Generalized inverses and similarity to partial isometries

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\textbf{A B S T R A C T}

We obtain some results related to the problems of Badea and Mbekhta (2005) \cite{1} concerning the similarity to partial isometries using the generalized inverses. Especially, we involve the Moore–Penrose inverses. Also a characterization for such a similarity is given in the terms of dilations similar to unitary operators, which leads to a new criterion for the similarity to an isometry and to a quasinormal partial isometry.

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1. Introduction

Throughout this paper $\mathcal{H}$ stands for a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ is the Banach algebra of all bounded linear operators on $\mathcal{H}$ where $I = I_\mathcal{H}$ is the identity operator. For $T \in \mathcal{B}(\mathcal{H})$ we write $T^*$ for its adjoint, and $\mathcal{R}(T)$, $\mathcal{N}(T)$ denote the range and the kernel of $T$, respectively.

If $T$, $S \in \mathcal{B}(\mathcal{H})$ such that $TST = T$ and $STS = S$ then $S$ (respectively, $T$) is called a generalized inverse of $T$ (respectively, $S$) (see \cite{1,3,8,9,11,12}). It is easy to see that $T$ has a generalized inverse if and only if $\mathcal{R}(T)$ is closed. In this case $T$ and $S$ are idempotents and $\mathcal{R}(T) = \mathcal{R}(TS)$, $\mathcal{N}(T) = \mathcal{N}(ST)$, and there exists a unique generalized inverse $T^\dagger$ of $T$ for which $TT^\dagger$ and $T^\dagger T$ are orthogonal projections. This operator $T^\dagger$ is called the Moore–Penrose inverse of $T$ \cite{3,9,11}.

When $T$ is a partial isometry, that is, $T^*$ is a generalized inverse of $T$, then the Moore–Penrose inverse of $T$ is just $T^*$. In particular this happens for every orthogonal projection $P_M$ (onto a closed subspace $M$ of $\mathcal{H}$), and every isometry $V$ ($V^*V = I$), coisometry $V^*$, and unitary operator ($VV^* = I$). An isometry has left inverses and a coisometry has right inverses.

It was proved in \cite{9} that a contraction $T$ on $\mathcal{H}$ (that is, $T^*T \leq I$) which has a contractive generalized inverse $S$ is a partial isometry. This result raises a natural question which was formulated by Badea and Mbekhta in \cite{1}, namely if the above result can be extended up to similarity (in hypothesis and conclusion). Another question was posed in \cite{1} in the context of power bounded operators which means that $\sup_{n \geq 1} \|T^n\| < \infty$, and particularly, for a power partial isometry which means that $T^n$ is a partial isometry for every integer $n \geq 1$.

Recall that two operators $T$ and $V$ on $\mathcal{H}$ are similar if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that $AT = VA$. In this case one says also that $T$ is similar to $V$ by the similarity $A$. The well-known Sz.-Nagy criterion states that $T$ is similar...
to an isometry if and only if $T$ and a left inverse $S$ of $T$ are power bounded operators. In particular it follows that an invertible operator $T$ is similar to a unitary operator if and only if $T$ and $T^{-1}$ are power bounded operators.

An analogous criterion of similarity to partial isometries is not true even if $T$ and $S$ (a generalized inverse of $T$) are polynomially bounded (by [1, Example 4.1]). The problem of [1] mentioned above refers to the case when $T$ and $S$ are similar to contractions, but also the case when $T$, $S$ are power bounded and $S^n$ is a generalized inverse of $T^n$ for any $n \geq 1$, for the similarity of $T$ to a partial isometry, is an open problem. Some results related to these problems were recently obtained in [10,12].

In this note we obtain some results concerning the similarity to partial isometries, using the generalized inverses. We consider a special case when $T$ and its Moore–Penrose inverse $T^\dagger$ have a particular form, suggested by the “model” of 2-quasi-isometries from [10]. Finally, we get a general criterion of similarity to partial isometries in terms of dilations similar to unitary operators.

