A Remark on the Kolmogorov–Stein Inequality*

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In this paper, essentially developing the Stein method, we prove the Kolmogorov–Stein inequality for any Orlicz norm (with the same constants).

1. INTRODUCTION

A. N. Kolmogorov has given the following result [1]: Let \( f(x), f'(x), \ldots, f^{(n)}(x) \) be continuous and bounded on \( \mathbb{R} \). Then

\[
\| f^{(k)} \|_n \leq C_{k,n} \| f \|_{n-k} \| f^{(n)} \|_n
\]

where \( 0 < k < n \), \( C_{k,n} = K_{n-k} / K_n \),

\[
K_i = \frac{4}{\pi} \sum_{j=0}^{\infty} (-1)^i/(2j + 1)^{i+1}
\]

for even \( i \), while

\[
K_i = \frac{4}{\pi} \sum_{j=0}^{\infty} 1/(2j + 1)^{i+1}
\]

for odd \( i \). Moreover the constants are best possible.

This result has been extended by E. M. Stein to the \( L_p \)-norm [2]. The Kolmogorov–Stein inequality and its variants are a problem of interest for

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many mathematicians and have various applications (see, for example, [3, 4] and their references).

In this paper, essentially developing the Stein method [2], we prove this inequality for an arbitrary Orlicz norm \( \| \cdot \|_\Phi \). The obtained result has been successfully applied to proving the corresponding imbedding theorems [5–7] for Sobolev–Orlicz spaces of infinite order and the result [8] for any Orlicz norm.

2. RESULTS

Let \( \Phi(t): [0, +\infty) \to [0, +\infty] \) be an arbitrary Young function [9–12], i.e., \( \Phi(0) = 0, \Phi(t) \geq 0, \Phi'(t) = 0, \) and \( \Phi(t) \) is convex. Denote by

\[ \Phi'(t) = \sup_{s \geq 0} \{ ts - \Phi(s) \} , \]

which is the Young function conjugate to \( \Phi(t) \) and \( L_\Phi(\mathbb{R}) \), the space of measurable functions \( u(x) \) such that

\[ |\langle u, v \rangle| = \left| \int u(x)v(x) \, dx \right| < \infty \]

for all \( v(x) \) with \( \rho(v, \Phi) < \infty \), where

\[ \rho(v, \Phi) = \int \Phi(|v(x)|) \, dx. \]

Then \( L_\Phi(\mathbb{R}) \) is a Banach space with respect to the Orlicz norm

\[ \| u \|_\Phi = \sup_{\rho(v, \Phi) \leq 1} \left| \int u(x)v(x) \, dx \right| , \]

which is equivalent to the Luxembung norm

\[ \| f \|_\Phi = \inf \left\{ \lambda > 0 : \int \Phi(|f(x)|/\lambda) \, dx \leq 1 \right\} < \infty. \]

We have the following results [9]:

**Lemma 1.** Let \( u \in L_\Phi(\mathbb{R}) \) and \( v \in L_\Phi(\mathbb{R}) \). Then

\[ \int |u(x)v(x)| \, dx \leq \| u \|_\Phi \| v \|_\Phi . \]

**Lemma 2.** Let \( u \in L_\Phi(\mathbb{R}) \) and \( v \in L_1(\mathbb{R}) \). Then

\[ \| u * v \|_\Phi \leq \| u \|_\Phi \| v \|_1 . \]

Recall that \( \| \cdot \|_p = \| \cdot \|_p \) when \( 1 \leq p < \infty \) and \( \Phi(t) = t^p ; \) and \( \| \cdot \|_\infty = \| \cdot \|_\infty \) when \( \Phi(t) = 0 \) for \( 0 \leq t \leq 1 \) and \( \Phi(t) = \infty \) for \( t > 1 \).
THEOREM 1. Let $\Phi(t)$ be an arbitrary Young function, $f(x)$ and its
generalized derivative $f^{(k)}(x)$ be in $L_{\Phi}(\mathbb{R})$. Then $f^{(k)}(x) \in L_{\Phi}(\mathbb{R})$ for all
$0 < k < n$ and

$$
\|f^{(k)}\|_{\Phi} \leq C_{k,n} \|f\|_{\Phi}^{n-k} \|f^{(n)}\|_{\Phi}^k.
$$

Proof. We begin to prove (1) with the assumption that $f^{(k)}(x) \in L_{\Phi}(\mathbb{R})$,
$0 \leq k \leq n$.

