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IRREPRESENTABILITY BY MULTIPLE INTERSECTION, OR WHY THE INTERVAL NUMBER IS UNBOUNDED

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We consider the following question: Given a family of sets, is there a positive integer, t, so that every graph is the intersection graph of sets each of which is the union of t sets from the given family? We show that the answer is 'no' precisely when some bipartite graph fails to be the intersection graph of sets from the given family. We are especially interested in the case where the given family of sets generalizes the family of real intervals. We extend our results to uniform hypergraphs and simplicial complexes.

1. Introduction and preliminaries

An *interval graph* is a graph for which it is possible to assign real intervals to the vertices so that adjacency of vertices corresponds exactly to non-empty intersection of the corresponding intervals. Not every graph is an interval graph. For example C_4, the 4-cycle, is not an interval graph, but it is a 2-*interval graph*, by which we mean it is the intersection graph of sets each of which is the union of two real intervals. By analogy we can define 3-interval, 4-interval, ..., t-interval graphs. This leads to the definition of a graph parameter: the *interval number* of a graph, G, is the least positive integer, t, for which G is a t-interval graph. How large can the interval number of a graph be? This question was answered easily when the interval number was introduced in [4] and [11] where simple counting arguments show that the interval number of the complete bipartite graph $K_{n,n}$ exceeds $\frac{1}{2}n$.

The purpose of this paper is to abstract the above process. We begin with a family of sets, Σ, and consider the resulting intersection graphs of sets in Σ. We proceed to define 2Σ-, 3Σ-, ..., $t\Sigma$-graphs leading to a parameter called the Σ-number. We will then ask if the Σ-number is bounded and show how to answer this question. See Fig. 1.

What kinds of families, Σ, will interest us? Our aim is to apply our program to sets which resemble intervals in some way. Although Σ may be arbitrary, we are

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Real intervals	$\leftrightarrow$	Family of sets, Σ
Interval graphs	$\leftrightarrow$	Σ-graphs
t-interval graphs	$\leftrightarrow$	$t\Sigma$-graphs
Interval number	$\leftrightarrow$	Σ-number
Bounded?	$\leftrightarrow$	Bounded?

Fig. 1. Our program.

most interested in the cases when Σ is the family of all:
(1) boxes in $\mathbb{R}^d$,
(2) curves in $\mathbb{R}^d$,
(3) convex sets in $\mathbb{R}^d$, or
(4) line segments in $\mathbb{R}^d$.
Notice that in case $d=1$ all the above reduce to $\Sigma=\{\text{real intervals}\}$.

By a *graph* we mean a finite undirected graph without loops or multiple edges. We write $v \sim w$ if vertices v and w are adjacent. Let Σ denote a collection of sets. A graph is said to have an *intersection representation* by sets in Σ (or a Σ-*representation*) if there exists a function $f: V(G) \to \Sigma$ with the property that for distinct vertices v and w we have $f(v) \cap f(w) \neq \emptyset$ if and only if $v \sim w$. We also say that the graph is a Σ-*graph*. We call f the Σ-representation of the graph, G. Notice that f need not be one-to-one. The class of all Σ-graphs is denoted $\Omega(\Sigma)$ and is called an *intersection class*. For example, if $\Sigma=\{[a,b]: a, b \in \mathbb{R}, a<b\}$, then $\Omega(\Sigma)$ is the class of interval graphs.

Let t be a positive integer. We define the family of sets $t\Sigma$ by

$$t\Sigma=\{S_1 \cup S_2 \cap \cdots \cup S_t : S_i \in \Sigma \text{ for } 1 \leqslant i \leqslant t\}.$$

If G is a graph, the least positive integer, t, for which $G \in \Omega(t\Sigma)$ is called the Σ-number of G and is denoted $\Sigma\#(G)$. (In case no such t exists we can either say $\Sigma\#(G)=\infty$ or that $\Sigma\#(G)$ is undefined.) In case Σ is the set of all real intervals, then $\Sigma\#(\mathrm{G})$ is exactly the interval number of G.

2. Ad hoc results on boxes

In this section we consider Σ to be the set of all *boxes*, that is, parallelpipeds with sides parallel to the coordinate axes, in $\mathbb{R}^d$. Thus a box is the cartesian product of real intervals, $[a_1, b_1] \times \cdots \times [a_d, b_d]$.

Theorem 2.1. *Let t and d be positive integers and let $\Sigma=\{\text{boxes in } \mathbb{R}^d\}$. The class $\Omega(t\Sigma)$ does not contain all graphs, i.e., there exist graphs, G, for which $\Sigma\#(G)>t$.*

Proof. Let t and d be fixed. In every intersection representation of a graph by boxes in $\mathbb{R}^d$ it is only the *order* of the end points of the defining intervals of the boxes which matters in determining which box intersects which. Thus if a graph has n vertices we lose no generality in forming a $t\Sigma$-representation for this graph if we assume that the coordinates of the corners of the boxes are integers in the set $\{1, 2, 3, \ldots, 2nt\}$.

Suppose $\Omega(t\Sigma)$ contains all graphs. Let n be a positive integer. The number of labeled graphs with n vertices is $2^{n(n-1)/2}$ and so the number of non-isomorphic (unlabeled) graphs on n vertices is at least $2^{n(n-1)/2}/n! \geq n^{-n}2^{n(n-1)/2}$.

The number of boxes in $\mathbb{R}^d$ with corners in the set $\{1, 2, \ldots, 2nt\}^d$ is at most $(2nt)^{2d}$ and therefore the number of non-isomorphic graphs with n vertices in $\Omega(t\Sigma)$ is at most $[(2nt)^{2d}]^{nt} = (2nt)^{2ndt}$. It follows that

$$(2nt)^{2ndt} \geq n^{-n}2^{n(n-1)/2}$$

or

$$n^n(2nt)^{2ndt} \geq 2^{n(n-1)/2}.$$

Taking logarithms to the base 2 we have

$$n(\log n) + 2ndt(\log 2nt) \geq \tfrac{1}{2}n(n-1)$$

or

$$(1 + 2td)(\log n) + 2td(\log 2t) \geq \tfrac{1}{2}(n-1).$$

This last inequality is obviouly false for n sufficiently large. $\square$

Corollary 2.2. *The interval number is unbounded.*

Proof. Let $d = 1$. $\square$

The *boxicity* of a graph G is the least positive integer d for which $G \in \Omega$ {boxes in $\mathbb{R}^d$} (see [1, 6, 10]). Like the interval number, it too is unbounded.

