Mappings of Subspaces into Subsets

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Let $V_n(q)$ denote the n-dimensional vector space over the finite field with $q$ elements, and $L_n(q)$ be the lattice of subspaces of $V_n(q)$. Two rank- and order-preserving maps from $L_n(q)$ onto the lattice of subsets of an n-set are constructed. Three equivalent formulations of these maps are given: an inductive procedure based on an elementary combinatorial interpretation of a well-known pair of difference equations satisfied by the Gaussian coefficients $\binom{\cdot}{\cdot}$, a direct set-theoretical definition, and, a direct definition involving a certain pair of modular chains in $L_n(q)$. The direct set-theoretical definition of one of these maps has already been given by Knuth. Knuth’s map, however, may be systematically discovered by means of the inductive procedure and the direct lattice-theoretic definition shows how it can be generalized. As a further application of the pair of difference equations satisfied by $\binom{\cdot}{\cdot}$, a direct-combinatorial proof of an identity of Carlitz that expands Gaussian coefficients in terms of binomial coefficients has been formulated.

1. INTRODUCTION

Let $GF(q)$ denote the finite field with $q$ elements, $V_n(q)$ the n-dimensional vector space over $GF(q)$, $L_n(q)$ the lattice of subspaces of $V_n(q)$, and $B_n$ the lattice of subsets of an n-set. Here, both $L_n(q)$ and $B_n$ are ordered by inclusion, and rank in $L_n(q)$ and $B_n$ is dimension and cardinality, respectively.

In [5], Goldman and Rota posed the problem of giving an explanation for the dual interpretation of various Eulerian formulas both as enumeration of subspaces or of linear transformations in vector spaces with given properties, and on the other hand, as partitions of a number with given properties. As a partial solution to this problem Knuth [6] discovered a rank- and order-preserving map (see Eq. (1.5)) from $L_n(q)$ onto $B_n$ which he used to relate partitions to the combinatorics of both finite-vector spaces and finite sets.

We shall construct two rank- and order-preserving maps $\phi_1$ and $\phi_2$ from $L_n(q)$ onto $B_n$. In Section 2 we shall give three equivalent formulations of

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each of these maps: an inductive procedure (see (2.11)) based upon an elementary combinatorial interpretation of (1.1) and (1.2); a direct set-theoretical definition (see (1.4) and (1.5)); and, a direct definition involving a certain pair of modular chains in $\mathcal{L}_n(q)$ (see (1.10) and (1.11)). The direct set-theoretical definition of $\phi_2$ has already been given by Knuth [6]. However, Knuth's map can be systematically discovered by means of our inductive procedure in (2.11), and our direct lattice-theoretic Definition 1.9 combined with Proposition 2.16 shows how it can be generalized.

Our starting point is the following well-known [2, p. 35] pair of difference equations satisfied by the Gaussian coefficients $\left[ \binom{n}{k} \right]$ which count the number of k-dimensional subspaces of $V_n(q)$.

\[
\begin{align*}
\begin{bmatrix} n+1 \\ k \end{bmatrix} &= \begin{bmatrix} n \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n \\ k \end{bmatrix} \\
\begin{bmatrix} n+1 \\ k \end{bmatrix} &= q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix} + \begin{bmatrix} n \\ k \end{bmatrix}.
\end{align*}
\]

These difference equations are analogous to the familiar recursion

\[
\left( \binom{n+1}{k} \right) = \left( \binom{n}{k-1} \right) + \left( \binom{n}{k} \right),
\]

where $\left( \binom{n}{k} \right)$ is the number of k-element sets of $B_n$.

First, we shall put together an inductive construction of $\phi_1$ by combining the combinatorial interpretation of (1.1a) in [1, 5] with the standard [4] interpretation of (1.2). A similar inductive procedure involving (1.1b) and (1.2) yields $\phi_2$. Our inductive construction of $\phi_1$ and $\phi_2$ immediately leads to the direct set-theoretical

**Definition 1.3.** Let $Z$ be a k-dimensional subspace of $V_n(q)$. The maps $\phi_1$ and $\phi_2$ are then determined by means of:

\[
\phi_1(Z) = \{m_1, \ldots, m_k\},
\]

where

\[
m_{\min} = \min \{ j \mid \exists (z_1, \ldots, z_n) \in Z, z_j \neq 0 \},
\]

\[
m_i = \min \{ j \mid \exists (z_1, \ldots, z_n) \in Z, z_{m_1} = 0, \ldots, z_{m_{i-1}} = 0, x_j \neq 0 \};
\]

and

\[
\phi_2(Z) = \{n_1, \ldots, n_k\},
\]
where
\[ n_1 = \max \{ j \mid \exists (z_1, \ldots, z_n) \in Z, z_j \neq 0 \} \],
\[ n_2 = \max \{ j \mid \exists (z_1, \ldots, z_n) \in Z, z_{n_1} = 0, \ldots, z_{n_i-1} = 0, z_j \neq 0 \}. \]

