Integral averaging technique for the interval oscillation criteria of certain second-order nonlinear differential equations

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Received 31 July 2003
Available online 23 September 2004
Submitted by R.E. O’Malley, Jr.

Abstract

We present new interval oscillation criteria related to integral averaging technique for certain classes of second-order nonlinear differential equations which are different from most known ones in the sense that they are based on the information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line. They generalize and improve some known results. Examples are also given to illustrate the importance of our results.

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Keywords: Interval oscillation criteria; Second order; Nonlinear differential equations; Integral averaging technique

1. Introduction

In this paper, we consider the oscillation behavior of solutions of the second-order nonlinear differential equation

$$\left(r(t)\Psi\left(y(t)\right)|y'(t)|^{\alpha-2}y'(t)\right)' + q(t)f\left(y(t)\right)g\left(y'(t)\right) = 0,$$

(1.1)

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where \( t \geq t_0 \), and functions \( r, \Psi, q, f \) and \( g \) are to be specified in the subsequent text and \( \alpha > 1 \) is a constant.

We recall that a function \( y : [t_0, t_1) \to (-\infty, \infty), t_1 > t_0 \), is called a solution of Eq. (1.1) if \( y(t) \) satisfies Eq. (1.1) for all \( t \in [t_0, t_1) \). In the sequel, it will be always assumed that solutions of Eq. (1.1) exist for any \( t_0 \geq 0 \). A solution \( y(t) \) of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

In the last decades, there has been an increasing interest in obtaining sufficient conditions for oscillation and/or nonoscillation of solutions for different classes of second-order differential equations [1–30]. There are great numbers of papers devoted to the particular cases of Eq. (1.1) such as the linear equation

\[
y''(t) + q(t)y(t) = 0, \tag{1.2}
\]

the half-linear equation

\[
(r(t)|y''(t)|^{\alpha - 2}y'(t))^\prime + q(t)|y(t)|^{\alpha - 2}y(t) = 0, \tag{1.3}
\]

the nonlinear equation

\[
(r(t)y'(t))' + q(t)f(y(t)) = 0 \tag{1.4}
\]

and the more general nonlinear equation

\[
(r(t)y'(t))' + q(t)f(y(t))g(y'(t)) = 0. \tag{1.5}
\]

Equation (1.5) was studied by Grace and Lalli [5]. Grace and Lalli [5] mentioned that, though the stability, boundedness, and convergence to zero of solutions Eq. (1.5) with \( r(t) \equiv 1 \) were investigated by Burton and Grimmer [2], Graef and Spikes [6,7], Lalli [14] and Wong and Burton [27], not much was known about the oscillatory behavior of Eq. (1.5) with \( r(t) \equiv 1 \) except for the results of Wong and Burton [27, Theorem 4] concerning the oscillatory behavior of Eq. (1.5) with \( r(t) \equiv 1 \) in connection with that of the corresponding linear equation (1.2).

In recent years, the so-called “interval criteria” for oscillation have been developed in several directions. One approach was initiated by Kong [10] for Eq. (1.3) with \( \alpha = 2 \), using the Riccati technique and the kernel functions of Philos’ type. These results have been extended by him to half-linear equations and linear systems of equations in [11,12], respectively. Later, further development of the “interval criteria” for oscillation have been obtained by many authors for both differential equations and delay differential equations in the same direction. Recently, Li and Agarwal [17] for Eq. (1.3) with a nondecreasing function \( r \in C^1([t_0, \infty) \cup (0, \infty)) \), Li and Huo [23] for Eq. (1.4), Li and Agarwal [19] and Rogovchenko [25] for Eq. (1.5) with \( r(t) \equiv 1 \) have presented new sufficient conditions that guarantee the oscillatory character. More recently, Li and Agarwal’s result [19] was extended by them to Eq. (1.5) in [20]. They are different from those of Grace and Lalli [5] and are applicable to other classes of equations that are not covered by the results of [5]. However, most oscillation results involve the integral of \( q \) and, hence, require the information on \( q \) on the entire half-line \([t_0, \infty)\). It is difficult to apply them to the cases where \( q \) has a “bad” behavior on a big part of \([t_0, \infty)\), e.g., when \( \int_{t_0}^{\infty} q(s) \, ds = -\infty \).

It follows from the Sturm Separation Theorem that oscillation is only an interval property, i.e., if there exists a sequence of subintervals \([a_i, b_i]\) of \([t_0, \infty)\) as \( a_i \to \infty \), such that
for each $i$ there exists a solution of Eq. (1.2) that has at least two zeros in $[a_i, b_i]$, then every solution of Eq. (1.2) is oscillatory, no matter how “bad” Eq. (1.2) is on the remaining parts of $[t_0, \infty)$.

Kwong and Zettl [13] partially applied this idea to oscillation and established a powerful “telescoping principle” that allows us to trim off the troublesome parts of $\int_{a_i}^{b_i} q(s) \, ds$ and apply the known criteria to the “good” parts. Unfortunately, this principle requires additional conditions for $q$ on the “bad” parts, i.e., $\int_{a_i}^{b_i+1} q(s) \, ds \geq 0$, $i \in N$, which does not reflect the interval oscillation property completely and restricts its applications.

El-Sayed [4] established an interval criterion for oscillation of a forced second-order equation, but the result is not very sharp, because a comparison with equations of constant coefficient is used in the proof. Later, Wong [28] proved a general result for a linear forced equation. Recently, Agarwal and Grace [1], Li and Agarwal [21] and Li and Cheng [22] established general results for nonlinear forced equations. More recently, Çakmak and Tiryaki [3], using general means along the lines given in [29], proved a more general result which extended the results of Wong [28] and Li and Cheng [22] to more general nonlinear forced equations.

In 1997, Huang [8] presented the following interval criteria for the oscillation and nonoscillation of the second-order linear differential equation (1.2), where $q(t) \geq 0$ for $t \in [t_0, \infty)$.

**Theorem A.** (i) If there exists $t_0 > 0$ such that, for every $n \in N$,

$$\int_{2^nt_0}^{2^{n+1}t_0} q(s) \, ds \leq \frac{\alpha_0}{2^{n+1}t_0},$$

(1.6)

where $\alpha_0 = 3 - 2\sqrt{2}$, then every solution of Eq. (1.2) is nonoscillatory.

