A new fractional modeling arising in engineering sciences and its analytical approximate solution

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Abstract The aim of this article is to introduce a new approximate method, namely homotopy perturbation transform method (HPTM) which is a combination of homotopy perturbation method (HPM) and Laplace transform method (LTM) to provide an analytical approximate solution to time-fractional Cauchy-reaction diffusion equation. Reaction diffusion equation is widely used as models for spatial effects in ecology, biology and engineering sciences. A good agreement between the obtained solution and some well-known results has been demonstrated. The numerical solutions obtained by proposed method indicate that the approach is easy to implement and accurate. Some numerical illustrations are given. These results reveal that the proposed method is very effective and simple to perform for engineering sciences problems.

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1. Introduction

The subject of fractional calculus (integral and derivatives of any arbitrary real or complex) was planted over 300 years ago. The theory of derivative and integrals of non-integer order goes back to Liouville, Leibnitz, Grunwald, Reimann and Letnikov. In the recent years, fractional calculus has played a very significant role in many areas in fluid flow, mechanics, viscoelasticity, biology, physics, science and engineering, and other applications [1–4]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [3–6].

The Cauchy-reaction diffusion equations describe a wide variety of nonlinear systems in physics, chemistry, ecology, biology and engineering [7–10]. Recently, Yildirim [11] have applied to obtain the solutions of the Cauchy-reaction diffusion equations by using homotopy perturbation method. The main aim of this article presents approximate analytical solutions of fractional model of Cauchy-reaction diffusion equations with fractional time derivative \( \alpha \) (0 < \( \alpha \) ≤ 1) in the form of a rapidly convergent series with easily computable components by using new homotopy perturbation transform method. Using the initial condition, the approximate analyti-
are He’s polynomials of \( C_0 \): by manipulating the homotopy perturbation method. The solutions of the linear and nonlinear fractional differential equations can be used to find analytical and approximate solutions. Therefore, there have been attempts to develop the new methods for obtaining analytical and approximate solutions of nonlinear fractional ordinary and partial differential equations. Recently, several methods have drawn special attention such as Adomian decomposition method [12–14], Variational iteration method [15,16], Homotopy analysis method [17–20], Differential transform method [21,22], Wavelet methods [23,24], and Homotopy perturbation method [25–36].

The main aim of this article is to illustrate how the Laplace transform can be used to find analytical and approximate solutions of the linear and nonlinear fractional differential equations by manipulating the homotopy perturbation method. The homotopy perturbation method introduced and applied by He [25–29]. Recently, many researchers [30–36] have obtained the series solution of the fractional differential equations and integral equation by using HPM. The proposed method is coupling of the Laplace transformation, the homotopy perturbation method and He’s polynomials mainly due to Ghorbani [37,38]. In the recent years, many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods with combined the Laplace transform. Among these are the Laplace decomposition methods [39,40], homotopy perturbation transform method [41–45].

**Definition 1.1.** The Laplace transform of function \( f(t) \) is defined by

\[
F(s) = L[f(t)] = \int_0^\infty e^{-st}f(t)dt. \tag{1.1}
\]

**Definition 1.2.** The Laplace transform \( L[f(t)] \) of the Riemann–Liouville fractional integral is defined as \[2\]

\[
L[I^s f(t)] = s^{-a}F(s). \tag{1.2}
\]

**Definition 1.3.** The Laplace transform \( L[f(t)] \) of the Caputo fractional derivative is defined as \[2\]

\[
L[D^s_{a}f(t)] = s^aF(s) - \sum_{k=0}^{n-1} s^{a-k-1}f^{(k)}(0), \quad n - 1 < n \alpha \leq n. \tag{1.3}
\]

**Definition 1.4.** The Mittag–Leffler function \( E_{\alpha}(z) \) with \( z > 0 \) is defined by the following series representation, valid in the whole complex plane [46]:

\[
E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}. \tag{1.4}
\]

### 2. Basic idea of newly fractional homotopy perturbation transform method

In order to elucidate the solution procedure of the fractional homotopy perturbation transform method, we consider the following nonlinear fractional differential equations:

\[
D^s_{a^+}u(x, t) + R[x]u(x, t) + N[x]u(x, t) = q(x, t), \quad t > 0,
\]

\[
\in \mathbb{R}, \quad n - 1 < n \alpha \leq n, u(x, 0) = h(x), \tag{2.1}
\]

where \( D^s_{a^+} \) is the linear operator in \( x \), \( N[x] \) is the general nonlinear operator in \( x \), and \( q(x, t) \) is continuous function. Now, the methodology consists of applying Laplace transform on both sides of Eq. (2.1), we get

\[
L[D^s_{a^+}u(x, t)] + L[R[x]u(x, t) + N[x]u(x, t)] = L[q(x, t)]. \tag{2.2}
\]