Corollary 2.3. *Boxicity is unbounded.*

Proof. Let $t = 1$. $\square$

Thus Theorem 2.1 can be viewed as a generalization of the previously known facts that interval number [3, 11] and boxicity [6] are unbounded.

3. Ad hoc results on curves and convex sets

Consider the following facts:

(1) Every graph is the intersection graph of curves in $\mathbb{R}^3$.

(2) Every graph is the intersection graph of convex sets in $\mathbb{R}^3$ [12].

Both of these statements are false in $\mathbb{R}^2$. The typical counterexample is the 'subdivision of K_5'-graph. If G is a graph, then the *subdivision* of G, denoted βG, is the graph formed from G by placing a vertex in the 'middle' of each edge forming two edges. See Fig. 2. More formally, if $G = (V, E)$ then βG is the

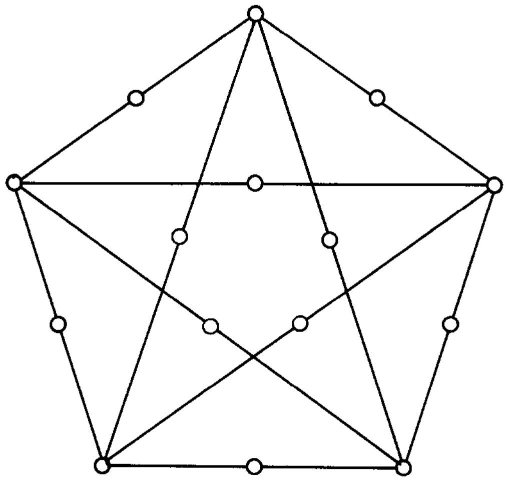

Fig. 2. Subdivision of K_5.

bipartite graph with vertices $V \cup E$ and for $v \in V$ and $e \in E$ we have $v \sim e$ if and only if vertex v is in edge e in G. It is instructive to review the argument in [2] or [9] that βK_5 is not the intersection graph of planar curves, since we will generalize their approach below.

Suppose βK_5 were the intersection graph of planar curves. Fix such an intersection representation in the plane. Shrink each of the five curves corresponding to the five 'original' vertices to points. The resulting configuration of five points and ten curves (corresponding to the ten vertices used to subdivide the edges) gives a planar embedding of βK_5 which is impossible. A similar argument shows that βK_5 is not the intersection graph of planar convex sets [12].

It may be the case, however, that every graph is the intersection graph of sets each of which is the union of two [or some other pre-specified number] planar curves or convex sets. This too, however, is false:

Theorem 3.1. *Let $\mathbb{M}^2$ be a two-dimensional manifold with finite Euler characteristic, χ, and let Σ be the set of all curves on $\mathbb{M}^2$, and let t be a positive integer. The family $\Omega(t\Sigma)$ does not contain all graphs, i.e.$\Sigma\#$ is unbounded.*

Proof. Our approach generalizes the above argument where $\mathbb{M}^2 = \mathbb{R}^2$ and $t = 1$. We specify a graph with the property that if it were in $\Omega(t\Sigma)$ then we could embed a second graph in $\mathbb{M}^2$. We then show that the second graph fails to embed in $\mathbb{M}^2$ by using the following well-known inequality for graphs, G, which embed in $\mathbb{M}^2$:

$$|E(G)| \leqslant 3(|V(G)| - \chi). \tag{1}$$

When k_1 and k_2 are positive integers, let $r(k_1, k_2)$ denote the classical Ramsey number.

We now construct a graph, G, which generalizes βK_5. Choose a positive integer

N large enough to that

$$\binom{N}{2} > 3(Nt - \chi). \tag{2}$$

Let $n = r(t+1, N)$. We define a bipartite graph, G, with $n + \binom{n}{t+1}$ vertices as follows: The n vertices in the first part are $\{v_1, \ldots, v_n\}$. The $\binom{n}{t+1}$ vertices in the second part are

$$\{w_I : I \subset \{1, 2, \ldots, n\} \text{ and } |I| = t+1\}.$$

That is, the vertices in the second part are indexed by the subsets of $\{1, 2, \ldots, n\}$ of cardinality $t+1$. Finally, we define vertex v_i to be adjacent to w_1 if and only if $i \in I$. This completes the definition of G. We claim that G is not in $\Omega(t\Sigma)$.

(In case $\mathbb{M}^2 = \mathbb{R}^2$ and $t = 1$ we observe that $\chi = 2$ and taking $N = 5$ we have

$$\binom{N}{2} = \binom{5}{2} = 10 > 9 = 3((5)(1) - 2) = 3(Nt - \chi).$$

Then $n = r(5, 2) = 5$ and we see that $G = \beta K_5$.)

Suppose $G \in \Omega(t\Sigma)$. Let $f: V(G) \to t\Sigma$ be a $t\Sigma$-representation. Let

$$f(v_i) = v_i^1 \cup \cdots \cup v_i^t \quad \text{and} \quad f(w_I) = w_I^1 \cup \cdots \cup w_I^t.$$

where the v_i^s and w_I^s with $1 \leq s \leq t$ are curves on $\mathbb{M}^2$.

We now assume that these curves satisfy the following properties:

(1) no curve intersects itself,

(2) no two curves assigned to the same vertex intersect, i.e., $v_i^p \cap v_i^q = \emptyset$ and $w_I^p \cap w_I^q = \emptyset$ whenever $p \neq q$, and

(3) if two curves intersect, the intersection is finite: $|v_i^p \cap w_I^q| < \infty$.

These three assumptions can be imposed on any collection of curves representing a $t\Sigma$-graph by making local adjustments to the curves. The topological details are not difficult, but they are tedious. The interested reader is referred to [8].

We say that curve w_I^s *joins* curves v_i^p and v_j^q if w_I^s meets both these curves and there is an arc of w_I^s whose end points lie in v_i^p and v_j^q and which meets no other curves. See Fig. 3.

Next, consider a complete graph K_n with vertex set $\{u_1, \ldots, u_n\}$. Color edge u_iu_j *red* if some v_i^p is joined to some v_j^q by some w_I^s. Color edge u_iu_j *blue* otherwise. Claim: there is no blue $(t+1)$-clique in this K_n. Suppose $u_{i_1}, \ldots, u_{i_{t+1}}$ formed a blue clique. Let $I = \{i_1, \ldots, i_{t+1}\}$. Consider the t curves $w_I^1, \ldots, w_I^t$. At least one of these w-curves must meet curves from two of $v_{i_1}, \ldots, v_{i_{t+1}}$. Suppose w_I^s meets curves from two different vertices. As we traverse w_I^s we encounter finitely many points of intersection with v-curves. It follows that some sequential pair must belong to v_j^p and v_k^q with $j \neq k$ and $j, k \in I$. Thus u_ju_k would be colored red.

