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Torsion-Free Abelian Groups of Finite Rank Projective as Modules over Their Endomorphism Rings*

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Since Richman and Walker [4] characterized modules over *PID*'s which are injective as modules over their endomorphism rings, various homological properties of abelian groups as modules over their endomorphism rings have been studied. In this paper we characterize the torsion-free abelian groups, A, of finite rank which are projective as modules over their endomorphism rings, E. The main results include the following.

THEOREM 2.4. *The following are equivalent*:

(a) *A is E-projective and generated by two elements*;

(b) *A is $\dot{E}$-projective*;

(c) *A is E-quasi-projective*;

(d) *for each prime $p \in Z$, A_p (A localized at p) is E_p (E localized at p) cyclic projective*;

(e) *A is a genus summand of E.*

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THEOREM 2.8. *A is E-projective if and only if*

(1) $\mathrm{Hom}_Z(C, A) = \mathrm{Hom}_C(C, A)$ *where* $C = \text{center } E$, *and*

(2) $A \cong J \oplus M$ *where J is a faithful projective ideal of C and M is a C-module.*

Finally, it is shown (Theorem 2.11) that A is E-projective if and only if A is nearly isomorphic to a group B which is cyclic and projective as a module over its endomorphism ring. Such groups B have been characterized by Reid and Niedzwicki [3].

1. Definitions and Preliminaries

Unless otherwise stated, all groups are abelian and torsion free of finite rank. In particular, A is always a torsion-free abelian group of finite rank, $E = E(A)$ is the endomorphism ring of A, and C the center of E. The concepts of quasi-isomorphism and near-isomorphism are used repeatedly.

DEFINITION 1.1. (a) The groups A and B are quasi-isomorphic if there is a monomorphism $f: B \to A$ such that $A/f(B)$ is finite.

(b) The groups A and B are nearly isomorphic if for each $n \in Z^+$, there exist $f_n \in \mathrm{Hom}_Z(B, A)$ and $g_n \in \mathrm{Hom}_Z(A, B)$ such that $f_n g_n = m \cdot 1_A$, $g_n f_n = m \cdot 1_B$ for some $m \in Z^+$ with $(n, m) = 1$.

Another important notion is that of an E-ring. There should be no confusion caused by this (unrelated) second use of the letter E.

DEFINITION 1.2. A ring S with identity is an E-ring provided every endomorphism of the abelian group $\langle S, + \rangle$ is left multiplication by some element of S.

These rings have been studied by Bowshell and Schultz [5, 6].

Finally, we quote for reference two useful results.

THEOREM 1.3 (Reid and Niedzwicki [3]). *Let B be a group. Then B is cyclic and projective as an E(B)-module iff* $B = S \oplus K$ *where S is an E-ring and K an S-module such that* $\mathrm{Hom}_Z(S, K) = \mathrm{Hom}_S(S, K)$.

Note. In this case $(1_S, 0)$ generates B as an E-module.

PROPOSITION 1.4 (Vinsonhaler [7]). *If A is E-quasi-projective, then A is quasi-isomorphic to some B which is E(B) cyclic and projective. Furthermore, A is generated by two elements as an E-module.*

Remark 1.5. These two results say that if A is E-projective, then there is $0 \neq n \in Z$ and $B = S \oplus K$ as in Theorem 1.3 such that $nA \subseteq B \subseteq A$.

2. THE CHARACTERIZATION

The first characterization of E-projective groups is a list of equivalent properties. This result, Theorem 2.4, is preceded by three preliminary ones.

LEMMA 2.1. *Let R be any ring with 1 and M a cyclic, quasi-projective R-module. If $M = Rm$ for some $m \in M$, and $L = \operatorname{ann}_R(m) = \{r \in R \mid rm = 0\}$, then for each $x \in M$, there is an $l \in L$ such that $L(1+l)x = 0$.*

Proof. Define $f: M \to M/Lx$ by $f(m) = x + Lx$. This is an R-homomorphism, which by quasi-projectivity lifts to $g \in \operatorname{Hom}_R(M, M)$. Then g has the form $g(m) = x + lx$ for some $l \in L$. It follows that $L(x + lx) = L(1+l)x = 0$.