To obtain our third formulation of \( \phi_1 \) and \( \phi_2 \) we first view \( V_n(q) \) as the linear combination over \( GF(q) \) of the \( n \) basis vectors \( x_1, x_2, \ldots, x_n \), where \( x_i \) is the vector \((0, \ldots, 0, 1, 0, \ldots)\) with 1 in the \( i \)-th coordinate and 0, otherwise. That is,
\[ V_n(q) = \langle x_1, \ldots, x_n \rangle, \]
\[ V_{n+1}(q) = V_n(q) \oplus \langle x_{n+1} \rangle, \]
\[ V_n(q) \subseteq V_{n+1}(q). \]

Now, consider the two chains \( S \) and \( T \) of subspaces of \( V_n(q) \) given by
\[ S = \{ \emptyset \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \equiv V_n(q) \}, \]
where,
\[ S_i = \langle x_{n-i+1}, \ldots, x_n \rangle, \]
and
\[ T = \{ \emptyset \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_n \equiv V_n(q) \}, \]
where,
\[ T_i = \langle x_{1}, \ldots, x_i \rangle. \]

This given, it is not difficult to show that the formulation of \( \phi_1 \) and \( \phi_2 \) given by Definition 1.3 is equivalent to

**Definition 1.9.** Let \( Z \) be a \( k \)-dimensional subspace of \( V_n(q) \) and \( S_i \) and \( T_i \) be given by (1.7b) and (1.8b), respectively. The maps \( \phi_1 \) and \( \phi_2 \) are then determined by
\[ (n - i + 1) \in \phi_1(Z) \] if and only if \( Z \cap S_i \neq Z \cap S_{i-1} \),
\[ i \in \phi_2(Z) \] if and only if \( Z \cap T_i \neq Z \cap T_{i-1} \).

It turns out that Definition 1.9 is essentially a special case of Stanley’s lattice-theoretical formulation of rank- and order-preserving maps that is implicit in [7]. In Section 2 we shall describe this situation in more detail.
Section 3 contains a further application of the pair of difference equations
given by (1.1). Indeed, we shall give a direct-combinatorial proof of an
identity of Carlitz [3] that relates Gaussian coefficients to binomial coeffi-
cients.

2. Rank—and Order—Preserving Maps

We first present the inductive construction of $\phi_1$ and $\phi_2$. To this end we
give the combinatorial interpretation of (1.1) and (1.2) that is found in
[1, 4, 5].

We start with the interpretation of (1.1a) from [5]. Choose a basis
$x_1, \ldots, x_{n+1}$ as in (1.6). Now let $U$ be a $k$-dimensional subspace
of $V_n(q)$.

There are two possibilities for $U$:

Case 1. $U$ includes the whole line spanned by $x_{n+1}$. If so, then
$U \cap V_n(q)$ is a subspace of dimension $k - 1$, and this intersection can be
chosen in $\binom{k-1}{1}$ ways, accounting for the first term on the right-hand side of
(1.1a).

Case 2. $U$ does not include the vector $x_{n+1}$. But then, the projection of $U$
onto $V_n(q)$ along the line $x_{n+1}$, is a subspace of dimension $k$, call it $W$, of
$V_n(q)$.

One then obtains $U$ by choosing such a $W$, and then “lifting it up,” e.g.,
choosing a basis $y_1, \ldots, y_k$ of $W$, and adding to each $y_i$ a multiple of $x_{n+1}$. There are altogether
$qk$ ways of performing the latter operation, and $\binom{n}{k}$ ways of performing the former. This accounts for the second term on the
right-hand side of (1.1a).