Since $n = r(t+1, N)$ and since there are no blue $(t+1)$-cliques, there must be a red N-clique. We may suppose that vertices $\{u_1, \ldots, u_N\}$ constitute a red clique.

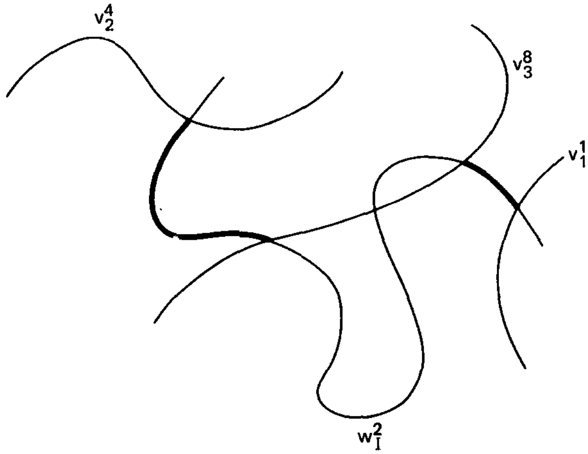

Fig. 3. Joining curves.

We now construct a new graph, H, as follows: H has tN vertices, $\{x_i^p: 1 \leq i \leq N, 1 \leq p \leq t\}$, with $x_i^p \sim x_j^q$ if and only if v_i^p is joined to v_j^q by some w-curve. Since $\{u_1, \ldots, u_N\}$ form a red clique, for all $i \neq j$ we have $x_i^p \sim x_j^q$ for some p, q. Thus $|E(H)| \geq \binom{N}{2}$. Contract each of these v-curves to distinct points which we call, with justifiable abuse of notation, x_i^p. See Fig. 4. Two points x_i^p and x_j^q are now joined by an arc of a w-curve if and only if vertices x_i^p and x_j^q are adjacent. Thus we have embedded H in $\mathbb{M}^2$. It follows from (1) that

$$|E(H)| \leq 3(|V(H)| - \chi);$$

thus

$$\binom{N}{2} \leq |E(H)| \leq 3(Nt - \chi)$$

contradicting (2). Thus $\Sigma\#(G) > t$. $\square$

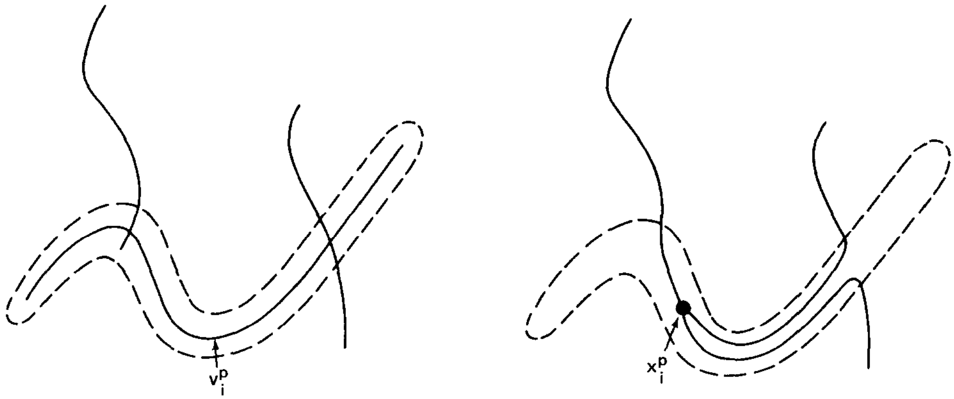

Fig. 4. Contracting curves to points.

Corollary 3.2. *Let $\mathbb{M}^2$ be a two-manifold with finite Euler characteristic, let Σ consist of all arc connected subsets of $\mathbb{M}^2$, and let t be a positive integer. There exist graphs which do not belong to $\Omega(t\Sigma)$, i.e., $\Sigma\#$ is unbounded.*

Proof. If Σ' is the family of all curves in $\mathbb{M}^2$, let G be a graph for which $\Sigma'\#(G) > t$. Suppose $G \in \Omega(t\Sigma)$. Let $S_1, S_2, S_3, \ldots$ be the arc connected sets used in the $t\Sigma$-representation of G. Choose a point $x_i \in S_i$. If S_i and S_j intersect, choose a point $y_{ij} \in S_i \cap S_j$. (Note: $y_{ij} = y_{ji}$.) For each i join point x_i to all points y_{ij} by curves contained in S_i. We can consider these curves radiating from x_i as one (perhaps self-intersecting) curve called z_i. Notice that if $S_i \cap S_j \neq \emptyset$ then $y_{ij} \in z_i \cap z_j$ and if $S_i \cap S_j = \emptyset$ then z_i and z_j cannot intersect. Thus in the $t\Sigma$-representation of G we can replace each arc connected set S_i by the corresponding curve z_i, giving a $t\Sigma'$-representation, which is impossible. □

We can proceed from Corollary 3.2 to replace 'arc connected' by 'open connected' or 'compact connected'. However, the only result in which we are interested is:

Corollary 3.3. *If Σ is the set of all planar convex sets, then $\Sigma\#$ is unbounded.*

Proof. Convex sets are clearly arc connected. □

Corollary 3.4. *Interval number is unbounded.*

4. General theory

Let Λ^d denote the family of all line segments in $\mathbb{R}^d$. We know that $\Lambda^d\#$ is unbounded in case $d = 1, 2$; however, the results of the previous section are no help when $d \geq 3$. Worse, the techniques do not extend: The enumerative/probabilistic method of Section 2 does not help because in determining if two line segments intersect we need to know more than the relative order of the coordinates of the end points. The embeddings technique of Section 3 does not help because all graphs embed in $\mathbb{R}^3$.

Is there an integer t so that $\Omega(t\Lambda^3)$ contains all graphs?