LEMMA 2.2. *Let R be any ring with 1 and M a cyclic, quasi-projective R-module. If (*), there is a finite set $X \subseteq M$ such that $\forall r \in R$, $rX = 0 \Rightarrow r = 0$, then M is projective.*

Proof. Let $M = Rm$, $m \in M$, and $L = \operatorname{ann}_R(m)$. It suffices to show $R \to^f Rm \to 0$, where $f(r) = rm$, has a splitting. Let $X = \{x_1, \dots, x_n\} \subseteq M$ satisfy (*). Using Lemma 2.1, pick $l_1 \in L$ such that $L(1 + l_1)x_1 = 0$, and inductively, $l_k \in L$ such that $L(1 + l_k)[(1 + l_{k-1}) \cdots (1 + l_1)x_k] = 0$ for $1 \leqslant k \leqslant n$. Then $(1 + l_n)(1 + l_{n-1}) \cdots (1 + l_1) = 1 + l$ for some $l \in L$. Note that $(1 + l)x_k = (1 + l')(1 + l_k)(1 + l_{k-1}) \cdots (1 + l_1)x_k$, where $(1 + l') = (1 + l_n) \cdots (1 + l_{k+1})$. Since $l' \in L$, $(1 + l)x_k = (1 + l')(1 + l_k) \cdots (1 + l_1)x_k = (1 + l_k) \cdots (1 + l_1)x_k$. Thus $L(1 + l)x_k = 0$ for $1 \leqslant k \leqslant n$. By (*), $L(1 + l) = 0$. Therefore $g: Rm \to R$ given by $g(m) = 1 + l$ is an R-homomorphism and $fg = 1$ on Rm.

The next lemma is a generalization of part of Theorem 3.1 of [8].

LEMMA 2.3. *If A is E-quasi-projective, then A/nA is E-cyclic for each $0 \neq n \in Z$.*

Proof. First we show that if I is a maximal ideal of finite index in C, the center of E, and k is a positive integer, then A/I^kA is E cyclic. To prove this it suffices to show A/IA is E cyclic, for if $A = Ea + IA$, then $IA = IEa + I^2A$ and $A = Ea + I^2A$. Repeating this argument gives $A = Ea + I^kA$ for all positive k.

Suppose $\{\bar{x}_i = x_i + IA \mid 1 \leqslant i \leqslant n\}$ is a minimal set of E generators for A/IA, $n > 1$. Note $|C/I| < \infty$ implies A/IA is bounded, thus finite, so minimal generating sets exist. Let H be given by $H/IA = E\bar{x}_1 \cap \sum_{i=2}^n E\bar{x}_i$. Then H is an E-submodule of A and $A/H = U \oplus V$, $U = E\bar{\bar{x}}_1$, and $V = \sum_{i=2}^n E\bar{\bar{x}}_i$, where $\bar{\bar{x}}_i = x_i + H$. This is a non-trivial E decomposition because of the minimality of $\{\bar{x}_i\}$.

Let u, v be the projections of A/H onto U, V and, by quasi-projectivity, choose u', $v' \in C$ with $\Pi u' = u\Pi$, $\Pi v' = v\Pi$, where $\Pi: A \to A/H$ is the natural map. Finally, let $T = \{f \in C \mid f(A) \subseteq H\}$. T is an ideal in C containing the maximal ideal I. But $T = C$ is impossible, for then $A = H$, contradicting the minimality of $\{\bar{x}_i\}$. Thus, $T = I$.

We have $u'v' \in T$, so $u' \in T$ or $v' \in T$. Thus, $U = (0)$ or $V = (0)$, a contradiction. This shows that A/IA and, hence, A/I^kA is cyclic for I a maximal ideal of finite index in C.

To complete the proof, let $n \in Z^+$, and J/nC be the Jacobson radical of C/nC. Then $J = I_1 I_2 \cdots I_k$, I_i distinct maximal ideals of C. Since C/nC is Artinian, $J^m = I_1^m \cdots I_k^m \subseteq nC$ for some $m \in Z^+$. Then $A/J^mA \cong \oplus_{i=1}^k A/I_i^mA$ is E cyclic since each summand is cyclic and the I_i's are distinct maximal ideals. Thus, A/nA is E cyclic.