In order to make use of the above interpretation of (1.1a) we need the sets
$S,(S)$ and $d,(S)$ given by

**Definition 2.1.** Let $S$ be any $k$-dimensional subspace of $V_n(q)$ and
$\langle y_1, \ldots, y_k \rangle$ any basis of $S$. The sets $\delta,(S)$ and $d,(S)$ are determined uniquely
by

$$
\delta_1(S) \equiv \{ \langle y_1, \ldots, y_k, x_{n+1} \rangle \} = \{ S \oplus \langle x_{n+1} \rangle \},
$$

and

$$
\delta_1(S) \equiv \{ \langle y_1 + a_1 x_{n+1}, \ldots, y_k + a_k x_{n+1} \rangle \mid a_i \in GF(q) \}.
$$

If $S = \emptyset$, then $d,(S)$ is also 0.
Remark 2.3. It is clear that \( \| \delta_1(S) \| = 1 \), \( \| \Delta_1(S) \| = q^k \), and each of the subspaces in \( \Delta_1(S) \) is \( k \)-dimensional. In addition, the definition of \( \delta_1(S) \) and \( \Delta_1(S) \) is independent of which basis \( \langle y_1, \ldots, y_k \rangle \) of \( S \) that we use. This independence is an immediate consequence of the fact that if \( \langle z_1, \ldots, z_k \rangle \) is any other basis of \( S \), then

\[
\sum_{j=1}^{k} c_{ij}(z_j + a_j y_{n+1}) = y_i + \left( \sum_{j=1}^{k} c_{ij} a_j \right) y_{n+1},
\]

where \( C = (c_{ij}) \) is the invertible \( k \times k \) matrix such that \( y_i = \sum_{j=1}^{k} c_{ij} z_j \).

The combinatorial interpretation of (1.1b) which is implicit in [1, 5] is similar to that of (1.1a). Indeed, for \( U \) a \( k \)-dimensional subspace of \( V_{n+1}(q) \) there are two possibilities:

Case 1. \( U \) is contained in \( V_n(q) \). If so, then \( U \cap V_n(q) \) can be chosen in \( \binom{n}{k} \) ways, accounting for the second term on the right-hand side of (1.1b).

Case 2. \( U \) is not contained in \( V_n(q) \). But then, \( U \cap V_n(q) \) is a subspace of dimension \( (k - 1) \), call it \( W \), of \( V_n(q) \). One then obtains \( U \) by choosing such a \( W \) and then adjoining a vector in \( V_{n+1}(q) - V_n(q) \) to \( W \). There are altogether \( q^{n-k+1} \) ways of performing the latter operation, and \( \binom{n}{k-1} \) ways of performing the former. This accounts for the first term on the right-hand side of (1.1b).

Just as above we define the sets \( \delta_2(S) \) and \( \Delta_2(S) \) by means of

**Definition** 2.4. Let \( S \) be any \( k \)-dimensional subspace of \( V_n(q) \). The sets \( \delta_2(S) \) and \( \Delta_2(S) \) are determined uniquely by

\[
\delta_2(S) = \{ S \oplus \langle x \rangle \mid x \in V_{n+1}(q) - V_n(q) \},
\]

and

\[
\Delta_2(S) = \{ S \}.
\]

Remark 2.6. It is clear that \( \| \Delta_2(S) \| = 1 \) and each subspace in \( \delta_2(S) \) is \( (k + 1) \)-dimensional. Since there are \( (q^{n+1} - q^n) \) vectors \( x \) in \( V_{n+1}(q) - V_n(q) \), and \( (q^{k+1} - q^k) \) of them lead to the same subspace \( S \oplus \langle x \rangle \), it is immediate that \( \| \delta_2(S) \| = (q^{n+1} - q^n)/(q^{k+1} - q^k) \approx q^{n-k} \).

The combinatorial interpretation of (1.2) is well known [4] and can be regarded as a degenerate case of (1.1). Now, for \( U \), a \( k \)-element subset of \( B_{n+1} \), there are two possibilities:

\[1\) If \( X \) is a set, then by \( \| X \| \) we mean the cardinality of \( X \).
Case 1. \( U \) contains the element \( \{n + 1\} \). If so, then \( U \cap B_n \) is a \( (k-1) \)-element set, and this intersection can be chosen in \( \binom{k}{k-1} \) ways, accounting for the first term on the right-hand side of (1.2).

Case 2. \( U \) does not contain the element \( \{n + 1\} \). If so, then \( U \cap B_n \) can be chosen in \( \binom{k}{k} \) ways, accounting for the second term on the right-hand side of (1.2).