A simple example [8] was able to show that $\Omega(\Lambda^3)$ does not contain all graphs. However, the difficulties encountered in answering this question led to the following development:

Our method flows from the following idea: For every graph in $\Omega(t\Sigma)$ one can construct a graph in $\Omega(\Sigma)$ with 't times as many vertices' as follows: If v is a vertex of the $t\Sigma$-graph assigned to $S_1 \cup \cdots \cup S_t$, we construct t vertices in the new graph $v_1, \ldots, v_t$ with v_i assigned to S_i. Note that $v \sim w$ if and only if for some i, j we have $v_i \sim w_j$. We abstract this process:

Let G and H be graphs. We say that $f: V(H) \to V(G)$ is a *simplicial cover*

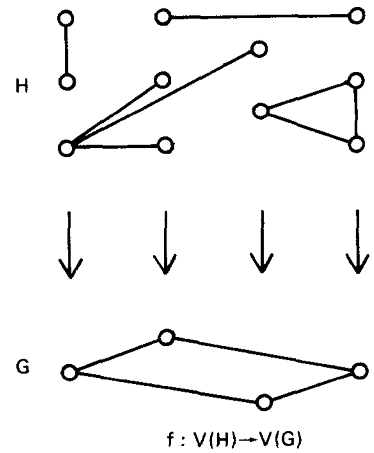

Fig. 5. A simplicial cover of multiplicity 3.

provided:

(1) f is onto,

(2) if $vw \in E(H)$ and $f(v) \neq f(w)$, then $f(v)f(w) \in E(G)$, and

(3) if $xy \in E(G)$ then there exists $vw \in E(H)$ with $f(v) = x$ and $f(w) = y$.

We say that such a function has *multiplicity* t if $\max\{|f^{-1}(v)|: v \in V(G)\} = t$, and we write $\text{mult}(f) = t$. See Fig. 5. If $\text{mult}(f) = 1$, then f is an isomorphism.

Let P be a family of graphs. We define the *P-number* of a graph, G, denoted $P\#(G)$, to be the least positive integer, t, so that there exists a graph $H \in P$ and a simplicial cover $f: V(H) \to V(G)$ of multiplicity t. (If no such cover exists, we can either say $P\#$ is undefined or infinite.)

The connection with our previous definitions is the following:

Proposition 4.1. *If P is an intersection class, $P = \Omega(\Sigma)$ for some Σ, and if G is a graph, then $P\#(G) = \Sigma\#(G)$.*

Proof. Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$.

Suppose $\Sigma\#(G) = t$. Let $f: V(G) \to t\Sigma$ be a $t\Sigma$-representation. Let H be a graph with nt vertices, $\{w_i^p\colon 1 \leq p \leq t, 1 \leq i \leq n\}$. We impose a Σ-representation, g, on H as follows: If $f(v_i) = S_i^1 \cup \cdots \cup S_i^t$ then $g(w_i^p) = S_i^p$. Thus, $w_i^p \sim w_j^q$ if and only if $S_i^p \cap S_j^q \neq \emptyset$. Thus H is a Σ-graph. Let $h: V(H) \to V(G)$ by $h(w_i^p) = v_i$. One easily checks that h is a simplicial cover of multiplicity t, hence $P\#(G) \leq t$.

Conversely, suppose $P\#(G) = t$. Let $f: V(H) \to V(G)$ be a simplicial cover of multiplicity t. Let $g: V(H) \to \Sigma$ be a Σ-respresentation of H. We define a map $h: V(G) \to t\Sigma$ as follows: Let $v \in V(G)$. Let $f^{-1}(v) = \{w_1, w_2, \ldots, w_s\}$ with $1 \leq s \leq t$. Put $h(v) = g(w_1) \cup \cdots \cup g(w_s) \cup g(w_s) \cup \cdots \cup g(w_s)$. (The $(t-s)$ extra $g(w_s)$ are written to show that $h(v) \in t\Sigma$.) One can easily verify that $h(v) \cap h(w) \neq \emptyset$ if and only if $v \sim w$. Thus $G \in \Omega(t\Sigma)$, i.e., $\Sigma\#(G) \leq t$. $\square$

Corollary 4.2. *If $\Omega(\Sigma)=\Omega(\Sigma')$ then for any graph G, we have $\Sigma\#(G)=\Sigma'\#(G)$.*

We study the boundedness of $P\#$ in the case where P is a *monotone* (or *hereditary*) class of graphs, i.e., a class of graphs with the property that if $G \in P$ and if H is an induced subgraph of G, then $H \in P$. Since all intersection classes, $\Omega(\Sigma)$, are necessarily monotone (see [7] for an easy proof) we loose no generality. For the sake of completeness, we begin with a simple result explaining when $P\#$ is well defined.

Denote by M_n, for positive integers, n, the 1-regular graph with $2n$ vertices. In other words, M_n has vertices $v_1, \ldots, v_n, w_1, \ldots, w_n$ and edges precisely of the form v_iw_i for $1 \leqslant i \leqslant n$.

Proposition 4.3. *Let P be a monotone class of graphs. $P\#(G)$ is defined (finite) for all graphs, G, if and only if $M_n \in P$ for all n.*

Proof. Suppose for some n the class P does not contain M_n. We claim $P\#(M_n)=\infty$. Suppose $P\#(M_n)=t<\infty$. Let $f: V(H) \to V(M_n)$ be a simplicial cover with $H \in P$. It is an easy exercise to show that H contains M_n as an induced subgraph, which is impossible.

Conversely, suppose $M_n \in P$ for all n. Let G be a graph and let $\Delta(G)$ denote the maximum degree (valence) of the vertices of G. We claim $P\#(G) \leqslant \Delta(G)$. Notice that all graphs which have maximum degree one are induced subgraphs of M_n for n sufficiently large and are therefore in P. Let $V(G)=\{v_1, \ldots, v_n\}$ and $E(G)=\{e_1, \ldots, e_m\}$. Define edges in a new graph, H, with $n\Delta(G)$ vertices $\{v_i^j: 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant \Delta(G)\}$ by the following algorithm:

Step 1. $i \leftarrow 1$. Label all vertices 'available'.

Step 2. Let v_p and v_q be the vertices in edge e_i. Find indices j, k so that v_p^j and v_q^k are both 'available'. Connect v_p^j and v_q^k by an edge and relabel both 'used'.

Step 3. $i \leftarrow i+1$. If $i \leqslant m$, then to to Step 2.

Step 4. Stop.