THEOREM 2.4. *Let A be a reduced torsion-free group of finite rank. The following are equivalent:*

(a) *A is E-projective and generated by two elements.*

(b) *A is E-projective.*

(c) *A is E-quasi-projective.*

(d) *For each prime $p \in Z$, A_p is E_p-cyclic and projective. Here $A_p = Z_p \otimes A$, $E_p = Z_p \otimes E$, denote the usual localizations.*

(e) *For each $n \in Z^+$ there exists $f \in \mathrm{Hom}_E(A, E)$ and $g \in \mathrm{Hom}_E(E, A)$ such that $gf = m \cdot 1_A$ where $m \in Z^+$ and $(m, n) = 1$. (A is said to be a genus summand of E—see* [1].)

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (d) First, A_p is E_p-quasi-projective. Given K an E_p-submodule of A_p and $f \in \mathrm{Hom}_{E_p}(A_p, A_p/K)$, use the fact that A is E-finitely generated (Proposition 1.4) to find an integer m with $(m, p) = 1$ such that $mf(A) \subseteq A + K/K \cong A/A \cap K$. By quasi-projectivity, mf lifts to $g \in C = \mathrm{Hom}_E(A, A)$. Then $(1/m)g \in \mathrm{Hom}_{E_p}(A_p, A_p)$ provides a lifting of f.

Second, A_p is E_p cyclic. To prove this assertion, first note that since A is reduced, there exists $k \in Z^+$ such that $kX \neq X$ for all non-zero subgroups X of A (Lady [2]). For any set of prime integers S, let $A_S = Z_S \otimes_Z A$, $E_S = Z_S \otimes_Z E$, where Z_S is the integers localized at S.

We show A_S is E_S cyclic for any finite set S such that $S \supseteq \{p \mid p \text{ divides } k\}$. Let $n_S = \Pi\{p \mid p \in S\}$. Then, for each $f \in E_S$, $1 - n_S f$ is a unit of E_S. To prove this write $f = (1/m)g$, $g \in E$, $m \in Z$, $(m, n_S) = 1$. Let $h = (m - n_S g) \in E$. Then h is monic, since $n_S \operatorname{Ker} h = \operatorname{Ker} h$. Furthermore, $h(A)$ is p-pure in A for all $p \in S$, so there is an $l \in Z$ with $(l, n_S) = 1$ such that $lA \subseteq h(A) \subseteq A$. (Note that since A is torsion free of finite rank, any monomorphism $h: A \to A$ has $A/h(A)$ finite.) Therefore, h, and hence

$1 - n_S f$, are units in E_S. Thus, $n_S E_S \subseteq J(E_S)$—the Jacobson radical of the ring E_S. (This last assertion, with a slightly different proof, appears in [9].)

By Lemma 2.3, $A/n_S A$ is E cyclic, so that $A_S/n_S A_S$ is E_S cyclic. Hence $A_S = E_S u + n_S A_S$ for some $u \in A_S$. Since A_S is finitely generated and $n_S E_S \subseteq J(E_S)$, it follows from Nakayama's lemma that $A_S = E_S u$.

Finally, let p be a prime and $S = S_k \cup \{p\}$, where $S_k = \{p \mid p \text{ divides } k\}$. Then $Z_p = (Z_S)_p$, $A_p = (A_S)_p$ and $E_p = (E_S)_p$. Consequently, A_p is E_p cyclic since A_S is E_S cyclic.

Now let X be a maximal Z-independent set in A, and apply Lemma 2.2 to conclude that A_p is E_p cyclic projective.

(d) ⇒ (e) We use induction on the number k of distinct prime factors of $n = p_1^{t_1} \cdots p_k^{t_k}$. Again, by [2] there exists an integer n_0 such that $n_0 X \neq X$ for all non-zero subgroups X of A. We can assume wolog that $n_0 \mid n$, so that (*): $nX \neq X$ for any non-zero subgroup X of A.

For $k = 1$, $n = p_1^t$, and there is a split exact sequence $0 \to K \to E_{p_1} \to A_{p_1} \to 0$ by (d). Hence there exist $f \in \mathrm{Hom}_E(A, E)$, $g \in \mathrm{Hom}_E(E, A)$, and a positive integer m such that $gf = m \cdot 1_A$ with $(m, p_1) = 1$.

