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NEWTON-LIKE METHODS FOR THE COMPUTATION OF FIXED POINTS

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Abstract—The celebrated Banach fixed point theorem provides conditions which assure that the method of successive substitution is convergent; the convergence, however, may take place very slowly so that it may be desirable to use a Newton-like method for the computation of the fixed point. If Newton's method itself is applied one ignores the additional information that the problem arises from a fixed point problem with a contraction mapping. In the present note some variants of Newton's method are discussed which make use of this contraction information; it turns out that the convergence of Newton's method can be accelerated without any relevant additional computational labour.

1. INTRODUCTION

Let X be a Banach space, $\bar{x} \in X$ a fixed point of the mapping $F: X \to X$,

$$\bar{x} = F(\bar{x}). \tag{1}$$

The particular form of this equation immediately suggests the method of successive substitution for the iterative computation of $\bar{x}$:

Given a starting point $x_0 \in X$; compute x_{i+1} by

$$x_{i+1} := F(x_i) \quad i = 0,1,2,3,\ldots. \tag{2}$$

The classical fixed point theorem of Banach gives conditions which assure the convergence of the sequence $\{x_i\}$ defined in (2). The convergence may be quite slow however if the contraction constant of F is close to 1; more rapid convergence can be achieved, e.g. by the application of a Newton-like method to the nonlinear equation

$$0 = x - F(x)$$

if F is, say, twice continuously differentiable. In the sequel we investigate the following iterative method:

Given a starting point $x_0 \in X$, $\gamma \in [0, 1]$, compute x_{i+1}:

$$\begin{aligned} y_i &:= F(x_i) \\ x_{i+1} &:= x_i - [I - F'(\gamma x_i + (1-\gamma) y_i)]^{-1}(x_i - F(x_i)) \\ i &= 0,1,2,3,\ldots. \end{aligned} \tag{3}$$

Obviously (3) corresponds to Newton's method if $\gamma = 1$; the choice $\gamma = 0$ leads to the largely unknown method of Stirling (see Rall[3]). Note that (3) requires one evaluation of F and one evaluation of F' per step independent of γ. It is therefore reasonable to ask for an optimal choice of the parameter γ, i.e. a parameter which maximizes the speed of convergence: it turns out that the choice $\gamma = 1/2$ is quite appropriate.

2. CONVERGENCE RESULTS

Let us first give a motivation which gives reasoning to the choice $\gamma = 1/2$.

If F is a contraction on X then, according to Banach's fixed point principle, $F(x_n)$ is a better approximation for $\bar{x}$ than x_n was. Let us now assume that F is continuously differentiable (in the Fréchet sense); then

$$x_n - F(x_n) = \bar{x} - F(\bar{x}) + \int_0^1 [I - F'(\bar{x} + t(x_n - \bar{x}))]\,\mathrm{d}t(x_n - \bar{x})$$

so that

$$\bar{x} = x_n - \left\{\int_0^1 [I - F'(\bar{x} + t(x_n - \bar{x}))]\,\mathrm{d}t\right\}^{-1}(x_n - F(x_n)) \tag{4}$$

(note that the contraction property implies the existence of the inverse in (4)). Approximation of the integral by

(i) I, yields the method of succesive substitutions,
(ii) $I - F'(x_n)$, yields Newton's method,
(iii) $I - F'(F(x_n))$, yields Stirling's method.

If we replace $\bar{x}$ in $\int_0^1[I - F'(\bar{x} + t(x_n - \bar{x}))]\,\mathrm{d}t$ by the best known approximation (which is after having computed the residual $x_n - F(x_n)$, $F(x_n)$) and if we choose the optimal one point quadrature rule, namely the mid-point rule, then we are led to the suggestion that $\gamma = 1/2$ is the appropriate choice of the free parameter γ in (3).

The following result contains theorem 2 and theorem 4 of Rall[3] as a special case:

PROPOSITION 1

Let $F \in C^{1,1}(X)$ be such that $\|F'(x)\| \le \alpha < 1$,

$$\|F'(x) - F'(y)\| \le L\|x - y\| \quad \text{for all } x,y \in X;$$

set

$$\kappa := L(1/2 + \alpha - (1+\alpha)\gamma + \gamma^2)/(1-\alpha),\ \theta := \kappa\|x_0 - \bar{x}\|,$$

where $\bar{x}$ is the unique solution of the equation $x = F(x)$ in X (whose existence is assured by Banach's fixed point principle).

