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Convergence of Nonhomogeneous Stochastic Chains with Countable States

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The work started by V. M. Maksimov [1970, *Theory Probab. Appl.* **15**, 604–618], and continued by A. Mukherjea [1980, *Trans. Amer. Math. Soc.* **263**, 505–520], is extended, and completed with respect to certain aspects. Infinite-dimensional stochastic chains are considered in the framework of Mukherjea [*loc. cit.*]; backward products of stochastic matrices and their convergence are also considered. The main theme centers around understanding how the convergence of products (backward and forward, finite and infinite dimensional) takes place and what it means in terms of various types of asymptotic behavior of the individual stochastic matrices in the chain. The study is based on establishing the existence of a basis for convergent chains. The basis then makes it possible to describe properly various aspects of convergence. All results are new; they are also complete at least in the sense they have been presented and suitable examples (or counter-examples) are presented to justify the assumptions involved.

INTRODUCTION

Let (P_n) be a sequence of finite or infinite-dimensional stochastic matrices with the same number of states such that for each positive integer k, the sequence $P_{k,n} = P_{k+1}P_{k+2} \cdots P_n$ $(k < n)$ converges to a stochastic matric Q_k. In this case, we call (P_n) a convergent forward stochastic chain (or briefly, an f.s.c.). If instead of $P_{k,n}$ we consider convergence of $P_{n,k} = P_n P_{n-1} \cdots P_{k+1}$ $(n > k)$, then we call (P_n) a convergent backward stochastic chain (b.s.c.).

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Two stochastic chains (P_n) and (P'_n) are called equivalent if the series $\sum_{n=1}^{\infty} |(P_n)_{ij} - (P'_n)_{ij}|$ is convergent for every i, j in the state space. In the finite-dimensional case, it is well-known that in the "forward" as well as in the "backward" case, two equivalent chains are either both convergent or both nonconvergent; when convergent, they have the same basis. For the definition of the basis in the finite and f.s.c. case, we refer the reader to [6]. Extension of the concept of the basis to the infinite-dimensional or the b.s.c. case is not immediate. We have considered this extension in Sections 1 and 2. Regarding equivalence of chains, we have shown that there are infinite chains, though equivalent, where one is convergent and the other one is not (Remark 3.8(c)). However, results that are expected of equivalent finite chains can be obtained for infinite chains under a uniformity condition (see Theorem 3.9). Also, results that are typical of equivalent chains can sometimes be obtained under conditions other than equivalence. This we show in Theorem 4.5.

A substantial portion of the paper is devoted to establishing results of the type $\sum_{n=1}^{\infty} (P_n)_{ij} < \infty$ for i, j in different classes of states in the basis and generalization of such results. These are useful in understanding the general problem of classifying the states of a nonhomogeneous Markov chain; they also help us to assess quickly the asymptotic behavior of the matrix products by replacing the chain by a simpler equivalent chain. The main results here are Theorems 3.3, 3.4, 3.6, 3.9, 4.4, 4.5, 4.6, and 4.7.

Throughout the paper, we have indicated differences and similarities in the "forward" and "backward" cases. The results have obvious implications in the study of general nonhomogeneous Markov chains. They also have applications in other apparently nonrelated areas such as the study of measures on semigroups; this will be taken up elsewhere. All our results are new. However, some of our results in the "forward" and finite-dimensional case can also be obtained using martingale convergence theorems and tail sigma-field considerations as followed in papers of Cohn. Our methods, however, are simpler and our results are best possible at least in the sense they have been presented. *More importantly, our methods can be adapted, as we have shown in this paper, to the "backward" as well as the infinite-dimensional situation*, and *even to the case of infinite nonnegative matrices as is shown in Lemma* 3.1. Detailed considerations of infinite nonnegative matrices and their products will be treated elsewhere.*

* *Note added in proof.* Finally, it should be pointed out that some of the results here including Theorems 3.3 and 3.6 were presented by the authors in the Proceedings of a 1981 conference in Oberwolfach [10]; later, Theorem 3.3 and other related results were tackled in [1] under tail-sigma field considerations.

1. Basis of a Convergent Forward Stochastic Chain

Establishing the existence of a basis for infinite-dimensional convergent f.s.c. is not immediate. The reason is, of course, that the pointwise limit of a sequence of stochastic matrices need not be a stochastic matrix; also, in this case, the joint continuity of matrix multiplication is no longer true. To establish our basis, let us first present a few simple (but necessary) lemmas.

Lemma 1.1. *Suppose that (A_n) and (B_n) are two sequences of stochastic matrices converging pointwise to, respectively, the matrices A and B. Suppose that A is stochastic. Then B is substochastic, and the sequence (A_nB_n) converges pointwise to the matrix AB. Unless B is stochastic, the sequence (B_nA_n) need not converge to the limit BA.*

Proof. The first part of the lemma is straightforward: We omit its proof. Let us illustrate the last assertion by the following example: Let $A_n = A\ \forall\, n$, where each row of A is $(1/2, 1/2^2, \ldots)$; let B_n be defined by

$$\begin{aligned}(B_n)_{ij} &= \frac{1}{n} \qquad \text{for } 1 \leqslant i,j \leqslant n;\\ &= 1 \qquad \text{for } i=j>n.\end{aligned} \tag{1}$$

Then $\lim_{n\to\infty} B_n = 0$, but $B_nA_n = A\ \forall\, n$. ∎

Lemma 1.2. *Suppose that for each positive integer k*

$$\lim_n P_{k,n} = Q_k \qquad \text{(pointwise)}$$

where each P_n as well as each Q_k is a stochastic matrix. Let Q be a limit point of the Q_k's with respect to pointwise convergence. Then for each k, we have

$$Q_kQ' = Q_k. \tag{2}$$

Proof. For $k < n < m$, $P_{k,m} = P_{k,n}P_{n,m}$. By Lemma 1.1,

$$Q_k = P_{k,n}Q_n.$$

Using Lemma 1.1 again, (2) follows. ∎

Lemma 1.3. *In Lemma 1.2 let k and t be such that $(Q_k)_{it} > 0$ for some i. Then we have*

$$\sum_{j=1}^{\infty} (Q')_{tj} = 1.$$

Proof. The lemma follows immediately from the following equality that follows from (2):

$$1=\sum_{j=1}^{\infty}(Q_k)_{ij}=\sum_{j=1}^{\infty}\sum_{s=1}^{\infty}(Q_k)_{is}\,Q'_{sj}$$
$$=\sum_{s=1}^{\infty}(Q_k)_{is}\cdot\left[\sum_{j=1}^{\infty}Q'_{sj}\right]. \quad \blacksquare$$

LEMMA 1.4. *Let Q' be as in Lemma* 1.2. *Then the entries in the* jth *row of Q' add up to* 1, *provided that the* jth *column of Q' is not a zero column.*

Proof. Immediate from Lemma 1.3. $\blacksquare$

LEMMA 1.5. *Let Q' be as in Lemma* 1.2. *Let*

$$T=\{j\colon Q'_{ij}=0 \text{ for each } i\};$$

that is, T is the set of zero columns of Q'. Then the matrix $(Q')^+$, defined as the restriction of Q' to the complement of T, is a stochastic idempotent matrix with no zero columns. (By Lemma 1.3, *the complement of T is nonempty.)*

Proof. Note that because of Eq. (2), the ith column of each Q_k, for i in T, is a zero column. Also, note that for $j\notin T$ and $k\notin T$, we have

$$(Q_p)_{jk}=(Q_pQ')_{jk}=\sum_{t=1}^{\infty}(Q_p)_{jt}Q'_{tk}=\sum_{t\notin T}(Q_p)_{jt}\,Q'_{tk}$$

so that for each positive integer p,

$$(Q_p)^+\cdot(Q')^+=(Q_p)^+.$$

Now notice that for any given $t\notin T$, there is a Q_p (for some p) such that the tth column of Q_p is not a zero column. It follows from (3), as in Lemma 1.3, that for each $t\notin T$,

$$\sum_{s\notin T}(Q')_{ts}=1.$$

Then it follows from Eq. (3) and Lemma 1.1 that $(Q')^+$ is a stochastic idempotent matrix. Finally, we show that this matrix has no zero columns. To this end, let T' be the set of zero columns of $(Q')^+$. Of course, $T'\subset T^c$ (the complement of T). For $j\in T'$ and for every i and k,

$$(Q_k)_{ij}=(Q_kQ')_{ij}=\sum_{t=1}^{\infty}(Q_k)_{it}Q'_{tj}=\sum_{t\notin T}(Q_k)_{it}\,Q'_{tj}=0,$$

since $j\in T'$. This means that $Q'_{ij}=0$ for each i so that $j\in T$. This is a contradiction. $\blacksquare$

LEMMA 1.6. *Let the Q_k's be as in Lemma 1.2. Let Q' and Q'' be any two (pointwise) limit points of the Q_k's. Then, the sets*

$$\{j\colon Q'_{ij} = 0 \text{ for each } i\} \qquad \text{and} \qquad \{j\colon Q''_{ij} = 0 \text{ for each } i\}$$

are the same. If we denote these sets by T, then Q' and Q'', when restricted to the complement of T, coincide. Each of these restrictions is a stochastic idempotent matrix with no zero columns.

Proof. Notice that by Eq. (2),

$$Q'_{ij} = 0 \text{ for each } i \qquad \text{iff } (Q_k)_{ij} = 0 \text{ for each } i \text{ and } k$$

for any limit point Q' of the Q_k's. This means that the "T" set is the same for both Q' and Q''. It follows from Eq. (3), Lemma 1.1, and Lemma 1.5 that when restricted to the complement of T,

$$Q''Q' = Q'' \qquad \text{and} \qquad Q'Q'' = Q'.$$

By using the structure theorem for stochastic idempotent matrices, it can be easily verified that $Q' = Q''$, when restricted to T^c. ∎

Because of Lemma 1.6, we can now define the basis of a convergent nonhomogeneous stochastic chain.

DEFINITION 1.7. Suppose that (P_n) is a convergent stochastic chain and that each Q_k, where $Q_k = \lim_n P_{k,n}$ (pointwise limit), is stochastic. Let Q' be a (pointwise) limit point of the Q_k's and let

$$T = \{j\colon Q'_{ij} = 0 \text{ for each } i\}.$$

Then Q', restricted to T^c, is a stochastic idempotent matrix with no zero columns. By the structure theorem for stochastic idempotent matrices (see [5]), there exists a partition $\{C_1, C_2, \ldots\}$ of T^c such that

$$\begin{aligned} Q'_{ij} &= 0 && \text{for } i,j \text{ in different } C\text{-classes;} \\ &= Q'_{kj}\ (>0) && \text{for } i,j, \text{ and } k \text{ in the same } C\text{-class.} \end{aligned}$$

The partition $\{T, C_1, C_2, \ldots\}$ remains the same for all limit points Q' of the Q_k's. This will be called the basis of the convergent chain (P_n) as well as the basis of Q'.