In Step 2 the algorithm is assured of finding suitable indices since there are $\Delta(G)$ 'available' vertices v_i^j for each i and each edge incident with v_i decreases the number of 'available' vertices by one. The resulting graph, H, has maximum degree 1 and is therefore in P. Let $f: V(H) \to V(G)$ by $f(v_i^j)=v_i$. One easily checks that f is simplicial and that $\mathrm{mult}(f) \leqslant \Delta(G)$. $\square$

We now come to our main result:

Theorem 4.4. *Let P be a monotone class of graphs. The following statements are equivalent:*

(1) *$P\#$ is bounded.*

(2) *For all graphs, G, $P\#(G) \leqslant 2$.*

(3) *P contains all bipartite graphs.*

The proof relies on the following result from Ramsey theory (see [3]).

Theorem 4.5. *Let G be a bipartite graph and let t be a postive integer. There exists a bipartite graph H with the property that for every t coloring of the edges of H there is an induced subgraph of H isomorphic to G with all edges colored the same.*

Proof of Theorem 4.4. Suppose P contains all bipartite graphs. Let G be a graph with $V(G)=\{v_1,\ldots,v_n\}$. Construct a bipartite graph H with $V(H)=\{x_1,\ldots,x_n,y_1,\ldots,y_n\}$ and $E(H)=\{x_iy_j\colon v_iv_j\in E(G)\}$. By hypothesis $H\in P$. Let $f\colon V(H)\to V(G)$ by $f(x_i)=f(y_i)=v_i$ for $1\leqslant i\leqslant n$. One checks that f is a simplicial cover of multiplicity two, and thus $P\#(G)\leqslant 2$.

Conversely, suppose P does not contain a bipartite graph, G. Let t be a positive integer and let $T=t^2$. By Theorem 4.5, there exists a graph G' such that for every T coloring of its edges, G' contains an induced monochromatic copy of G. We show that $P\#(G')>t$.

Suppose $P\#(G')\leqslant t$, and so there exists a simplicial cover $f\colon V(H')\to V(G')$ with $H'\in P$ and $\mathrm{mult}(f)\leqslant t$. Since G' is bipartite let $V(G')=X\cup Y$ be a partition of its vertices into independent sets. We color the edges of G' with $T=t^2$ colors (each color is an ordered pair of integers (p,q) with $1\leqslant p,\ q\leqslant t$) as follows: For $x_i\in X$ let $f^{-1}(x_i)=\{x_i^1,\ldots,x_i^t\}$ and for $y_i\in Y$ let $f^{-1}(y_i)=\{y_i^1,\ldots,y_i^t\}$. If $x_i\sim y_j$ then for some (p,q) we have $x_i^p\sim y_j^q$. Color edge x_iy_j with color (p,q). (In case there is more than one choice, select one color arbitrarily.)

By construction, there is an induced copy of G in G' all of whose edges have the same color. Let that color be (p,q). We may assume, without loss of generality, that the (p,q)-color induced copy of G in G' occurs on vertices $x_1,\ldots,x_n,y_1,\ldots,y_m$. Denote by H the induced subgraph of H' on vertices $x_1^p,\ldots,x_n^p,y_1^q,\ldots,y_m^q$. We claim that H is isomorphic to G; indeed $f\mid V(H)$ is an isomorphism. First, it is clear that $f\mid V(H)$ is a bijection: $x_i^p\leftrightarrow x_i$ and $y_j^q\leftrightarrow y_j$. Second, if $x_i\sim y_j$, then, since x_iy_j is colored (p,q), we known that $x_i^p\sim y_j^q$. Conversely, if $x_i^p\sim y_j^q$, then $x_i\sim y_j$ since f is a simplicial cover. Thus H' contains an induced subgraph, H, isomorphic to G. Since $H'\in P$ and P is monotone, we have $G\in P$, contrary to hypothesis. Thus $P\#(G')>t$, and the equivalence of the three statements is immediate. □

Corollary 4.6. *The inverval number is unbounded.*

Proof. Since C_4, the 4-cycle, is bipartite and not an interval graph, there must exist graphs with arbitrarily high interval number. □

5. Applications

This result gives us a quick proof that for $\Sigma=\{$planar curves$\}$ or $\Sigma=\{$planar convex sets$\}$, $\Sigma\#$ is unbounded. In either case, $P=\Omega(\Sigma)$ does not contain βK_5, which is bipartite. Thus $P\#$ is unbounded, but since $P\# = \Sigma\#$, we know that $\Sigma\#$ is unbounded.

It remains to analyze $\Lambda^3\#$. To show that $\Lambda^3\#$ is unbounded, we must find a *bipartite* graph which fails to be in $\Omega(\Lambda^3)$. This requires some geometry:

Theorem 5.1. *Given three mutually skew lines in $\mathbb{R}^3$ there exists a unique doubly ruled surface containing those lines. Moreover, any line intersecting all three of the given lines must lie in that surface.*

Theorem 5.1 is classical. See [5] for a discussion and definitions. We now construct a bipartite graph, G, by taking the complete bipartite graph $K_{3,3}$ and the subdivision of the K_5-graph, βK_5, and coloring the vertices of each black and white so that adjacent vertices have different colors. Next we connect all black (respectively white) vertices of $K_{3,3}$ to all white (resp. black) vertices of βK_5. See Fig. 6. This completes the construction of G. Clearly the original black/white colorings of $K_{3,3}$ and βK_5 gives a valid 2-coloring for G and therefore G is bipartite. We claim that G is not in $\Omega(\Lambda^3)$.

Suppose $G \in \Omega(\Lambda^3)$. Choose a Λ^3-representation, $f: V(G) \to \Lambda^3$. Consider the six line segments corresponding to the $K_{3,3}$ portion of G. If any two non-intersecting segments of these six are coplanar, one checks that all six are coplanar. In which case all the line segments representing the βK_5 part lie in this plane as well. This implies $\beta K_5 \in \Omega(\Lambda^2)$ which is false.