For $k \geqslant 2$, let $n_1 = p_1^{t_1}$, $n_2 = p_2^{t_2} \cdots p_k^{t_k}$. By induction there exist $f_1, f_2 \in \mathrm{Hom}_E(A, E)$, and $g_1, g_2 \in \mathrm{Hom}_E(E, A)$, and positive integers m_1 and m_2 such that $g_i f_i = m_i \cdot 1_A$ with $(m_i, n_i) = 1$, $i = 1, 2$. Let $r = (m_1 n_2^2, m_2 n_1^2)$. Then $(r, n) = 1$ and there exist $a, b \in Z$ such that $am_1 n_2^2 + bm_2 n_1^2 \equiv 1 \pmod{n}$. Let $f = an_2 f_1 + bn_1 f_2$ and $g = n_2 g_1 + n_1 g_2$. Then $(gf - 1)(A) \subseteq nA$. This says that Ker gf is divisible by n. Since Ker gf is pure, n Ker gf = Ker gf and Ker $gf = 0$ by (*). Furthermore, $(gf - 1)(A) \subseteq nA$ implies that $gf(A)$ is p-pure for all $p \mid n$. Thus there exists $m \in Z$, $(n, m) = 1$ such that $mA \subseteq gf(A) \subseteq A$. Then $(gf)^{-1}$ is defined on mA and $[m(gf)^{-1} g] f = m \cdot 1_A$. This completes the proof.

(e) ⇒ (a) Let $f_1, f_2 \in \mathrm{Hom}_E(A, E)$, $g_1, g_2 \in \mathrm{Hom}_E(E, A)$ satisfy $g_1 f_1 = m_1 \cdot 1_A$, $g_2 f_2 = m_2 \cdot 1_A$ with $(m_1, m_2) = 1$. Write $am_1 + bm_2 = 1$ for some $a, b \in Z$. Now define $f: A \to E \oplus E$ by $f(x) = (af_1(x), bf_2(x))$ and $g: E \oplus E \to A$ by $g(y_1, y_2) = g_1(y_1) + g_2(y_2)$. Then f and g are E-maps and $gf = 1_A$. Thus A is a summand of $E \oplus E$, and therefore is E-projective, and generated by two elements.

The idea of the last argument (e) ⇒ (a) is due to Lady.

We next obtain an "internal" characterization of the E-projective groups involving faithful projective ideals in E-rings. Again some preliminary lemmas precede the main results.

The next lemma holds for an arbitrary abelian group A.

LEMMA 2.5. *Regard A as a right C module via $ac = ca$ for all $a \in A$, $c \in C$. Then for $F \in \mathrm{Hom}_E(A, E)$ there exists $f \in \mathrm{Hom}_C(A, C)$ such that $F(x)(y) = xf(y)$ for all $x, y \in A$. Conversely, if $f \in \mathrm{Hom}_C(A, C)$, then $F(x)(y) = xf(y)$ defines $F \in \mathrm{Hom}_E(A, E)$.*

Proof. Define $f(y): A \to A$ by $f(y)(x) = F(x)(y)$. A routine check shows that $f(y) \in C$ and that $y \to f(y)$ is a right C-module homomorphism. The converse is clear.

We are grateful to R. Wiegand for calling our attention to the result in the next lemma. He has proved a more general result using a different argument.

LEMMA 2.6. *Suppose there exist $x_1 \cdots x_e \in A$ and $f_1 \cdots f_e \in \mathrm{Hom}_C(A, C)$ such that (1) $\sum_{i=1}^{e} f_i(x_i) = 1 \in C$. Suppose also that (2) $n(R_1 \times \cdots \times R_t) \subseteq C \subseteq R_1 \times \cdots \times R_t$ for some $n \in Z^+$ with each R_i a Dedekind domain. Then $A \cong J \oplus K$ where J is a faithful projective ideal in C and K is a C-submodule of A.*

Proof. Let $R = R_1 \times \cdots \times R_t$ and, for $1 \leqslant j \leqslant t$, let $\Pi_j: R \to R_j$ be the natural projections and $P_j = (\mathrm{Ker}\, \Pi_j) \cap C$. Then P_j is a proper prime ideal of C. Let $I = \{r \in C \mid rR \subseteq C\}$. Clearly I is an ideal of C and $n \in I$. Hence C/I is finite. Let $M_1, M_2, \ldots, M_s$ be a finite list of distinct maximal ideals of C such that

(a) If M is a maximal ideal of C and $I \subseteq M$, then $M = M_k$ for some k.

(b) For each $1 \leqslant j \leqslant t$, there is a k such that $P_j \subseteq M_k$.