Just as before it is convenient to define the sets \( \delta_3(S) \) and \( \Delta_3(S) \) by

**Definition 2.7.** Let \( S \) be any \( k \)-element subset of \( B_n \). The sets \( \delta_3(S) \) and \( \Delta_3(S) \) are determined uniquely by

\[
\delta_3(S) = \{S \cup \{n + 1\}\}, \tag{2.8a}
\]

and

\[
\Delta_3(S) = \{S\}. \tag{2.8b}
\]

Thinking of \( B_n \) as \( \mathcal{L}_n(1) \) and \( \mathcal{L}_n(q) \) as its underlying collection of subspaces it is not hard to see from the above discussion that \( \mathcal{L}_{n+1}(q) \) is the disjoint union

\[
\mathcal{L}_{n+1}(q) = \sum_{S \in \mathcal{S}_n(q)} \delta_i(S) \oplus \Delta_i(S), \tag{2.9}
\]

where \( q = 1 \) or a prime power, and \( i = 1, 2, 3 \).

If \( \phi_1 \) is a map from \( \mathcal{L}_n(q) \) onto \( B_n \), then by considering (2.2) and (2.8) simultaneously we are naturally led to a mapping of \( \mathcal{L}_{n+1}(q) \) onto \( B_{n+1} \), which extends \( \phi_1 \). Similarly, considering (2.5) and (2.8) simultaneously leads to a mapping of \( \mathcal{L}_{n+1}(q) \) onto \( B_{n+1} \), which extends \( \phi_2; \mathcal{L}_n(q) \to B_n \). Indeed, it is immediate from (2.9) and the construction of the sets \( \delta_i(S) \) and \( \Delta_i(S) \) that the following inductive definition of \( \phi_1 \) and \( \phi_2 \) is well defined:

**Definition 2.10 (Inductive construction of \( \phi_1 \) and \( \phi_2 \)).** In what follows \( i = 1, 2 \). Choose a basis \( x_1, \ldots, x_{n+1} \) as in (1.6). The mapping \( \phi_i; \mathcal{L}_{n+1}(q) \to B_{n+1} \) which extends \( \phi_i; \mathcal{L}_n(q) \to B_n \) is uniquely determined by means of

\[
\phi_i(\emptyset) = \emptyset, \tag{2.1a}
\]

\[
\phi_i(T) = \{\phi_i(S)\} \cup \{n + 1\}, \quad \text{if } T \in \delta_i(S), \tag{2.1b}
\]

\[
= \phi_i(S), \quad \text{if } T \in \Delta_i(S). \tag{2.1c}
\]

We are now ready to prove one of the main results of this section.

**Theorem 2.12.** Each of the relations (1.4), (1.10), and (2.11) uniquely
determines the same mapping \( \phi_1 : \mathcal{L}_n(q) \rightarrow B_n \). Also, each of the relations (1.5), (1.11), and (2.11) determines the same mapping \( \phi_2 : \mathcal{L}_n(q) \rightarrow B_n \). 

Proof. For \( i = 1, 2 \) let \( \phi_i^1, \phi_i^2 \), and \( \phi_i^3 \) be the mappings of \( \mathcal{L}_n(q) \) onto \( B_n \) determined by Definitions 2.10, 1.3, and 1.9, respectively.

First, we shall show that \( \phi_1^i = \phi_2^i \) on each \( \mathcal{L}_n(q) \). We shall proceed by induction on \( n \). Clearly, \( \phi_1^1(\mathcal{O}) = \phi_2^1(\mathcal{O}) = 0 \). Given that the mappings \( \phi_1^i \) and \( \phi_2^i \) of \( \mathcal{L}_n(q) \) onto \( B_n \) are the same it is not hard to see that \( \phi_1^i \) and \( \phi_2^i \) are equal on \( \mathcal{L}_{n+1}(q) \) as well. By (2.9) it is clear that we need only consider two possibilities for \( T \in \mathcal{L}_{n+1}(q) \).

Case 1. \( T \in \delta_1(S) \) for some \( S \in \mathcal{L}_n(q) \). Since \( x_{n+1} \) only affects the \((n+1)\)th coordinates of the vectors in \( \delta_1(S) \) it is clear that if \( T \in \delta_1(S) \), then \( \phi_1^1(T) = \{ \phi_1^2(S) \} \cup \{ n + 1 \} \). On the other hand, since every vector \( x \in \mathcal{V}_{n+1}(q) \) has a nonzero \((n + 1)\)th coordinate it is immediate that if \( T \in \delta_2(S) \), then \( \phi_2^2(T) = \{ \phi_2^2(S) \} \cup \{ n + 1 \} \). But now both \( \phi_2^1 \) and \( \phi_1^i \) satisfy (2.1 lb) and the inductive hypothesis implies that \( \phi_1^1(T) = \phi_2^1(T) \) provided that \( T \in \delta_1(S) \).