We may therefore assume that any non-intersecting pair of line segments representing the $K_{3,3}$ portion are non-coplanar. By Theorem 5.1, it follows that

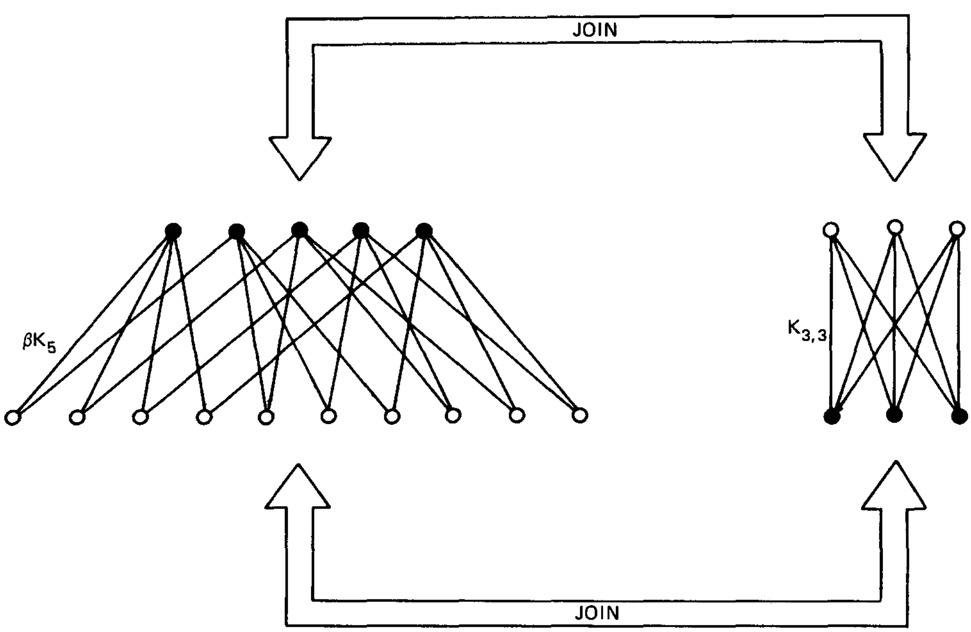

Fig. 6. A bipartite graph not in $\Omega(\Lambda^3)$.

there is a unique doubly-ruled surface, $\mathbb{M}^2$, containing *all* the line segments representing G. Since all doubly-ruled surfaces are homeomorphic to a subset of the plane, let $\phi:\mathbb{M}^2 \to \mathbb{R}^2$ be one-to-one and continuous. Thus $\phi \circ f \mid V(\beta K_5)$ is a planar curve representation of βK_5, which is impossible. Thus G is not in $\Omega(\Lambda^3)$. By Theorem 4.4, it follows that $\Lambda^3\#$ is unbounded.

Naturally, it remains to decide if $\Lambda^d\#$ for $d>3$ is bounded. Fortunately, we can reduce to dimension three by the following:

Proposition 5.2. *For $d>3$ we have $\Omega(\Lambda^d)=\Omega(\Lambda^{d-1})$.*

Proof. Suppose $G\in\Omega(\Lambda^d)$ for $d>3$ and let $f:V(G)\to\Lambda^d$ be a representation. Consider the set of all orthogonal projections $\pi:\mathbb{R}^d\to\mathbb{R}^{d-1}$. This set of projections can be identified with $\mathbb{RP}^{d-1}$, real projective space of dimension $d-1$. Of these projections there are some which will cause non-intersecting line segments in G's representation to intersect. However, the set of all such projections is at most a two-dimensional subset of the set of all projections which, in case $d>3$, has dimension exceeding 2. Thus for almost all orthogonal projections, π, the map $\pi\circ f: V(G)\to\mathbb{R}^{d-1}$ is a Λ^{d-1}-representation of G. $\square$

Corollary 5.3. *For all $d\geqslant 3$, $\Lambda^d\#=\Lambda^3\#$.*

For every intersection class, P, we have examined, the corresponding parameter, $P\#$, is unbounded. Is this true in general? Clearly not, since the class of all graphs is an intersection class and the corresponding parameter is identically one. Indeed, there are non-trivial examples such as the class of perfect graphs. It is known that perfect graphs form an intersection class (see [7]) and since all bipartite graphs are perfect, the corresponding parameter is bounded by two.

6. Extension to uniform hypergraphs and simplicial complexes

In this section we extend the results of Section 4 to hypergraphs and to simplicial complexes.

For a positive integer, k, we say that a hypergraph is *k-uniform* provided every edge of H has cardinality k. One defines induced subhypergraphs and monotone families of hypergraphs by analogy to the graph case. A hypergraph is called *k-partite* if it is possible to partition its vertex set into k parts so that no edge contains two vertices from the same part.

If H and K are k-uniform hypergraphs we say $f:V(K)\to V(H)$ is a *simplicial cover* provided:

(1) f is onto,

(2) if $e\in E(K)$ and $f\mid e$ is one-to-one, then $f(e)\in E(H)$, and

(3) if $\{v_1,\dots,v_k\}\in E(H)$ then there exist $\{w_1,\dots,w_k\}\in E(K)$ such that $f(w_i)=v_i$ for $1\leqslant i\leqslant k$.

In case $k=2$ this definition is indentical with the graph case. Multiplicity is defined as in the graph case, $\text{mult}(f)=\max\{|f^{-1}(v)|: v\in V(H)\}$. Likewise, if P is a family of k-uniform hypergraphs and if H is a k-uniform hypergraph, the *P-number* of H, denoted $P\#(H)$, is the lease positive integer, t, such that there exists $K\in P$ and a simplicial cover $f: V(K)\to V(H)$ with $\text{mult}(f)=t$.

We have the following generalization of main theorem, Theorem 4.4:

Theorem 6.1 (uniform hypergraph version). *If P is a monotone family of k-uniform hypergraphs, then the following statements are equivalent:*

(1) *$P\#$ is bounded.*

(2) *For any k-uniform hypergraph, H, we have $P\#(H)\leq k$.*

(3) *P contains all k-partite k-uniform hypergraphs.*

We omit the proof of this theorem as it is analogous to the proof of Theorem 4.4 and depends on a k-uniform hypergraph analogue of Theorem 4.5. See [8] for details.

The theorem is of interest for two reasons. First, it removes some of the mystery behind the appearance of bipartite graphs in the Theorem 4.4: *bi*partite graphs and the universal upper bound of 2 appear because graphs have *two* vertices in each edge. Second, it is needed in extending our results to simplicial complexes, which we do presently.

A *simplicial complex* is a hypergraph, K, satisfying:

(1) if $v\in V(K)$, then $\{v\}\in E(K)$, and

(2) if $\emptyset\neq e'\subset e\in E(K)$ and $e'\in E(K)$.