Now, using the differentiation property of the Laplace transform, we have

\[
L[u(x, t)] = s^{-a}h(x) + s^{-n}L[q(x, t)] - s^{-n}L[R[x]u(x, t)] + N[x]u(x, t)]. \tag{2.3}
\]

Operating the inverse Laplace transform on both sides in Eq. (2.3), we get

\[
u(x, t) = \sum_{n=0}^{\infty} p^n u_n(x, t), \tag{2.5}
\]

where the homotopy parameter \( p \) is considered as a small parameter \((0 < p < 0, 1])\). The nonlinear term can be decomposed as

\[
N[u(x, t)] = \sum_{n=0}^{\infty} p^n H_n(u), \tag{2.6}
\]

where \( H_n \) are He’s polynomials of \( u_0, u_1, u_2, \ldots, u_n \) and it can be calculated by the following formula

\[
H_n(u_0, u_1, u_2, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ N\left( \sum_{n=0}^{\infty} p^n u_i \right) \right] \bigg|_{p=0}, \quad n = 0, 1, 2, \ldots .
\]

Substituting Eqs. (2.5) and (2.6) in Eq. (2.4) and using HPM [25–29], we get

\[
\sum_{n=0}^{\infty} p^n u_n(x, t) = G(x, t) - pL^{-1} \left[ s^{-n} L \left( R\sum_{n=0}^{\infty} p^n u_n(x, t) + \sum_{n=0}^{\infty} p^n H_n(u) \right) \right]. \tag{2.7}
\]

This is coupling of the Laplace transform and homotopy perturbation method by using He’s polynomials. Now, equating the coefficient of corresponding power of \( p \) on both sides, the following approximations are obtained as

\[
p^0 : u_0(x, t) = G(x, t),
\]

\[
p^1 : u_1(x, t) = L^{-1}(s^{-n} L[R[x]u_0(x, t) + H_0(u)]),
\]

\[
p^2 : u_2(x, t) = L^{-1}(s^{-n} L[R[x]u_1(x, t) + H_1(u)]),
\]

\[
p^3 : u_3(x, t) = L^{-1}(s^{-n} L[R[x]u_2(x, t) + H_2(u)]).
\]

Proceeding in this same manner, the rest of the components \( u_n(x, t) \) for all \( n > 3 \) can be completely obtained and the series solution is thus entirely determined.
Finally, we approximate the analytical solution \( u(x, t) \) by truncated series

\[
u(x, t) = \lim_{N \to -\infty} \sum_{n=0}^{N} u_n(x, t).
\]  

(2.8)

The above series solutions generally converge very rapidly.

3. Numerical examples

In this section, four examples of time-fractional Cauchy-reaction diffusion equations are solved to demonstrate the performance and efficiency of the HPM with coupling of Laplace transform and Hé’s polynomials.

Example 1. We consider the following linear time-fractional linear Cauchy-reaction diffusion equation \([11]\) as follows:

\[
D^\alpha_t w(x, t) = D^{\beta}_x w(x, t) - w(x, t) \\
0 < \alpha, \beta \leq 1, \quad (x, t) \in \Omega \subset \mathbb{R}^2,
\]

(3.1)

with initial and boundary conditions

\[
w(x, 0) = e^{-x} + x = g(x), \quad w(0, t) = f_0(t),
\]

\[
\frac{\partial w(0, t)}{\partial n} = e^{-t} - 1 = f_1(t), \quad x, t \in \mathbb{R}.
\]

(3.2)

The methodology consists of applying Laplace transform first on both sides in Eq. (3.1) and using the differentiation property of Laplace transform, we get

\[
L[w(x, t)] = s^{-\alpha}(e^{-x} + x) + s^{-\beta}L[D^\beta_x w - w].
\]

(3.3)

The inverse Laplace transform on both sides implies that

\[
w(x, t) = (e^{-x} + x) + L^{-1}(s^{-\beta}L[D^\beta_x w - w]).
\]

(3.4)

Now, we apply the homotopy perturbation method \([25-29]\), we get

\[
\sum_{n=0}^{\infty} p^n w_n(x, t) = (e^{-x} + x) + p \left[ L^{-1} \left( s^{-\beta}L \left( \sum_{n=0}^{\infty} p^n H_n(w) \right) \right) \right],
\]