Case 2. \( T \in \mathcal{A}_1(S) \) for some \( S \in \mathcal{L}_n(q) \). Let \( \langle y_1, \ldots, y_k \rangle \) be a basis of \( S \) and consider \( A_1(S) \). For suitable choice of the \( a_0, a_1, \ldots, a_k \) \( \mathcal{V}_n(q) \) is a basis of \( \mathcal{T} \). It is now immediate from (1.4) that \( \phi_1^1(S) \subseteq \phi_1^2(T) \) and \( \phi_2^1(T) \subseteq \phi_2^2(S) \) is either \( \langle n + 1 \rangle \) or 0. If \( \langle n + 1 \rangle \in \phi_2^1(T) \) we must have \( x_{n+1} \in T \). This implies, however, that we can find a, \( x_{n+1} \in GF(q) \) which are not all 0 such that

\[
x_{n+1} = \sum_{\nu=1}^{k} a_{\nu} (y_\nu + a_\nu x_{n+1}).
\] (2.13)

Since \( y, \ldots, y_k \in \mathcal{V}_n(q) \) are independent, (2.13) clearly gives \( a_0 = \ldots = a_k = 0 \) and \( a_1 a_i + \ldots + a_k a_{n+1} = 1 \) which is a contradiction. Thus, we must have \( \phi_1^1(T) = \phi_2^1(S) \) if \( T \in \mathcal{A}_1(S) \). On the other hand, since \( A_1(S) = \{ S \} \) it is immediate that \( \phi_2^2(T) = \phi_2^2(S) \) if \( T \in \mathcal{A}_1(S) \). Thus, both \( \phi_2^1 \) and \( \phi_1^i \) satisfy (2.1 lc) and the inductive hypothesis implies that \( \phi_1^1(T) = \phi_2^1(T) \) if \( T \in \mathcal{A}_1(S) \).

We complete the proof of Theorem 2.12 by showing that \( \phi_1^i = \phi_3^i \). It is not hard to see that (1.4) is equivalent to (1.10). Indeed, since \( Z \cap S_i \supseteq Z \cap S_{i-1} \) it is immediate from (1.7) and (1.10) that \( (n-i+1)E \phi_3^1(Z) \) if and only if there exists a vector \( z = (z_1, z_2, \ldots, z_n) \in Z \) with \( z_{n-i+1} \neq 0 \) and \( z_1 = z_2 = \cdots = z_{n-i} = 0 \). Starting with \( i = n \) and working down to \( i = 1 \) it is clear by (1.4) that \( (n-i+1)E \phi_3^1(Z) \) if and only if \( (n-i+1)E \phi_3^1(Z) \). That is, \( \phi_3^1 = \phi_1^i \).

By a similar argument it follows that (1.4) is equivalent to (1.11). Just note that (1.8) and (1.11) imply that \( i \in \phi_3^1(Z) \) if and only if there exists a vector \( z = (z_1, z_2, \ldots, z_n) \in Z \) with \( z_i \neq 0 \) and \( z_{i+1} = z_{i+2} = \cdots = z_n = 0 \).
Again, starting with $i = n$ and decreasing to $i = 1$ it is clear from (1.5) that $i \in \phi_2^1(Z)$ if and only if $i \in \phi_2^2(Z)$. Thus $\phi_2^1 = \phi_2^2$. Q.E.D.

The three formulations of $\phi_1$ and $\phi_2$ given by Theorem 2.12 lead to a very simple proof of

**Theorem 2.14.** As determined by either Definition 2.10, 1.3, or 1.9 the mappings $\phi_j: \mathcal{L}_n(q) \to B_n$ are both rank and order preserving. That is,

\[ \| \phi_1(Z) \| = \dim Z \]  
\[ W \not\subseteq Z \text{ implies that } \phi_1(W) \subseteq \phi_1(Z). \]

**Proof.** The rank-preserving property (2.15a) follows immediately by induction from (2.9), Definition 2.10, and Theorem 2.12. Indeed, just recall that for $i = 1.2$ we have

\[ \dim T = 1 + \dim S, \quad \text{if } T \in \delta_1(S), \]
\[ = \dim S, \quad \text{if } T \in \delta_1(S). \]

By (1.11) we have $W \cap T_i \neq W A T_{i-1}$. But then, $W \not\subseteq Z$ implies that $W \cap (Z \cap T_i) \neq W \cap (Z \cap T_{i-1})$, which is clearly equivalent to $(Z \cap T_i) \neq (Z \cap T_{i-1})$. Thus, again by (1.1), $i \in \phi_2(Z)$ and we obtain $\phi_2(W) \subseteq \phi_2(Z)$. For $\phi_1$ the proof is the same. Just replace $i$ by $n-i+1$.