(ii) If there exist $t_0 > 0$ and $\alpha > \alpha_0$ such that, for every $n \in N$,

$$\int_{2^nt_0}^{2^{n+1}t_0} q(s) \, ds \geq \frac{\alpha}{2^{n+1}t_0},$$

(1.7)

where $\alpha_0 = 3 - 2\sqrt{2}$, then every solution of Eq. (1.2) is oscillatory.

As an application, Huang [8] obtained the following corollary.

**Corollary A.** (i) If

$$\lim_{t \to \infty} \int_{t}^{2t} q(s) \, ds = \alpha < \frac{\alpha_0}{2},$$

(1.8)

then every solution of Eq. (1.2) is nonoscillatory.
(ii) If

$$\lim_{t \to \infty} t \int_t^{2t} q(s) \, ds = \alpha > \alpha_0,$$  \hspace{1cm} (1.9)

then every solution of Eq. (1.2) is oscillatory, where $\alpha_0 = 3 - 2\sqrt{2}$.

We note that the above result seems surprisingly interesting because the interval $(\alpha_0/2, \alpha_0)$ is not covered by conditions (1.6) and (1.7). In particular, if $q(t) = \gamma/t^2$, where $\gamma > 0$ is a constant, then

$$\lim_{t \to \infty} t \int_t^{2t} \frac{\gamma}{s^2} \, ds = \frac{\gamma}{2} < \frac{\alpha_0}{2} \quad \Rightarrow \quad \gamma < 3 - 2\sqrt{2} < \frac{1}{4},$$

and

$$\lim_{t \to \infty} t \int_t^{2t} \frac{\gamma}{s^2} \, ds = \frac{\gamma}{2} = \alpha > \alpha_0 \quad \Rightarrow \quad \gamma > 6 - 4\sqrt{2} > \frac{1}{4}.$$

This implies that Huang’s result remains open for $\gamma \in (3 - 2\sqrt{2}, 6 - 4\sqrt{2})$. That is to say, Huang’s oscillation criterion is not sharp. In fact, the Euler equation

$$y''(t) + \frac{\gamma}{t^2}y(t) = 0$$

is oscillatory if $\gamma > 1/4$, and nonoscillatory if $\gamma \leq 1/4$ [15,16].

Note that, Kong [10,11], Li and Agarwal [17–20] and Li and Huo [23] employed the technique developed in the work of Philos [24] and obtained several interval oscillation results. However, they cannot be applied to the nonlinear differential equation (1.1).

Motivated by the ideas of Kong [10,11], Li and Agarwal [17–20], Li and Huo [23] and Rogovchenko [25], in this paper we obtain, by using averaging functions and a generalized Riccati technique due to Zheng [30], several new interval criteria for oscillation, that is, criteria given by the behavior of Eq. (1.1) only on a sequence of subintervals of $[t_0, \infty)$.

Our results involve the Karamete-type condition and improve and extend the results of Huang [8], Kamenev [9] and Philos [24], and can be applied to extreme cases such as $\int_{t_0}^{\infty} q(s) \, ds = -\infty$ in the special cases. We believe that this study is more general than recent results of Kong [10,11], Li and Agarwal [17,19,20] and Li and Huo [23]. Finally, examples are also given to illustrate the importance of our results.

Hereinafter, we assume that

\begin{enumerate}
  \item[(H1)] the function $r : [t_0, \infty) \to (0, \infty)$ is continuous;
  \item[(H2)] the function $q : [t_0, \infty) \to [0, \infty)$ is continuous and $q(t) \not\equiv 0$ on any ray $[T, \infty)$ for some $T \geq t_0$;
  \item[(H3)] the functions $\Phi, f : R \to R$ are continuous and $\Phi(y) > 0, yf(y) > 0$ for $y \neq 0$;
  \item[(H4)] the function $g : R \to R$ is continuous and $g(y) \geq K > 0$ for $y \neq 0$, where $K$ is a constant.
\end{enumerate}
We say that a function $H = H(t, s)$ belongs to a function class $X$, denoted by $H \in X$, if $H \in C(D, R^+)$, where $D = \{(t, s): -\infty < s \leq t < \infty\}$, which satisfies

$$H(t, t) = 0, \quad H(t, s) > 0 \quad \text{for } t > s,$$

and has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on $D$ such that

$$\frac{\partial H}{\partial t} = \lambda_1(t, s)H(t, s)^{1/\beta} \quad \text{and} \quad \frac{\partial H}{\partial s} = -\lambda_2(t, s)H(t, s)^{1/\beta},$$

where $1/\alpha + 1/\beta = 1$; $\lambda_1$ and $\lambda_2$ are nonnegative continuous functions on $D$.

Let any positive function $\rho \in C^1([t_0, \infty))$ and we take the integral operators $A^\rho(\cdot; \tau, t)$ and $B^\rho(\cdot; t, \tau)$, which are defined in [30], in terms of $H(t, s)$ and $\rho(s)$ as

$$A^\rho(h; \tau, t) = \int_{\tau}^{t} \rho(s)H(t, s)h(s)\, ds, \quad t \geq \tau, \quad (1.12)$$

$$B^\rho(h; t, \tau) = \int_{\tau}^{t} \rho(s)H(s, t)h(s)\, ds, \quad t \leq \tau, \quad (1.13)$$

where $h \in C([t_0, \infty))$. It is easy to verify that $A^\rho(\cdot; \tau, t)$ and $B^\rho(\cdot; t, \tau)$ are linear operators and satisfy

$$A^\rho(h'; \tau, t) = -\rho(\tau)H(t, \tau)h(\tau) - A^\rho\left(\left[-\lambda_2H^{-1/\alpha} + \frac{\rho'}{\rho}\right]h; \tau, t\right)$$

$$\geq -\rho(\tau)H(t, \tau)h(\tau) - A^\rho\left(\left[\lambda_2H^{-1/\alpha} - \frac{\rho'}{\rho}\right]|h|; \tau, t\right), \quad t \geq \tau, \quad (1.14)$$

$$B^\rho(h'; t, \tau) = \rho(\tau)H(\tau, t)h(\tau) - B^\rho\left(\left[\lambda_1H^{-1/\alpha} + \frac{\rho'}{\rho}\right]|h|; t, \tau\right)$$

$$\geq \rho(\tau)H(\tau, t)h(\tau) - B^\rho\left(\left[\lambda_1H^{-1/\alpha} + \frac{\rho'}{\rho}\right]|h|; t, \tau\right), \quad t \leq \tau \quad (1.15)$$

with $\lambda_1 = \lambda_1(s, t)$ and $\lambda_2 = \lambda_2(t, s)$.