Fix $0 < k < n$. It is known that

$$
\rho(v, \Phi) = 1 \text{ if and only if } \|v\|_{\Phi} = 1.
$$

Therefore, by the definition we get

$$
\|f^{(k)}\|_{\Phi} = \sup_{\|v\|_{\Phi} \leq 1} \left| \int_{-\infty}^{\infty} f^{(k)}(x)v(x) \, dx \right|.
$$

Let $\epsilon > 0$. We choose a function $v_{\epsilon}(x) \in L_{\Phi}(\mathbb{R})$ such that $\|v_{\epsilon}\|_{\Phi} = 1$ and

$$
\left| \int_{-\infty}^{\infty} f^{(k)}(x)v_{\epsilon}(x) \, dx \right| \geq \|f^{(k)}\|_{\Phi} - \epsilon.
$$

Put

$$
F_{\epsilon}(x) = \int_{-\infty}^{\infty} f(x + y)v_{\epsilon}(y) \, dy.
$$

Then $F_{\epsilon}(x) \in L_{\Phi}(\mathbb{R})$ by virtue of Lemma 1, and

$$
F_{\epsilon}^{(r)}(x) = \int_{-\infty}^{\infty} f^{(r)}(x + y)v_{\epsilon}(y) \, dy, \quad 0 \leq r \leq n.
$$

Actually, for every function $\varphi(x) \in C_0^\infty(\mathbb{R})$ it follows from the assumption
and Lemma 1 that

$$
\langle F_{\epsilon}^{(r)}(x), \varphi(x) \rangle = (-1)^r \langle F_{\epsilon}(x), \varphi^{(r)}(x) \rangle
$$

$$
= (-1)^r \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x + y)v_{\epsilon}(y) \, dy \right) \varphi^{(r)}(x) \, dx
$$

$$
= (-1)^r \int_{-\infty}^{\infty} v_{\epsilon}(y) \left( \int_{-\infty}^{\infty} f(x + y) \varphi^{(r)}(x) \, dx \right) \, dy
$$

$$
= \int_{-\infty}^{\infty} v_{\epsilon}(y) \left( \int_{-\infty}^{\infty} f^{(r)}(x + y) \varphi(x) \, dx \right) \, dy
$$

$$
= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f^{(r)}(x + y)v_{\epsilon}(y) \, dy \right) \varphi(x) \, dx
$$

$$
= \left( \int_{-\infty}^{\infty} f^{(r)}(x + y)v_{\epsilon}(y) \, dy, \varphi(x) \right).
$$

So we have proved (3).
For all $x \in \mathbb{R}$, clearly,

$$|F_{x}^{(r)}(x)| \leq \|f^{(r)}(x + \cdot)\|_{\Phi} \|v_{\epsilon}\|_{\mathcal{S}} = \|f^{(r)}\|_{\Phi}.$$  

Now we prove continuity of $F_{x}^{(r)}(x)$ on $\mathbb{R}$ ($0 \leq r \leq n$). We show this for $r = 0$ by contradiction: Assume that for some $\epsilon > 0$, point $x^{0}$, and subsequence $|t_{k}| \to 0$

$$\left| \int_{-\infty}^{\infty} (f(x^{0} + t_{k} + y) - f(x^{0} + y))v_{\epsilon}(y)dy \right| \geq \epsilon, \quad k \geq 1. \quad (4)$$

Since $f \in L_{\Phi}$ we get easily $f \in L_{1/\epsilon}(\mathbb{R})$. Then for any $n = 1, 2, \ldots$, $f(t_{k} + y) \to f(y)$ in $L_{1}((-n, n))$. Therefore, there exists a subsequence, denoted again by $(t_{k})$, such that $f(t_{k} + y) \to f(y)$ a.e. in $(-n, n)$. Therefore, there exists a subsequence (for simplicity of notation we assume that it is coincident with $(t_{k})$) such that $f(x^{0} + t_{k} + y) \to f(x^{0} + y)$ a.e. in $(-\infty, \infty)$.