If $\theta < 1$ then the iterative method (3) converges to $\bar{x}$ with Q-order 2 at least:

$$\|x_{n+1} - \bar{x}\| \le \kappa\|x_n - \bar{x}\|^2 \le \theta^{2^{n+1}-1}\|x_0 - \bar{x}\|. \tag{5}$$

Proof. Let $A_n(t) := F'(\bar{x} + t(x_n - \bar{x}))$, $B_n(t) := F'(F(x_n) + t(x_n - F(x_n)))$, $t \in [0, 1]$: then (3) implies that

$$x_{n+1} - \bar{x} = [I - B_n(\gamma)]^{-1}[F(x_n) - F(\bar{x}) - B_n(\gamma)(x_n - \bar{x})].$$

As $\|B_n(\gamma)\| \le \alpha$ the inverse $(I - B_n(\gamma))^{-1}$ exists and satisfies the estimate

$$\|(I - B_n(\gamma))^{-1}\| \le \frac{1}{1-\alpha};$$

furthermore

$$\|F(x_n) - F(\bar{x}) - B_n(\gamma)(x_n - \bar{x})\| = \left\|\int_0^1 (A_n(t) - B_n(\gamma))(x_n - \bar{x})\,\mathrm{d}t\right\|$$

$$\le \left\|\int_0^1 (A_n(t) - A_n(\gamma))(x_n - \bar{x})\,\mathrm{d}t\right\| + \|(A_n(\gamma) - B_n(\gamma))(x_n - \bar{x})\|$$

$$\le L\int_0^1 |t - \gamma|\,\mathrm{d}t\,\|x_n - \bar{x}\|^2 + L\|(1-\gamma)(\bar{x} - F(x_n))\|\,\|x_n - \bar{x}\|$$

$$\le L\left(\gamma^2 - \gamma + \frac{1}{2}\right)\|x_n - \bar{x}\|^2 + L\alpha(1-\gamma)\|x_n - \bar{x}\|^2 \tag{6}$$

so that $\|x_{n+1} - \bar{x}\| \le \kappa \|x_n - \bar{x}\|^2$. A sufficient condition for the convergence of (3) thus is

$$\kappa \|x_0 - \bar{x}\| < 1;$$

the second inequality in (5) then follows by induction. □

Remark

(a) If no additional information concerning the location of the exact solution is known one can use

$$\|x_0 - F(x_0)\|/(1 - \alpha)$$

instead of $\|x_0 - \bar{x}\|$ in the above proposition.

(b) Some typical values for the quantity $\kappa = \kappa(\gamma)$ are:

γ	κ
0 (Stirling's method)	$(1 + 2\alpha)\frac{L}{2(1-\alpha)}$
$\frac{1}{2}$	$\left(\frac{1}{2} + \alpha\right)\frac{L}{2(1-\alpha)}$
$\frac{1}{2}(1+\alpha)$	$\left(\frac{1}{2} + \alpha - \frac{1}{2}\alpha^2\right)\frac{L}{2(1-\alpha)}$
1 (Newton's method)	$\frac{L}{2(1-\alpha)}$

Note that κ is minimal for $\gamma = (1 + \alpha)/2$, i.e. this choice minimizes the error estimates of proposition 1 (but not necessarily the actual error!). □

The asymptotical behaviour of (3) is described in the following.

PROPOSITION 2

If, in addition to the assumptions of proposition1, F is twice continuously differentiable then

$$\limsup_{n\to\infty} \frac{\|x_{n+1} - \bar{x}\|}{\|x_n - \bar{x}\|^2} \le \|(I - F'(\bar{x}))^{-1}\| \, \|F''(\bar{x})\| \, \left\|(1-\gamma)F'(\bar{x}) + \left(\gamma - \frac{1}{2}\right)I\right\| \qquad (7)$$

("$\le$" can be replaced by "$=$" if $X = \mathbb{R}$).