2. Oscillation results for $f(x)$ with monotonicity

In this section, we always assume the following condition holds:

$$(H5) \quad f \text{ is differentiable and} \quad \frac{f'(y)}{(\Psi(y)|f(y)|^{\alpha-2})^{1/(\alpha-1)}} \geq M > 0 \quad (2.1)$$

for $y \neq 0$, where $M$ is a constant.

First, we prove two lemmas, which will be useful for establishing oscillation criteria for Eq. (1.1).
Lemma 2.1. Suppose that conditions (H1)–(H5) are satisfied and \( y \) is a solution of Eq. (1.1) such that \( y(t) > 0 \) on \([c, b]\). Let
\[
\begin{align*}
    u(t) &= \frac{r(t)\Psi(y(t))y'(t)^{\alpha-2}y'(t)}{f(y(t))} \quad (2.2)
\end{align*}
\]
on \([c, b]\). Then for any \( H \in X \) and any positive function \( \rho \in C^1([t_0, \infty)) \),
\[
    A^\rho(Kq; c, b) \leq \rho(c)H(b, c)u(c) + A^\rho \left( \frac{r}{M^\alpha-1} \left\{ \frac{1}{\alpha} \left| \lambda_2 H^{-1/\alpha} - \frac{\beta'}{\rho} \right| \right\}^{\alpha} : c, b \right). \quad (2.3)
\]

Proof. From (1.1) and (2.2), we have for \( s \in [c, b] \),
\[
    u'(t) = -q(t)g(y'(t)) - \frac{1}{r(t)^{\beta-1}}(\Psi(y(t)))^{|2\alpha-1|/\alpha}|u(t)|^\beta. \quad (2.4)
\]
Because of conditions (2.1) and (H4), we obtain by the above inequality that
\[
    u'(t) \leq -Kq(t) - \frac{M}{r(t)^{\beta-1}}|u(t)|^\beta. \quad (2.5)
\]
Applying the operator \( A^\rho(\cdot; c, t) \ (c \leq t < b) \) to (2.5) and using (1.14), we obtain
\[
    A^\rho(Kq; c, t) \leq \rho(c)H(t, c)u(c) + A^\rho \left( \frac{r}{r\beta-1} \left| \lambda_2 H^{-1/\alpha} - \frac{\beta'}{\rho} \right| |u| - \frac{M}{r\beta-1} |u|^\beta ; c, t \right). \quad (2.6)
\]
For given \( t \), set
\[
    F(V_2) := \left| \lambda_2 H^{-1/\alpha} - \frac{\beta'}{\rho} \right| V_2 - \frac{M}{r\beta-1} V_2^\beta, \quad V_2 > 0.
\]

\( F(V_2) \) obtains its maximum at
\[
    V_2 = \left\{ \frac{r\beta-1}{\beta M} \left| \lambda_2 H^{-1/\alpha} - \frac{\beta'}{\rho} \right| \right\}^{1/(\beta-1)}
\]
and
\[
    F(V_2) \leq F_{\max} = r \left| \lambda_2 H^{-1/\alpha} - \frac{\beta'}{\rho} \right|^{\alpha} \left( \frac{1}{\alpha} \right)^{\alpha} \frac{1}{M^\alpha-1}. \quad (2.7)
\]
Then we get, by using (2.7),
\[
    A^\rho(Kq; c, t) \leq \rho(c)H(t, c)u(c) + A^\rho \left( \frac{r}{M^\alpha-1} \left| \lambda_2 H^{-1/\alpha} - \frac{\beta'}{\rho} \right| \right)^{\alpha} ; c, t \right). \quad (2.8)
\]
Let \( t \to b^- \) in the above, we obtain (2.3). The proof is complete. \( \square \)

Lemma 2.2. Suppose that conditions (H1)–(H5) are satisfied and \( y \) is a solution of Eq. (1.1) such that \( y(t) > 0 \) on \([a, c]\). Let \( u(t) \) be defined by (2.2) on \([a, c]\). Then for any \( H \in X \) and any positive function \( \rho \in C^1([t_0, \infty)) \),
\[
    B^\rho(Kq; a, c) \leq -\rho(c)H(c, a)u(c) + B^\rho \left( \frac{r}{M^\alpha-1} \left| \lambda_1 H^{-1/\alpha} + \frac{\beta'}{\rho} \right| \right)^{\alpha} ; a, c \right). \quad (2.9)
\]
Proof. We proceed as in the proof of Lemma 2.1 and return to inequality (2.5). Applying the operator $B^\rho(\cdot; t, c) (a < t \leq c)$ to (2.5) and using (1.15), we get

$$\begin{align*}
B^\rho(Kq; t, c) &\leq -\rho(c)H(c, t)u(c) + B^\rho\left(\left|\lambda_1 H^{-1/\alpha} + \frac{\rho'}{\rho}\right| |u| - \frac{M}{r^{\beta-1}} |u|^\beta ; t, c \right).
\end{align*}$$

(2.10)

Similarly, for given $t$, set

$$F(V_1) := \left|\lambda_1 H^{-1/\alpha} + \frac{\rho'}{\rho}\right| V_1 - \frac{M}{r^{\beta-1}} V_1^\beta, \quad V_1 > 0.$$

$F(V_1)$ obtains its maximum at

$$V_1 = \left|\frac{\rho'}{\rho M}\right| \left(\frac{1}{\alpha} \left|\lambda_1 H^{-1/\alpha} + \frac{\rho'}{\rho}\right| \right)^{1/(\beta-1)}.$$

and

$$F(V_1) \leq F_{\max} = r \left|\lambda_1 H^{-1/\alpha} + \frac{\rho'}{\rho}\right| \left(\frac{1}{\alpha} \frac{1}{M^{\alpha-1}} \right).$$

(2.11)

Then we obtain, by using (2.11),

$$\begin{align*}
B^\rho(Kq; t, c) &\leq -\rho(c)H(c, t)u(c) + B^\rho\left(\frac{r}{M^{\alpha-1}} \left|\lambda_1 H^{-1/\alpha} + \frac{\rho'}{\rho}\right| \right)^\alpha ; t, c) .
\end{align*}$$

(2.12)

Let $t \to a^+$ in the above, we obtain (2.9). The proof is complete.