2. Similarity and generalized inverses

We remark that any operator $T \in B(\mathcal{H})$ has a matrix representation on $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ of the form

$$T = \begin{pmatrix} C & S \\ 0 & 0 \end{pmatrix}$$

with $C = T|_{\mathcal{R}(T)} \in B(\mathcal{R}(T))$ and $S \in B(\mathcal{N}(T^*), \mathcal{R}(T))$.

For an operator $T$ with its polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and $U$ is the partial isometry with $\mathcal{N}(T) = \mathcal{N}(U)$, the operators $\tilde{T} = |T|U$ and $\Delta(T) = |T|^{1/2}U|T|^{1/2}$ are called the Duggal, respectively Aluthge transform of $T$ (see [6]).

In [1] has been posed the following question: if $T, S \in B(\mathcal{H})$ are two operators similar to contractions such that $S$ is a generalized inverse of $T$, then $T$ is similar to a partial isometry? We remark that the hypothesis implies that there exist a contraction $C$ and a (positive) invertible operator $A$ in $B(\mathcal{H})$ such that $C = ATA^{-1}$. Then the operator $B = ASA^{-1}$ will be a generalized inverse of $C$ and $B$ is similar to a contraction on $\mathcal{H}$ because $S$ is such an operator. Thus the above problem on $T$ and $S$ can be reduced to the same problem for the operators $C$ and $B$, that is, to the case when $T$ is a contraction.

The answer to the above problem is not known, but some partial results were given in [12]. In the sequel we also obtain some facts related to this problem and the results of [12].

Notice firstly that if $T$ is similar to a quasinormal partial isometry, then $T|_{\mathcal{R}(T)}$ is similar to an isometry, hence $T|_{\mathcal{R}(T)}$ is injective, that is, $\mathcal{R}(T) \cap \mathcal{N}(T) = \{0\}$. Now, we give a partial converse of this statement in the terms of Duggal and Aluthge transforms $\tilde{T}$ and $\Delta(T)$ of $T$. This completes Theorem 1.4 from [10].

**Theorem 2.1.** Let $T \in B(\mathcal{H})$ having the range closed such that $\mathcal{R}(T) \cap \mathcal{N}(T) = \{0\}$. If either $\tilde{T}$, or $\Delta(T)$ are power bounded with the range closed and the Moore–Penrose inverse of $\tilde{T}$, respectively of $\Delta(T)$, is power bounded, then $T$ is similar to a quasinormal partial isometry.

**Proof.** Suppose firstly that $\tilde{T}$ is power bounded with $\mathcal{R}(\tilde{T})$ closed, such that its Moore–Penrose inverse $\tilde{T}^\dagger$ is power bounded. We have $\tilde{T} = \tilde{T}_1 \oplus 0$ on $\mathcal{H} = \mathcal{R}(T^*) \oplus \mathcal{N}(T)$ and $U_0\tilde{T}_1 = T_0U_0$, where $\tilde{T}_1 = |T|T_1T^{\dagger}_1$, $U_0 = U|_{\mathcal{R}(T^*)}$, and $T_0 = T|_{\mathcal{R}(T)}$, $U$ being the partial isometry from the polar decomposition of $T$. By hypothesis $T_0$ is injective, hence $T_1$ is injective ($U_0$ being unitary), and $\mathcal{R}(\tilde{T}_1) = \mathcal{R}(\tilde{T})$ is closed. If $\tilde{T}_1^\dagger$ is the Moore–Penrose inverse of $\tilde{T}_1$ then it is easy to see that $\tilde{T}^\dagger = \tilde{T}_1^\dagger \oplus 0$ on $\mathcal{R}(T^*) \oplus \mathcal{N}(T)$ is the Moore–Penrose inverse of $\tilde{T}$. Since $\tilde{T}^\dagger$ is power bounded by hypothesis, it follows that $\tilde{T}_1^\dagger$ is power bounded.

As $\tilde{T}_1^\dagger$ is a left inverse of $\tilde{T}_1$, by the well-known theorem of Sz.-Nagy [13] it follows that $\tilde{T}_1$ is similar to an isometry. Consequently, $T$ is similar to a quasinormal partial isometry by Corollary 1.6 from [10].