Proof. We prove an asymptotic estimate for

$$F(x_n) - F(\bar{x}) - F'(\gamma x_n + (1-\gamma)F(x_n))(x_n - \bar{x})$$

which is different from (6):

$$F(x_n) - F(\bar{x}) - F'(\gamma x_n + (1-\gamma)F(x_n))(x_n - \bar{x}) = \left\{\int_0^1 \left[F'(\bar{x} + t(x_n - \bar{x})) - F'\left(\frac{x_n + \bar{x}}{2}\right)\right] dt + \left[F'\left(\frac{x_n + \bar{x}}{2}\right) - F'(\gamma x_n + (1-\gamma)F(x_n))\right]\right\}(x_n - \bar{x})$$

Note that the integral-term is of order $o(\|x_n - \bar{x}\|^2)$ since F is twice continuously differentiable; it may be therefore neglected in our asymptotic considerations. From

$$F'\left(\frac{x_n + x}{2}\right) - F'(\gamma x_n + (1-\gamma)F(x_n)) = \int_0^1 F''\left(\gamma x_n + (1-\gamma)F(x_n) + \tau\left(\frac{x_n + \bar{x}}{2} - \gamma x_n - (1-\gamma)F(x_n)\right)\right) d\tau \left(\frac{x_n + \bar{x}}{2} - \gamma x_n - (1-\gamma)F(x_n)\right)$$

and

$$\left(\frac{x_n+\bar{x}}{2}-\gamma x_n-(1-\gamma)F(x_n)\right)=\left(\frac{1}{2}-\gamma\right)(x_n-\bar{x})-(1-\gamma)(F(x_n)-F(\bar{x}))$$

$$=\left\{\left(\frac{1}{2}-\gamma\right)I-(1-\gamma)\int_0^1 F'(\bar{x}+t(x_n-\bar{x}))\,dt\right\}(x_n-\bar{x})$$

one easily concludes the validity of the estimate (7). □

Remark

The asymptotic estimate (7) suggests the following strategy: choose $\gamma \in [0, 1]$ such that $\|(1-\gamma)F'(\bar{x})+(\gamma-1/2)I\|$ is minimal. This minimization problem is easily solvable if $X=\mathbb{R}^m$ and $\|\cdot\|$ is the Frobenius norm of a matrix: $\|M\|_F^2:=tr\,M^TM$. If we set $M:=I-F'(\bar{x})$ then

$$\gamma_{\text{opt}}:=\max\left\{1-\frac{1}{2}\operatorname{tr}M/\operatorname{tr}M^TM,\ 0\right\}.$$

Note that, due to the contraction property of F, $\operatorname{tr} M>0$, so that $\gamma_{\text{opt}}<1$. This strategy thus prefers Stirling's method if $\operatorname{tr}M\le 2\operatorname{tr}M^TM$ which is the case, e.g. if $F'(\bar{x})$ is a symmetric matrix whose spectrum is contained in the interval $[1/2, 1)$. In practice we replace the unknown matrix $F'(\bar{x})$ by the last Jacobian of F which was computed in the course of the iteration, i.e. we use a variable parameter γ instead of a fixed one. Later we refer to this variant of (3) as the *variable γ method*. For other norms or in the infinite dimensional case the choice $\gamma=1/2$ seems to be an adequate compromise. □

The *global* contraction property of F and the *global* Lipschitz continuity of F' are too restrictive in applications; the above results remain true however if the assumption are fulfilled in a ball

$$B(x_0;r):=\{x\in X|\ \|x-x_0\|\le r\}$$

which contains the sequence of iterates generated by (3):

PROPOSITION 3

Let $D\subseteq X$, $F\in C^{1,1}(D)$, $x_0\in D$,

$$\alpha(r):=\sup\{\|F'(x)\|\ |x\in B(x_0;r)\cap D\},$$

$$\|F'(x)-F'(y)\|\le L(r)\|x-y\|\quad\text{for all}\quad x,y\in B(x_0;r)\cap D,$$

with a nondecreasing function L. Assume that there are real numbers r_0, s_0 such that: (i) $F(B(x_0;r_0))\subseteq B(x_0;r_0)$; (ii) $r_0<s_0<2r_0$; (iii) $\alpha(s_0)<1$; (iv) $B(x_0;s_0)\subseteq D$; (v) $(1+\theta(s_0))r_0\le s_0$

where

$$\theta(s):=L(s)\left\{\frac{1}{2}+\alpha(s)-(1+\alpha(s))\gamma+\gamma^2\right\}\frac{r_0}{1-\alpha(s)}.$$

Then the sequence generated by (3) remains in $B(x_0;s_0)$ and converges with Q-order 2 at least to a fixed point $\bar{x}\in B(x_0;r_0)$ of F. $\bar{x}$ is the unique fixed point of F in $B(x_0;s_0)$.