Theorem 2.1. Assume that (H1)–(H5) hold and that for some $c \in (a, b)$, $H \in X$ and any positive function $\rho \in C_1([t_0, \infty))$,

$$\begin{align*}
\frac{1}{H(c, a)} B^\rho(Kq; a, c) + \frac{1}{H(b, c)} A^\rho(Kq; c, b) &> \frac{1}{M^{\alpha-1} a^\alpha} \left[\frac{1}{H(c, a)} B^\rho\left(\left|\lambda_1 H^{-1/\alpha} + \frac{\rho'}{\rho}\right| ; a, c \right) \right. \\
&+ \frac{1}{H(b, c)} A^\rho\left(\left|\lambda_2 H^{-1/\alpha} - \frac{\rho'}{\rho}\right| ; c, b \right) \right].
\end{align*}$$

(2.13)

Then every solution of Eq. (1.1) has at least one zero in $(a, b)$.

Proof. Suppose the contrary. Then without loss of generality we may assume that there is a solution $y(t)$ of Eq. (1.1) such that $y(t) > 0$ for $t \in (a, b)$. From Lemmas 2.1 and 2.2, we see that both (2.3) and (2.9) hold. Dividing (2.3) and (2.9) by $H(b, c)$ and $H(c, a)$, respectively, and then adding them, we have that

$$\begin{align*}
\frac{1}{H(c, a)} B^\rho(Kq; a, c) + \frac{1}{H(b, c)} A^\rho(Kq; c, b) &\leq \frac{1}{M^{\alpha-1} a^\alpha} \left[\frac{1}{H(c, a)} B^\rho\left(\left|\lambda_1 H^{-1/\alpha} + \frac{\rho'}{\rho}\right| ; a, c \right) \right. \\
&+ \frac{1}{H(b, c)} A^\rho\left(\left|\lambda_2 H^{-1/\alpha} - \frac{\rho'}{\rho}\right| ; c, b \right) \right].
\end{align*}$$
\[ + \frac{1}{H(b, c)} A^\rho \left( \left| \lambda_2 H^{-1/\alpha} - \frac{\rho'(s)}{\rho(s)} \right|_{c, b} \right), \]

which contradicts the assumption (2.13) and completes the proof. □

**Theorem 2.2.** Suppose that (H1)–(H5) hold. If, for every \( T \geq t_0 \), there exist \( H \in X \), any positive function \( \rho \in C^1([t_0, \infty)) \) and \( a, b, c \in \mathbb{R} \) such that \( T \leq a < c < b \) and (2.13) holds, then every solution of Eq. (1.1) is oscillatory.

**Proof.** Pick a sequence \( \{T_i\} \subset [t_0, \infty) \) such that \( T_i \to \infty \) as \( i \to \infty \). By the assumption, for each \( i \in \mathbb{N} \), there exist \( a_i, b_i, c_i \in \mathbb{R} \) such that \( a_i < c_i < b_i \) and (2.13) holds, where \( a, b, c \) are replaced by \( a_i, b_i, c_i \), respectively. By virtue of Theorem 2.1, every solution \( y(t) \) has at least one zero, \( t_i \in (a_i, b_i) \). Noting that \( t_i > a_i \geq T_i, i \in \mathbb{N} \), we see that every solution has arbitrary large zeros. Thus, every solution of Eq. (1.1) is oscillatory. The theorem is proved. □

**Theorem 2.3.** Assume that (H1)–(H5) hold. If

\[
\limsup_{t \to \infty} \int_t^l H(s, l) \left[ K \rho(s) q(s) - \frac{\rho(s) r(s)}{M^{\alpha-1}} \left\{ \frac{1}{\alpha} \left| \lambda_1(s, l) H(s, l)^{-1/\alpha} + \frac{\rho'(s)}{\rho(s)} \right| \right\}^\alpha \right] ds > 0
\]

(2.14)

and

\[
\limsup_{t \to \infty} \int_l^t H(l, s) \left[ K \rho(s) q(s) - \frac{\rho(s) r(s)}{M^{\alpha-1}} \left\{ \frac{1}{\alpha} \left| \lambda_2(l, s) H(l, s)^{-1/\alpha} - \frac{\rho'(s)}{\rho(s)} \right| \right\}^\alpha \right] ds > 0,
\]

(2.15)

for some \( H \in X \), any positive function \( \rho \in C^1([t_0, \infty)) \) and for each \( l \geq t_0 \), then every solution of Eq. (1.1) is oscillatory.

**Proof.** For any \( T \geq t_0 \), let \( a = T \). In (2.14) we choose \( l = a \). Then there exists \( c > a \) such that

\[
\int_a^c H(s, a) \left[ K \rho(s) q(s) - \frac{\rho(s) r(s)}{M^{\alpha-1}} \left\{ \frac{1}{\alpha} \left| \lambda_1(s, a) H(s, a)^{-1/\alpha} + \frac{\rho'(s)}{\rho(s)} \right| \right\}^\alpha \right] ds > 0.
\]

(2.16)

In (2.15), we choose \( l = c \). Then there exists \( b > c \) such that

\[
\int_c^b H(b, s) \left[ K \rho(s) q(s) - \frac{\rho(s) r(s)}{M^{\alpha-1}} \left\{ \frac{1}{\alpha} \left| \lambda_2(b, s) H(b, s)^{-1/\alpha} - \frac{\rho'(s)}{\rho(s)} \right| \right\}^\alpha \right] ds > 0.
\]

(2.17)

Combining (2.16) and (2.17), we obtain (2.13). The required conclusion thus comes from Theorem 2.2. The proof is complete. □
For the case where $H := H(t - s) \in X$, we have that $\lambda_1(t - s) = \lambda_2(t - s)$ and denote them by $\lambda(t - s)$. The subclass of $X$ containing such $H(t - s)$ is denoted by $X_0$. Applying Theorem 2.2 to $X_0$, we obtain the following theorem:

