$ is very close to the exact one: $u(x) = \ln(\frac{2}{x^2+1})$. In order to show the efficiency of the IADM in obtaining more accurate numerical solutions than the ADM-Padé technique used in [37], we evaluated the numerical results for the absolute errors $|y(x) - \text{Padé}[5/5](x)|$ and $|u(x) - \Phi_{10}(x)|$ in Table 5. Obviously, a good approximation is achieved using 10-terms of our algorithm without any need for the Padé-approximant.

6. Conclusions

In this paper, an efficient approach (IADM) is proposed for solving linear and non-linear singular two-point BVPs. Moreover, the IADM not only used in a direct way but also requires less computational work in comparison with the other methods. The IADM is verified by several linear and non-linear singular second-order boundary value problems.

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