Remark 1.8. The pointwise convergence considered above is certainly weaker than the usual norm convergence, where

$$\|P - Q\| = \sup_i \sum_{j=1}^{\infty} |P_{ij} - Q_{ij}|,$$

even when the pointwise limit is a stochastic matrix. For example, consider (P_n) defined by

$$\begin{aligned}(P_n)_{ij} &= 1-(1/n^{1/i}) && \text{for } j=i\leqslant n;\\ &= 1/n^{1/i} && \text{for } j=i+1\leqslant n+1;\\ &= 1 && \text{for } j=i>n;\\ &= 0 && \text{otherwise.}\end{aligned}$$

Then P_n converges to the identity matrix I pointwise, but $P_n \not\to I$ in norm as n tends to infinity.

Remark 1.9. There are a number of interesting differences in the convergence behavior of forward and backward chains. Here we point out only one. Consider the chain (P_n) defined by

$$\begin{aligned}(P_n)_{ij} &= 0 && \text{for } j=1,2,\ldots,n \text{ and all } i\geqslant 1;\\ &= 1/2^k && \text{for } j=n+k \text{ and all } i\geqslant 1.\end{aligned}$$

For $n>k$, define $P_{n,k}=P_nP_{n-1}\cdots P_{k+1}$. Then, $P_{n,k}=P_k$ for all $n>k$. In this case, unlike the forward chain case,

$$\lim_k \lim_n P_{n,k} = 0.$$

2. Basis of a Convergent Backward Stochastic Chain

Here we consider the basis of a convergent b.s.c.. This case, as has been already pointed out in Remark 1.9, is somewhat different. The following example will clarify this even more.

Example. Consider infinite-dimensional stochastic matrices A and (B_n) such that the first row of A is $(1\ 0\ 0\ \cdots)$, all other rows of A are $(0\ 1\ 0\ \cdots)$, and every entry in the first n columns of B_n is $1/n$. Then $AB_n=B_n$ and $B_mB_n=B_n$ (for any m, n). Also, $B_nA=C_n$, where each row in C_n is $(1/n\ \ 1-1/n\ 0\ \cdots)$. Define the sequence (P_n) such that for each positive integer k,

$$P_{2k}=A, \qquad P_{2k+1}=B_k.$$

Then $P_nP_{n-1}\cdots P_{2k}=C_k$ and $P_nP_{n-1}\cdots P_{2k+1}=B_k$ (for each $n\geqslant 2k+1$). In this case then $R_{2k-1}=C_k$ and $R_{2k}=B_k$ so that one of the limit points of the R_k's is stochastic whereas the other limit point is the zero matrix. Thus, the b.s.c. case is quite different from the f.s.c. case. [Here $R_k=\lim_{n\to\infty} P_{n,k}$.]

Now we present a number of basic results in the b.s.c. case.

LEMMA 2.1. *Suppose that for each positive integer* k, $R_k = \lim_{n\to\infty} P_{n,k}$ *and each* R_k *is stochastic. If* R' *is a (pointwise) limit point of the* R_k's *and* R' *is stochastic, then for each positive integer* k

$$R'R_k = R_k. \tag{4}$$

If R'' is another limit point of the R_k's, then $R'R'' = R''$. (R'', of course, need not be stochastic.)

Proof. The proof is immediate from Lemma 1.1 and the observation that for $k < m$, $R_k = R_m P_{m,k}$. ∎

PROPOSITION 2.2. *Let* A *and* B *be two infinite-dimensional stochastic idempotent matrices such that the bases of* A *and* B *are respectively*

$$A: \{T, C_1, C_2, \ldots\} \qquad \text{and} \qquad B: \{T', C_1', C_2', \ldots\}.$$

Then, $AB = B$ *and* $BA = A$ iff *the following hold*:

(i) *For each* i $(1 \leqslant i < \infty)$, *there exist* j *and* k $(1 \leqslant j, k < \infty)$ *such that* $C_i \subset C_j' \cup T'$ *and* $C_i' \subset C_k \cup T$;

(ii) $A_{ij} = 0 = B_{ij}$ *whenever* i *and* j *belong to two different* C-*components in the basis of* A *or in the basis of* B;

(iii) *if* $i \in C_k'$ $(1 \leqslant k < \infty)$ *and* $t \in T'$, *then* $A_{it} > 0 \Rightarrow$ *for each* j, $A_{ij} = A_{tj}$, *and* $B_{ij} = B_{tj}$. A similar result also holds for B.

Proof. First we assume that $AB = B$ and $BA = A$. This means that A has two rows identical iff the corresponding two rows of B are also identical. Hence, if $\{a, b\} \subset C_i$, then the ath and bth rows of A (and therefore, also of B) must be identical. This means that there is a j such that $\{a, b\} \subset C_j' \cup T'$. It is now clear that (i) holds. To establish (ii) and (iii), let $\{i, j\} \subset C_k'$. Then $AB = B$ implies

$$0 < B_{ij} = \sum_{c \in C_k'} A_{ic} B_{cj} + \sum_{t \in T'} A_{it} B_{tj}.$$

Since $B_{ij} = B_{ij} \cdot \sum_{t \in T'} A_{it} + B_{ij} \cdot \sum_{c \in C_k'} A_{ic} + B_{ij} \cdot \sum_{c \notin C_k' \cup T'} A_{ic}$ and since $B_{ij} = B_{cj}$ for $c \in C_k'$ and $B_{tj} \leqslant B_{ij}$ for $t \in T'$, it follows easily that (ii) and (iii) hold.

Now we prove the converse part. Let us then assume (i), (ii), and (iii). It is enough to show that $AB = B$. Let $t \in T'$. Then

$$(AB)_{it} = \sum A_{ij} B_{jt} = 0 = B_{it}.$$

Now let $i \in C'_{k_1}$ and $j \in C'_{k_2}$ ($k_1 \neq k_2$). Then

$$(AB)_{ij} = \sum_{t \in T'} A_{it} B_{tj} + \sum_{s \in C'_{k_2}} A_{is} B_{sj} = 0 \qquad \text{(by (ii) and (iii))}$$
$$= B_{ij}.$$

Finally, let $\{i,j\} \subset C'_{k_1}$. Then

$$(AB)_{ij} = \sum_{t \in T'} A_{it} B_{tj} + \sum_{s \in C'_{k_1}} A_{is} B_{sj}$$
$$= B_{ij} \cdot \sum_{t \in T'} A_{it} + B_{ij} \cdot \sum_{s \in C'_{k_1}} A_{is} = B_{ij} \cdot \sum_{s} A_{is} = B_{ij}. \quad \blacksquare$$

Now we consider a convergent b.s.c. where $\lim_{n \to \infty} P_{n,k} = R_k$ and all the limit points of the R_k's are stochastic. By Lemma 2.1, these limit points then satisfy the equation $R'R'' = R''$ and, therefore, each such limit point is also idempotent. If, furthermore, none of them has a zero column, then they all have the same basis by Proposition 2.2. In case some of these limit points have zero columns, then we define the basis of the chain in the following manner.

DEFINITION 2.3. A partition $\{S_1, S_2, \ldots\}$ of the positive integers is called an S-basis of a b.s.c. when i and j are in the same S-class iff the ith and jth rows of any limit point of the R_k's are identical. (Recall that whenever any two rows of such a limit point are identical, the corresponding two rows are also identical for all other limit points.)

It is clear that any C-component in the basis of a limit point is completely contained in some S-class and, also, at most one such component can be contained by an S-class. We now present a nonobvious property of an S-basis.

PROPOSITION 2.4. *Let $\{S_1, S_2, \ldots,\}$ be an S-basis of a connvergent* b.s.c. (P_n), where $R_k = \lim_{n \to \infty} P_{n,k}$ *and each limit point of the* R_k's *is stochastic. Let R' and R'' be two such limit points with bases respectively*

$$R' \colon \{T, C_1, C_2, \ldots\} \qquad \textit{and} \qquad R'' \colon \{T', C'_1, C'_2, \ldots,\}.$$

Then the following holds: $S_i \subset T \Rightarrow S_i \subset T'$.

Proof. Suppose that $S_i \subset T$ and $S_i \not\subset T'$. Then, $S_i \cap C'_j$ is nonempty for some j. But this means that $C'_j \subset S_i \subset T$. Let $\{a, b\} \subset C'_j$. (If C'_j has only one element, then $a = b$.) Since $R'' = R'R''$,

$$R''_{ab} = \sum_{k \in T' \cup C'_j} R'_{ak} R''_{kb} = \sum_{k \in T'} R'_{ak} R''_{kb} \qquad \text{(since } C'_j \subset T\text{)}$$

so that

$$1=\sum_{b\in C_j'} R''_{ab}=\sum_{k\in T'} R'_{ak}\cdot\left[\sum_{b\in C_j'} R''_{kb}\right].$$

This equation shows that there is a $k\in T'$ such that $R'_{ak}>0$ and also for this k, $\sum_{b\in C_j'} R''_{kb}=1$. Since $a\in C_j'$, it immediately follows from the structure of an idempotent stochastic matrix that the kth and ath rows of R'' are identical. Since $a\in C_j'\subset S_i$, k also must be in S_i. Since $S_i\subset T$, $k\in T$. This is a contradiction since $R'_{ak}>0$. The proposition now follows. ∎

We now make another useful definition.

DEFINITION 2.5. Let $R_k=\lim_{n\to\infty} P_{n,k}$. Assume that the limit points of the R_k's are all stochastic. A state i is called strongly recurrent for the convergent b.s.c. if $R'_{ii}>0$ for each limit point R' of the R_k's. If this holds for at least one limit point, then i is called weakly recurrent.

One of the main themes of this paper is to decide when results like $\sum_{n=1}^{\infty}(P_n)_{ij}<\infty$ hold for convergent stochastic chains. As will be shown in the next section, such a result holds for any f.s.c. whenever i and j belong to two different C-components in its basis (i.e. the basis of any limit point of the Q_k's, $Q_k=\lim_{n\to\infty} P_{k,n}$). The situation is again somewhat different in the b.s.c. case even for finite-dimensional chains. The following example illustrates this.