For each $1 \leqslant j \leqslant s$, choose $b_j \in (\bigcap_{k \neq j} M_k) \backslash M_j$. Also, using property (1), we can choose i_k such that $f_{i_k}(x_{i_k}) \notin M_k$. Define $g = \sum_{k=1}^{s} b_k f_{i_k} \in \mathrm{Hom}_C(A, C)$, and $J = g(A)$. It suffices to prove J is a faithful, projective ideal of C since $A \to^{g} J \to 0$ will then split.

Note that for all k, (*) $g(x_{i_k}) = b_k f_{i_k}(x_{i_k}) \neq 0 \pmod{M_k}$. In particular, by (b), $J \not\subseteq P_j$ for $1 \leqslant j \leqslant t$, so that J is faithful: $rJ = 0 \Rightarrow rJ \subseteq P_j \Rightarrow r \in P_j$ or $J \subseteq P_j \Rightarrow r \in P_j$ for all j, and $r = 0$. To prove that J is projective, it suffices to prove that the localizations J_M of J are projective in C_M for all maximal ideals M of C. If $M = M_k$ for some k, then by (*), $J \not\subseteq M$. Thus, $J_M = C_M$ in this case. If $M \neq M_j$ for all j, then by (a) $I \not\subseteq M$. Consequently, J_M is an ideal of $C_M = R_M$. Since R is hereditary, so is R_M, and J_M is therefore projective.

LEMMA 2.7. *Let S be a commutative ring with J and K ideals of S such that*

(1) *J is a projective generator of the class of S-modules.*

(2) *K is projective and S/K is Artinian.*

Then as S-modules, $J \oplus K \cong S \oplus JK$.

Proof. Let $\{M_1, \ldots, M_t\}$ be the set of maximal ideals of S containing K. By (1) there exist $a_j \in J$, $f_j \in \mathrm{Hom}_S(J, S)$ with $f_j(a_j) \notin M_j$. Choose $b_j \in \bigcap_{i \neq j} M_i \backslash M_j$ and let $g = \sum_{j=1}^{t} b_j f_j$. Let $h = (g, i): J \oplus K \to S$, where i is the inclusion map. Then $\mathrm{Im}\, h \supseteq K$ and, for all j, $\mathrm{Im}\, h \not\subseteq M_j$. Hence, $\mathrm{Im}\, h = S$. Thus, $J \oplus K \cong S \oplus B$ for some S-module B.

Let P be a prime ideal of S. Then $J_P \oplus K_P \cong S_P \oplus B_P$ as S_P-modules. Since J_P and K_P are projective and S_P is local, J_P and K_P are free. Clearly, $\operatorname{rank}_{S_P} J_P \leqslant 1$, $\operatorname{rank}_{S_P}(K_P) \leqslant 1$. Thus, $\operatorname{rank}_{S_P} B_P \leqslant 1$.

Let $X = J$, K, S or B and Λ^2 be the second exterior power. Then $[\Lambda_S^2(X)]_P = \Lambda_{S_P}^2(X_P) = (0)$ for all P, so $\Lambda_S^2(X) = (0)$. We have, therefore, $\Lambda_S^2(J \oplus K) \cong J \otimes_S K \cong JK$, $\Lambda_S^2(S \oplus B) \cong S \otimes_S B \cong B$. But, $\Lambda_S^2(J \oplus K) \cong \Lambda_S^2(S \oplus B)$. Thus, $B \cong JK$.

THEOREM 2.8. *A is E-projective iff*

(1) $\operatorname{Hom}_Z(C, A) = \operatorname{Hom}_C(C, A)$,

(2) *$A \cong J \oplus K$ where J is a faithful projective ideal of C and K is a C-module.*

Proof. Assume (1) and (2) are satisfied. Note that C/J is Artinian since, as before, $nC \subseteq J$. By Lemma 2.7, $A \oplus A \cong J \oplus J \oplus K \oplus K \cong C \oplus J^2 \oplus K \oplus K$. Apply $\operatorname{Hom}_C(\ \ , A)$ to get $E \oplus E = \operatorname{Hom}_C(A, A) \oplus \operatorname{Hom}_C(A, A) \cong A \oplus \operatorname{Hom}_C(J^2, A) \oplus \operatorname{Hom}_C(K, A) \oplus \operatorname{Hom}_C(K, A)$ as E-modules. Thus A is an E-module summand of $E \oplus E$, hence projective.