**Theorem 2.4.** Suppose that (H1)–(H5) hold. If, for each $T \geq t_0$, there exist $H \in X_0$, any positive function $\rho \in C^1([t_0, \infty))$ and $a, c \in \mathbb{R}$ such that $T \leq a < c$ and

$$
\int_a^c H(s - a) K\left[\rho(s)q(s) + \rho(2c - s)q(2c - s)\right] ds
$$

$$
> \frac{1}{M^{1-1/\alpha}} \left\{ \int_a^c \rho(s) H(s - a)r(s) \left| \lambda(s - a) H(s - a)^{-1/\alpha} + \frac{\rho'(s)}{\rho(s)} \right|^\alpha ds \right. 
$$

$$
+ \left. \int_a^c \rho(2c - s) H(s - a)r(2c - s) \left| \lambda(s - a) H(s - a)^{-1/\alpha} + \frac{\rho'(2c - s)}{\rho(2c - s)} \right|^\alpha ds \right\},
$$

(2.18)

then every solution of Eq. (1.1) is oscillatory.

**Proof.** Let $b = 2c - a$. Then $H(b - c) = H(c - a) = H((b - a)/2)$, and for any $v \in L[a, b]$, we have

$$
\int_a^c v(s) ds = \int_a^c v(2c - s) ds.
$$

Hence

$$
A\rho(Kq; c, b) = \int_a^c \rho(s) H(b - s) Kq(s) ds = \int_a^c \rho(2c - s) H(s - a) Kq(2c - s) ds
$$

and

$$
A\rho\left( r, H^{-1/\alpha} - \frac{\rho'}{\rho} \right; c, b) = \int_a^b \rho(s) H(b - s)r(s) \left| \lambda(b - s) H(b - s)^{-1/\alpha} - \frac{\rho'(s)}{\rho(s)} \right|^\alpha ds
$$

$$
= \int_a^c \rho(2c - s) H(s - a)r(2c - s) \left| \lambda(s - a) H(s - a)^{-1/\alpha} + \frac{\rho'(2c - s)}{\rho(2c - s)} \right|^\alpha ds.
$$

Thus that (2.18) holds, implies that (2.13) holds for $H \in X_0$, any positive function $\rho \in C^1([t_0, \infty))$ and therefore every solution of Eq. (1.1) is oscillatory by virtue of Theorem 2.2. The theorem is proved.
From above oscillation criteria, one can obtain different sufficient conditions for oscillation of all solutions of Eq. (1.1) by different choices of $H(t,s)$.

Let

$$H(t,s) = (t - s)^\theta, \quad t \geq s \geq t_0,$$

where $\theta > \alpha - 1$ is a constant.

**Corollary 2.1.** Assume that (H1)–(H5) hold. Then every solution of Eq. (1.1) is oscillatory provided that for every $l \geq t_0$ and for some $\theta > \alpha - 1$, there exists a positive function $\rho \in C^1([t_0, \infty))$ such that the following two inequalities hold:

$$\limsup_{t \to \infty} \frac{1}{t^{\theta-\alpha+1}} \int_l^t (s - l)^\theta \left[ K\rho(s)q(s) - \frac{\rho(s)r(s)}{M^{\alpha-1} \alpha^\alpha} \left| \frac{\rho'(s)}{\rho(s)} \right|^{\alpha} \right] ds > 0 \quad (2.19)$$

and

$$\limsup_{t \to \infty} \frac{1}{t^{\theta-\alpha+1}} \int_l^t (t - s)^\theta \left[ K\rho(s)q(s) - \frac{\rho(s)r(s)}{M^{\alpha-1} \alpha^\alpha} \left| \frac{\rho'(s)}{\rho(s)} \right|^{\alpha} \right] ds > 0. \quad (2.20)$$

The proof is similar to that of Theorem 2.3, we omit it here.

Define

$$R(t) = \int_l^t \frac{1}{r(s)^{1/(\alpha-1)}} ds, \quad t \geq l \geq t_0,$$

and set

$$H(t,s) = \left[ R(t) - R(s) \right]^\theta, \quad t \geq t_0$$

where $\theta > \alpha - 1$ is a constant.

If we take $\rho(t) = 1$, then, by Theorem 2.3, we have the following important oscillation criterion, which extends Theorem 2.3(i) of Kong [10], Theorem 2.5 of Li and Agarwal [17,19,20] and Theorem 2.5 of Li and Huo [23].

**Theorem 2.5.** Suppose that (H1)–(H5) hold and $\lim_{t \to \infty} R(t) = \infty$. Then every solution of Eq. (1.1) is oscillatory provided that for each $l \geq t_0$ and for some $\theta > \alpha - 1$, the following two inequalities hold:

$$\limsup_{t \to \infty} \frac{M^{\alpha-1}}{R^{\theta-\alpha+1}(t)} \int_l^t \left[ R(s) - R(l) \right]^\theta Kq(s) ds > \frac{\theta^\alpha}{\alpha^\alpha (\theta - \alpha + 1)} \quad (2.21)$$

and

$$\limsup_{t \to \infty} \frac{M^{\alpha-1}}{R^{\theta-\alpha+1}(t)} \int_l^t \left[ R(t) - R(s) \right]^\theta Kq(s) ds > \frac{\theta^\alpha}{\alpha^\alpha (\theta - \alpha + 1)}. \quad (2.22)$$
Proof. Since $H(t, s) = [R(t) - R(s)]^\theta$, then

$$
\lambda_1(t, s) = \theta \left[ R(t) - R(s) \right]^{(\theta - \alpha)/\alpha} \frac{1}{r(t)^{1/(\alpha - 1)}}
$$

and

$$
\lambda_2(t, s) = \theta \left[ R(t) - R(s) \right]^{(\theta - \alpha)/\alpha} \frac{1}{r(t)^{1/(\alpha - 1)}}.
$$