EXAMPLE 2.6. Let $(u_n, v_n, w_n, x_n, y_n, z_n)$ be a sequence of 6-tuples such that $u_n+v_n+w_n\to 1$ and $x_n+y_n+z_n\to 0$ as $n\to\infty$, all these entries are nonnegative, and $\sum_n x_n=\sum_n y_n=\sum_n z_n=\infty$. Now consider the following stochastic matrices:

$$A=\begin{bmatrix} 0 & a & 1-a & & \text{zeros} & \\ 0 & a & 1-a & & & \\ 0 & a & 1-a & & & \\ & & & 0 & d & 1-d \\ & \text{zeros} & & 0 & d & 1-d \\ & & & 0 & d & 1-d \end{bmatrix},\qquad B=\begin{bmatrix} a & 0 & 1-a & & \text{zeros} & \\ a & 0 & 1-a & & & \\ a & 0 & 1-a & & & \\ & & & d & 0 & 1-d \\ & \text{zeros} & & d & 0 & 1-d \\ & & & d & 0 & 1-d \end{bmatrix},$$

$$A_n=\begin{bmatrix} u_n & v_n & w_n & x_n & y_n & z_n \\ 0 & a & 1-a & 0 & 0 & 0 \\ 0 & a & 1-a & 0 & 0 & 0 \\ e_n & f_n & g_n & h_n & p_n & r_n \\ 0 & 0 & 0 & 0 & d & 1-d \\ 0 & 0 & 0 & 0 & d & 1-d \end{bmatrix},\qquad B_n=\begin{bmatrix} a & 0 & 1-a & 0 & 0 & 0 \\ u_n & v_n & w_n & x_n & y_n & z_n \\ a & 0 & 1-a & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 1-d \\ e_n & f_n & g_n & h_n & p_n & r_n \\ 0 & 0 & 0 & d & 0 & 1-d \end{bmatrix},$$

where the 6-tuples $(e_n, f_n, g_n, h_n, p_n, r_n)$ have the same properties as the sequence $(u_n, v_n, w_n, x_n, y_n, z_n)$ and $0 < a, d < 1$.

For all nonnegative integers n, let us define

$$P_{2n} = A, \qquad P_{2n+1} = B, \qquad P'_{2n} = A_n, \qquad \text{and} \qquad P'_{2n+1} = B_n.$$

Then it can be verified easily that

$$P_n P'_n = P_n, \qquad P_n P_k = P_k, \qquad \text{and} \lim_n P'_n P_k = P_k.$$

Now we consider the stochastic chain (L_n) defined by

$$L_{2n} = P_n \qquad \text{and} \qquad L_{2n-1} = P'_n.$$

After some easy computations, it follows that for every k,

$$\lim_n L_{n,k} \text{ exists and} = P_k.$$

Notice that each P_k is either A or B and, therefore, the limit points of the P_k's are only A and B with bases, respectively,

$$A\colon T = \{1, 4\}, C_1 = \{2, 3\}, C_2 = \{5, 6\};$$

$$B\colon T = \{2, 5\}, C_1 = \{1, 3\}, C_2 = \{4, 6\}.$$

The S-basis of this convergent backward chain (L_n) is

$$\{S_1 = \{1, 2, 3\}, S_2 = \{4, 5, 6\}\}.$$

Clearly each of $\{1, 2, 4, 5\}$ is weakly recurrent, and both 3 and 6 are strongly recurrent. It is evident that here for i weakly recurrent,

$$\sum_{n=1}^{\infty} (L_n)_{ij} = \infty \text{ for each } j.$$

We now present our last result in this section.

PROPOSITION 2.7. *Let (P_n) be a convergent* b.s.c. *such that* $\lim_{n\to\infty} P_{n,k} = R_k$ *and the limit points of the R_k's are all stochastic. Let i be a strongly recurrent state such that $i \in S_a$, a member of the S-basis of the chain. Then* $\lim_{n\to\infty} \sum_{j\notin S_a} (P_n)_{ij} = 0$.

Proof. Since for $m < n$, $P_{n,m} P_m = P_{n,m-1}$, we have

$$(P_{n,m})_{ii} \cdot \sum_{j\notin S_a} (P_m)_{ij} \leqslant \sum_{j\notin S_a} (P_{n,m-1})_{ij}.$$

The proposition now follows from the observation that

$$\lim_{k\to\infty} \sum_{j\notin S_a} (R_k)_{ij} = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \inf(R_k)_{ii} > 0.$$

3. Main Results

First, we establish a basic inequality that will be crucial in the discussions to follow. We present the inequality in a general form.

LEMMA 3.1. *Let (P_n) be a sequence of infinite-dimensional nonnegative matrices. Let i, j, k, n_1, n_2, n be fixed positive integers such that $k < n_1 < n_2 < n$. Suppose that the matrix products $P_{r,s}$ are all well-defined for $r < s$. Then the following holds*:

$$\begin{aligned}(P_{k,n})_{ij} \geqslant & \sum_{m=n_1}^{n_2} (P_{k,m})_{ii} (P_{m+1})_{ij} (P_{m+1,n})_{jj} \\ & - \sum_{n_1 \leqslant m < m' \leqslant n_2} (P_{k,m})_{ii} (P_{m+1})_{ij} (P_{m+1,m'})_{ji} \\ & \times (P_{m'+1})_{ij} (P_{m'+1,n})_{jj}. \end{aligned} \tag{5}$$

A similar inequality also holds for backward products.

Proof. For each $n-k-1$-tuple of positive integers $(s_1, s_2, ..., s_{n-k-1})$, consider an element $x(s_1, s_2, ..., s_{n-k-1})$ and let all such elements (distinct elements correspond to distinct tuples) form some set A. We define a discrete measure β on A such that

$$\beta(\{x(s_1, s_2, ..., s_{n-k-1})\}) = (P_{k+1})_{is_1} (P_{k+2})_{s_1 s_2} \cdots (P_n)_{s_{n-k-1} j}.$$

For $k < m < n$, define the sets A_m by

$$A_m = \{x(s_1, s_2, ..., s_{n-k-1}) \in A : s_{m-k} = i \text{ and } s_{m-k+1} = j\}.$$

Notice that $\beta(A) = (P_{k,n})_{ij}$ and $A \supset \bigcup_{m=n_1}^{n_2} A_m$; also,

$$\beta\left(\bigcup_{m=n_1}^{n_2} A_m\right) \geqslant \sum_{m=n_1}^{n_2} \beta(A_m) - \sum_{n_1 \leqslant m < m' \leqslant n_2} \beta(A_m \cap A_{m'}).$$

The inequality (5) now follows easily from this inequality. It is also clear that a similar inequality also holds for backward products and can be derived similarly. ∎

LEMMA 3.2. *Let f be a real-valued function on* $[0, \infty)$ *and* (a_n) *be a sequence of nonnegative reals such that* $\lim_{n\to\infty} a_n = 0$. *Consider the following assertions*:

(i) $\sum_{n=1}^{\infty} a_n < \infty$;

(ii) *given* $\varepsilon > 0$, *there exists a positive integer* $N(\varepsilon)$ *such that for any positive integers* n_1, n_2 *with* $n_2 > n_1 \geqslant N(\varepsilon), f(v) < \varepsilon$, *where* $v = \sum_{m=n_1}^{n_2} a_n$.

If $f(0) = 0$ *and f is upper semi-continuous, then* (i) $\Rightarrow$ (ii). *If* $f(u) > 0$ *for some positive u and f is lower semi-continuous, then* (ii) $\Rightarrow$ (i).

Proof. The first part of the lemma is obvious. For the second part, assume (ii) and suppose that $\sum_{n=1}^{\infty} a_n = \infty$. Let $0 < \varepsilon < f(u)$. Let $N(\varepsilon)$ be any positive integer. By lower semi-continuity of f at u, there is a $t > 0$ such that

$$u - t < x < u + t \Rightarrow f(x) > \varepsilon.$$

Choose $n_0 > N(\varepsilon)$ such that $a_n < t$ for $n \geqslant n_0$. Let $n_1 > n_0$. By our assumption, $\sum_{n=n_1}^{\infty} a_n = \infty$. Let n_2 be the smallest positive integer such that

$$\sum_{n=n_1}^{n_2} a_n > u.$$

Then it is clear that $u < \sum_{n=n_1}^{n_2} a_n < u + t$. This means that $f(v) > \varepsilon$ if $v = \sum_{n=n_1}^{n_2} a_n$. This contradicts the assumption of (ii). The lemma follows. ∎

Now we present our first theorem which proves the conjecture in [5].

THEOREM 3.3. *Let* (P_n) *be a convergent* f.s.c. *with basis* $\{T, C_1, C_2, \ldots\}$. *Then if i and j belong to two different C-components of the basis, the series* $\sum_{n=1}^{\infty} (P_n)_{ij}$ *must converge.*

Proof. Let $\lim_{n\to\infty} P_{k,n} = Q_k$. Let i and j belong to two different C-components of the basis. Then for any two limit points Q' and Q'' of the sequence (Q_k), we have

$$Q'_{ij} = Q''_{ij} = 0,\ Q'_{ii} = Q''_{ii} > 0 \qquad \text{and} \qquad Q'_{jj} = Q''_{jj} > 0.$$

Notice that there exist $d > 0$ and positive integers k_0 and $n(k)$ such that $k \geqslant k_0$ and $n > n(k) \Rightarrow (P_{k,n})_{ii} > d$, $(P_{k,n})_{jj} > d$. Then given $\varepsilon > 0$, there exist integers $N(\varepsilon) > k_0$ and $N(k) > n(k)$ such that $(P_{k,n})_{ij} < \varepsilon$ whenever $k > N(\varepsilon)$

and $n > N(k)$. Now let $k > N(\varepsilon)$ and $n > n_2 > n_1 > N(k)$. Then by (5) in Lemma 3.1,

$$\begin{aligned}\varepsilon &> d^2 \cdot \sum_{m=n_1}^{n_2} (P_{m+1})_{ij} - \sum_{n_1 \leqslant m < m' \leqslant n_2} (P_{m+1})_{ij} (P_{m'+1})_{ij} \\ &\geqslant d^2 \cdot \sum_{m=n_1}^{n_2} (P_{m+1})_{ij} - \tfrac{1}{2} \cdot \left[\sum_{m=n_1}^{n_2} (P_{m+1})_{ij} \right]^2. \end{aligned} \tag{6}$$

Observe that $\lim_{n\to\infty} (P_n)_{ij} = 0$. (This can be easily shown by a proof similar to that of Proposition 2.7.) Then the theorem follows immediately from Lemma 3.2 by taking $f(v) = d^2 v - (1/2)\, v^2$. ∎

A similar result holds for convergent b.s.c. Let us state such a result. We will omit its proof since the proof is almost identical to that of Theorem 3.3.

