HAMILTONIAN CYCLES IN VERTEX SYMMETRIC
GRAPHS OF ORDER $2p^2$

Dragan MARUŠIČ

Department of Mathematics, University of California, Santa Cruz, CA 95064, U.S.A.
(Vojke Šmuc 12, 66000 Koper, Yugoslavia)

Received 29 February 1984
Revised 25 September 1986

We prove that every connected vertex symmetric graph of order $2p^2$, where $p$ is a prime, is
Hamiltonian.

1. Introduction

L. Lovász has conjectured that every connected vertex symmetric graph has a
Hamiltonian path [5]. This conjecture has been verified for graphs of order $p$, $2p$, $3p$, $p^2$ and $p^3$ (in which case the graph has a Hamiltonian cycle, unless it is the
Petersen graph), and $4p$, $5p$, where we always let $p$ denote a prime. (See [1,
6–9].) In this paper we shall prove that every connected vertex symmetric graph
of order $2p^2$ is Hamiltonian (Theorem 2.5).

We shall assume that the reader is familiar with the basic graph theory
terminology. If $G$ is a graph and $X, Y \subseteq V(G)$ we let $X \cup Y$ denote the set of all
edges of $G$ having one end-vertex in $X$ and the other end-vertex in $Y$ and we let
$[X, Y]$ denote the subgraph of $G$ whose vertex set is $X \cup Y$ and whose edge set is
$X \cup Y$. The subgraph of $G$ induced by $X$ is the graph $[X] = [X, X]$. If $\mathcal{V}$ is a
partition of $V(G)$ then the factor graph $G/\mathcal{V}$ of $G$ with respect to $\mathcal{V}$ has the vertex
set $\mathcal{V}$ and $X, Y \in \mathcal{V}$ are adjacent iff $X \cup Y \neq \emptyset$. If $x, y \in V(G)$, we let $N(x)$ denote
the set of neighbours of $x$ and we let $m(x, y) = |N(x) \cap N(y)|$.

Let $n$ be an integer. We shall use the notation $\mathbb{Z}_n$ for the ring of residue classes
of integers mod $n$ and $\mathbb{Z}_n^*$ for the group of units of $\mathbb{Z}_n$. By $C_n$ we shall denote any
cyclic group of order $n$. For convenience we let CVSG$(n)$, HG denote the classes
of connected vertex symmetric graphs of order $n$, and Hamiltonian graphs
respectively.

For the group-theoretic concepts not defined here we refer the reader to [10].
Let $V$ be a finite set, $\Gamma$ be a permutation group on $V$, $W$ be an orbit of $\Gamma$, $v \in V$
and $\gamma \in \Gamma$. By $\Gamma(v)$, $\Gamma_w$, $\gamma^w$, $\mathcal{V}(\Gamma)$, $\mathcal{V}(\gamma)$ we shall denote the orbit of $\Gamma$
containing $v$, the restriction of $\Gamma$ on $W$, the restriction of $\gamma$ on $W$, the set of orbits
of $\Gamma$ and the set of orbits of $\gamma$ respectively. We say that $\gamma$ is $(m, n)$-homogeneous
if it has $m$ orbits of length $n \geq 2$ and no other orbits.

If $G$ is a graph, $\Gamma \leqslant \text{Aut } G$ and $X, Y \in \mathcal{V}(\Gamma)$ have equal length, then the graphs $[X, Y], [X]$ are regular of some valency $d(X, Y) = d(Y, X), d(X)$ respectively.

2. Hamiltonian properties

The following propositions will be needed to obtain our main result.

**Proposition 2.1** ([10, Theorem 3.4']). Let $W$ be an orbit of a permutation group $\Gamma$. If $p^k$ is the highest power of a prime $p$ dividing $|W|$ and $\Pi$ is a Sylow $p$-subgroup of $\Gamma$, then every shortest orbit of $\Pi$ in $W$ has cardinality $p^k$.

**Lemma 2.2.** Let $V$ be a finite set of cardinality $p^k$ and $\Pi$ be a Sylow $p$-subgroup of a transitive permutation group $\Gamma$ on $V$. Then $Z(\Pi)$ contains a $(p^{k-1}, p)$-homogeneous element.

**Proof.** $\Pi$ is transitive by Proposition 2.1. Since every finite $p$-group has a non-trivial centre [4, Theorem 4.3.1] we can select an element $\alpha$ of $Z(\Pi)$ of order $p$. Since $\alpha \neq 1$ there is $v \in V$ such that $\alpha(v) \neq v$. If $w \in V$, then $\rho(v) = w$ for some $\rho \in \Pi$ (since $\Pi$ is transitive) and so $\alpha(w) = \alpha(v) = \rho v \neq \rho(v) = w$. Hence $\alpha$ has no fixed vertex, i.e., $\alpha$ is $(p^{k-1}, p)$-homogeneous. $\square$

**Proposition 2.3** ([4, p. 51, II]). There are just two non-isomorphic groups of order $p^2$; a cyclic group of order $p^2$ and a direct product of two cyclic groups of order $p$.