On the other hand, without loss of generality we may assume that $\rho(2f, \Phi) < \infty$. Therefore by the Young inequality we get

$$\left| f(x^{0} + t_{k} + y) - f(x^{0} + y) \right| v_{\epsilon}(y) \leq \Phi(\left| f(x^{0} + t_{k} + y) - f(x^{0} + y) \right|) + \bar{\Phi}(\left| v_{\epsilon}(y) \right|)$$

$$\leq \frac{1}{2}\Phi(2\left| f(x^{0} + y) \right|) + \frac{1}{2}\Phi(2\left| f(x^{0} + t_{k} + y) \right|) + \bar{\Phi}(\left| v_{\epsilon}(y) \right|).$$

The last expression belongs to $L_{1}(\mathbb{R})$, therefore by Lebesgue's theorem we have

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} \left| f(x^{0} + t_{k} + y) - f(x^{0} + y) \right| v_{\epsilon}(y)dy = 0,$$

which contradicts (4). The cases $1 \leq r \leq n$ are proved similarly. The continuity of $F_{x}^{(r)}(x)$ has been proved.

The functions $F_{x}^{(r)}(x)$ are continuous and bounded on $\mathbb{R}$. Therefore, it follows from the Kolmogorov inequality and (2)–(3) that

$$\left( \|f^{(k)}\|_{\Phi} - \epsilon \right)^{n} \leq |F_{x}^{(k)}(0)|^{n} \leq \|F_{x}^{(k)}\|^{n}_{\infty}$$

$$\leq C_{k, n}\|F_{x}\|_{\infty}^{n-k}\|F_{x}^{(n)}\|_{\infty}^{k}.$$  

(5)

On the other hand,

$$\|F_{x}\|_{\infty} \leq \|f(x + y)\|_{\Phi}\|v_{\epsilon}(y)\|_{\mathcal{S}} = \|f\|_{\Phi},$$

(6)

$$\|F_{x}^{(n)}\|_{\infty} \leq \|f^{(n)}(x + y)\|_{\Phi}\|v_{\epsilon}(y)\|_{\mathcal{S}} = \|f^{(n)}\|_{\Phi}.$$  

(7)
Combining (5)–(7), we get

$$\left( |f^{(k)}|_{\Phi} - \epsilon \right)^n \leq C_{k,n} \|f\|_{\Phi}^{n-k} \|f^{(n)}\|_{\Phi}^k.$$  

By letting $\epsilon \to 0$ we have (1).

To complete the proof, it remains to show that $f^{(k)} \in L_\Phi(\mathbb{R})$, $0 < k < n$ if $f, f^{(n)} \in L_\Phi(\mathbb{R})$.

Let $\psi_k(x) \in C_0^\infty(\mathbb{R})$, $\psi_k(x) \geq 0$, $\psi_k(x) = 0$ for $|x| \geq \lambda$ and $\int \psi_k(x) \, dx = 1$. We put $f_k = f * \psi_k$. Then $f_k \in C^\infty(\mathbb{R})$ because of $f \in L_1(\omega, \mathbb{R})$. Therefore, $f_k^{(k)} = f * \psi_k^{(k)}$, $k \geq 0$, and it is easy to check that $f_k^{(n)} = f^{(n)} * \psi_k$.

On the other hand, it follows from Lemma 2 that $f_k^{(k)} = f * \psi_k^{(k)} \in L_\Phi(\mathbb{R})$, $k \geq 0$. Therefore, by the fact proved above, we have

$$\|f_k^{(k)}\|_{\Phi} \leq C_{k,n} \|f_k\|_{\Phi}^{n-k} \|f_k^{(n)}\|_{\Phi}^k, \quad 0 < k < n.$$  

Therefore, since

$$\|f_k\|_{\Phi} \leq \|f\|_{\Phi} \cdot \|\psi_k\|_1 = \|f\|_{\Phi}, \quad \|f_k^{(n)}\|_{\Phi} \leq \|f^{(n)}\|_{\Phi} \|\psi_k\|_1 = \|f^{(n)}\|_{\Phi}$$

we get that, for any $0 \leq k \leq n$, the sequence $f_k^{(k)}$ is bounded in $L_\Phi(\mathbb{R})$.