Assume now that $\Delta(T)$ is power bounded with $\mathcal{R}(\Delta(T))$ closed, such that its Moore–Penrose inverse is power bounded. We have $\Delta(T) = T_\ast \oplus 0$ on $\mathcal{R}(T^*) \oplus \mathcal{N}(T^*)$ and $U_1T_\ast = T_0U_1$, where $T_\ast = |T|T_1T^{\dagger}_1$, $U_1 = U|_{\mathcal{R}(T^*)}$ and $T_0 = T|_{\mathcal{R}(T)}$ is an invertible operator in $B(\mathcal{R}(T^*))$. As $T_0$ is injective, $T_\ast$ is also injective, and $\mathcal{R}(T_\ast) = \mathcal{R}(\Delta(T))$ is closed. Using the same argument as above, one can show that $T_\ast$ is similar to an isometry, hence $T_0$ is similar to an isometry and by Proposition 1.5 from [10], $T$ is similar to a quasinormal partial isometry. This ends the proof. \[\square\]

**Remark 2.2.** A converse statement of the previous proposition, in the following sense, is true. More exactly, let us assume that $T$ is similar to a quasinormal partial isometry. By Proposition 1.5 [10], $T_0$ is similar to an isometry and consequently (see the previous proof) $\tilde{T}_1$ and $T_\ast$ are similar to an isometry. Then clearly $\tilde{T} = \tilde{T}_1 \oplus 0$ and $\Delta(T) = T_\ast \oplus 0$ on $\mathcal{R}(T^*) \oplus \mathcal{N}(T)$ is similar to a quasinormal partial isometry, and by Theorem 3.6 [12], $\tilde{T}$ and respectively $\Delta(T)$ are power bounded and they have a power bounded generalized inverse (which can be different to the Moore–Penrose inverse of $\tilde{T}$ and $\Delta(T)$, respectively). In fact, by Theorem 3.15 [12] one can find a generalized inverse $S$ of $\tilde{T}$ (or $\Delta(T)$) with $\mathcal{R}(\tilde{T}) \subset \mathcal{R}(S)$ (respectively, $\mathcal{R}(\Delta(T)) \subset \mathcal{R}(S)$). In the sequel we see other conditions under which a generalized inverse $S$ of $\tilde{T}$ with $\mathcal{R}(\tilde{T}) \subset \mathcal{R}(S)$ exists.
Theorem 2.3. Let $T \in B(H)$, then $\mathcal{R}(T) + \mathcal{N}(T)$ is a closed subspace and $\mathcal{R}(T) \cap \mathcal{N}(T) = \{0\}$ if and only if there exists a generalized inverse $S$ of $T$ such that $\mathcal{R}(T) \subset \mathcal{R}(S)$. Furthermore, in this case $T$ is similar to a quasinormal partial isometry if and only if $T$ and $S|_{\mathcal{R}(T)}$ are power bounded operators.

Proof. Let $\mathcal{M}_0$ be a topological complement of $\mathcal{R}(T) + \mathcal{N}(T)$ in $\mathcal{H}$, that is, $\mathcal{M}_0$ is a closed subspace such that $\mathcal{H} = \mathcal{M}_0 + \mathcal{R}(T) + \mathcal{N}(T)$ and $\mathcal{M}_0 \cap (\mathcal{R}(T) + \mathcal{N}(T)) = \{0\}$. We remark that the hypothesis yields a result of [7] that the range $\mathcal{R}(T)$ is closed, and we can see that the subspace $\mathcal{M}_0 + \mathcal{R}(T)$ is closed using the angle between two subspaces [3]. Indeed we have

$$
c_0(\mathcal{M}_0, \mathcal{R}(T)) := \sup \{ |\langle h, k \rangle| : h \in \mathcal{M}_0, k \in \mathcal{R}(T), \|h\| = \|k\| = 1 \} \leq c_0(\mathcal{M}_0, \mathcal{R}(T) + \mathcal{N}(T)).$$