Noting that

$$
\int_{l}^{t} H(s, l) r(s) \left| \lambda_1(s, l) H(s, l) - \frac{1}{\alpha} \right|^{\alpha} ds
$$

$$
= \int_{l}^{t} r(s) \theta^\alpha \left[ R(t) - R(s) \right]^{\theta - \alpha} \frac{1}{r(s)^{\alpha/(\alpha - 1)}} ds = \frac{\theta^\alpha}{\theta - \alpha + 1} \left[ R(t) - R(l) \right]^{\theta - \alpha + 1},
$$

and

$$
\int_{l}^{t} H(t, s) r(s) \left| \lambda_2(t, s) H(t, s) - \frac{1}{\alpha} \right|^{\alpha} ds
$$

$$
= \int_{l}^{t} r(s) \theta^\alpha \left[ R(t) - R(s) \right]^{\theta - \alpha} \frac{1}{r(s)^{\alpha/(\alpha - 1)}} ds = \frac{\theta^\alpha}{\theta - \alpha + 1} \left[ R(t) - R(l) \right]^{\theta - \alpha + 1}.
$$

In view of $\lim_{t \to \infty} R(t) = \infty$, we have

$$
\lim_{t \to \infty} \frac{1}{\alpha^\theta R^{\theta - \alpha + 1}(t)} \int_{l}^{t} H(s, l) r(s) \left| \lambda_1(s, l) H(s, l) - \frac{1}{\alpha} \right|^{\alpha} ds = \frac{\theta^\alpha}{\alpha^\theta (\theta - \alpha + 1)}.
$$

(2.23)

and

$$
\lim_{t \to \infty} \frac{1}{\alpha^\theta R^{\theta - \alpha + 1}(t)} \int_{l}^{t} H(t, s) r(s) \left| \lambda_2(t, s) H(t, s) - \frac{1}{\alpha} \right|^{\alpha} ds = \frac{\theta^\alpha}{\alpha^\theta (\theta - \alpha + 1)}.
$$

(2.24)

From (2.23) and (2.24), we have that

$$
\limsup_{t \to \infty} \frac{M^{\alpha - 1}}{R^{\theta - \alpha + 1}(t)} \int_{l}^{t} \left[ R(s) - R(l) \right]^\theta \left\{ Kq(s) - \frac{r(s) \theta^\alpha}{M^{\alpha - 1} \alpha^\theta} \left| \lambda_1(s, l) H(s, l) - \frac{1}{\alpha} \right|^{\alpha} \right\} ds
$$

$$
= \limsup_{t \to \infty} \frac{M^{\alpha - 1}}{R^{\theta - \alpha + 1}(t)} \int_{l}^{t} \left[ R(s) - R(l) \right]^\theta Kq(s) ds
$$
\[
- \frac{1}{\alpha^2} \lim_{t \to \infty} \frac{1}{R^{\alpha-\alpha+1}(t)} \int_{t}^{\infty} r(s) \left[ R(t) - R(s) \right]^{\alpha-\alpha} \frac{1}{r(s)^{\alpha/(\alpha-1)}} d\theta \\
= \limsup_{t \to \infty} M^{\alpha-1} \int_{t}^{\infty} \left[ R(s) - R(l) \right]^{\theta} Kq(s) ds - \frac{\theta^\alpha}{\alpha^\alpha (\theta - \alpha + 1)} > 0,
\]

i.e., (2.14) holds. Similarly, (2.22) implies that (2.15) holds. By Theorem 2.3, every solution of Eq. (1.1) is oscillatory. The proof is complete. 

**Remark 2.1.** Although Li and Agarwal [17] studied on Eq. (1.3) with a positive non-decreasing function \( \rho \in C^1([t_0, \infty)) \), it seems that there is no restriction on the sign of \( \rho'(t) \) for the half linear equation. In this point we emphasize the importance of this study. When \( \Psi(y(t)) \equiv 1 \), \( f(y(t)) = |y(t)|^{\alpha-2} y(t) \) and \( g(y(t)) \equiv 1 \) in Eq. (1.1) and \( \rho'(t) \geq 0 \), our results reduce to the results of Li and Agarwal [17].

**Remark 2.2.** If \( g(y(t)) \equiv 1 \), then the sign condition on \( q(t) \) can be dropped. We will see this by Example 1 in Section 4.

**Remark 2.3.** When \( \Psi(y(t)) \equiv 1 \), \( \alpha = 2 \), \( f(y(t)) = y(t) \) and \( g(y(t)) \equiv 1 \) in Eq. (1.1) and \( \rho(t) = 1 \), our results reduce to the results of Kong [10].

3. Oscillation results for \( f(x) \) without monotonicity

In this section, we consider the oscillation of Eq. (1.1) with \( \alpha = 2 \) when the function \( f(y) \) is not monotone. In this case, we always assume that the following condition holds:

\[ \text{(H5')} \quad f(y)/y \geq M_0 > 0 \text{ and } 0 < \Psi(y) \leq d \text{ for } y \neq 0, \text{ where } M_0 \text{ and } d \text{ are constants.} \]

**Lemma 3.1.** Suppose that conditions (H1)–(H4) and (H5') are satisfied and \( y \) is a solution of Eq. (1.1) with \( \alpha = 2 \) such that \( y(t) > 0 \) on \([c, b)\). Let

\[
w(t) = \frac{r(t) \Psi(y(t)) y'(t)}{y(t)}
\]

on \([c, b)\). Then for any \( H \in X \) and any positive function \( \rho \in C^1([t_0, \infty)) \),

\[
A_p(KM_0q; c, b) \leq \rho(c) \left( H(b, c) w(c) + A_p \left( \frac{dr}{4} \left( \kappa H^{-1/2} \frac{\rho'}{\rho} \right)^2 ; c, b \right) \right).
\]

**Proof.** From (1.1) with \( \alpha = 2 \) and (3.1), we have for \( s \in [c, b)\):

\[
w'(t) = -q(t) \frac{f(y(t))}{y(t)} g(y'(t)) - \frac{1}{r(t) \Psi(y(t))} w^2(t). \]
In view of \( f(y)/y \geq M_0 > 0 \), \( g(y') \geq K > 0 \) and \( 0 < \Psi(y) \leq d \), we obtain by the above equality:

\[
 w'(t) \leq -KM_0 q(t) - \frac{1}{d^r(t)} w^2(t).
\] (3.4)

The rest of the proof is similar to that of Lemma 2.1, so Lemma 3.1 is proved. □

**Lemma 3.2.** Suppose that conditions (H1)–(H4) and (H5') are satisfied and \( y \) is a solution of Eq. (1.1) with \( \alpha = 2 \) such that \( y(t) > 0 \) on \( (a, c] \). Let \( w(t) \) be defined by (3.1) on \( (a, c] \).