(3.5)

where \( H_n(w) \) are Hé’s polynomials \([36,37] \) that represent the nonlinear terms. The components of Hé’s polynomials, for given example are obtained by recursive relation \( H_0(w) = D^\beta_x w - w, \forall n \in N \). Now, by equating the coefficient of corresponding power of \( p \) on both sides, the following approximations are obtained as follows:

\[
p^0 : w_0(x, t) = e^{-x} + x,
\]

\[
p^1 : w_1(x, t) = L^{-1}(s^{-\beta}L[H_0(w)]) = x \left( \frac{-p^0}{\Gamma(\alpha+1)} \right),
\]

\[
p^2 : w_2(x, t) = L^{-1}(s^{-\beta}L[H_1(w)]) = x \left( \frac{-p^0}{\Gamma(2\alpha+1)} \right),
\]

\[
p^3 : w_3(x, t) = L^{-1}(s^{-\beta}L[H_2(w)]) = x \left( \frac{-p^0}{\Gamma(3\alpha+1)} \right),
\]

\[
\vdots
\]

\[
p^\alpha : w_\alpha(x, t) = L^{-1}(s^{-\beta}L[H_{\alpha-1}(w)]) = x \left( \frac{-p^0}{\Gamma(\alpha \alpha+1)} \right).
\]

Then the series solution expression by HPM can be written in the form:

\[
w(x, t) = \sum_{n=0}^{\infty} p^n w_n(x, t).
\]

(3.6)

Using the above terms, the solution \( w(x, t) \) is given as

\[
w(x, t) = e^{-x} + x \left( \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \right).
\]

Finally, we approximate the analytical solution \( w(x, t) \) by truncated series

\[
\lim_{n \to \infty} \sum_{n=0}^{N} w_n(x, t).
\]

(3.7)

subject to the initial condition \( w(x, 0) = e^{-x} \) and exact solution \( w(x, t) = e^{x^2t}. \)

By applying the aforementioned homotopy perturbation method, we have

\[
\sum_{n=0}^{\infty} p^n w_n(x, t) = e^{x^2} + p \left[ L^{-1} \left( s^{-\beta}L \left( \sum_{n=0}^{\infty} p^n H_n(w) \right) \right) \right].
\]

(3.8)

Equating the coefficient of like power of \( p \) on both sides, we get

\[
p^0 : w_0(x, t) = e^{x^2},
\]

\[
p^1 : w_1(x, t) = L^{-1}(s^{-\beta}L[H_0(w)]) = e^{x^2} \left( \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \right),
\]

\[
p^2 : w_2(x, t) = L^{-1}(s^{-\beta}L[H_1(w)]) = e^{x^2} \left( \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} \right),
\]

\[
p^3 : w_3(x, t) = L^{-1}(s^{-\beta}L[H_2(w)]) = e^{x^2} \left( \frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)} \right),
\]

\[
\vdots
\]

\[
p^\alpha : w_\alpha(x, t) = L^{-1}(s^{-\beta}L[H_{\alpha-1}(w)]) = e^{x^2} \left( \frac{\Gamma(\alpha \alpha+1)}{\Gamma(\alpha \alpha+1)} \right).
\]

(3.9)

Now, the approximate solution in a series is given as

\[
w(x, t) = \lim_{n \to \infty} \sum_{n=0}^{\infty} p^n w_n(x, t) = e^{x^2} \left( 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \right).
\]

Now, the standard case \( \alpha = 1 \), this series has the closed form of the solution \( w(x, t) = e^{x^2t} \), which is an exact solution of the given Cauchy-reaction diffusion Eq. (3.7) for \( \alpha = 1 \) and complete agreement with Yildirim \([11]\).

Example 3. We consider the following time-fractional Cauchy-reaction diffusion equation \([11]\) as follows:

\[
D^\theta_t w(x, t) = D^{\beta}_x w(x, t) - \left( 1 + 4x^2 \right) w(x, t) \\
0 < \theta, \beta \leq 1, \quad (x, t) \in \Omega \subset \mathbb{R}^2.
\]

(3.10)

subject to the initial condition \( w(x, 0) = e^x \) and exact solution \( w(x, t) = e^{x^2t}. \)

By applying the aforementioned homotopy perturbation method, we have

\[
\sum_{n=0}^{\infty} p^n w_n(x, t) = e^{x^2} + p \left[ L^{-1} \left( s^{-\beta}L \left( \sum_{n=0}^{\infty} p^n H_n(w) \right) \right) \right].
\]

(3.11)