Since $\mathcal{R}(T) + \mathcal{N}(T)$ is closed and its intersection with $\mathcal{M}_0$ is reduced to $\{0\}$ one has $c_0(\mathcal{M}_0, \mathcal{R}(T) + \mathcal{N}(T)) < 1$. Hence $c_0(\mathcal{M}_0, \mathcal{R}(T)) < 1$, which means that $\mathcal{M}_0 + \mathcal{R}(T)$ is closed and $\mathcal{M}_0 \cap \mathcal{R}(T) = \{0\}$. Also, we have $[\mathcal{M}_0 + \mathcal{R}(T)] \cap \mathcal{N}(T) = \{0\}$. Indeed, if $h \in \mathcal{N}(T)$ and $h = h_0 + h_1$ with $h_0 \in \mathcal{M}_0, h_1 \in \mathcal{R}(T)$ then $h_0 + h_1 - h = 0$, hence $h_0 = 0$ and $h = h_1 \in \mathcal{R}(T) \cap \mathcal{N}(T)$ that is $h = 0$. It follows that $\mathcal{M} = \mathcal{M}_0 + \mathcal{R}(T)$ is a topological complement of $\mathcal{N}(T)$, and so there exists a generalized inverse $S$ of $T$ such that $\mathcal{R}(S) = \mathcal{M} \supset \mathcal{R}(T)$.

Conversely, if there exists $S$ a generalized inverse of $T$ with $\mathcal{R}(T) \subset \mathcal{R}(S)$ then obviously $\mathcal{R}(T) \cap \mathcal{N}(T) = \{0\}$. Moreover, since $\mathcal{M}$ is invariant for $T$, $S$ one has $(S|_{\mathcal{M}})\dagger T|_{\mathcal{M}} S = I|_{\mathcal{M}}$ which yields $\mathcal{R}(T|_{\mathcal{M}} S|_{\mathcal{M}})$ is closed for any $n \geq 1$. Therefore, $\mathcal{R}(T^n) = \mathcal{R}(T(T|_{\mathcal{M}} S|_{\mathcal{M}})^n)$ is closed for every $n \geq 1$, which assures by a result of [7] that $\mathcal{R}(T) + \mathcal{N}(T)$ is closed.

For the second part, we use the above remark that $S|_{\mathcal{M}}$ is a left inverse of $T|_{\mathcal{M}}$ in $B(\mathcal{M})$. Thus, if $T$ and $S|_{\mathcal{M}}$ are power bounded operators then $T|_{\mathcal{M}}$ is similar to an isometry [13], hence $T|_{\mathcal{R}(T)}$ is also similar to an isometry and by Proposition 1.5 [10] it follows that $T$ is similar to a quasinormal partial isometry. The converse assertion is obvious and this ends the proof. \qed

In the previous theorem we have $\mathcal{R}(T) = \mathcal{R}(S)$ if and only if $TS = ST$, which means that $S$ is the Drazin inverse of $T$, and in this case $T$ will be similar to a normal partial isometry (see [7,12]).

From a result of [5] we remark that if $T$ satisfies the equivalent conditions from the first part of Theorem 2.3, on a separable Hilbert space having the spectrum in the open unit disc and $\dim(\mathcal{R}(T)) = \dim(\mathcal{N}(T)) = \dim(\mathcal{N}(T^*)) = \chi_0$, then $T$ is similar to a partial isometry which, obviously, is not quasinormal. In this case the operator $S$ from Theorem 2.3 is not power bounded.

Corollary 2.4. Let $T \in B(H)$ with $\mathcal{R}(T)$ closed and $\mathcal{N}(T) \subset \mathcal{N}(T^*)$. If $T$ and the Moore–Penrose inverse $T^\dagger$ of $T$ are power bounded operators then $T$ is similar to a quasinormal partial isometry.

Proof. We have $\mathcal{R}(T) \cap \mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) \subset \mathcal{R}(T^*) = \mathcal{R}(T^\dagger)$. So the conclusion of corollary follows from the second part of the previous theorem. \qed

This corollary can be also obtained from Theorem 3.15 [12].