THEOREM 3.4. *Let (P_n) be a convergent* b.s.c. *such that for each positive integer k, $\lim_{n\to\infty} P_{n,k} = R_k$ and all (pointwise) limit points of the R_k's are stochastic. Let $\{S_1, S_2, \ldots\}$ be the S-basis of the chain. Then for i and j not in the same S-class, such that i is strongly recurrent and j weakly recurrent, $\sum_{n=1}^{\infty} (P_n)_{ij} < \infty$.*

Proof. The proof can be given as in Theorem 3.3 using the backward analog of (5), Proposition 2.7, and Lemma 3.2. ∎

Though Theorem 3.3 is interesting, a stronger version of this result is desirable especially when there are an infinite number of C-classes in the basis of the convergent f.s.c.. We would like to have for $i \in C_k$ (a C-class),

$$\sum_{n=1}^{\infty} \sum_{j \notin C_k \cup T} (P_n)_{ij} < \infty. \tag{7}$$

If we assume some kind of uniformity in convergence, then (7) is possible. That some condition is surely needed to obtain (7) follows from the following example.

EXAMPLE 3(a). Consider the following sequence of infinite-dimensional stochastic matrices given by

$$(P_n)_{11} = (P_n)_{22} = 1 - \frac{1}{n}; \qquad (P_n)_{1,n+2} = (P_n)_{2,n+2} = \frac{1}{n};$$

$$(P_n)_{ii} = 1 \text{ for } 2 < i \leqslant n; \qquad (P_n)_{i1} = (P_n)_{i2} = \frac{1}{2} \text{ for } i > n.$$

Then we show that $\lim_{n\to\infty} P_{k,n} = Q_k$ and $\lim_{k\to\infty} Q_k = Q$, where the matrices Q_k and Q are given by

(i) $(Q_k)_{11} = (Q_k)_{12} = (Q_k)_{21} = (Q_k)_{22} = \frac{1}{2}$; $(Q_k)_{ii} = 1$ for $2 < i \leqslant k+1$; $(Q_k)_{i1} = (Q_k)_{i2} = \frac{1}{2}$ for $i > k+1$.

(ii) $Q_{11} = Q_{12} = Q_{21} = Q_{22} = \frac{1}{2}$; $Q_{ii} = 1$ for $i > 2$.

Before we prove the convergence of the chain, notice that the basis of the chain is here the basis of Q, which is

$$\{C_1 = \{1, 2\},\ C_2 = \{3\},\ C_3 = \{4\}, \ldots\}.$$

Note that here $1 \in C_1$, but $\sum_{n=1}^{\infty} \sum_{s \in C_{n+1}} (P_n)_{1s} = \infty$, so that (7) does not hold.

Now to prove its convergence, we first notice that for $2 < i \leqslant k+1$, $(P_{k,n})_{ii} \geqslant (P_{k+1})_{ii} \cdots (P_n)_{ii} = 1$ so that $\lim_{n\to\infty} (P_{k,n})_{ii} = 1$. Also, for $n > k$, $i > 2$ (but $i \neq n+2$), and $s \geqslant 2$,

$$(P_{k,n})_{1i} = 0 = (P_{k,n})_{k+s,i}. \tag{8}$$

Moreover,

$$(P_{k,n+1})_{11} = (P_{k,n})_{11}\left(1 - \frac{1}{n+1}\right) + \frac{1}{2} \cdot (P_{k,n})_{1,n+2} \tag{9}$$

and

$$(P_{k,n+1})_{12} = (P_{k,n})_{12} \cdot \left(1 - \frac{1}{n+1}\right) + \frac{1}{2} \cdot (P_{k,n})_{1,n+2} \tag{10}$$

Now $(P_{k,n})_{1,n+2} = \sum_i (P_{k,n-1})_{1i} (P_n)_{i,n+2} = (1/n) \cdot [(P_{k,n-1})_{11} + (P_{k,n-1})_{12}] \to 0$; also,

$$(P_{k,n+1})_{11} - (P_{k,n+1})_{12} = \left(1 - \frac{1}{n+1}\right)[(P_{k,n})_{11} - (P_{k,n})_{12}]. \tag{11}$$

It follows from the above equations that

$$\lim_{n\to\infty} (P_{k,n})_{11} = \lim_{n\to\infty} (P_{k,n})_{12} = \tfrac{1}{2}.$$

Similarly, for $s \geqslant 2$, $\lim_{n\to\infty} (P_{k,n})_{k+s,1} = \lim_{n\to\infty} (P_{k,n})_{k+s,2} = \frac{1}{2}$.

We will now show that (7) is guaranteed by the following uniformity condition, which holds trivially for finite-dimensional matrices.

CONDITION (U). A convergent f.s.c. (P_n) is said to satisfy condition (U), if for each j in each C-class in the basis

$$\lim_{k\to\infty} \sup_{i\notin T} |(Q_k)_{ij} - Q'_{ij}| = 0 \tag{12}$$

where, as usual, Q' is a limit point of the Q_k's and $Q_k = \lim_{n\to\infty} P_{k,n}$.

Note that the chain in Example 3(a) does not satisfy (12). We are now ready to prove that (12) implies (7). First, a lemma.

LEMMA 3.5. *Let (P_n) be a convergent* f.s.c. *with basis $\{T, C_1, C_2, \ldots\}$. Let $D_s = \bigcup \{C_j : j \neq s\}$. Then the following results hold*:

(a) *Let $i \in C_s$. Then* $\lim_{n\to\infty} \sum_{j\in D_s\cup T} (P_n)_{ij} = 0$.

(b) *Suppose condition* (U) *holds. Then given $\varepsilon > 0$, there exists a positive integer $k(\varepsilon)$ such that for $n > k > k(\varepsilon)$, $\forall d \in D_s$*

$$\sum_{j\in D_s\cup T} (P_{k,n})_{dj} > 1 - \varepsilon. \quad \blacksquare$$

Proof. (a) For $i \in C_s$, given $\varepsilon > 0$, it is clear that there is a $\beta > 0$ and $k(\varepsilon)$ such that for $k \geqslant k(\varepsilon)$,

$$(Q_k)_{ii} > \beta \text{ and } \sum_{j\in D_s\cup T} (Q_k)_{ij} < \beta\varepsilon. \tag{13}$$

Since $P_{k,n-1}P_n = P_{k,n}$ (for $k < n-1$), we also have

$$(P_{k,n-1})_{ii} \cdot \sum_{j\in D_s\cup T} (P_n)_{ij} \leqslant \sum_{j\in D_s\cup T} (P_{k,n})_{ij}. \tag{14}$$

Now the assertion in (a) follows immediately from (13) and (14) by taking n sufficiently large.

(b) For $k < n$, $Q_k = P_{k,n}Q_n$. Let $i \in C_s$. Then for any $d \in D_s$,

$$(Q_k)_{di} \geqslant \sum_{j\in C_s} (P_{k,n})_{dj} (Q_n)_{ji}. \tag{15}$$

Notice that for any limit point Q' of the Q_n's, $Q'_{ii} > 0$ and $Q'_{di} = 0$. Choose any ε such that $0 < \varepsilon < Q'_{ii}$. By (12), there exists $k(\varepsilon)$ such that for $n > k > k(\varepsilon)$, we have from (15),

$$\tfrac{1}{2}\varepsilon \cdot Q'_{ii} \geqslant \tfrac{1}{2} \cdot Q'_{ii} \cdot \sum_{j\in C_s} (P_{k,n})_{dj}.$$

Part (b) of the lemma now follows. $\blacksquare$

THEOREM 3.6. *For every convergent* f.s.c., *condition* (U) *implies* (7), *where* $\{T, C_1, C_2, \ldots\}$ *is the basis of the chain.* ∎

Proof. Let $i \in C_s$ and $D_s = (T \cup C_s)^c$. Then using the same method as used in Lemma 3.1, we have for $k < n_1 < n_2 < n$:

$$(P_{k,n})_{ia} \geqslant \sum_{\substack{m=n_1 \\ d \in D_s}}^{n_2} (P_{k,m})_{ii} (P_{m+1})_{id} (P_{m+1,n})_{da}$$
$$- \sum_{\substack{n_1 \leqslant m < m' \leqslant n_2 \\ d,d' \in D_s}} (P_{k,m})_{ii} (P_{m+1})_{id} (P_{m+1,m'})_{di}$$
$$\times (P_{m'+1})_{id'} (P_{m'+1,n})_{d'a}.$$

It is now clear that there is a $c > 0$ such that given $\varepsilon > 0$, there is a $k(\varepsilon)$ so that for any n_1, n_2 with $k < n_1 < n_2$ and n sufficiently large,

$$\varepsilon > \sum_{a \in D_s \cup T} (P_{k,n})_{ia}$$
$$\geqslant c \cdot \sum_{\substack{m=n_1 \\ d \in D_s}}^{n_2} (P_{m+1})_{id} \cdot \left[\sum_{a \in D_s \cup T} (P_{m+1,n})_{da} \right]$$
$$- \sum_{\substack{n_1 \leqslant m < m' \leqslant n_2 \\ d,d' \in D_s}} (P_{m+1})_{id} (P_{m'+1})_{id'}.$$

Now we use condition (U). By Lemma 3.5(b), we now have: given $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that for any $n_2 > n_1 > N(\varepsilon)$,

$$\varepsilon \geqslant \frac{c}{2} \cdot \sum_{\substack{m=n_1 \\ d \in D_s}}^{n_2} (P_{m+1})_{id} - \frac{1}{2} \cdot \left[\sum_{\substack{m=n_1 \\ d \in D_s}}^{n_2} (P_{m+1})_{id} \right]^2.$$

It follows from Lemma 3.2 (by taking $a_n = \sum_{d \in D_s} (P_{n+1})_{id}$ there so that $\lim_{n \to \infty} a_n = 0$ by Lemma 3.5(a), and by choosing $f(x) = (c/2)\,x - (1/2)\,x^2$ that the inequality (7) holds. ∎

It is not clear how a result like Theorem 3.6 can be obtained for a convergent b.s.c. The main difficulty here lies in obtaining a suitable analog of Lemma 3.5(b). Under a different condition, we have the following proposition.