Proof. Banach's fixed point principle implies the existence of a fixed point $\bar{x}\in B(x_0;r_0)$ which is unique in $B(x_0;s_0)$. Assume that x_n, $F(x_n)\in B(x_0;s_0)$; then (compare the proof of proposition 1 and observe, that θ is nondecreasing)

$$\begin{aligned}\|_{n+1}-x_0\| &\le \|x_{n+1}-\bar{x}\|+\|\bar{x}-x_0\|\\ &\le(1+\theta(s_0))r_0\\ &\le s_0,\ \text{i.e. } x_{n+1}\in B(x_0;s_0),\end{aligned}$$

$$
\begin{aligned}
\|F(x_{n+1}) - x_0\| &\leq \|F(x_{n+1}) - F(\bar{x})\| + \|\bar{x} - x_0\| \\
&\leq \alpha(s_0)\|x_{n+1} - \bar{x}\| + \|\bar{x} - x_0\| \\
&\leq (1 + \alpha(s_0)\theta(s_0))r_0 \\
&< s_0.
\end{aligned}
$$

Hence x_{n+1}, $F(x_{n+1})$, $\gamma x_{n+1} + (1-\gamma)F(x_{n+1}) \in B(x_0; s_0)$; from (ii) and (v) one easily derives that $\theta(s_0) < 1$. The assertion then follows similar as in proposition 1. □

3. STEFFENSEN'S METHOD

The motivation for the choice $\gamma = 1/2$ was stimulated by two approximations in (4); $\bar{x}$ was approximated by $F(x_n)$ and the integral was replaced by a quadrature formula, namely the midpoint rule. In the scalar case, however, the approximation of the integral can be avoided since

$$\int_0^1 F'(F(x_n) + t(x_n - F(x_n)))\,\mathrm{d}t = \frac{F(F(x_n)) - F(x_n)}{F(x_n) - x_n}; \tag{8}$$

the resulting iterative method then is the well-known *Steffensen method* (see [6]):

$$x_{n+1} := x_n - \frac{x_n - F(x_n)}{1 - \dfrac{F(F(x_n)) - F(x_n)}{F(x_n) - x_n}}$$

$$n = 0,1,2,3,\ldots.$$

For this method one can improve the estimates of proposition 1:

PROPOSITION 4

Let $F \in C^{1,1}(\mathbb{R})$ be such that $|F'(x)| \leq \alpha < 1$,

$$|F'(x) - F'(y)| \leq L|x - y| \quad \text{for all } x,y \in \mathbb{R}.$$

Set

$$\theta := \frac{1}{2}\frac{\alpha L}{1-\alpha}|x_0 - \bar{x}|$$

where $\bar{x}$ is the unique fixed point of F in $\mathbb{R}$. If $\theta < 1$ then Steffensen's method converges to $\bar{x}$ with Q-order 2 at least:

$$|x_{n+1} - \bar{x}| \leq \frac{1}{2}\frac{\alpha L}{1-\alpha}|x_n - \bar{x}|^2 \leq \theta^{2^{n+1}-1}|x_0 - \bar{x}|.$$

If furthermore F is twice continuously differentiable then

$$\lim_{n\to\infty} \frac{x_{n+1} - \bar{x}}{(x_n - \bar{x})^2} = \frac{1}{2}\frac{F'(\bar{x})}{1 - F'(\bar{x})}F''(\bar{x}).$$

Proof. Use (8) to replace (6) by

$$
\begin{aligned}
&\left|F(x_n) - F(\bar{x}) - \int_0^1 F'(F(x_n) + t(x_n - F(x_n)))\,\mathrm{d}t\,(x_n - \bar{x})\right| = \left|\int_0^1 \{F'(\bar{x} + t(x_n - \bar{x}))\right. \\
&\quad \left. - F'(F(x_n) + t(x_n - F(x_n)))\}\,\mathrm{d}t\,(x_n - \bar{x})\right| \leq \frac{L}{2}|\bar{x} - F(x_n)||x_n - \bar{x}| \leq \frac{1}{2}\alpha L|x_n - \bar{x}|^2.
\end{aligned}
$$

The assertion then follows by arguments similar to those used in proposition 1 and proposition 2. □

Remark

(a) Note that under the above assumptions Steffensen's method locally converges faster than Newton's method.

(b) Obviously proposition 3 is valid for Steffensen's method, too, if the definition of θ is modified appropriately.