Now we prove that, for any $0 \leq k \leq n$, there exists a subsequence, which is $*$-convergent to some $g_k \in L_\Phi(\mathbb{R})$. (We say that $h_k$ is $*$-convergent to $h$, where $h_k, h \in L_\Phi(\mathbb{R})$, if $\int h_k \, v \to \int h \, v$ for all $v \in L_\Psi(\mathbb{R})$.) We will show, for example, the fact that $f_k$ is $*$-convergent to $f$ by contradiction: Assume that for some $\epsilon_0 > 0$, $v \in L_\Psi(\mathbb{R})$ and a subsequence $\lambda_k \to 0$,

$$\left| \int (f_{\lambda_k}(x) - f(x)) \, v(x) \, dx \right| \geq \epsilon_0, \quad k \geq 1. \quad (8)$$

Then, it is known that $f_{\lambda_k} \to f$, $\lambda \to 0$ in $L_{1,1,\omega}(\mathbb{R})$. Therefore, there exists a subsequence $\{k_m\}$ (for simplicity we assume that $k_m = m$) such that $f_{\lambda_k}(x) \to f(x)$ a.e.

We may assume that $\rho(2f, \Phi) < \infty$. Then it follows from Young's inequality that

$$|f_{\lambda_k}(x) - f(x)| \, |v(x)| \leq \frac{1}{2} \Phi(2|f_{\lambda_k}(x)|) + \frac{1}{2} \Phi(2|f(x)|) + \tilde{\Phi}(|v(x)|),$$

moreover, the right side of the last inequality belongs to $L_1(\mathbb{R})$. Therefore, by virtue of Lebesgue's theorem we get

$$\lim_{k \to \infty} \int |f_{\lambda_k}(x) - f(x)| \, |v(x)| \, dx = 0$$

because of $f_{\lambda_k}(x) \to f(x)$ a.e., which contradicts (8).
Finally, it follows from ∗-convergence $f_\lambda \to f$ that for any $\varphi \in C_0^\infty(\mathbb{R})$

$$\langle f_\lambda^{(k)}(x), \varphi(x) \rangle = (-1)^k \langle f_\lambda(x), \varphi^{(k)}(x) \rangle$$

$$\to (-1)^k \langle f(x), \varphi^{(k)}(x) \rangle = \langle f^{(k)}(x), \varphi(x) \rangle.$$

Therefore, since the ∗-convergence of some subsequence of $\{f_\lambda^{(k)}\}$ to $g_k \in L_\Phi(\mathbb{R})$, we get $f^{(k)} = g_k \in L_\Phi(\mathbb{R}) (0 < k < n)$. So we have proved the fact that $f^{(k)} \in L_\Phi(\mathbb{R})$ for all $0 < k < n$ if $f, f^{(n)} \in L_\Phi(\mathbb{R})$. The proof is complete.

Remark 1. To obtain Theorem 1 we have developed the Stein method because, for example, the property $[g(x + h) - g(x)]/h \to g'(x)$ in the $L_p$ mean ($1 \leq p < \infty$), which is used in [2], holds for $L_\Phi$ only if $\Phi(t)$ satisfies the $\Delta_2$-condition (see [10]).

Remark 2. For periodic functions we have:

**Theorem 2.** Let $\Phi(t)$ be an arbitrary Young function, $f(x)$ and its generalized derivative $f^{(n)}(x)$ be in $L_\Phi(\mathbb{T})$. Then $f^{(k)}(x) \in L_\Phi(\mathbb{T})$ for all $0 < k < n$ and

$$\|f^{(k)}\|_\Phi \leq C_{k,n} \|f\|_\Phi^{n-k} \|f^{(n)}\|_\Phi^k,$$

where $\mathbb{T}$ is the torus and $\|\cdot\|_\Phi$ is the corresponding norm.

Remark 3. By the representation [11, p. 135]

$$\|u\|_{(\Phi)} = \sup_{\|v\|_{(\Phi)} \leq 1} \left| \int u(x)v(x) \, dx \right|,$$

it is easy to see that the obtained results still hold for any Luxemburg norm.

**REFERENCES**