PROPOSITION 3.7. *Let* (P_n) *be a convergent* b.s.c. *such that* $\lim_{n \to \infty} P_{n,k} = R_k$ *and* $\lim_{k \to \infty} R_k = R$, *where the* R_k*'s and* R *are all stochastic. Let the basis of* R *be* $\{T, C_1, C_2, \ldots\}$. *Let* C_s *be a C-class such that* $\inf_{j \in C_s} R_{ij} > 0$

for $i \in C_s$. Then, given $\varepsilon > 0$, there is a positive integer $k(\varepsilon)$ such that for $n > k > k(\varepsilon)$, $\forall d \notin C_s$

$$\sum_{j \in C_s} (P_{n,k})_{jd} < \varepsilon.$$

Proof. Let $A_n = (P_n)^{\text{tr}}$, $B_n = (R_n)^{\text{tr}}$ and $B = R^{\text{tr}}$. Then for $n > k$, $B_k = A_{k,n} B_n$. Let $i \in C_s$ and $d \notin C_s$. Then,

$$(B_k)_{di} \geqslant \sum_{j \in C_s} (A_{k,n})_{dj} (B_n)_{ji}.$$

Notice that $\lim_{k \to \infty} \sum_{a=1}^{\infty} |(R_k)_{ia} - R_{ia}| = 0$ and $R_{id} = 0$. Since $\inf_{j \in C_s} B_{ji} > 0$, the proposition now follows easily. ∎

Now we present several important remarks and a few necessary examples.

Remark 3.8. (a) In the finite-dimensional case, one important result for convergent chains where the "T" set in its basis is empty is that each subchain obtained by normalizing the restriction of the original chain to any C-class in its basis is strongly ergodic. See Theorem 8 in [5]. This result is false in the infinite dimensional situation. This is illustrated by Example 3(a), where the restriction of P_n to C_1, after being normalized, is $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$.

(b) Theorem 2 in [6] is false in the infinite-dimensional case. This is illustrated by the following example. Consider the chain (P_n) where the first n rows of P_n are $(0, 1/2, 1/2^2, \ldots)$ and the remaining rows are all $(1, 0, 0, \ldots)$. Then it is easily verified that for each positive integer k, $\lim_{n \to \infty} P_{k,n} = Q$, where Q has identical rows and each row of Q is $(0, 1/2, 1/2^2, \ldots)$. Notice that $P_{k,n}$ acutally converges as $n \to \infty$ to Q in the norm. The basis of the chain is, of course, $\{T = \{1\}, C = \{2, 3, \ldots\}\}$. For $i \in C$ and $j \in T$, the series $\sum_{n=1}^{\infty} (P_n)_{ij} < \infty$. But we cannot get a stochastic matrix by normalizing the restriction of P_n to the C-class.

(c) An important property for finite-dimensional stochastic chains is that two such chains, when equivalent, are either both convergent or both divergent, and in case of convergence, they have the same basis. This property fails to be true in the infinite-dimensional case. As an example, consider the chain (P_n) defined as follows: For $k \geqslant 1$, let $a_k = 2^{4(k-1)}$; then if $a_k \leqslant n < a_{k+1}$,

$$\begin{aligned}
(P_{3n-2})_{ij} &= 1/2^j && \text{if } i = 0 \text{ and } j = 1, 2, 3, \ldots;\\
&= 1 - (1/2^k) && \text{if } i = j = 1, 2, \ldots, k;\\
&= 1 && \text{if } i = j = k+1, k+2, \ldots;\\
&= 1/2^k && \text{if } j = n \text{ and } i = 1, 2, \ldots, k,
\end{aligned}$$

$$(P_{3n-1})_{ij} = 1 \qquad \text{if } i=j=0, 1, 2,..., k;$$
$$= 1 \qquad \text{if } j=n \text{ and } i>k,$$

and

$$(P_{3n})_{ij} = 1 \qquad \text{if } i=n \text{ and } j=0;$$
$$= 1 \qquad \text{if } i=j \neq n.$$

Here our state space is $\{0, 1, 2,...\}$. We write $P'_n = P_{3n-2}P_{3n-1}P_{3n}$. Let us first show that the stochastic chain (P'_n) is strongly ergodic. To that end, let $u = (u_0, u_1, u_2,...)$ be any probability vector. We claim that for each positive integer m,

$$\lim_{n\to\infty} uP'_{m,n} = x = (0, 1/2, 1/2^2,...). \tag{16}$$

To prove (16), let us write $u_n = uP'_{m,n}$ for $n > m$ (m fixed). Let k be such that $a_k \leqslant n < a_{k+1}$. Then since $u_n = u_{n-1}P'_n$, we have

$$(u_n)_j = 0 \text{ for } j > k;$$
$$= (u_{n-1})_0 \cdot (1/2^j) + (u_{n-1})_j \cdot [1-(1/2^k)] \qquad \text{for } j = 1, 2,..., k.$$

Summing both sides of $(u_n)_j$ over $j = 1, 2,..., k$, we see that $(u_n)_0 = 1/2^k$. It then follows immediately that for $a_k < n < a_{k+1}$ and $1 \leqslant j \leqslant k$,

$$(u_n)_j - \frac{1}{2^j} = \left(1 - \frac{1}{2^k}\right)\left[(u_{n-1})_j - \frac{1}{2^j}\right]. \tag{17}$$

When $n = a_{k+1}$ and $1 \leqslant j \leqslant k$, we have

$$(u_n)_j - \frac{1}{2^j} = \left(1 - \frac{1}{2^{k+1}}\right)\left[(u_{n-1})_j - \frac{1}{2^j} + \left(\frac{1}{2^{j+k+1}}\right)\right]. \tag{18}$$

Also it can be verified easily that

$$\left(1 - \frac{1}{2^k}\right)^{a_{k+1}-a_k} \leqslant \exp\left\{-\frac{15}{16}\cdot 2^{3k}\right\}. \tag{19}$$

Now it follows from (17), (18), and (19) that $\lim_{n\to\infty} u_n = x$. This establishes our claim (16).

Now we redefine P_{3n} as simply the identity matrix. Write $P^*_n = P_{3n-2}P_{3n-1}I$. Then it easily follows from (19) that $(P^*_{m,n})_{jj}$ converges to zero as $n \to \infty$, for large m. This means that the chain (P^*_n) cannot be convergent, since, as we have seen in Section 1, for j in a C-class, $\lim_{n\to\infty} (P^*_{m,n})_{jj} > 0$ for large m.

The above example establishes the following: Suppose we define the chains (S_n) and (S_n^*) as

$$\begin{aligned} S_n = S_n^* &= P_{3m-2} \qquad \text{if } n = 3m-2; \\ &= P_{3m-1} \qquad \text{if } n = 3m-1. \\ S_{3m} = P_{3m} \qquad &\text{and} \qquad S_{3m}^* = I. \end{aligned}$$

Then the two chains are clearly equivalent. But the chain (S_n^*) is not convergent, whereas the chain (S_n) is convergent. (Notice that $\lim_{n\to\infty} P_{3n-1} = I$ and $x \cdot (\lim_{n\to\infty} P_{3n-2}) = x$, x as in (16).

In the finite-dimensional case for a convergent chain (P_n), when we replace $(P_n)_{ij}$, for each n, by zero for all i, j belonging to different C-classes in the basis of the chain, the resulting chain (after being normalized) remains convergent with the same basis as a consequence of equivalence. Though the property of finite equivalent chains does not extend to the infinite case as our example 3.8(c) shows, a similar result on convergence exists under condition (U) in the infinite case. Our next theorem illustrates this.

THEOREM 3.9. *Let (P_n) be a convergent chain with condition* (U) *(as described in* (12)). *Define for each n the nonnegative matrix P_n^* (not necessarily stochastic) by*

$$\begin{aligned} (P_n^*)_{ij} &= 0 \qquad \text{if } i,j \text{ belong to two different } C\text{-} \\ &\qquad\quad\ \text{classes in the basis of the chain;} \\ &= (P_n)_{ij} \quad \text{otherwise.} \end{aligned}$$

Then for every positive integer k, $P_{k,n}^* = P_{k+1}^* P_{k+2}^* \cdots P_n^*$ converges to some Q_k^* as $n \to \infty$, and for every i,j,

$$\lim_{k\to\infty} [(Q_k)_{ij} - (Q_k^*)_{ij}] = 0.$$

Furthermore, if C_s is a C-class in the basis of (P_n) and if $A = T \cup C_s \cup \{a\}$ is the state space of a new stochastic chain (B_n) such that a is an absorbing state and $B_n|_{T\cup C_s} = P_n|_{T\cup C_s}$, then the chain (B_n) is also convergent with basis $\{T, C_s, \{a\}\}$.

Proof. Let (X_n) be the Markov chain induced by (P_n). Now for any i, j, $0 \leqslant (P_{k,n}^*)_{ij} \leqslant (P_{k,n})_{ij}$ so that for $j \in T$, $\lim_n (P_{k,n}^*)_{ij} = 0$. Assume now that j belongs to some C-class C and let $D = (C \cup T)^c$. Notice that for any i, we can write

$$\begin{aligned}(P_{k,n})_{ij} &= \Pr(X_n = j \mid X_k = i) \\ &= \Pr(X_n = j, X_s \in D \quad \text{for some } s,\, k < s < n \mid X_k = i) \\ &\quad + \Pr(X_n = j, X_s \notin D \quad \text{for each } s \text{ with } k < s < n \mid X_k = i) \\ &= (P_{k,n})_{ij}^D + \sum (P_{k+1})_{is_1} (P_{k+2})_{s_1 s_2} \cdots (P_n)_{s_{n-k-1},j},\end{aligned}$$

where the first term represents the probability of transition from i to j through a state in D and the summation in the second term is over all $s_1, s_2, \ldots, s_{n-k-1}$ in $C \cup T$. It follows that

(i) for $i \in C \cup T$,

$$(P_{k,n})_{ij} - (P_{k,n}^*)_{ij} \leqslant (P_{k,n})_{ij}^D, \tag{20}$$

(ii) for $i \in D$, $Q'_{ij} = 0$ so that

$$\lim_n (P_{k,n}^*)_{ij} \leqslant \lim_n (P_{k,n})_{ij} \to 0 \qquad \text{as } k \to \infty.$$

Now it can be verified that

$$(P_{k,n})_{ij}^D \leqslant \sum_{m=k+1}^{n-1} (P_{k,m})_{iD}^f \cdot \sup_{d \in D} (P_{m,n})_{dj} \tag{21}$$

where

$$(P_{k,m})_{iD}^f = Pr(X_m \in D, X_t \notin D \text{ for } k < t < m \mid X_k = i).$$

Since $\sum_{m=k+1}^{\infty} (P_{k,m})_{iD}^f \leqslant 1$, given $\varepsilon > 0$, there exists N such that

$$(P_{k,n})_{ij}^D \leqslant \sum_{m=k+1}^{N} (P_{k,m})_{iD}^f \cdot \sup_{d \in D} (P_{m,n})_{dj} + \varepsilon \quad \text{(for all } n).$$

It follows by condition (U) that if k is sufficiently large, then there exists a positive integer $n(k)$ such that for $n > n(k)$,

$$(P_{k,n})_{ij}^D < 2\varepsilon. \tag{22}$$

From (20) and (22), and from Theorem 3.6, we now have

Given $\varepsilon > 0$, there exists k_0 such that for $k > k_0$ and $n > n(k)$, $(P_{k,n})_{ij} - (P_{k,n}^*)_{ij} < \varepsilon$. *The integers* k_0 *and* $n(k)$'s *depend on* i *and* j.