Conversely, assume A is E-projective. By Proposition 1.4, A is quasi-isomorphic to a group B which is cyclic and projective over $E' = E(B)$. As in Remark 1.5, assume $kA \subseteq B \subseteq A$ for some $0 \neq k \in Z$. Let $C' = \text{center } E'$. By Theorem 1.3, $\operatorname{Hom}_Z(C', B) = \operatorname{Hom}_{C'}(C', B)$. If $\phi \in \operatorname{Hom}_Z(C, A)$, then $k\phi \in \operatorname{Hom}_Z(C', B) = \operatorname{Hom}_{C'}(C', B)$. Furthermore, $kC' \subset C$. It follows easily that $\phi \in \operatorname{Hom}_C(C, A)$. This proves (1).

To prove (2) we use Lemmas 2.5 and 2.6. Since A is E-projective and finitely generated (Theorem 2.4), there exist $x_1, x_2, \ldots, x_l \in A$ and $F_1, F_2, \ldots, F_l \in \operatorname{Hom}_E(A, E)$ such that for all $y \in A$, $y = \sum_{j=1}^{l} F_j(y)(x_j)$. By Lemma 2.5, there exist $f_j \in \operatorname{Hom}_C(A, C)$ such that $F_j(y)(x) = yf_j(x)$ for all $x, y \in A$, $1 \leqslant j \leqslant l$. Consequently, $y = \sum_{j=1}^{l} yf_j(x_j)$ for all $y \in A$, and hence $\sum_{j=1}^{l} f_j(x_j) = 1$.

By Theorem 1.3, C' is an E-ring. Thus, C is an E-ring and there exist a positive integer n and Dedekind domains $R_1, \ldots, R_t$ such that $n(R_1 \times \cdots \times R_t) \subseteq C \subseteq R \times \cdots \times R_t$ (Bowshell and Schultz [6]). We now apply Lemma 2.6 to get (2).

COROLLARY 2.9. *Let S be a finite rank, torsion-free E-ring, J a faithful projective ideal of S, and K a finite rank torsion-free S-module satisfying* $\operatorname{Hom}_Z(S, K) = \operatorname{Hom}_S(S, K)$. *Then $A = J \oplus K$ is E-projective. Moreover, every E-projective group arises in this way.*

Proof. Let $QS = Q \otimes_Z S$, $QE(S) = Q \otimes_Z E(S)$. For $s \in QS$, define f_s: $QS \to QS$ by $f_s(x) = sx$. Since S is an E-ring, the mapping $s \to f_s$ is a ring isomorphism from QS onto $QE(S)$ which contracts to a ring isomorphism from S onto $E(S)$.

S is a finite rank torsion-free ring and J is a faithful ideal of S. It is easy to show that $nS \subseteq J$ for some $n \in Z^+$. Consequently, $nE(J) \subseteq E(S)$. Thus, every element of $E(J)$ has the form $f_{r/n}$, where $r \in S$ and $(r/n)J \subseteq J$.

Since J is projective, $S = \mathrm{Tr}(J) = \sum_{g \in \mathrm{Hom}_Z(J,S)} g(J)$. Thus for r/n as above, $(r/n)S = (r/n)\,\mathrm{Tr}(J) = \mathrm{Tr}((r/n)J) \subseteq \mathrm{Tr}(J) = S$. That is, $r/n \in S$. Therefore, $E(J) = E(S) = \{f_s \mid s \in S\}$.

A routine calculation shows that the center of $E(J \oplus K)$ consists of the multiplications by elements of S. The corollary follows from Theorem 2.8.

Our last result shows that any group A which is $E(A)$ projective is nearly isomorphic to a group B which is $E(B)$ cyclic projective. First, we need a lemma.

LEMMA 2.10. *Let S be a torsion free finite-rank E-ring and J an ideal of S with $mS \subseteq J$ for some $m \in Z^+$. If J is projective as an S-module, then J is nearly isomorphic to S.*

Proof. We have shown, in the proof of Corollary 2.9, that $E(J) = E(S) = S$. Thus, J is $E(J)$ projective and, by Theorem 2.4, is a genus summand of $E(J)$. Since rank J = rank S = rank $E(J)$, J is nearly isomorphic to S.