The *dimension* of a simplicial complex is given by

$$\dim(K)=\max\{|e|: e\in E(K)\}-1.$$

When k is a positive integer, the *k-skeleton* of a simplicial complex is denoted $\text{sk}_k K$ and is defined by

$$V(\text{sk}_k K)=V(K) \quad \text{and} \quad E(\text{sk}_k K)=\{e\in E(K): |e|\leq k+1\}.$$

The *barycentric subdivision* of a simplicial complex, K, is denoted βK and is defined by

$$V(\beta K)=E(K) \quad \text{and} \quad E(\beta K)=\{\{e_1, e_2, \ldots, e_t\}: e_1\subset e_2\subset\cdots\subset e_t\}.$$

Note that the barycentric subdivision of a k-dimensional simplicial complex is a $(k+1)$-partite k-dimensional simplicial complex.

Let Σ be a family of non-empty sets. We say that a simplicial complex, K, is a *nerve* of sets in Σ provided there exists a function $f: V(K)\to\Sigma$ so that $\{v_1,\ldots,v_s\}\in E(K)$ if and only if $f(v_1)\cap\cdots\cap f(v_s)\neq\emptyset$. The family of all such nerves is denoted $N(\Sigma)$. We will ask: when does $N(t\Sigma)$ contain all k-dimensional simplicial complexes?

Given simplicial complexes K and L we call a function $f: V(L)\to V(K)$ a *simplicial cover* provided:

(1) f is onto,

(2) if $e \in E(L)$, then $f(e) \in E(K)$, and

(3) if $\{v_1, \ldots, v_s\} \in E(K)$ then there exists $\{w_1, \ldots. w_s\} \in E(L)$ such that $f(w_i) = v_i$ for $1 \leq i \leq t$.

If P is a family of simplicial complexes and if K is a simplicial complex we define $P\#(K)$ to be the least positive integer, t, such that there exists $L \in P$ and a simplicial cover $f: V(L) \rightarrow V(K)$ with $\text{mult}(f) = t$.

Proposition 6.2. *If $P = N(\Sigma)$ and K is a simplicial complex, then $P\#(K) = t$ if and only if t is the least integer for which $K \in N(t\Sigma)$.*

Let k be a fixed positive integer. We want to derive a nerve version of our main Theorem 4.4 from the hypergraph version, Theorem 6.1, but the correlation between k-dimensional simplicial complexes and $(k+1)$-uniform hypergraphs is somewhat tedious. Please bear with us as we present some more definitions.

If Σ is a family of non-empty sets, define the following classes:

$\Omega^*(\Sigma)$ is the family of all $(k+1)$-uniform hypergraphs, H, for which there exists a function $f: V(H) \rightarrow \Sigma$ such that $\{v_0, \ldots, v_k\} \in E(H)$ if and only if $f(v_0) \cap \cdots \cap f(v_k) \neq \emptyset$.

$\Omega'(\Sigma)$ is the subclass of $\Omega^*(\Sigma)$ of those hypergraphs whose representing functions have the additional property that for any $k+2$ vertices $v_0, \ldots, v_{k+1}$ we have $f(v_0) \cap \cdots \cap f(v_{k+1}) = \emptyset$.

$N^*(\Sigma) = \{\text{sk}_k K : K \in N(\Sigma)\}$.

$N'(\Sigma) = \{K \in N(\Sigma) : \dim(K) \leq k\}$.

A simplicial complex is called *k-equimaximal* if and only if each of its maximal edges has cardinality $k+1$. If $\dim(K) \leq k$ we define a new complex, εK, as follows: Let the maximal edges of K be $\{e_1, \ldots, e_p\}$. Let $q_i = k+1-|e_i|$ and let $f_i = \{w_{i,1}, \ldots, w_{i,q_i}\}$ be new vertices, i.e., all $w_{i,j}$ are distinct vertices not already in K. (In case $q_i = 0$ we put $f_i = \emptyset$.) The vertices of εK are $V(K) \cup f_1 \cup \cdots \cup f_p$ and the maximal edges are $\{e_1 \cup f_1, e_2 \cup f_2, \ldots, e_p \cup f_p\}$. See Fig. 7. Note that εK is k-equimaximal.

Let $\mathcal{S}$ denote the class of all simplicial complexes of dimension at most k, $\mathcal{U}$ denote the class of all $(k+1)$-uniform hypergraphs, and $\mathcal{E}$ denote the class of all k-equimaximal simplicial complexes. There is a natural bijection between $\mathcal{E}$ and $\mathcal{U}$ given by the following pair of functions:

Let $\mu: \mathcal{E} \rightarrow \mathcal{U}$ be defined as follows: If K is a k-equimaximal simplicial complex, then $V(\mu K) = V(K)$ and $E(\mu K) = \{e \in E(K) : |e| = k+1\}$.

Let $\sigma: \mathcal{U} \rightarrow \mathcal{E}$ be defined as follows: If H is a $(k+1)$-uniform hypergraph, let $V(\sigma H) = V(H)$ and $E(\sigma H) = \{e : \emptyset \neq e \subset e' \in E(H)\}$.

Clearly σ and μ are inverses of one another. We now present the relationship between Ω^* and N^* and between Ω' and N'.

Proposition 6.3. *Let $H \in \mathcal{U}$. Then $H \in \Omega^*(\Sigma)$ if and only if $\sigma H \in N^*(\Sigma)$ and $H \in \Omega'(\Sigma)$ if and only if $\sigma H \in N'(\Sigma)$.*

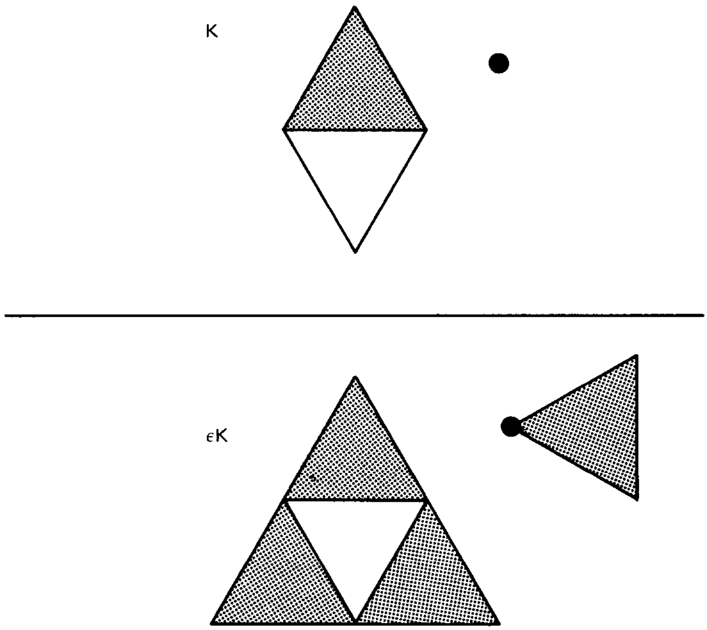

Fig. 7. Forming k-equimaximal simplicial complex, εK.