4. FIXED POINTS OF NONCONTRACTIVE MAPPINGS

If a fixed point of a noncontractive mapping F is to be computed it is near at hand to apply the previously introduced method (3) to

$$\mathbb{F}(x) := x - (I - F'(x_0))^{-1}(x - F(x));$$

it is well known that $\mathbb{F}$ locally is a contraction if x_0 is a sufficiently good approximation of the solution. In this case (3) can be written as follows:

$$\begin{aligned} &\text{given } x_0, y_0; \text{ compute } x_{n+1}, y_{n+1} \text{ according to} \\ &x_{n+1} := x_n - \{I - F'(\gamma x_n + (1-\gamma) y_n)\}^{-1}(x_n - F(x_n)) \\ &y_{n+1} := x_{n+1} - \{I - F'(x_0)\}^{-1}(x_{n+1} - F(x_{n+1})) \\ &n = 0,1,2,3,\ldots \end{aligned} \tag{9}$$

As y_{n+1} should be as good an approximation for the solution as can be computed without additional function evaluations it is reasonable to replace $(I - F'(x_0))^{-1}$ by

$$(I - F'(\gamma x_n + (1-\gamma) y_n))^{-1}.$$

Note that one must solve two linear equations with the same linear mapping then; this can be done economically if $X = \mathbb{R}^m$: in this case one computes a LR-decomposition of the Jacobian so that both linear systems can be solved by back substitution. If $\gamma = 1/2$ the resulting method is of R-order $1 + \sqrt{2}$ if F is sufficiently smooth:

PROPOSITION 5

Let $F \in C^{1,1}(X)$, $F(\bar{x}) = \bar{x}$, $I - F'(\bar{x})$ nonsingular; then the iterative method given x_0, y_0; compute x_{n+1}, y_{n+1} according to

$$\begin{aligned} x_{n+1} &:= x_n - \left[I - F'\left(\frac{x_n + y_n}{2}\right)\right]^{-1}(x_n - F(x_n)) \\ y_{n+1} &:= x_{n+1} - \left[I - F'\left(\frac{x_n + y_n}{2}\right)\right]^{-1}(x_{n+1} - F(x_{n+1})) \\ n &= 0, 1, 2, 3, \ldots \end{aligned} \tag{10}$$

locally converges Q-quadratically to $\bar{x}$. If furthermore $F \in C^2(X)$ then the convergence is superquadratic; if $F \in C^{2,1}(X)$ then (10) converges locally with R-order $1 + \sqrt{2}$ at least.

For a *proof* we refer to [7].

Remark

If F is contractive then our previous considerations show that the choice $y_0 := F(x_0)$ is appropriate; otherwise use $y_0 := x_0$. □

Now we restrict our interest to the case $X = \mathbb{R}^m$; the quality of y_{n+1} depends on how good $I - F'(x_n + y_n)/2$ approximates $I - F'(x_n + \bar{x})/2$. We therefore apply a rank one correction:

$$y_{n+1}:=x_{n+1}-\left[I-F'\left(\frac{x_n+y_n}{2}\right)+u_nv_n^T\right]^{-1}(x_{n+1}-F(x_{n+1}))$$

where u_n, $v_n\in\mathbb{R}^m$ are determined such that

$$\left(I-F'\left(\frac{x_n+y_n}{2}\right)+u_nv_n^T\right)(x_{n+1}-x_n)=(x_{n+1}-F(x_{n+1}))-(x_n-F(x_n));$$

u_n and v_n are chosen according to Broyden's method (see Schwetlick[5], p. 139 ff., Broyden[2]):

$$u_n=x_{n+1}-F(x_{n+1})$$

$$v_n:=\frac{x_{n+1}-x_n}{(x_{n+1}-x_n)^T(x_{n+1}-x_n)}.$$

Using the Sherman–Morrison inversion formula we then get the *updated* $1+\sqrt{2}$ *order method*,

given x_0, y_0; compute x_{n+1}, y_{n+1} as follows:

$$\begin{aligned} x_{n+1}&=x_n-\left[I-F'\left(\frac{x_n+y_n}{2}\right)\right]^{-1}(x_n-F(x_n))\\ y_{n+1}&=x_{n+1}-\mu_n\left[I-F'\left(\frac{x_n+y_n}{2}\right)\right]^{-1}(x_{n+1}-F(x_{n+1}))\end{aligned}\tag{11}$$

where

$$\mu_n:=\left(v_n^T\left[I-F'\left(\frac{x_n+y_n}{2}\right)\right]^{-1}(x_{n+1}-F(x_{n+1}))+1\right)^{-1}.$$

The following result justifies this modification:

LEMMA 6

Let $A_n:=I-F'\left(\frac{x_n+y_n}{2}\right)$, $B_n:=\int_0^1\left[I-F'(x_n+t(x_{n+1}-x_n))\right]dt$, $F(x_{n+1})\neq x_{n+1}$;

then $\|A_n+u_nv_n^T-B_n\|_F<\|A_n-B_n\|_F$.