Q.E.D.

We shall now describe how Definition 1.9 is essentially a special case of Stanley’s lattice-theoretical formulation of rank- and order-preserving maps that is implicit in [7].

Let $L$ be a finite lattice and $C = \{ \vec{0} < C_1 < C_2 < \ldots < C_n = \vec{1} \}$ a maximal chain of $L$, where $\vec{0}$ denotes the bottom element and $\vec{1}$ the top element of $L$. If for every chain $K$ of $L$ the sublattice generated by all the meets and joints of $K$ and $C$ is distributive, then $C$ is known as a modular or M-chain and we call $(L, C)$ a supersolvable lattice (or SS-lattice).

Now, if $L$ is an SS-lattice whose M-chain $C$ has length $n$ (or cardinality $n + 1$), then every maximal chain $K$ of $L$ has length $n$ since all maximal chains of the distributive lattice generated by $C$ and $K$ have the same length. Thus, $L$ has defined on it a unique rank function $r$: $L \to \{0, 1, 2, \ldots, n \}$ satisfying $r(\vec{0}) = 0$, $r(\vec{1}) = n$, $r(y) = r(x) + 1$ if $y$ covers $x$.

This given, it can be shown that Stanley’s construction of the sets $\Gamma(I, J)$ in [7] implies

**Proposition 2.16.** Let $L$ be an SS-lattice with M-chain $C = \{ \vec{0} < C, <$
\(C_2 < \cdots < C_n = 1\), and rank function \(r\). Let \(B_n\) be the lattice of subsets of \(\{1, 2, \ldots, n\}\) ordered by inclusion. Define \(\phi : L \to B_n\) by means of

\[
i \in \phi(x) \quad \text{if and only if} \quad x \land C_i \neq x \land C_{i-1}.
\]

We then have

\[
\|\phi(x)\| = r(x)
\]

(2.18a)

\(x \leq y\) implies that \(\phi(x) \subseteq \phi(y)\). (2.18b)

**Remark 2.19.** If \(\sigma\) is any permutation of \(\{1, 2, \ldots, n\}\), then let \(\hat{\sigma} : B_n \to B_n\) be determined by \(\hat{\sigma}(\{i_1, \ldots, i_k\}) = \{\sigma(i_1), \ldots, \sigma(i_k)\}\). If \(\phi\) is defined by (2.17), then it is immediate that \(\|\hat{\sigma}(\phi(x))\| = r(x)\) and \(x \leq y\) implies that \(\hat{\sigma}(\phi(x)) \subseteq \hat{\sigma}(\phi(y))\). That is, the composition \(\hat{\sigma} \circ \phi\) is also rank and order preserving.

It turns out that the chains \(S\) and \(T\) of \(\mathcal{L}_n(q)\) given by (1.7) and (1.8) are both modular chains. Thus, by setting \(L = \mathcal{L}_n(q)\) and \(C = T\) it is clear from (1.8) and (1.11) that the rank- and order-preserving map \(\phi_2\) is a special case of Proposition 2.6. On the other hand, the map \(\phi_1\) is a \textit{special case} of Remark 2.19. Just set \(L = \mathcal{L}_n(q)\), \(C = S\), and define \(\sigma\) by \(u(i) = n - i + 1\).

A natural question to ask at this point is whether or not all rank- and order-preserving maps of \(\mathcal{L}_n(q)\) onto \(B_n\) can be obtained as in Remark 2.19.