Then for any \( H \in X \) and any positive function \( \rho \in C^1([t_0, \infty)) \),

\[
 B^\rho(KM_0q; a, c) \leq -\rho(c)H(c, a)w(c) + B^\rho \left( \frac{dr}{4} \left( \lambda_1 H^{-1/2} + \frac{\rho'}{\rho} \right)^2 ; a, c \right). \] (3.5)

The following theorem is an immediate result from Lemmas 3.1 and 3.2.

**Theorem 3.1.** Assume that (H1)–(H4) and (H5') hold and that for some \( c \in (a, b) \), \( H \in X \) and any positive function \( \rho \in C^1([t_0, \infty)) \),

\[
 \frac{1}{H(c, a)}B^\rho(KM_0q; a, c) + \frac{1}{H(b, c)}A^\rho(KM_0q; c, b)
 \geq \frac{1}{4H(c, a)}B^\rho \left( dr \left( \lambda_1 H^{-1/2} + \frac{\rho'}{\rho} \right)^2 ; a, c \right)
 + \frac{1}{4H(b, c)}A^\rho \left( dr \left( \lambda_2 H^{-1/2} - \frac{\rho'}{\rho} \right)^2 ; c, b \right). \] (3.6)

Then every solution of Eq. (1.1) with \( \alpha = 2 \) has at least one zero in \( (a, b) \).

**Theorem 3.2.** Suppose that (H1)–(H4) and (H5') hold. If, for every \( T \geq t_0 \), there exist \( H \in X \), any positive function \( \rho \in C^1([t_0, \infty)) \) and \( a, b, c \in \mathbb{R} \) such that \( T \leq a < c < b \) and (3.6) holds, then every solution of Eq. (1.1) with \( \alpha = 2 \) is oscillatory.

**Theorem 3.3.** Assume that (H1)–(H4) and (H5') hold. If

\[
 \limsup_{t \to \infty} \int_t^l H(s, l) \left[ KM_0 \rho(s)q(s) - \frac{d\rho(s)r(s)}{4} \left( \lambda_1(s, l) H(s, l)^{-1/2} + \frac{\rho'(s)}{\rho(s)} \right)^2 \right] ds > 0 \] (3.7)

and

\[
 \limsup_{t \to \infty} \int_t^l H(t, s) \left[ KM_0 \rho(s)q(s) - \frac{d\rho(s)r(s)}{4} \left( \lambda_2(t, s) H(t, s)^{-1/2} - \frac{\rho'(s)}{\rho(s)} \right)^2 \right] ds > 0 \] (3.8)

for some \( H \in X \), any positive function \( \rho \in C^1([t_0, \infty)) \) and for each \( l \geq t_0 \), then every solution of Eq. (1.1) with \( \alpha = 2 \) is oscillatory.
Theorem 3.4. Suppose that (H1)–(H4) and (H5′) hold. If, for each \( T \geq t_0 \), there exist \( H \in X_0 \), any positive function \( \rho \in C^1([t_0, \infty)) \) and \( a, c \in \mathbb{R} \) such that \( T \leq a < c \) and
\[
\int_a^c H(s-a)KM_0\left[\rho(s)q(s) + \rho(2c-s)q(2c-s)\right]ds
\]
\[
> \frac{1}{4}\left\{ \int_a^c \rho(s)H(s-a)dr(s)\left(\lambda(s-a)H(s-a)^{-1/2} + \frac{\rho'(s)}{\rho(s)}\right)^2 ds + \int_a^c \rho(2c-s)H(s-a)dr(2c-s)\left(\lambda(s-a)H(s-a)^{-1/2} + \frac{\rho'(2c-s)}{\rho(2c-s)}\right)^2 ds \right\},
\]
then every solution of Eq. (1.1) with \( \alpha = 2 \) is oscillatory.

Corollary 3.1. Assume that (H1)–(H4) and (H5′) hold. Then every solution of Eq. (1.1) with \( \alpha = 2 \) is oscillatory provided that for every \( l \geq t_0 \) and for some \( \theta > 1 \), there exists a positive function \( \rho \in C^1([t_0, \infty)) \) such that the following two inequalities hold:
\[
\limsup_{t \to \infty} \frac{1}{t^{\theta-1}} \int_l^t (t-s)^{\theta} \left[ K\rho^2(s)q(s) - \frac{d}{4}\rho(s)r(s)\left(\frac{\theta}{s-t} + \frac{\rho'(s)}{\rho(s)}\right)^2 \right] ds > 0
\]
and
\[
\limsup_{t \to \infty} \frac{1}{t^{\theta-1}} \int_l^t (t-s)^{\theta} \left[ K\rho^2(s)q(s) - \frac{d}{4}\rho(s)r(s)\left(\frac{\theta}{t-s} - \frac{\rho'(s)}{\rho(s)}\right)^2 \right] ds > 0.
\]

By Theorem 3.3, we have the following oscillation criterion.

Theorem 3.5. Suppose that (H1)–(H4) and (H5′) hold and \( \lim_{t \to \infty} R(t) = \infty \). Then every solution of Eq. (1.1) with \( \alpha = 2 \) is oscillatory provided that for each \( l \geq t_0 \) and for some \( \theta > 1 \), the following two inequalities hold:
\[
\limsup_{t \to \infty} \frac{1}{R^{\theta-1}(t)} \int_l^t \left[ R(t) - R(l) \right]^\theta K\rho^2(s)q(s) ds > \frac{d\theta^2}{4(\theta - 1)} \tag{3.9}
\]
and
\[
\limsup_{t \to \infty} \frac{1}{R^{\theta-1}(t)} \int_l^t \left[ R(l) - R(s) \right]^\theta K\rho^2(s)q(s) ds > \frac{d\theta^2}{4(\theta - 1)}. \tag{3.10}
\]

Remark 3.1. When \( \Psi(y(t)) = 1 \), \( \alpha = 2 \) and \( g(y(t)) = 1 \) in Eq. (1.1), our results in Section 3 reduce to the results of Li and Huo [23].
Remark 3.2. When $\Psi(\gamma(t)) \equiv 1$ and $\alpha = 2$ in Eq. (1.1), our results reduce to the results of Li and Agarwal [20].