Proof. If $\bar{A}:=A+(y-As)s^T/s^Ts$, $A\in\mathbb{R}^{m,m}$, y, $s\in\mathbb{R}^m$, $s\neq 0$, then for any $B\in\mathbb{R}^{m,m}$:

$$\|\bar{A}-B\|_F^2=\|A-B\|_F^2-\frac{\|(A-B)s\|_2^2}{s^Ts}+\frac{\|y-Bs\|_2^2}{s^Ts}$$

(see Schwetlick[5], p. 142; Broyden[2], lemma 4). Set $A:=A_n$,

$$y:=(x_{n+1}-F(x_{n+1}))-(x_n-F(x_n)),$$

$B:=B_n$, $s:=x_{n+1}-x_n$; then $y-As=x_{n+1}-F(x_{n+1})$, $\bar{A}=A+u_nv_n^T$, $y-Bs=0$, $(A-B)s=-(x_{n+1}-F(x_{n+1}))\neq 0$, so that

$$\|A_n+u_nv_n^T-B_n\|_F^2=\|A_n-B_n\|_F^2-\frac{\|x_{n+1}-F(x_{n+1})\|_2^2}{s^Ts}<\|A_n-B_n\|_F^2.$$

Remark

In the scalar case, i.e. $X = \mathbb{R}$, the variable γ method introduced previously coincides with the above $1+\sqrt{2}$ order method as long as the parameters γ which are used remain positive. The updated $1+\sqrt{2}$ order method reads in this case:

$$x_{n+1} = x_n - \frac{x_n - F(x_n)}{1 - F'\left(\dfrac{x_n + y_n}{2}\right)}$$

$$y_{n+1} = x_{n+1} - \frac{x_{n+1} - F(x_{n+1})}{\dfrac{(x_{n+1} - F(x_{n+1})) - (x_n - F(x_n))}{x_{n+1} - x_n}}$$

$$n = 0,1,2,3,\ldots.$$

The order of convergence of the $1+\sqrt{2}$ order method is not affected by the updating procedure.

5. AN APPLICATION

Consider the nonlinear integral equation

$$x(s) = 1 - \frac{1}{2}\lambda \int_0^1 \frac{s}{t + s\,x(t)}\,\frac{1}{}\mathrm{d}t, \quad s \in [0,\ 1],\ \lambda \in [0,\ 1] \text{ fixed}, \tag{12}$$

in the space $C([0,\ 1])$ of continuous functions equipped with the sup-norm $\|\cdot\|$. Note that (12) is a version of the so called H-equation which arises in the theory of radiative transfer (see Rall[4], p. 74 ff. and the references given there). In the sequel we use the following notations:

(i) $F\colon D \to C([0,\ 1]),\ D := \{x \in C([0,\ 1]),\ x \text{ positive}\}$

$$[F(x)](s) := 1 - \frac{1}{2}\lambda \int_0^1 \frac{s}{t + s\,x(t)}\,\mathrm{d}t$$

(ii) $$\mu := \frac{1}{2}\lambda \ln(2)$$

(iii) $$r_0 := \frac{1}{2} - \sqrt{\frac{1}{4} - \mu}.$$

(iv) $x_0(s) = 1 \quad \text{for } s \in [0,\ 1].$

For the application of proposition 3 we need the quantities α, L, $\|x_0 - F(x_0)\|$:

LEMMA 7

If $\mu < 1/4$ then the following assertions are valid:

(a) (12) has a solution in the ball $B(x_0; r_0)$ which is unique in the interior of $B(x_0; 1 - \sqrt{\mu})$.

(b) $\|F'(x)\| \le \mu/(1-r)^2 =: \alpha$ if $x \in B(x_0; r)$, $r < 1$.

(c) $\|F'(x) - F'(y)\| \le 2\mu/(1-r)^3\, \|x - y\|$ if $x, y \in B(x_0; r)$, $r < 1$.