Under some conditions it is enough to improve the hypothesis of Corollary 2.4 only for $T|_{\mathcal{R}(T)}$ to obtain the similarity of $T$ to a partial isometry. Such a case appears in the following

Theorem 2.5. Let $T \in B(H)$ with $\mathcal{R}(T)$ closed and having the matrix representation $(2.1)$. Suppose $C^* S = 0, \mathcal{N}(C) \subset \mathcal{N}(C^*)$ and that $C$ and its Moore–Penrose inverse are power bounded operators. Then $T$ is similar to a partial isometry.

Proof. The condition $C^* S = 0$ ensures that $\mathcal{R}(T) = \mathcal{R}(C) \oplus \mathcal{R}(S)$. Since $\mathcal{R}(T)$ is closed, $\mathcal{R}(C)$ will be also closed and by Corollary 2.4 it follows that $C$ is similar to a quasinormal partial isometry $W_0$ of $\mathcal{R}(T)$. Using the same argument as in the proof of Theorem 1.1 [10] one can obtain that $T$ is similar to an operator $W$ on $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ of the form

$$W = \begin{pmatrix} W_0 & W_1 \\ 0 & 0 \end{pmatrix}$$

with $W_1 \in B(\mathcal{N}(T^*), \mathcal{R}(T))$.

Since $W_0$ is a quasinormal partial isometry one has $\mathcal{R}(W_0) \subset \mathcal{R}(W_0^\dagger)$ and so $W_0$ is an isometry on $\mathcal{R}(W_0)$. But $\mathcal{R}(W_0^\dagger) \subset W_0 \mathcal{R}(T) = \mathcal{R}(W_0)$ and $W_0 \mathcal{R}(W_0^\dagger) \subset W_0 \mathcal{R}(W) = W \mathcal{R}(W) = \mathcal{R}(W^2)$. Thus $\mathcal{R}(W^2)$ is an invariant subspace for $W_0$ and $W_0|_{\mathcal{R}(W^2)} = W_0|_{\mathcal{R}(W^2)}$ is an isometry.

As $\mathcal{R}(T)$ and $\mathcal{R}(T^2) = \mathcal{R}(C)$ are closed, $\mathcal{R}(W)$ and $\mathcal{R}(W^2)$ are also closed (by similarity). Then by Corollary 2.9 [10], $T$ is similar to a partial isometry. \qed
Another version of Theorem 2.5 is the following

**Theorem 2.6.** Let $T \in B(\mathcal{H})$ with $\mathcal{R}(T)$ closed such that $C^*S = 0$ in the matrix representation (2.1) of $T$. Let $C^\dagger$ and $S^\dagger$ be the Moore–Penrose inverses of $C$ and $S$, respectively. If there exist positive invertible operators $A_0$ on $\mathcal{R}(T)$ and $A_1$ on $\mathcal{N}(T^*)$ such that $A_0C = C^\dagger A_0$ and $A_0S = S^\dagger A_1$, then $T$ is similar to a partial isometry. If furthermore $C$ and $C^\dagger$ are contractions, then $C = C^\dagger S^\dagger$ is a partial isometry.

**Proof.** Since $\mathcal{R}(T) = \mathcal{R}(C) \oplus \mathcal{R}(S)$ (by $C^*S = 0$) and $\mathcal{R}(T)$ is closed, $\mathcal{R}(C)$ and $\mathcal{R}(S)$ are also closed; therefore, there exist $C^\dagger$ and $S^\dagger$ as in the hypothesis. Define $T^\dagger$ on $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ by

$$T^\dagger = \begin{pmatrix} C^\dagger & 0 \\ S^\dagger & 0 \end{pmatrix}.$$ 

We have

$$TT^\dagger = \begin{pmatrix} CC^\dagger + SS^\dagger & 0 \\ 0 & 0 \end{pmatrix}, \quad T^\dagger T = \begin{pmatrix} C^\dagger C & C^\dagger S \\ S^\dagger C & S^\dagger S \end{pmatrix}.$$