Remark 3.3. When $r(t) \equiv 1$, $\Psi(\gamma(t)) \equiv 1$ and $\alpha = 2$ in Eq. (1.1), our results reduce to the results of Li and Agarwal [19].

4. Examples

In this section, we will show the applications of our oscillation criteria by two examples. We will see that the equations in the examples are oscillatory based on the results in Sections 2 and 3, though the oscillation cannot be demonstrated by the results of Kong [10,11], Li and Agarwal [17,19,20] and Li and Huo [23], and most other known criteria.

Example 1. Consider the nonlinear differential equation

$$
((1 + \sin^2 t)|y(t)|^{\alpha - 2}|y'(t)|^\alpha + q(t)y^3(t))' = 0
$$

(4.1)

where

$$
q(t) = \begin{cases} 
3(t - 3n), & 3n \leq t \leq 3n + 1, \\
3(-t + 3n + 2), & 3n + 1 < t \leq 3n + 2, \\
-\sin \pi t, & 3n + 2 < t \leq 3n + 3,
\end{cases}
$$

$n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $1 < \alpha \leq 2$. For any $T \geq 0$, there exists $n \in \mathbb{N}_0$ such that $3n \geq T$. Let $\alpha = 3n$, $c = 3n + 1$ and $\rho(t) \equiv 1$. We choose $H(t - s) = (t - s)^2$ then $\lambda(t - s) = 2(t - s)^{(2-\alpha)/\alpha}$. Since $f(y) = y^3$, $\Psi(y) = |y|^{4-\alpha}$ then $M = 3$, we have

$$
\int_a^c H(s - a)\left[q(s) + q(2c - s)\right]ds = \int_{3n}^{3n+1} (s - 3n)^2\left[3(s - 3n) + 3(6n + 2 - s - 3n)\right]ds = 6 \int_{3n}^{3n+1} (s - 3n)^2 ds = 2
$$

and

$$
\frac{1}{M^{\alpha-1}c^\alpha} \int_a^c (r(s) + r(2c - s))(\lambda(s - a))^\alpha ds \leq \frac{2^{\alpha+2}}{3^{\alpha-1}c^\alpha} \int_{3n}^{3n+1} (s - 3n)^{2-\alpha} ds = \frac{2^{\alpha+2}}{3^{\alpha-1}(3 - \alpha)c^\alpha} < 2.
$$

This implies that (2.18) holds and, hence, every solution of Eq. (4.1) is oscillatory by virtue of Theorem 2.4 with Remark 2.2. Note that we have $\int_0^\infty q(s) ds = -\infty$ in this equation.
When \( \alpha = 2 \) then it is easy to see that \( 4/3 < 2 \), so Eq. (4.1) with \( \alpha = 2 \) is also oscillatory. However, the results of Kong [10,11], Li and Agarwal [17,19,20] and Li and Huo [23] fail to apply to Eq. (4.1).

Example 2. Consider the nonlinear differential equation

\[
\left( 1 + \sin^2 t \right) \frac{1}{1 + y^2(t)} y'(t)'
\]

\[
+ \frac{2(1 + \sin^2 t)}{(3 + \sin^2 t)(1 + \cos^2 t)} \left( \frac{y(t)}{2} + \frac{y(t)}{1 + y^2(t)} \right) \left( 1 + (y'(t))^2 \right) = 0
\]

where \( \alpha \geq 1 \). Note that

\[
f(y) = y \left( \frac{1}{2} + \frac{1}{1 + y^2} \right),
\]

let \( \rho(t) = t^2 \) and \( \theta = 2 \). We can prove the oscillatory character of Eq. (4.2) by using Corollary 3.1. Taking into account that, for all \( y \in \mathbb{R} \setminus \{0\} \),

\[
\frac{f(y)}{y} = \frac{1}{2} + \frac{1}{1 + y^2} \geq \frac{1}{2} = M_0, \quad \Psi(y) = \frac{1}{1 + y^2} \leq 1 = d
\]

and \( g(y) = 1 + y^2 \geq 1 = K \), we get

\[
\lim_{t \to \infty} \frac{1}{t} \int_1^t (s-l)^2 \left[ \frac{1}{2} \rho(s) q(s) - \frac{1}{4} \rho(s) r(s) \left( \frac{2}{s-l} + \frac{\rho'(s)}{\rho(s)} \right)^2 \right] ds
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \int_1^t (s-l)^2 \left[ \frac{1}{2} r(s) \rho(s) - \frac{1}{4} r(s) \left( \frac{2}{s-l} + \frac{\rho'(s)}{\rho(s)} \right)^2 \right] ds
\]

\[
\geq \lim_{t \to \infty} \frac{1}{t} \int_1^t \left[ \frac{1}{8} (s-l)^2 s^2 - \frac{1}{2} (4s - 2l)^2 \right] ds = \infty
\]

and

\[
\lim_{t \to \infty} \frac{1}{t} \int_1^t (t-s)^2 \left[ \frac{1}{2} \rho(s) q(s) - \frac{1}{4} \rho(s) r(s) \left( \frac{2}{t-s} - \frac{\rho'(s)}{\rho(s)} \right)^2 \right] ds
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \int_1^t (t-s)^2 \left[ \frac{1}{2} \rho(s) q(s) - \frac{1}{4} \rho(s) r(s) \left( \frac{2}{t-s} - \frac{\rho'(s)}{\rho(s)} \right)^2 \right] ds
\]

\[
\geq \lim_{t \to \infty} \frac{1}{t} \int_1^t \left[ \frac{1}{8} (t-s)^2 s^2 - \frac{1}{2} (4s - 2t)^2 \right] ds = \infty.
\]

Therefore, Eq. (4.2) is oscillatory by Corollary 3.1. Observe that \( y(t) = \sin t \) is an oscillatory solution of Eq. (4.2).
Acknowledgments

The author thanks the referees for their valuable suggestions and useful comments.

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