Finally, we leave the proof of the following straightforward proposition to the reader.

**Proposition 2.4.** Let $G$ be a graph, $x, y \in V(G)$ and $\alpha \in \text{Aut } G$. Then $m(x, y) = m(\alpha(x), \alpha(y))$.

We can now prove the main result of this paper.

**Theorem 2.5.** CVSG$(2p^2) \subset \text{HG}$.

**Proof.** Let $G \in \text{CVSG}(2p^2)$. Since CVSG $(8) \subset \text{HG}$ (see [8]) we may assume that $p \geq 3$. Let $\Pi$ be a Sylow $p$-subgroup of $\text{Aut } G$. By Proposition 2.1, $\Pi$ has two orbits $X, Y$ of length $p^2$. Since $G$ is connected

$$d(X, Y) \geq 1. \quad (1)$$

We shall distinguish two different cases.
Case 1. $H$ contains a semiregular subgroup $\Gamma$ such that $V(\Gamma) = \{X, Y\}$.

Subcase 1(a). $\Gamma = C_p$. Then $G$ has a $(2, p^2)$-homogeneous automorphism $\alpha$ with orbits $X$ and $Y$. Let us first assume that either $[X]$ or $[Y]$ is connected. Then $G$ contains a subgraph isomorphic to a generalized Petersen graph $G(p^2, r)$ for some $r \in \{1, 2, \ldots, \frac{1}{2}(p^2 - 1)\}$ and since $p^2 \equiv 5 \pmod{6}$ it follows by $[2]$ that $G \subset HG$. Suppose now that both $[X]$ and $[Y]$ are disconnected. Let $x \in X$. By (1) there exists $y \in Y \cap N(x)$. Let $T = \{t \in \mathbb{Z}_{p^2} : x \sim \alpha(t)\}$. The connectedness of $G$ then implies that $T \cap \mathbb{Z}_{p^2}^* \neq \emptyset$. Let $t \in T \cap \mathbb{Z}_{p^2}^*$ and let $P_i = \alpha(t) \alpha'(x)$. Then $P_0 P_1 P_2 \ldots P_{(p^2-1)y}$ is a Hamiltonian cycle in $G$.

Subcase 1(b). $\Gamma = C_p \times C_p$. Let $x \in X$. By (1) there exists $y \in Y \cap N(x)$. Define the following sets:

$S = \{\sigma \in \Gamma : x \sim \sigma(x)\}$, \quad \quad $S' = \{\sigma \in \Gamma : y \sim \sigma(y)\}$,

and

$T = \{\sigma \in \Gamma : x \sim \sigma(y)\}$.

(Note that since $G$ is a graph, $S = S^{-1}$ and $S' = (S')^{-1}$.) Clearly, for each $\gamma \in \Gamma$, $N(\gamma(x)) = \{\sigma\gamma(x) : \sigma \in S\} \cup \{\sigma\gamma(x) : \sigma \in T\}$ and $N(\gamma(y)) = \{\sigma\gamma(y) : \sigma \in S'\} \cup \{\sigma^{-1}\gamma(x) : \sigma \in T\}$. The connectedness of $G$ implies that $\langle S \cup S' \cup T \rangle = \Gamma$.

Suppose first that $\langle T \rangle = \Gamma$. Then there exist $\rho, \tau \in T$ such that $\langle \rho, \tau \rangle = \Gamma$. Let, for each $\gamma \in \Gamma$, $P(\gamma)$ denote the path $\gamma(\gamma')\gamma(x)$. Furthermore, for each $i \in \mathbb{Z}_p$, let

$P_i = P(\rho^i\tau^{p-i})P(\rho^i\tau^{p-i+1})P(\rho^i\tau^{p-i+2}) \ldots P(\rho^i\tau^{p-i+p-1})$.

Then $P_0 P_1 P_2 \ldots P_{p-1} x$ is a Hamiltonian cycle in $G$.

Suppose now that $\langle S \cup S' \rangle = \Gamma$. Then there are $\rho \in S$ and $\tau \in S'$ such that $\langle \rho, \tau \rangle = \Gamma$. For each $i \in \mathbb{Z}_p$, define the following paths in $G$:

$P_i = \rho^i\tau^i(x)\rho^{i-1}\tau^i(x)\rho^{i-2}\tau^i(x) \ldots \rho^{i-(p-1)}\tau^i(x)$,

and

$Q_i = \rho^{i+1}\tau^i(y)\rho^{i+1}\tau^{i-1}(y)\rho^{i+1}\tau^{i-2}(y) \ldots \rho^{i+1}\tau^{i-(p-1)}(y)$.