Since $CC^\dagger = P_{\mathcal{R}(C)}$ and $SS^\dagger = P_{\mathcal{R}(S)}$, one has $CC^\dagger + SS^\dagger = I_{\mathcal{R}(T)}$ and so $TT^\dagger = P_{\mathcal{R}(T)}$ and $TT^\dagger T = T$. On the other hand, we have $\mathcal{N}(C^\dagger) = \mathcal{N}(C^*C) = \mathcal{R}(S)$ and $\mathcal{N}(S^\dagger) = \mathcal{N}(S^*S) = \mathcal{R}(C)$, hence $C^\dagger S = 0$ and $S^\dagger C = 0$. Also $C^\dagger C = P_{\mathcal{R}(C)}$, $S^\dagger S = P_{\mathcal{R}(S)}$, and it follows that $T^\dagger T = P_{\mathcal{R}(T)}$ and $T^\dagger T T^\dagger = T^\dagger$. We conclude that $T^\dagger$ is the Moore–Penrose of $T$.

In the case when $C$ and $C^\dagger$ are contractions on $\mathcal{R}(T)$, by Theorem 3.1 [9] it follows that $C$ is a partial isometry and $C^\dagger = C^*$. In this case $T|_{\mathcal{R}(T)} = C = T^\dagger|_{\mathcal{R}(T)}$ where $\mathcal{R}(T) = \mathcal{R}(T^*)$, and this fact suggests that $T$ can be similar to $T^\dagger$.

In general, if $T$ is similar to $T^\dagger S$ by a positive operator $A \in B(\mathcal{H})$ then by Theorem 3.1 [1], $T$ will be similar to a partial isometry. To get such an operator $A$, we write $A$ on $\mathcal{H} = \mathcal{R}(T) \oplus \mathcal{N}(T^*)$ in the form

$$A = \begin{pmatrix} A_0 & A_2 \\ A_2^* & A_1 \end{pmatrix},$$

with $A_0$, $A_1$ positive operators on $\mathcal{R}(T)$, $\mathcal{N}(T^*)$ respectively, and $A_2 \in B(\mathcal{N}(T^*), \mathcal{R}(T))$. Now the condition $AT = T^\dagger S$ leads to the equations

$$A_0C = C^\dagger A_0 + S^\dagger A_2^*, \quad A_0S = CA_2 + S^\dagger A_1, \quad A_1^* C = 0, \quad A_2^* S = 0.$$

Since $\mathcal{R}(T) = \mathcal{R}(C) \oplus \mathcal{R}(S)$, the last two conditions imply $A_2 = 0$. Thus $A$ has the form $A = A_0 \oplus A_1$ and $A$ is invertible if and only if $A_0$ and $A_1$ are invertible. In this case, the above conditions become $A_0C = C^\dagger A_0$ and $A_0S = S^\dagger A_1$, which ensure that $T$ is similar to $T^\dagger$ by $A$. This ends the proof. \qed

**Remark 2.7.** The problem concerning the similarity to partial isometries mentioned in this section remains open even for a quasicontraction $T$ (i.e. $T|_{\mathcal{R}(T)}$ is a contraction) which has its Moore–Penrose inverse $T^\dagger$ a quasicontraction (even if $C^*S = 0$ in the matrix (2.1) of $T$). In this case $T$ and $T^\dagger$ are similar to contractions (by Theorem 4.1 [2]), a more general context than $T$ and $T^\dagger$ contractions, where a positive answer to the problem was given in Theorem 3.1 [9]. Even the case when $T|_{\mathcal{R}(T)}$ is a partial isometry remains an interesting open problem regarding the similarity of $T$ to a partial isometry. The previous theorems give only sufficient conditions for similarity to partial isometries. Another positive answer in a particular case can be given using a result of [5]:

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. If $T = \begin{pmatrix} W & S \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$, a closed range operator, with $W$ a partial isometry and $W^* S = 0$, having the spectrum in the open unit disc and $\dim(\mathcal{R}(T)) = \dim(\mathcal{R}(W)) = \dim(\mathcal{N}(S)) = \chi_0$, then $T$ is similar to a partial isometry.

3. Similarity by dilations

We obtain now an extension of the concept of dilation used in [4].