Proof. The same function, f, can be used to show membership in either class in either case. $\square$

Proposition 6.4. *$\Omega^*(\Sigma) = \mathcal{U}$ if and only if $N^*(\Sigma) = \mathcal{S}$, and $\Omega'(\Sigma) = \mathcal{U}$ if and only if $N'(\Sigma) = \mathcal{S}$.*

Proof. Suppose $\Omega^*(\Sigma) = \mathcal{U}$. Let $K \in \mathcal{S}$. We know that $\mu\varepsilon K \in \mathcal{U} = \Omega^*(\Sigma)$. Thus $\sigma\mu\varepsilon K = \varepsilon K \in N^*(\Sigma)$. One easily checks that $N^*(\Sigma)$ is monotone and that K is an induced subcomplex of εK concluding that $K \in N^*(\Sigma)$. Thus $N^*(\Sigma) = \mathcal{S}$.

Conversely, if $N^*(\Sigma) = \mathcal{S}$ and $H \in \mathcal{U}$ we know $\sigma H \in \mathcal{S} = N^*(\Sigma)$. Thus $H \in \Omega^*(\Sigma)$ by the preceding proposition, and $\Omega^*(\Sigma) = \mathcal{U}$.

The proof for Ω' and N' is analogous. $\square$

Theorem 6.5. *Let $P = N^*(\Sigma)$. There is a positive integer, t, with the property that $N^*(t\Sigma) = \mathcal{S}$ if and only if P contains all $(k+1)$-partite k-dimensional simplicial complexes. (Respectively for N'.)*

Proof. Let $P = N^*(\Sigma)$ and $Q = \Omega^*(\Sigma)$. Suppose P contains all $(k+1)$-partite k-dimensional simplicial complexes. It follows that if H is any $(k+1)$-partite $(k+1)$-uniform hypergraph that $\sigma H \in N^*(\Sigma)$, hence $H \in Q$. Thus Q contains all

$(k+1)$-partite $(k+1)$-uniform hypergraphs and so one can prove $\Omega^*((k+1)\Sigma)=\mathcal{U}$. Thus $N^*((k+1)\Sigma)=\mathcal{S}$.

Conversely, suppose K is a $(k+1)$-partite k-dimensional simplicial complex not in P. Since K is an induced subcomplex of εK, we know εK is not in P. Thus $\mu\varepsilon K$ is not in Q. Since $\mu\varepsilon K$ is $(k+1)$-partite we know for all t we can prove $\Omega^*(t\Sigma)\neq\mathcal{U}$. Thus for all t we have $N^*(t\Sigma)\neq\mathcal{S}$.

An analogous proof holds for N'. □

Theorem 6.6 (nerve version). *The following statements are equivalent:*

(1) *There exists an integer t such that $N(t\Sigma)$ contains all k-dimensional simplicial complexes.*

(2) *There exists an integer t such that every k-dimensional simplicial complex is the k-skeleton of an element of $N(t\Sigma)$.*

(3) *$N(\Sigma)$ contains all $(k+1)$-partite k-dimensional simplicial complexes.*

Proof. This follows immediately from Theorem 6.5. The equivalence of (1) and (3) follows from considering N' and the equivalence of (2) and (3) follows by considering N^*. □

At this point it would be natural to apply these results to some specific family of sets, Σ. Wegner has shown that if Σ is the family of all convex sets in $\mathbb{R}^{2k+1}$, then $N(\Sigma)$ contains all k-dimensional simplicial complexes. We would like to show that if Σ is the set of all convex sets in $\mathbb{R}^{2k}$ then for no integer t does $N(t\Sigma)$ contain all k-dimensional simplicial complexes. To do this we need 'only' demonstrate the exixtence of a $(k+1)$-partite k-dimensional simplicial complex not in $N(\Sigma)$. To date no such simplicial complex has been demonstrated, but it is conjectured to exist. Wegner [12] posited:

Conjecture 6.7. *If K is a k-dimensional complex which does not embed in $\mathbb{R}^{2k}$ (and such simplicial complexes are known to exist) then βK is not the nerve of convex sets in $\mathbb{R}^{2k}$.*

This conjecture seems very reasonable. Since βK would be a $(k+1)$-partite k-dimensional simplicial complex, it would provide the necessary example.

Acknowledgments

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References

[1] M.B. Cozzens and F.S. Roberts, Computing the boxicity of a graph by covering its complement by cointerval graphs, Discrete Appl. Math. 6 (1983) 217–228.

[2] G. Ehrlich, S. Even and R.E. Tarjan, Intersection graphs of curves in the plane, J. Combin. Theory Ser. B 21 (1976) 8–20.

[3] R.L. Graham, B.L. Rothschild and J.H. Spencer, Ramsey Theory (Wiley, New York, 1980).

[4] J.R. Griggs and D.B. West, Extremal values of the interval number of a graph, SIAM J. Algebraic Discrete Meths. 1 (1980) 1–7.

[5] D. Hilbert and S. Cohn-Vossen, Geometry and the Imagination, translated by P. Nemenyi (Chelsea, New York, 1952).

[6] F.S. Roberts, On the boxicity and cubicity of a graph, in: W.T. tuttle, ed., Recent Progress in Combinatorics (Academic Press, New York, 1968) 301–310.

[7] E.R. Scheinerman, Characterizing intersection classes of graphs, Discrete Math. 55 (1985) 185–193.

[8] E.R. Scheinerman, Intersection classes and multiple intersection parameters of graphs, Doctoral Dissertation, Princeton University, 1984.

[9] F.W. Sinden, Topology of thin film RC circuits, Bell System Tech. J. (1966) 1639–1662.

[10] W.T. Trotter, Jr., A characterization of Roberts' inequality for boxicity, Discrete Math. 28 (1979) 303–313.

[11] W.T. Trotter, Jr. and F. Harary, On double and multiple interval graphs, J. Graph Theory 3 (1979) 205–211.

[12] G. Wegner, Eigenschaften der Nerven homolgisch-eigenfacher Familien im $\mathbb{R}^n$, Doctoral Dissertation, University of Gottingen, 1967.