We shall close this section by noting that there is \textit{no} permutation \(\sigma\) of \(\{1, 2, \ldots, n\}\) such that \(\phi_1 = \hat{\sigma} \circ \phi_2\). To see this let \(x_i\) be as in (1.6), \(k < n\), and \(\{i_1, r, i_2, \ldots, i_k\}\) a \((k + 1)\)-element set of \(\{1, 2, \ldots, n\}\) such that \(i_1 < r < i_2 < \cdots < i_k\). Now it is clear that \(\phi_1(\langle x_{i_1}, x_{i_2}, \ldots, x_{i_k} \rangle) = \{i_1, i_2, \ldots, i_k\}\) which is also \(\phi_2(\langle x_{i_1}, x_{i_2}, \ldots, x_{i_k} \rangle)\). If there were a \(\sigma\) such that \(\phi_1 = \hat{\sigma} \circ \phi_2\), then \(\hat{\sigma}\) must take \(\{i_1, i_2, \ldots, i_k\}\) onto itself. These relations, however, cannot both occur since

\[
\phi_1(\langle x_{i_1}, x_r + x_{i_2}, x_{i_3}, \ldots, x_{i_k} \rangle) = \{i_1, r, i_3, \ldots, i_k\},
\]

while

\[
\phi_2(\langle x_{i_1}, x_r + x_{i_2}, x_{i_3}, \ldots, x_{i_k} \rangle) = \{i_1, i_2, \ldots, i_k\}.
\]

Thus, no such \(\sigma\) exists and \(\phi_1\) cannot be trivially obtained as a permutation composed with \(\phi_2 \circ \phi_1\), however, can be thought of as a "projective-space dual" of \(\phi_2\).

### 3. An Expansion of Gaussian Coefficients in Terms of Binomial Coefficients

It is well known that \(\lim_{q \to 1} \left[ \binom{n}{k} \right] = \binom{n}{k}\). On the other hand there is a relation between Gaussian and binomial coefficients that does not involve limits.
Indeed, as part of an investigation of a certain problem in Abelian groups Carlitz [3] discovered the identity

\[ \binom{n}{k} = \sum_{j=k}^{n} \binom{n}{j} F(j, k), \]  

(3.1)

where,

\[ F(n + 1, k) = F(n, k - 1) + (q^k - 1) F(n, k), \]  

(3.2a)

with

\[ F(0, 0) = 1 \quad \text{and} \quad F(n, k) = 0 \quad \text{unless} \quad 0 \leq k \leq n. \]  

(3.2b)

In order to give a direct-combinatorial proof of (3.1) we shall first put together a combinatorial interpretation of the array \((F(n, k))\) determined by (3.2). To this end we need

**DEFINITION 3.3.** Let \(x_i\) be as in (1.6) so that \(V_i(q) = \langle x_1, \ldots, x_n \rangle\). A subspace \(S\) of \(V_n(q)\) is an \((n - j)\)-dimensional coordinate plane of \(V_n(q)\) if and only if there exist \(0 < i_1 < i_2 < \cdots < i_j < n\) such that

\[ s = \{(z_1, \ldots, z_n) \in V_n(q) \mid z_{i_1} = z_{i_2} = \cdots = z_{i_j} = 0\}. \]  

(3.4)

The vector space \(V_i(q)\) is an \(n\)-dimensional coordinate plane of itself.

This given, we have

**THEOREM 3.5.** Let \(F(n, k)\) be defined as in (3.2). Then, \(F(n, k)\) counts the number of \(k\)-dimensional subspaces \(S\) of \(V_i(q)\) such that there is no \((n - 1)\)-dimensional coordinate plane of \(V_n(q)\) which contains \(S\).

**Proof.** Let \(\Omega_1(n, k)\) be the set of \(k\)-dimensional subspaces of \(V_i(q)\) which are not contained in any \((n - 1)\)-dimensional coordinate plane of \(V_n(q)\) and let \(\Omega_2(n, k)\) denote the \(k\)-dimensional subspaces of \(V_i(q)\) that are contained in an \((n - 1)\)-dimensional coordinate plane of \(V_n(q)\), It is clear that

\[ \Omega_1(n, k) \cap \Omega_2(n, k) = \emptyset, \]  

(3.6a)

\[ \|\Omega_1(n, k)\| + \|\Omega_2(n, k)\| = \binom{n}{k}. \]  

(3.6b)