Recall that an operator $U \in B(\mathcal{K})$ with $\mathcal{K} \supset \mathcal{H}$ is a dilation of $T \in B(\mathcal{H})$ if $T^n h = PU^n h$ for $h \in \mathcal{H}$ and $n \in \mathbb{N}$, where $P$ is the orthogonal projection onto $\mathcal{H}$. In this case $U$ is an extension of $T$ if $U \mathcal{H} \subset \mathcal{H}$, and $U$ is a lifting of $T$ if $TU = PU$.

**Theorem 3.1.** Let $T \in B(\mathcal{H})$. Then $T$ is similar to a partial isometry if and only if $T$ has a dilation $U \in B(\mathcal{K})$ which is similar to a unitary operator by a positive invertible operator $B$ on $\mathcal{K}$, such that $\mathcal{H}$ reduces $B$ and $U^{-1}$ is a dilation of a generalized inverse of $T$.

**Proof.** Suppose that $T$ is similar to a partial isometry. By Theorem 3.1 [1] there exist a generalized inverse $S$ of $T$ and a positive invertible operator $A_0$ on $\mathcal{H}$ such that $T^* A_0 T \leq A_0$ and $S^* A_0 S \leq A_0$. Equivalently, there exist two contractions $V$ and $W$ on $\mathcal{H}$ such that $AT = VA$ and $AS = WA$, where $A = A_0^{1/2}$. Clearly, $W = ASA^{-1}$ is a generalized inverse of $V = ATA^{-1}$, and since $V$ and $W$ are contractions, by Theorem 3.1 [9] we have $W = V^*$ and $V$ is a partial isometry.
Now let $\tilde{V} \in B(\mathcal{K})$ be a unitary dilation of $V$, therefore $\tilde{V}^*$ is a unitary dilation of $W$. This means that for $h \in \mathcal{H}$ and $n \in \mathbb{N}$ one has

$$V^n h = P\tilde{V}^n h, \quad W^n h = P\tilde{V}^n h,$$

whence

$$T^n h = A^{-1} P\tilde{V}^n Ah, \quad S^n h = A^{-1} P\tilde{V}^n Ah.$$

Defining the operator $B$ on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$ by $B = A^{-1} \oplus I_{\mathcal{H}^\perp}$, it follows that $B$ is positive invertible on $\mathcal{K}$ and $\mathcal{H}$ is a reducing subspace for $B$, hence $BP = PB$. Then we obtain

$$T^n h = BP\tilde{V}^n B^{-1} h = PB\tilde{V}^n B^{-1} h = PU^n h,$$

where $U = B\tilde{V} B^{-1}$ is similar to a unitary operator. Analogously, we find $S^n h = PU^{-n} h$ since $B\tilde{V}^n B^{-1} = B\tilde{V}^{-n} B^{-1} = U^{-n}$, for $h \in \mathcal{H}$ and $n \in \mathbb{N}$. Consequently, $U$ and $U^{-1}$ are dilations for $T$ and $S$ respectively, and the necessary part of the theorem is proved.

Conversely, we assume that for $T$ there exist $S$, $U$ and $B$ in $B(\mathcal{K})$ as above, that is, $U$, $U^{-1}$ are dilations for $T$, $S$ respectively, and $U$ is similar to a unitary operator $\tilde{U}$ by $B$ such that $BP = PB$. We define a new Hilbert norm in $\mathcal{K}$ by $\|k\|_B = \|Bk\|$, $k \in \mathcal{K}$. Then for $h \in \mathcal{H}$ we have

$$\|Th\|_B = \|BTh\| = \|BPuh\| = \|PUh\| = \|Bu\| = \|Bh\| = \|h\|_B,$$

hence $T$ is a contraction in the norm $\| \cdot \|_B$ on $\mathcal{H}$. Similarly, one obtains that $S$ is a contraction in the norm $\| \cdot \|_B$ and finally, by Theorem 3.1 [1] it follows that $T$ is similar to a partial isometry. $\square$

**Corollary 3.2.** An operator $T \in B(\mathcal{H})$ is similar to an isometry if and only if $T$ has an extension $U \in B(\mathcal{K})$ similar to a unitary operator by a positive invertible operator $B$ on $\mathcal{K}$, such that $\mathcal{H}$ reduces $B$ and $U^{-1}$ is a lifting for a left inverse of $T$.