The first part of the theorem will be proved once we show that for each k, $\lim_n P_{k,n}^* = Q_k^*$ exists. To this end, notice that

$$(P_{k,n}^* - P_{k,n'}^*)_{ij} = \sum (P_{k,m}^*)_{is} [P_{m,n}^* - P_{m,n'}^*]_{sj} \tag{23}$$

Given $\varepsilon > 0$, there exists m_0 and N such that $m \geqslant m_0 \Rightarrow$

$$\sum_{s=1}^{N} (P_{k,m})_{is} > 1 - (\varepsilon/2),$$

which means

$$\sum_{s=N+1}^{\infty} (P^*_{k,m})_{is} < \varepsilon/2. \tag{24}$$

By the assertion following (18), there also exists $m_1 \geqslant m_0$ such that for $m > m_1$ and $n > n(m)$, we have

$$(P_{m,n})_{sj} - (P^*_{m,n})_{sj} < \varepsilon/2^N \tag{25}$$

for $s = 1, 2, \ldots, N$.

From (23), (24), (25), and the fact that $\lim_n P_{m,n}$ exists, the convergence of $P^*_{k,n}$ as $n \to \infty$ follows.

Now for the last part of the theorem, notice that as in (20), for $i \in C_s \cup T$ and $j \in C_s$,

$$(P_{k,n})_{ij} - (B_{k,n})_{ij} \leqslant (P_{k,n})^D_{ij}.$$

Then by the same argument as before, the convergence of the chain (B_n) follows. ∎

Now we present another result similar to that of Theorem 3.6 using a condition different from condition (U).

THEOREM 3.10. *Let (P_n) be a convergent* f.s.c. *with basis $\{T, C_1, C_2, \ldots\}$. Suppose that there is a $t \in T$ and some C_s such that* $\lim_{n\to\infty} \inf \sum_{i \notin C_s} (Q_n)_{ti} > 0$. *Then for each $i \in C_s$, $\sum_{n=1}^{\infty} (P_n)_{it} < \infty$. If the* "lim inf" *condition is strengthened by*

$$\liminf_{n\to\infty} \left[\inf_{t \in T} \sum_{i \notin C_s} (Q_n)_{ti} \right] > 0,$$

then for each i in C_s,

$$\sum_{n=1}^{\infty} \sum_{t \in T} (P_n)_{it} < \infty.$$

Proof. The proof follows easily from the fact that for any i in C_s, $\lim_{n\to\infty} \sum_{t \in T} (P_n)_{it} = 0$ and from an argument similar to the one used in the proof of Theorem 3.6. We omit the details. ∎

We remark that an analog of Theorem 3.10 can also be given for convergent b.s.c. However, a similar analog of Theorem 3.9 is not immediately clear; but it will not be difficult to find out what type of results for b.s.c. are available along the lines of Theorem 3.9 since the general method is already available in the proof of this theorem. We will not discuss these things any further in this paper; instead, we go to the next section to present some interesting results concerning the basis of a finite-dimensional convergent f.s.c. or b.s.c. All these results are new.

4. Convergent Stochastic Chains with a Finite Number of States

Let us first state an interesting result on convergent b.s.c. similar to Theorem 8 in [5] given for convergent f.s.c.

THEOREM 4.1. *Suppose that* $\lim_{n\to\infty} P_{n,k} = R_k$ *and* $\lim_{k\to\infty} R_k = R$, *where* (P_n) *is a convergent* b.s.c. *with finite number of states. Suppose that* R *has the basis* $\{T, C_1, C_2, \ldots, C_p\}$ *and that for* $i \notin T$ *and* $j \in T$, *the series* $\sum_{n=1}^{\infty} (P_n)_{ij} < \infty$. *For* $1 \leqslant i \leqslant p$, *let* $(P_n(C_i))$ *be a new stochastic chain obtained by considering the normalized restriction of the* P_n*'s to the* C_i*-block. Then the* b.s.c. $(P_n(C_i))$ *is strongly ergodic.*

Proof. The proof is based on the concept of equivalence for finite chains. It uses Theorem 3.4 and is exactly the same as that of Theorem 8 in [5]. We omit the proof. ∎

We now give an example showing that the assumption of convergence of $\sum_{n=1}^{\infty} (P_n)_{ij}$ for $i \notin T, j \in T$ in Theorem 4.1 cannot be removed.

EXAMPLE 4(a). Consider the stochastic chain (P_n) given by

$$P_n = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/n & 0 & 1-1/n \\ 1/n & 1-1/n & 0 \end{pmatrix}.$$

Let us write $L_m = P_{2m} P_{2m-1}$. Then Bernstein's condition for weak ergodicity (see [9, p. 105]) is easily seen to hold for the chain (L_n). It is also known that weak and strong ergodicity for backward products are equivalent. Examining a few products, it follows easily that (P_n) is a convergent b.s.c. and $\lim_{k\to\infty} \lim_{n\to\infty} P_{n,k}$ is a matrix where each row is (0 1/2 1/2). The basis of the chain is $\{T = \{1\},\ C = \{2, 3\}\}$. But notice that the backward chain $(P_n(C))$ is not even convergent.

In [6], a converse to Theorem 8 in [5], has been given. A similar converse does not exist for backward products as can be verified easily by considering the chain (P_n)

$$P_n = \begin{pmatrix} 0 & a_n & 1-a_n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where (a_n) is a nonconvergent sequence of reals. Notice that there for $n > k$, $P_{n,k} = P_n$. However, a result in the same direction can be given as below.

THEOREM 4.2. *Let (P_n) be a given finite stochastic chain. Suppose that there is a partition $\{T, C_1, C_2, \ldots, C_p\}$ of the state space such that the following hold*:

(i) $\sum_{n=1}^{\infty} (P_n)_{ij} < \infty$ for ($j \in T$ and $i \notin T$) *and for i and j in two different C-classes in the given partition*;

(ii) *for each i, $1 \leqslant i \leqslant p$, the chain $(P_n(C_i))$ is strongly ergodic for backward products with no zero entry in* $\lim_k \lim_n P'_{n,k}$, $P'_n = P_n(C_i)$.

(iii) $\lim (P_n)_{ij}$ *exists for $i \in T$ and any j and this limit is 0 for $i \in T$, $j \in T$. Then the* b.s.c. (P_n) *is convergent with basis $\{T, C_1, \ldots, C_p\}$.*

Proof. Because of (i), we can assume with no loss of generality using equivalence of chains that each P_n is of the form

	T	C_1	C_2
T			
C_1	0		0
C_2	0	0	

By condition (ii), it follows easily that for $i \notin T$, $\lim_{n\to\infty} (P_{n,k})_{ij} = (R_k)_{ij}$ exists and also $\lim_{k\to\infty} (R_k)_{ij}$ exists. For $i \in T, j \in T$,

$$(P_{n+1,k})_{ij} = \sum_{t\in T} (P_{n+1})_{it} (P_{n,k})_{tj}$$

and therefore, by (iii), $\lim_{n\to\infty} (P_{n,k})_{ij} = 0$. Now for $i \in T$ and j in a C-class, we have

$$(P_{n+1,k})_{ij} = \sum_{t\in T} (P_{n+1})_{it} (P_{n,k})_{tj} + \sum_{r\notin T} (P_{n+1})_{ir} (P_{n,k})_{rj}.$$

The theorem now follows by condition (iii). ∎

Before we go into more serious results, it is relevant to point out one more difference between forward and backward products. The forward products, we know, correspond to multi-step transition probabilities of some Markov chain. Also, a Markov chain (X_n), when observed in reverse order, is a Markov chain and if L_n is a stochastic matrix so that

$$P(X_{n-1}=j \mid X_n=i)=(L_n)_{ij} \tag{26}$$

whenever the left side is defined, then for $n>k$

$$P(X_k=j \mid X_n=i)=(L_{n,k})_{ij}$$

whenever the left side is defined. So the question is: When does a given stochastic chain induce such a reversed Markov chain? The following result answers this. (Though it must be well-known, we have not succeeded in finding a reference for it. We only state it.)

THEOREM 4.3. *Let (L_n) be a stochastic chain. Then a necessary and sufficient condition for the existence of a Markov chain (X_n) satisfying* (26) *is that there exists a sequence (π_n) of probability vectors satisfying $\pi_{n-1}=\pi_n L_n$ for $1 \leqslant n < \infty$.*

A simple example illustrating the above result is the following: Consider the chain (L_n) such that

$$L_n=\begin{bmatrix} b_n & 1-b_n \\ c_n & 1-c_n \end{bmatrix}, \qquad c_n=a(1-b_n)/(1-a), 0<b_n<1, a \leqslant \tfrac{1}{2}.$$

Then $(a, 1-a)=\pi$ satisfies $\pi L_n=\pi$ for each n.

Two of our results in the previous section were in proving the convergence of the series $\sum_{n=1}^{\infty} (P_n)_{ij}$ for convergent f.s.c. and b.s.c. when i and j are in two different C-classes of the basis. While such a result does not hold for i in a C-class and j in the T class in the general case, as can be seen easily by Example 4(a) in both the f.s.c. and b.s.c. cases, we will present below several interesting results in this direction.