Then $P_0 Q_0 P_1 Q_1 \ldots P_{p-1} Q_{p-1} x$ is a Hamiltonian cycle in $G$.

Finally, suppose that neither $\langle T \rangle = \Gamma$ nor $\langle S \cup S' \rangle = \Gamma$. Then there exist $p \in S$, $\tau \in T$ such that $\langle \rho, \tau \rangle = \Gamma$ and $\rho^r \in S'$ for some $r \in \mathbb{Z}_{p}^*$. For each $i \in \mathbb{Z}_p$, define the following paths in $G$:

$P_i = \rho^{i(r+1)}\tau^i(x)\rho^{i(r+1)-1}\tau^i(x)\rho^{i(r+1)-2}\tau^i(x) \ldots \rho^{i(r+1)-(p-1)}\tau^i(x)$,

and

$Q_i = \rho^{i(r+1)+1}\tau^{i+1}(y)\rho^{i(r+1)+1-r}\tau^{i+1}(y)\rho^{i(r+1)+1-2r}\tau^{i+1}(y) \ldots \rho^{i(r+1)+1-(p-1)r}\tau^{i+1}(y)$.

Then $P_0 Q_0 P_1 Q_1 \ldots P_{p-1} Q_{p-1} x$ is a Hamiltonian cycle in $G$.

Case 2. There is no semiregular subgroup $\Gamma$ of $H$ such that $V(\Gamma) = \{X, Y\}$. 

Since every vertex symmetric graph of order \(p^2\) is a Cayley graph \([8]\) it follows by Proposition 2.3 that both \([X]\) and \([Y]\) are Cayley graphs of an Abelian group.

Let us first assume that both \([X]\) and \([Y]\) are connected. If \(d = d(X) = d(Y) = 2\), then \([X]\) and \([Y]\) are cycles of length \(p^2\) and therefore \(\Pi^X\) and \(\Pi^Y\) are cyclic groups of order \(p^2\). This implies the existence of a \((2, p^2)\)-homogeneous element of \(\Pi\), a contradiction. Thus we may assume that \(d \geq 3\). Then, by \([3]\), \([X]\) and \([Y]\) are Hamiltonian connected. By (1) there exist distinct vertices \(x, x' \in X\) and \(y, y' \in Y\) such that \(x \sim y\) and \(x' \sim y'\). There are Hamiltonian paths \(P, Q\) in \([X], [Y]\) with origins \(x, y'\) and termini \(x', y\) respectively. Clearly \(PQx\) is a Hamiltonian cycle in \(G\). We may therefore assume that

either \([X]\) or \([Y]\) is disconnected

and so

\(d \leq p - 1\).

Moreover, we shall also assume that

\([X, Y] \notin HG\).

Let us first prove the following result.

There exists a \((2p, p)\)-homogeneous element \(\alpha \in \Pi\) with orbits \(X_0, X_1, \ldots, X_{p-1} \subset X\) and \(Y_0, Y_1, \ldots, Y_{p-1} \subset Y\) and an element \(\beta \in \Pi\) such that, for each \(i \in \mathbb{Z}_p\), \(X_i = \beta^i(x_0), Y_i = \beta^i(y_0), [X_i, Y_i] = K_{p,p}\) and \([X, Y] = [X_0, Y_0] + [X_1, Y_1] + \cdots + [X_{p-1}, Y_{p-1}]\).

Suppose first that \(\Pi\) contains no element of order \(p^2\). By \([4, \text{Theorem 4.3.1}]\), \(Z(\Pi)\) contains an element \(\sigma\) of order \(p\). It is not difficult to see that \(\sigma\) cannot be \((2p, p)\)-homogeneous for otherwise (the orbits of \(\sigma\) being blocks of \(\Pi\)) there would exist a semi-regular subgroup \((= C_p \times C_p)\) of \(\Pi\) with orbits \(X\) and \(Y\). Hence we may assume by Lemma 2.2 that, with no loss of generality, \(\sigma^X\) is \((p, p)\)-homogeneous and \(\sigma^Y = 1\). Since the orbits of \(\sigma\) in \(X\) are blocks of \(\Pi\) there exists \(\tau \in \Pi\) such that \(X\) is an orbit of \(\langle \sigma, \tau \rangle = C_p \times C_p\). If \(\tau\) has a fixed vertex in \(Y\), then \([X, Y] = K_{p^2, p^2}\), which contradicts (4). Therefore \(\tau\) is \((2p, p)\)-homogeneous. Since \(\sigma^Y = 1\), it follows that \(\sigma^Y(x) \sim u_i (i \in \mathbb{Z}_p)\). Since \(\sigma^Y = 1\), it follows that \(\sigma^Y(x) \sim u_i\) for any two \(i, j \in \mathbb{Z}_p\). Let \(X_0 = \langle \sigma \rangle(x), Y_0 = \{u_0, u_1, \ldots, u_{p-1}\}\) and \(X_i = \tau^i(X_0), Y_i = \tau^i(Y_0)(i \in \mathbb{Z}_p)\). Clearly, \([X_i, Y_i] = K_{p,p}\) \((i \in \mathbb{Z}_p)\).