**Proof.** Suppose that $T$ is similar to an isometry. Preserving the notations from the previous proof, we have that $V = AT A^{-1}$ is an injective partial isometry, that is, an isometry. Hence we can choose $\tilde{V}$ to be a unitary extension on $\mathcal{K} \supseteq \mathcal{H}$ of $V$ such that $\tilde{V}$ is similar to $U$ by $B = A^{-1} \oplus I_{\mathcal{H}^\perp}$ on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. One has

$$Uh = B\tilde{V} B^{-1} h = B\tilde{V} Ah = A^{-1} V Ah = Th,$$

for $h \in \mathcal{H}$, hence $U$ is an extension of $T$. From $B\tilde{V} = UB$ we infer $Bu = \tilde{V}^* B$ and $B^{-1} U^{-1} = \tilde{V}^* B^{-1}$ whence it follows that $B^2 Ub = (U^{-1})^*$. Since the subspace $\mathcal{H}$ is invariant for $B$, $B^{-1}$ and $U$, it follows that it is also invariant for $(U^{-1})^*$. But $U^{-1}$ is a dilation of a generalized (necessarily, left) inverse $S$ of $T$, so by the previous remark $(U^{-1})^*$ will be an extension of $S^*$ and, in turn, $U^{-1}$ is a lifting of $S$.

Conversely, if there exist $U$, $B$ and $S$ as in corollary, then, by Theorem 3.1, $T$ is similar to a partial isometry $W$ on $\mathcal{H}$. Also, $U$ is injective being similar to a unitary operator, so $T$ and $W$ are injective, hence $W$ is an isometry. This ends the proof. $\square$

**Remark 3.3.** Suppose that $T$ in Theorem 3.1 is a contraction and that $T$ is similar to a partial isometry $V$. Then by the commutant lifting theorem of Sz.-Nagy and Foias [14] the minimal unitary dilations of $T$ and $V$ will be unitarily equivalent. So, choosing $\tilde{V}$ the minimal unitary dilation of $V$ in the proof of Theorem 3.1, we infer that the corresponding dilation $U$ of $T$ is similar to the minimal unitary dilation of $T$.

Also, we remark that the generalized inverse $S$ of $T$ invoked in the last two results is necessarily similar to $T^*$. A dual version of Corollary 3.2 is the following

**Corollary 3.4.** An operator $T \in B(\mathcal{H})$ is similar to a coisometry if and only if $T$ has a lifting $U \in B(\mathcal{K})$ similar to a unitary operator by a positive invertible operator $B$ on $\mathcal{K}$, such that $\mathcal{H}$ reduces $B$ and $U^{-1}$ is an extension of a right inverse of $T$.

Finally, we can give a criterion of similarity to a quasinormal partial isometry as follows:

**Corollary 3.5.** An operator $T \in B(\mathcal{H})$ is similar to a quasinormal partial isometry if and only if $T$ has an extension $U \in B(\mathcal{K})$ similar to a normal partial isometry by a positive invertible operator $B$ on $\mathcal{K}$, such that $\mathcal{H}$ reduces $B$ and $U^{-1}$ is a lifting of a generalized inverse of $T$. 

Proof. Our assumption implies that the operator $V$ from the proof of Theorem 3.1 is a quasinormal partial isometry, that is, it has the form $V = V_0 \oplus 0$ with $V_0$ an isometry on an appropriate subspace $\mathcal{H}_0$ of $\mathcal{H}$. Taking $\tilde{V}_0 \in B(K_0)$ a unitary extension of $V_0$ and $\tilde{V} = \tilde{V}_0 \oplus 0$ on $K = K_0 \oplus (\mathcal{H} \ominus \mathcal{H}_0)$, one has that $\tilde{V}$ is a normal partial isometry on $K$. As in the proof quoted above it follows that $\tilde{V}$ is similar to an extension of $T$ by an operator $B$ on $K$ having the required properties.

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References