THEOREM 4.4. *Let (P_n) be a convergent f.s.c. with basis $\{T, C_1, \ldots, C_p\}$. Let $t \in T$, $c_1 \in C_1$ and $c_2 \in C_2$, Then we have*

$$\sum_{n=1}^{\infty} \min\{(P_n)_{c_1 t}, (P_n)_{c_2 t}\} < \infty.$$

Proof. Let $Q_m=\lim_{n \to \infty} P_{m,n}$. Let k be the number of states and

$$N_j=\{m: (Q_m)_{tj} \geqslant 1/k\}.$$

Then $N = \bigcup_{j=1}^{k} N_j$. Suppose that $j \notin C_1$. For $n_1 < n_2 < n-2$,

$$\begin{aligned}(P_{k,n})_{c_1 j} &= P(X_n = j \mid X_k = c_1)\\ &\geqslant P\Bigg(\bigcup_{\substack{m=n_1\\ m+1\in N_j}}^{n_2} \{X_n = j, X_{m+1} = t, X_m = c_1\} \mid X_k = c_1\Bigg)\end{aligned}$$

Using (5), we then get

$$\begin{aligned}(P_{k,n})_{c_1 j} \geqslant & \sum_{\substack{m=n_1\\ m+1\in N_j}}^{n_2} (P_{k,m})_{c_1 c_1} (P_{m+1})_{c_1 t} (P_{m+1,n})_{tj}\\ & - \sum (P_{k,m})_{c_1 c_1} (P_{m+1})_{c_1 t} (P_{m+1,m^*})_{t c_1}\\ & \times (P_{m^*+1})_{c_1 t} (P_{m^*+1,n})_{tj},\end{aligned}$$

where the second summation is over

$$\{(m, m^*): n_1 \leqslant m < m^* \leqslant n_2, m+1 \in N_j, m^*+1 \in N_j\}.$$

Hence, proceeding as in (6), there exist $d > 0$ and a positive integer $N(\varepsilon)$ such that for any n_1, n_2 with $N(\varepsilon) < n_1 < n_2$,

$$\varepsilon > d \cdot \sum_{\substack{m=n_1\\ m+1\in N_j}}^{n_2} (P_{m+1})_{c_1 t} - \tfrac{1}{2} \cdot \Bigg(\sum_{\substack{m=n_1\\ m+1\in N_j}}^{n_2} (P_{m+1})_{c_1 t}\Bigg)^2.$$

Therefore, as in Theorem 3.3, since $\lim_{n\to\infty} (P_n)_{c_1 t} = 0$, it follows from Lemma 3.2 that

$$\sum_{\substack{m=1\\ m\in N_j}}^{\infty} (P_m)_{c_1 t} < \infty \qquad (\text{for } j \notin C_1).$$

Similarly, we also have

$$\sum_{\substack{m=1\\ m\in N_j}}^{\infty} (P_m)_{c_2 t} < \infty \ (\text{for } j \notin C_2).$$

Since $C_1 \cap C_2 = \varnothing$, the theorem now follows. ∎

We remark that a similar result can also be stated for b.s.c. We also remark that because of Theorem 4.4, the following is true:

Let (P_n) be a convergent f.s.c. with basis $\{T, C_1, \ldots, C_p\}$. Define (P_n^*) as follows:

$$
\begin{aligned}
(P_n^*)_{ij} &= 0 && \text{if } i, j \text{ belong to two different } C\text{-classes;} \\
&= 0 && \text{if } j \in T,\, i \in C_s \text{ and } (P_n)_{ij} \leqslant (P_n)_{kj} \text{ for some} k \notin T \cup C_s; \\
&= (P_n)_{ij} && \text{otherwise.}
\end{aligned}
$$

If we wish, we can also normalize the P_n^* to make them stochastic. In any case, by Theorems 3.3 and 4.4, the series

$$\sum_{n=1}^{\infty} \|P_n - P_n^*\|$$

is convergent. Therefore, the chain (P_n^*) is also convergent with the same basis. A similar remark also applies for b.s.c.. All these lead to another interesting question: In Theorem 4.4, is the series $\sum_{n=1}^{\infty} \min\{(P_n)_{tc_1}, (P_n)_{tc_2}\}$ convergent? An example in [6] immediately gives the answer in the negative. The situation in the b.s.c. case is the same, as will be evidenced by the example at the end of this section. However, a result similar to the above remark holds in this situation. This result is by no means obvious. We present it below.

THEOREM 4.5. *Let (P_n) be a convergent f.s.c. with basis $\{T, C_1, C_2, \ldots, C_p\}$. Let $t \in T$, $c_1 \in C_1$, and $c_2 \in C_2$. Let $A = \{n\colon (P_n)_{tc_1} \geqslant (P_n)_{tc_2}\}$. For $n \in A$, define P_n^* such that*

$$
\begin{aligned}
(P_n^*)_{ij} &= (P_n)_{ij} && \text{if } (i,j) \neq (t, c_2); \\
&= 0 && \text{if } (i,j) = (t, c_2).
\end{aligned}
$$

Let $P_n^* = P_n$ for $n \notin A$. We also assume that an additional absorbing state d has been introduced so that for $n \in A$, P_n and P_n^* look like

P_n:

	t			d
t				0
				0
d	0		0	1

P_n^*:

	t			d
t				$(P_n)_{tc/2}$
				0
d	0		0	1

Then, for each positive integer k, $\lim_{n\to\infty} P_{k,n}^* = Q_k^*$ exists. Moreover, for $i \notin T$, $\lim_{k\to\infty} (Q_k^*)_{ij} = \lim_{k\to\infty} \lim_{n\to\infty} (P_{k,n})_{ij}$.

Proof. Clearly, for $j \neq d$ and any i,

$$(P^*_{k,n})_{ij} \leqslant (P_{k,n})_{ij}. \tag{27}$$

Therefore,

$$\lim_{n \to \infty} (P^*_{k,n})_{ij} = 0 \qquad (\text{for } j \in T \text{ and any } i). \tag{28}$$

We now claim the following: (29)

Let $(i,j) \notin T \times C_2$. Given $\varepsilon > 0$, there exists a positive integer k_0 such that for each $k > k_0$, there exists a positive integer $n(k)$ such that $n > n(k) \Rightarrow |(P_{k,n})_{ij} - (P^*_{k,n})_{ij}| < \varepsilon$.

To prove the claim, first notice that if for some i, (29) holds for all $j \neq d$, then it also holds for $j = d$. Now we consider the following cases:

(i) (29) is immediate for $i = d$.

(ii) Let i and j belong to two different C-classes. Then (29) follows from (27) since then $\lim_{k \to \infty} \lim_{n \to \infty} (P_{k,n})_{ij} = 0$.

(iii) Let $j \in T$. Then again (29) is immediate.

(iv) Let $j \notin C_2 \cup T$. Then

$$(P_{k,n})_{ij} - (P^*_{k,n})_{ij} \leqslant \sum_{k<m<n} (P_{k,m})^f_{ic_2} (P_{m,n})_{c_2 j},$$

where

$$(P_{k,m})^f_{ic_2} = P(X_m = c_2, X_s \neq c_2 \text{ for } k < s < m \mid X_k = i).$$

Since $\sum_{m=k+1}^{\infty} (P_{k,m})^f_{ic_2} \leqslant 1$ and $\lim_{m \to \infty} \lim_{n \to \infty} (P_{m,n})_{c_2 j} = 0$, (29) follows.

(v) Let $j \in C_2$ and $i \notin C_1 \cup T$.

In this case, let us first define

$$(P_{k,m})^t_{is} = P((X_{m-1}, X_m) = (t, s), (X_{n-1}, X_n) \neq (t, s) \\ \text{for } k < n < m \mid X_k = i).$$

Then for $j \neq d$, we have:

$$(P_{k,m})_{ij} - (P^*_{k,m})_{ij} \leqslant \sum_{\substack{n \in A \\ k < n-1 < n \leqslant m}} (P_{k,n})^t_{ic_2} (P_{n,m})_{c_2 j}. \tag{30}$$

Also we have

$$(P_{k,n})_{ic_1} \geqslant \sum_{r \in A, k<r<n} (P_{k,r})^t_{ic_1} (P_{r,n})_{c_1 c_1}$$

so that by taking n to ∞ we see that there is a $\beta > 0$ such that for k sufficiently large,

$$(Q_k)_{ic_1} \geqslant \beta \cdot \sum_{r \in A, r > k} (P_{k,r})^t_{ic_1} \geqslant \beta \cdot \sum_{r \in A, r > k} (P_{k,r})^t_{ic_2}.$$

Since $i \notin C_1 \cup T$, it follows that $\lim_{k \to \infty} (Q_k)_{ic_1} = 0$ and therefore

$$\lim_{k \to \infty} \sum_{r \in A, r > k} (P_{k,r})^t_{ic_2} = 0.$$

Now (29) follows immediately from (30). This completes the proof of our claim (29). The proof of the theorem will now be complete once we establish the convergence of $P^*_{k,n}$. To this end, we observe that for any i, j, and $k < r < n < m$,

$$\begin{aligned}
|(P^*_{k,n})_{ij} - (P^*_{k,m})_{ij}| &= |(P^*_{k,r} P^*_{r,n})_{ij} - (P^*_{k,r} P^*_{r,m})_{ij}| \\
&\leqslant \sum_{s \in T} (P^*_{k,r})_{is} + \sum_{s \notin T} (P^*_{k,r})_{is} \cdot |(P^*_{r,n})_{sj} - (P^*_{r,m})_{sj}| \\
&\leqslant \sum_{s \in T} (P^*_{k,r})_{is} + \sum_{s \notin T} |(P^*_{r,n})_{sj} - (P_{r,n})_{sj}| \\
&\quad + \sum_{s \notin T} |(P_{r,n})_{sj} - (P_{r,m})_{sj}| + \sum_{s \notin T} |(P_{r,m})_{sj} - (P^*_{r,m})_{sj}|.
\end{aligned}$$

It follows that for each k, the sequence $(P^*_{k,n})_{ij}$ is Cauchy. This completes the proof of the theorem. ∎

We remark that one can also state and prove a result similar to the above for b.s.c.'s. Now we will present the last two results in this paper. We will assume the following condition in our first result:

$$\sum_{j \in C_1} (Q')_{tj} < 1 \tag{31}$$

for some $t \in T$, for each limit point Q' of the Q_k's, where $Q_k = \lim_{n \to \infty} P_{k,n}$ and (P_n) is a convergent f.s.c. with basis $\{T, C_1, ..., C_p\}$.