Now, let \(S\) be any \(k\)-dimensional subspace of \(V_n(q)\) and \((y_1, \ldots, y_k)\) any basis of \(S\). It is not hard to see that: \(\langle y_1, \ldots, y_k \rangle \in \Omega_1(n, k)\) if and only if \(\langle y_1, \ldots, y_k, x_{n+1} \rangle \in \Omega_1(n + 1, k + 1)\), \(\langle y_1 + 0x_{n+1}, \ldots, y_k + 0x_{n+1} \rangle \in \Omega_2(n + 1, k)\), and \(\langle y_1, \ldots, y_k \rangle \in \Omega_2(n, k)\) if and only if \(\langle y_1 + a_1 x_{n+1}, \ldots, y_k + a_k x_{n+1} \rangle \in \Omega_1(n + 1, k)\) and \(a_1 a_2 \cdots a_k \neq 0\).
A direct induction based on these observations, (3.6), (2.2), Remark 2.3, and (2.9), yields

\[ \| \Omega_1(n + 1, k) \| = \| \Omega_1(n, k - 1) \| + (q^k - 1) \| \Omega_1(n, k) \|, \]  

with

\[ \| \Omega_1(0, 0) \| = 1 \quad \text{and} \quad \| \Omega_1(n, k) \| = 0 \quad \text{unless} \quad 0 \leq k \leq n. \]  

(3.7a)  

(3.7b)

Theorem 3.5 follows at once from (3.7) since (3.2) has a unique solution.

Q.E.D.

In view of Theorem 3.5 it is interesting to note that if \( S_q(n, k) = F(n, k) q^{k/2} (q - 1)^{-(n-k)} \), then the polynomial \( S_q(n, k) \) is the q-analog of the Stirling numbers of the second kind determined by

\[ S_q(n + 1, k) = q^{(k-1)} S_q(n, k - 1) + ((q^k - 1)/(q - 1)) S_q(n, k). \]

We now give our combinatorial proof of the identity in (3.1). To this end, let \( \mathcal{L}_{n,k}(q) \) be the set of all \( k \)-dimensional subspaces of \( V_n(q) \) and \( S \) any element of \( \mathcal{L}_{n,k}(q) \). It is immediate that if \( S \) is contained in two different coordinate planes \( T_1 \) and \( T_2 \) of \( V_n(q) \), then

\[ T_1 \cap T_2 \text{ is a coordinate plane of both } T_1 \text{ and } T_2, \]  

\[ \dim(T_1 \cap T_2) < \min\{\dim T_1, \dim T_2\}, \]  

\[ S \subseteq T_1 \cap T_2. \]

(3.8a)  

(3.8b)  

(3.8c)

Since \( V_n(q) \) is an \( n \)-dimensional coordinate plane of itself and \( S \in \mathcal{L}_{n,k}(q) \) it is clear from (3.8) that there is a unique coordinate plane \( T(S) \) of minimal dimension \( j \) such that \( S \subseteq T(S) \) and \( k \leq j \leq n \).

Now, denote by \( CP_{n,j}(q) \) the set of all \( j \)-dimensional coordinate planes of \( V_n(q) \). Clearly, if we set \( S_1 \sim S_2 \) if and only if \( T(S_1) = T(S_2) \), then \( \sim \) is an equivalence relation on \( \mathcal{L}_{n,k}(q) \) whose equivalence classes are of the form

\[ \psi_1(T_0) = \{ S \in \mathcal{L}_{n,k}(q) \mid T(S) = T_0 \}, \]

(3.9)

where \( T_0 \) is a \( j \)-dimensional coordinate plane of \( V_n(q) \) with \( k \leq j \leq n \). It then follows that \( \mathcal{L}_{n,k}(q) \) is the disjoint union

\[ \mathcal{L}_{n,k}(q) = \sum_{j=k}^{n} \sum_{T_0 \in CP_{n,j}(q)} \psi_1(T_0). \]

(3.10)

Using the basis \( \langle x_1, \ldots, x_n \rangle \) of \( V_n(q) \) to define the coordinate planes of the \( j \).
dimensional coordinate plane \( T_0 \) it is not hard to see from (3.8) that \( \psi_1(T_0) \) is equal to the set

\[
\psi_2(T_0) \equiv \{ S \in \mathcal{L}_{n,k}(q) : S \subseteq T_0 \text{ and } S \text{ is not contained in any } (j-1)\text{-dimensional coordinate plane of } T_0 \} \tag{3.11}
\]

Since \( T_0 \) is isomorphic to \( V_j(q) \), Theorem 3.5 implies that

\[
\| \psi_1(T_0) \| = \| \psi_2(T_0) \| = F(j, k). \tag{3.12}
\]

As there are clearly \( \binom{r}{j} \) \( j \)-dimensional coordinate planes of \( V_j(q) \) and \( \| \mathcal{L}_{n,k}(q) \| = \left[ \frac{q^k}{1 - q^k} \right] \), identity (3.1) follows immediately from (3.10), (3.11), and (3.12). Q.E.D.

REFERENCES