THEOREM 2.11. *Let A and B be torsion-free abelian groups of finite rank.*

(1) *If A is nearly isomorphic to B, then A is $E(A)$-projective iff B is $E(B)$-projective.*

(2) *If A is $E(A)$-projective, then there exists B nearly isomorphic to A such that B is $E(B)$ cyclic and projective.*

Proof. Part (1) follows from a result of Lady and the following:

Claim. For $n \geqslant 1$, A *is* E-projective if and only if A^n is $E(A^n)$-projective.

If A^n is $E(A^n)$-projective, then $A^n \oplus X = E(A^n)^k$ for some $k > 0$ and $E(A^n)$-module X. But $E(A^n) \cong E^{n^2}$ as E-modules. Thus A is an E-summand of E^{n^2k}, hence E-projective.

Conversely, if A is E-projective, then by Theorem 2.8, $A \cong J \oplus K$ where J is a faithful projective ideal of C and $\mathrm{Hom}_Z(C, A) = \mathrm{Hom}_C(C, A)$. Note that $E(A^n)$ is the ring of $n \times n$ matrices over E, so that the center of $E(A^n)$ is also C. Moreover, $A^n \cong J \oplus (J^{n-1} \oplus K^n)$ and $\mathrm{Hom}_Z(C, A^n) = \mathrm{Hom}_Z(C, A)^n = \mathrm{Hom}_C(C, A)^n = \mathrm{Hom}_C(C, A^n)$. Therefore, by Theorem 2.8, A^n is $E(A^n)$-projective.

By Lady's result (see [2]) if A and B are nearly isomorphic, then $A^n \cong B^n$ for some $n \geqslant 1$. Thus, by the claim, A is E-projective if and only if B is $E(B)$-projective.

For part (2), write $A \cong J \oplus K$ as in Theorem 2.8, and let $B = S \oplus K$. By Lemma 2.10, B is nearly isomorphic to A, and is cyclic projective over $E(B)$ by Theorem 1.3.

We close the paper by giving some examples.

EXAMPLE 2.12 (Douglas and Farahat). Let p_i, $i = 1, 2, 3$, be odd primes and $A_i = Z[1/p_i]$, the subring of Q generated by Z and $1/p_i$, $i = 1, 2, 3$. Let G be the subgroup of Q^3 generated by $A = \oplus_{i=1}^{3} A_i$ and all elements of the form $(a_1, a_2, a_3)/2$ with $a_1 + a_2 + a_3 \in 2Z_2$, where Z_2 is the localization of Z at 2. Then A is an E-ring and $2G \subseteq A \subseteq G$ but G is not $E(G)$ projective—in fact G has infinite projective dimension over $E(G)$. This example and Theorem 2.11 point out the distinction between near- and quasi-isomorphism as related to E-projectivity.

EXAMPLE 2.13. Let R be an E-ring which is a principal ideal domain such that $p = p_1^3$, where p is a prime in Z, p_1 is a prime in R. Let $q \neq p$ be an integral prime with $qR \neq R$ and let $R' = Z[1/q] \otimes_Z R$. Let G be the subgroup of $Q \otimes (R \oplus R')$ generated by $R \oplus R'$ and the element $g = (p_1^2, p_1)/p$. Then, it is easy to check that G is E cyclic with generator g. But G is not E-projective. If $K = pR \oplus p_1^2 R'$, then $\alpha \in \mathrm{Hom}_E(G, G/K)$ defined by $\alpha(g) = (p_1, 1 - p_1) + K$ is not induced by any endomorphism of G.

EXAMPLE 2.14. Let p be an integral prime with $-5 = x^2(p)$. Let $S = \{(a + b\sqrt{-5})/p^l \mid a,\ b,\ l \in Z,\ l \geqslant 0,\ p^l \mid a^2 + 5b^2\}$. Then S is a subring of $Q(\sqrt{-5})$ such that S is a Dedekind domain and S is strongly indecomposable. Thus, S is an E-ring [6]. Let I be the (non-principal) ideal of S generated by $2 + \sqrt{-5}$, $2 - \sqrt{-5}$. Then, $9S \subseteq I$ and $E(I) = E(S) = S$ as in the proof of Corollary 2.9. Thus, I is $E(I)$-projective (S is Dedekind) but not $E(I)$ cyclic. (Compare Theorem 2.4(d).)

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