It may be helpful to remark that the condition (31) is weaker than the condition $\sum_{n=1}^{\infty} (P_n)_{sj} < \infty$ for each $s \in T$ and each $j \in C_1$. This is because under the latter condition, due to equivalence, we can assume with no loss of generality that $(P_n)_{ij} = 0$ whenever $i \in T$ and $j \in C_1$ or i, j belong to two different C-classes. Then, in the products $P_{k,n}$ the entries in the $T \times C_1$ position are all zero, so that the sum in (31) is then zero for each s in T.

THEOREM 4.6. *Consider a convergent* f.s.c. (P_n) *with basis* $\{T, C_1, C_2, \ldots, C_p\}$. *Suppose that* (31) *holds for some* $t \in T$. *Then for each* j *in* C_1, *we have*: $\sum_{n=1}^{\infty} (P_n)_{jt} < \infty$. *If* (31) *holds for each* t *in* T, *then the* f.s.c. $(P_n(C_1))$ *is strongly ergodic.*

Proof. The proof is similar to that of Theorem 3.3. The inequality to use in this case is the following: For $j \in C_1$ and $s \notin C_1$,

$$(P_{k,n})_{js} \geqslant \sum_{m=n_1}^{n_2} (P_{m+1,n})_{ts} (P_{m+1})_{jt} (P_{k,m})_{jj} - \sum_{n_1 \leqslant m < m' \leqslant n_2} (P_{m+1})_{jt} (P_{m'+1})_{jt}$$

so that summing the terms on both sides over all $s \notin C_1$, we have:

$$\sum_{s \notin C_1} (P_{k,n})_{js} \geqslant \sum_{m=n_1}^{n_2} \left(\sum_{s \notin C_1} (P_{m+1,n})_{ts} \right) (P_{m+1})_{jt} (P_{k,m})_{jj} - u \cdot \sum_{n_1 \leqslant m < m' \leqslant n_2} (P_{m+1})_{jt} (P_{m'+1})_{jt},$$

where u is the number of states of the chain.

The rest of the proof now follows as in Theorem 3.3. ∎

We now present the backward analog of Theorem 4.6. We will omit its proof.

THEOREM 4.7. *Let* (P_n) *be a convergent* b.s.c. *with* $R_k = \lim_{n \to \infty} P_{n,k}$ *and S-basis as* $\{S_1, S_2, \ldots, S_p\}$. *Choose any* S_i *in this basis and let* t *be a state not in this* S_i. *Suppose that* $\sum_{j \in S_i} R'_{tj} < 1$ *for each limit point* R' *of the* R_k*'s. Then if* s *is a strongly recurrent state (if there is any) in* S_i, *it follows that* $\sum_{n=1}^{\infty} (P_n)_{st} < \infty$. *Thus, if* $\lim_{k \to \infty} R_k$ *exists, a result similar to the last part of Theorem* 4.6 *also holds for the backward chain.*

It is relevant to point out that unless the R_k's converge in Theorem 4.7, the convergent b.s.c. may not have any strongly recurrent state. For example, consider the chain where the matrices are alternately $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

We now end this section with the following example. For this example, the f.s.c. and the b.s.c. are both convergent and have the same basis. This example shows that in Theorems 4.6 and 4.7, the conditions assumed for the limit points Q' and R' cannot be removed. The f.s.c. case of this example appeared earlier in [6]. Since the proof of convergence in the b.s.c. case is quite different, we present it below.

EXAMPLE 4(b). Let $0 < a_n < 1$ and

$$P_n = \begin{vmatrix} 0 & a_n & 1-a_n \\ 0 & 1 & 0 \\ a_n & 0 & 1-a_n \end{vmatrix}.$$

If $\sum_{n=1}^{\infty} a_n < \infty$, then clearly (P_n) is a convergent b.s.c. and $\lim_{k\to\infty} \lim_{n\to\infty} P_{n,k}$ is

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Suppose now that $\sum_{n=1}^{\infty} a_n = \infty$. Let $a_n \to 0$ as $n \to \infty$. Then we claim that the b.s.c. (P_n) is convergent; in particular, the b.s.c. is strongly ergodic if $\sum_{n=1}^{\infty} a_n a_{n+1} = \infty$. Otherwise, it has basis $\{T = \{1\}, C_1 = \{2\}, C_2 = \{3\}\}$.

Proof of the claim. Write

$$P_{n,k} = \begin{vmatrix} a_{n,k} & b_{n,k} & c_{n,k} \\ 0 & 1 & 0 \\ d_{n,k} & e_{n,k} & f_{n,k} \end{vmatrix}.$$

Computing the product $P_{n+1} P_{n,k} = P_{n+1,k}$, we have

(i) $a_{n+1,k} = d_{n,k}(1 - a_{n+1})$

(ii) $d_{n+1,k} = a_{n+1} a_{n,k} + d_{n,k}(1 - a_{n+1})$

(iii) $b_{n+1,k} = a_{n+1} + (1 - a_{n+1})\, e_{n,k}$

(iv) $e_{n+1,k} = a_{n+1} b_{n,k} + (1 - a_{n+1})\, e_{n,k}$.

From (i) and (ii),

$$\begin{aligned} a_{n+1,k} \leqslant d_{n+1,k} &\leqslant a_{n+1} d_{n,k} + d_{n,k}(1 - a_{n+1}) \\ &= d_{n,k} \leqslant d_{n-1,k} \leqslant \cdots \leqslant a_k. \end{aligned}$$

This means that $\lim_{n\to\infty} d_{n,k}$ exists. Also, from (iii) and (iv), we have:

$$b_{n+1,k} \geqslant e_{n+1,k};$$
$$e_{n+1,k} \geqslant a_{n+1} e_{n,k} + (1 - a_{n+1})\, e_{n,k} = e_{n,k}.$$

Thus, $\lim_{n\to\infty} e_{n,k}$ exists. It follows easily that $\lim_{n\to\infty} P_{n,k}$ exists. Since $a_n \to 0$, each limit point of the R_k's, $R_k = \lim_{n\to\infty} P_{n,k}$, is of the form

$$\begin{vmatrix} 0 & b & c \\ 0 & 1 & 0 \\ 0 & b & c \end{vmatrix}.$$

Since this limit point must be idempotent, $c = c^2$ so that $c = 0$ or 1. In the ergodic case, b must be 1 and $c = 0$; otherwise, $b = 0$ and $c = 1$. In the case when $\sum_{n=1}^{\infty} a_n a_{n+1} = \infty$, by direct computations it can be verified that Bernstein's condition holds for the chain $(P_{n+1} P_n)$ and therefore the b.s.c. is weakly (and therefore strongly, being a b.s.c.) ergodic. Now consider the case when $\sum_{n=1}^{\infty} a_n a_{n+1} < \infty$. Notice that from (iii) and (iv) we have:

$$\begin{aligned} e_{n+1,k} &= a_{n+1}[a_n + (1 - a_n) e_{n-1,k}] + (1 - a_{n+1}) e_{n,k} \\ &\leqslant a_n a_{n+1} + (1 - a_n a_{n+1}) e_{n,k} \qquad (\text{since } e_{n-1,k} \leqslant e_{n,k}). \end{aligned}$$

Thus, $e_{n+1,k} \leqslant \sum_{i=k+1}^{n+1} a_i a_{i+1}$ and therefore $\lim_{k\to\infty} \lim_{n\to\infty} e_{n,k} = 0$. The claim is now justified. ∎

5. Applications

It is always relevant to point out applications. As we already mentioned, our results can be applied to obtain significant results on measures on semigroups. Instead of taking up these applications, let us point out a few applications in the mainstream of probability in the context of classification of states for nonhomogeneous Markov chains (both discrete and continuous time). We will present only two theorems (omitting their proofs which follow rather easily from results in this paper, especially Theorem 3.3). In what follows, we consider a finite state space only in Proposition 5.1.

Let $E = \{1, 2, ..., s\}$ be the state space and i, j be in E. We say $i \to j$ iff there exist $j_1, j_2, ..., j_n$ in E with $j_1 = i$ and $j_n = j$ such that for each k, $1 \leqslant k < n$, $\sum_{m=1}^{\infty} (P_m)_{j_k j_{k+1}} = \infty$. We say that i is essential iff $i \to j \Rightarrow j \to i$; otherwise, i will be called nonessential. Note that in the subset $F = \{i: i \leftrightarrow i\}$, the relation $\leftrightarrow$ is an equivalence relation. If $i \notin F$, i is obviously nonessential. In each equivalence class of F (note that F is always nonempty), either all states are essential, or none is (It turns out that for convergent chains and even for more general chains with a basis, the nonessential states are precisely the transient or nonrecurrent states iff $\sum_{n=1}^{\infty} (P_n)_{ji} < \infty$ for $j \notin T$ and $i \in T$, where T is the "T" class of the basis of the chain.) We also have the following proposition.

PROPOSITION 5.1. *Let (P_n) be a convergent chain with basis*

$$\{T, C_1, C_2, \dots, C_p\}.$$

Then the following two statements are equivalent:

(a) *Each C-class is an essential class; the T class is a non-essential class. Moreover, the classes can be identified in the following manner: for $1 \leqslant i \leqslant p$, $C_i = \{k \in E: j \to k\}$ for each $j \in C_i$.*

(b) $\sum_{n=1}^{\infty} (P_n)_{jt} < \infty$ *for each $j \notin T$ and $t \in T$.*

It follows from Proposition 5.1 that the basis of the chain can be determined without computing the products $P_{k,n}$ once we come up with a simple mechanism to determine the T states. Several theorems in this regard are given along with Proposition 5.1 in [8].

Our second application uncovers an interesting structure of a continuous parameter nonhomogeneous Markov chain with countable states and separately continuous transition probability $P(s, t)$ such that

(i) $P(s, u) P(u, t) = P(s, t), s \leqslant u \leqslant t$;

(ii) $\lim_{s \to t} P(s, t) = I$ (the identity matrix).

PROPOSITION 5.2. *Let $t_n < t_{n+1} \to t$ as $n \to \infty$. Then for $i \neq j$,*

$$\sum_{n=1}^{\infty} P_{ij}(t_n, t_{n+1}) < \infty.$$

Also if $s_n < s_{n-1} \to s$ as $n \to \infty$, then for $i \neq j$,

$$\sum_{n=1}^{\infty} P_{ij}(s_{n+1}, s_n) < \infty.$$

If $\lim_{s \to t} P_{ij}(s, t) = 0$ uniformly in all i (different from j, for each j), then for $t_n < t_{n+1} \to t$, we have: for each i,

$$\sum_{n=1}^{\infty} \sum_{k \neq i} P_{ik}(t_n, t_{n+1}) < \infty.$$

The proof follows immediately from Theorems 3.3, 3.4, and 3.6.

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