Equating the coefficient of like power of \( p \) on both sides, we get

\[
p^0 : w_0(x, t) = e^{x^2},
\]

\[
p^1 : w_1(x, t) = L^{-1}(s^{-\beta}L[H_0(w)]) = e^{x^2} \left( \frac{\Gamma(\beta+1)}{\Gamma(2\beta+1)} \right),
\]

\[
p^2 : w_2(x, t) = L^{-1}(s^{-\beta}L[H_1(w)]) = e^{x^2} \left( \frac{\Gamma(2\beta+1)}{\Gamma(3\beta+1)} \right),
\]

\[
p^3 : w_3(x, t) = L^{-1}(s^{-\beta}L[H_2(w)]) = e^{x^2} \left( \frac{\Gamma(3\beta+1)}{\Gamma(4\beta+1)} \right),
\]

\[
\vdots
\]

\[
p^\beta : w_\beta(x, t) = L^{-1}(s^{-\beta}L[H_{\beta-1}(w)]) = e^{x^2} \left( \frac{\Gamma(\beta \beta+1)}{\Gamma(\beta \beta+1)} \right).
\]

(3.12)

Now, the approximate solution in a series is given as

\[
w(x, t) = \lim_{n \to \infty} \sum_{n=0}^{\infty} p^n w_n(x, t) = e^{x^2} \left( 1 + \frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \cdots \right).
\]

(3.13)

Now for the standard case \( \beta = 1 \), this series has the closed form of the solution \( w(x, t) = e^{x^2t} \), which is an exact solution of the given Cauchy-reaction diffusion Eq. (3.7) for \( \beta = 1 \) and complete agreement with Yildirim \([11]\).
subject to the initial condition \( w(x,0) = e^{x^2} \) and exact solution \( w(x,t) = e^{e^{xt^2}} \).

By applying the aforesaid method subject to initial condition, we have
\[
w(x,t) = e^{x^2} + L^{-1}(s^{-2} L[w_0 + 2tw]).
\]

Now, we apply homotopy perturbation method [25–29], we have
\[
\sum_{n=0}^{\infty} p^n w_n(x,t) = e^{x^2} + p \left( L^{-1} \left( s^{-2} L \left[ \sum_{n=0}^{\infty} p^n H_n(w) \right] \right) \right).
\]

Comparing the coefficient of like power of \( p \) on both sides, we get
\[
p^0 : w_0(x,t) = e^{x^2},
\]
\[
p^1 : w_1(x,t) = L^{-1}(s^{-2} L[H_0(w)]) = e^{x^2} \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2t^{2\alpha+1}}{\Gamma(\alpha + 2)} \right),
\]
\[
p^2 : w_2(x,t) = L^{-1}(s^{-2} L[H_1(w)]) = e^{x^2} \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{2(2+\alpha)t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + \frac{4(2+\alpha)t^{2\alpha+2}}{\Gamma(2\alpha + 3)} \right),
\]
\[
p^3 : w_3(x,t) = L^{-1}(s^{-2} L[H_2(w)]) = e^{x^2} \left( \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{6(\alpha+1)t^{2\alpha+1}}{\Gamma(\alpha + 2)} + \frac{4(2+\alpha)(2\alpha+3)t^{2\alpha+2}}{\Gamma(3\alpha+3)} + \frac{8(\alpha+2)(2\alpha+3)t^{2\alpha+3}}{\Gamma(3\alpha+4)} \right).
\]

Proceeding in this manner, the rest of the components \( w_n(x,t) \) for \( n \geq 4 \) can be completely obtained and the series solutions are thus entirely determined. Finally, we approximate the analytical solution \( w(x,t) \) by truncated series
\[
w(x,t) = \lim_{N \to \infty} \sum_{n=1}^{N} w_n(x,t).
\]

Now for the standard case \( \alpha = 1 \), this series has the closed form of the solution \( w(x,t) = e^{e^{xt^2}} \), which is an exact solution of the given Cauchy-reaction diffusion Eq. (3.10) for \( \alpha = 1 \). The above result is in complete agreement with Yıldırım [11].

**Example 4.** In this example, we consider the following time-fractional Cauchy-reaction diffusion equation [11] as follows:
\[
D_t^\alpha w(x,t) = D_x^\alpha w(x,t) - (4x^2 - 2t + 2)w(x,t) \quad 0 < \alpha \leq 1,
\]
subject to the initial condition \( w(x,0) = e^{x^2} \) and exact solution \( w(x,t) = e^{e^{xt^2}} \).