(d) $\|x_0 - F(x_0)\| = \mu$.

Proof. Let $x \in B(x_0; r)$, $r_0 \le r \le 1/2 + \sqrt{1/4 - \mu}$; then $x(t) \ge 1 - r$ for $t \in [0,\ 1]$.

Hence

$$|1 - [F(x)](s)| = \frac{1}{2}\lambda \int_0^1 \frac{s}{t + s\,x(t)}\,\mathrm{d}t \le \frac{\mu}{1-r} \le r,$$

i.e. F maps $B(x_0; r)$ into itself for all r in the indicated range.

If $h \in C([0, 1])$, then

$$[F'(x)h](s) = -\frac{1}{2}\lambda \int_0^1 \frac{s}{t+s}\frac{h(t)}{x(t)^2}\,dt$$

so that $\alpha(r) := \mu/(1-r)^2$, $L(r) := 2\mu/(1-r)^3$; F is a contraction for $r \in [r_0, 1-\sqrt{\mu})$.

Numerical example

(a) Let us apply proposition 3 in the case $\lambda = 1/2$; then $\mu = \ln(2)/4 \doteq 0.1733$, $r_0 \doteq 0.2231$. We show that the assumptions of proposition 3 are fulfilled for

$$s_0 := 1.3\ r_0 \doteq 0.2899:$$

as $L(s_0) \doteq 0.9681$, $\alpha(s_0) \doteq 0.3437$ we get

$$\theta(s_0) \leq 0.2777.$$

Hence $(1 + \theta(s_0))r_0 \leq 0.2851 < s_0$, so that by proposition 3 the iterative method (3) converges for any $\gamma \in [0, 1]$ to a solution of (12).

(b) For the numerical computations we replaced the integral in (12) by the composite trapezoidal rule with mesh size $1/m$, i.e. (12) is replaced by a finite dimensional fixed point problem

$$x = F_m(x)$$

whose solution is denoted by $(\bar{x}^{(0)}, \bar{x}^{(1)}, \ldots, \bar{x}^{(m)})^T$. The following tables contain the errors

$$\max_{0 \leq i \leq m} \left|x_n^{(i)} - \bar{x}^{(i)}\right|$$

for the iterates $(x_n^{(0)}, \ldots, x_n^{(m)})^T$ generated by various iterative methods:

I	Stirling's method
II	(3) with $\gamma = \frac{1}{2}$
III	Newton's method
IV	the variable γ method
V	the $1+\sqrt{2}$ order method (10)
VI	the updated $1+\sqrt{2}$ order method (11)

In V, VI we used $y_0 := F(x_0)$ as an additional starting value.

Table 1. $\lambda = \frac{1}{2}$, $m = 20$, $x_0^{(i)} = 1$, $i = 0\ (1)\ 20$

n	I	II	III	IV	V	VI
1	$4.18_{10}-3$	$9.23_{10}-4$	$4.80_{10}-3$	$9.23_{10}-4$	$9.23_{10}-4$	$9.23_{10}-4$
2	$1.98_{10}-6$	$2.07_{10}-8$	$3.69_{10}-6$	$1.99_{10}-8$	$1.41_{10}-8$	$1.36_{10}-8$
3	$4.33_{10}-13$	$1.04_{10}-17$	$2.17_{10}-12$	$9.11_{10}-18$	$2.71_{10}-18$	$2.49_{10}-18$

The variable γ method IV started with $\gamma_0 := 0.5$; then $\gamma_1 = 0.4965$ and $\gamma_2 = 0.4957$ were computed.

For a less accurate initial value x_0 we obtained the following results:

Table 2. $\lambda = 1/2$, $m = 20$, $x_0^{(i)} = 1.2$, $i = 0(1)20$

n	I	II	III	IV	V	VI
1	$1.85_{10}-2$	$4.11_{10}-3$	$1.75_{10}-2$	$4.11_{10}-3$	$4.11_{10}-3$	$4.11_{10}-3$
2	$4.11_{10}-5$	$4.24_{10}-7$	$5.03_{10}-5$	$4.11_{10}-7$	$3.19_{10}-7$	$3.01_{10}-7$
3	$1.88_{10}-10$	$4.33_{10}-15$	$4.05_{10}-10$	$3.87_{10}-15$	$1.41_{10}-15$	$1.25_{10}-15$

The numerical results do agree quite well with our previous analysis which showed that Newton's method is not optimal for the computation of fixed points of contractive mappings.

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