By applying the aforesaid homotopy perturbation method, we have
\[
\sum_{n=0}^{\infty} p^n w_n(x,t) = e^{x^2} + p \left( L^{-1} \left( s^{-2} L \left[ \sum_{n=0}^{\infty} p^n H_n(w) \right] \right) \right).
\]

Equating the coefficient of like power of \( p \) on both sides, we get
\[
p^0 : w_0(x,t) = e^{x^2},
\]
\[
p^1 : w_1(x,t) = L^{-1}(s^{-2} L[H_0(w)]) = 2e^{x^2} \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)},
\]
\[
p^2 : w_2(x,t) = L^{-1}(s^{-2} L[H_1(w)]) = 2^2 e^{x^2} \frac{(2+\alpha)t^{2\alpha+1}}{\Gamma(2\alpha + 3)},
\]
\[
p^3 : w_3(x,t) = L^{-1}(s^{-2} L[H_2(w)]) = 2^3 e^{x^2} \frac{(2+\alpha)(2\alpha+3)t^{2\alpha+2}}{\Gamma(3\alpha + 4)}.
\]

In the similar way, the rest of the components \( w_n(x,t) \) for \( n \geq 4 \) can be completely obtained and the series solutions are thus entirely determined. Finally, we approximate the analytical solution \( w(x,t) \) by truncated series
\[
w(x,t) = \lim_{N \to \infty} \sum_{n=1}^{N} w_n(x,t).
\]

Now for the standard case \( \alpha = 1 \), this series has the closed form of the solution \( w(x,t) = e^{e^{xt^2}} \), which is an exact solution of the given Cauchy-reaction diffusion Eq. (3.10) for \( \alpha = 1 \). The above result is in complete agreement with Yıldırım [11].

**4. Numerical result and discussion**

In this section, we have discussed the behavior of the approximate solutions for different values \( \alpha \) for examples 1–4. The error analysis between the exact solution and approximate solution obtained by the present method is depicted through the figures.

Fig. 1 shows approximate solution for different fractional Brownian motions \( \alpha = 0.7, 0.8, 0.9 \) and also for the standard motion \( \alpha = 1 \). It is shown in Fig. 1 that the solutions obtained by present method (HPTM) decrease very rapidly with the increase in \( t \) at the value of \( x = 1 \). The accuracy of the result can be improved by introducing more terms of the approximate solutions.

Figs. 2 and 3 show the same behavior of the approximate solutions \( w(x,t) \) for different values of \( \alpha = 0.7, 0.8, 0.9 \) and for standard Cauchy-reaction diffusion equation i.e. at \( \alpha = 1 \) for both example 2 and 3. It is shown in Figs. 2 and 3 that the solutions obtained by present method (HPTM) increase very rapidly with the increase in \( x \) at the value of \( t = 1 \).

The simplicity and accuracy of the proposed method for example 4 are illustrated by computing the absolute errors \( E_n(x,t) = |w(x,t) - \hat{w}_n(x,t)| \), where \( w(x,t) \) is the exact solutions and \( \hat{w}_n(x,t) \) is approximate solutions obtained by present
method (HPTM) by truncating the respective solution series (3.16) at the level \(n = 10\). Fig. 4 represents the absolute errors between the exact solution and approximate solution obtained by homotopy perturbation transform method for example 4. Here, during the all numerical computation for example 4 only ten order term of the series solution is considered. It is shown in Fig. 4 that the analytical solution obtained by the present method is nearly identical to the exact solution of the standard Cauchy-reaction diffusion equation i.e. for the standard motion \(\alpha = 1\). It achieves a high level of accuracy in only ten order term of approximations. The behavior of the approximate solutions depicted through graphically.

Fig. 5 shows the behavior of the approximate solution \(w(x, t)\) for different values of \(\alpha = 0.7, 0.8, 0.9\) and for standard Cauchy-reaction diffusion equation i.e. at \(\alpha = 1\) for example 4. It is shown in Fig. 5 that the solution obtained by present method (HPTM) increases very rapidly with the increases in \(x\) at the value of \(t = 1\). The accuracy of the result can be improved by introducing more terms of the approximate solutions.

5. Concluding remarks

In this paper, the homotopy perturbation transform method is applied to obtain approximate analytical solutions of the time-fractional partial differential equations such as a Cauchy-reaction diffusion equation, which is widely used as models for spatial effects in ecology and engineering sciences. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that
the method is capable of reducing the volume of the computational work as compared to the classical methods with high accuracy of the numerical result and will considerably benefit mathematicians and scientists working in the field of partial differential equations. The main advantage of the method is its fast convergence to the solution. The numerical results obtained here, conform to its high degree of accuracy. It may be concluded that the HPTM methodology is very powerful and efficient in finding approximate solutions as well as analytical solutions. The computations associated with the example in this paper are performed using Mathematica 7.

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