Suppose that \(\tau^r(u_i) \in N(x)\) for some \(r \in \mathbb{Z}_p^*, j \in \mathbb{Z}_p\). Then \([X_0, \{\tau^r(u_j)\}] = K_{p,1}\) and therefore \([X_i, \{\tau^{i+r}(u_j)\}] = K_{p,1}\) for each \(i \in \mathbb{Z}_p\). It is then not difficult to see that \([X, Y] \in HG\), which contradicts (4). Therefore \([X, Y] = [X_0, Y_0] + [X_1, Y_1] + \cdots + [X_{p-1}, Y_{p-1}]\). Since \(Y \in V(\Pi)\) there exists \(\rho \in \Pi\) such that \(\rho(u_0) \in Y \setminus \{u_0\}\). Suppose that \(\rho(u_i) \notin Y_0\) for some \(i\). Since \(X_0 \subseteq N(u_0) \cap N(u_i)\), it follows that \(m(u_0, u_i) \geq p\). On the other hand, \(m(\rho(u_0), \rho(u_i)) \leq p - 1\) by (3).
This contradicts Proposition 2.4. Therefore $\rho(u_i) \in Y_0$ for each $i \in Z_p$, i.e., $Y_0 \in \mathcal{V}(\rho)$. Similarly, $\rho(Y_i) = Y_i$ for each $i \in Z_p$. Letting $\beta = \tau$ and defining the automorphism $\alpha$ by

$$\alpha(v) = \begin{cases} \rho(v), & \text{if } v \in Y, \\ \sigma(v), & \text{if } v \in X, \end{cases}$$

we complete the proof of (5) for the case when $\Pi$ has no element of order $p^2$.

If $\Pi$ contains an element $\tau$ of order $p^2$, then $\tau$ must have an orbit of length $p^2$, say $X_0$, and $p$ orbits $U_0, U_1, \ldots, U_{p-1} \subseteq Y$ of length $p$ by (4). Let $x \in X$. By (1) it follows that, for each $i \in Z_p$, there exists $u_i \in U_i \cap N(x)$. Let $X_0 = \langle \tau^p \rangle$ (x), $Y_0 = \{u_0, u_1, \ldots, u_{p-1}\}$ and $X_i = \tau'(X_0)$, $Y_i = \tau'(Y_0)$ ($i \in Z_p$). Clearly, $[X_i, Y_i] = K_{p,p}$ ($i \in Z_p$). We now follow the argument used in the paragraph above (for the case when $\Pi$ had no automorphism of order $p^2$) and complete the proof of (5).

Let $\mathcal{V} = \{X_i : i \in Z_p\}$ and $\mathcal{W} = \{Y_i : i \in Z_p\}$. It follows by (5) that $[X]/\mathcal{V}$, $[Y]/\mathcal{W}$ and each of $[X_i]$, $[Y_i]$ are vertex symmetric graphs of order $p$. Let us assume that $[X_i]$, $[Y_i]$ ($i \in Z_p$) are totally disconnected. Then (since a vertex symmetric graph of prime order is either connected or totally disconnected) it follows that $[X]/\mathcal{V}$ and $[Y]/\mathcal{W}$ are connected graphs.

In view of (2) we lose no generality in assuming that $d(Y_i, Y_j) \leq 1$ for all $i, j \in Z_p$. Suppose that $X_0$ is not a block of Aut $G$. Then there exist $\rho \in \text{Aut } G$, $x, x' \in X_0$, $j \in Z_p^*$ such that $\rho(x) \in X_0$ and $\rho(x') \in X_j \cup Y_j$. Since $Y_0 \subseteq N(x) \cap N(x')$ it follows that $m(x, x') \geq p$. Therefore, by Proposition 2.4 $m(\rho(x), \rho(x')) \geq p$. Hence, $\rho(x') \in Y_j$ and so $p \leq m(\rho(x), \rho(x')) = d(X_0, X_j) + d(Y_0, Y_j) \leq d(X_0, X_j) + 1$. Therefore by (3), $d(X_0, X_j) = p - 1$ and so $m(x, x') = 2p - 2 > m(\rho(x), \rho(x'))$, a contradiction. Thus $X_0$ is a block of Aut $G$ and so each $X_i$ is a block of Aut $G$. This, (3), (6) and the fact that $[X_i, Y_j] = K_{p,p}$ together imply that $\{X, Y\}$ is a block system of Aut $G$. Thus $[X] \cong [Y]$ and by (2) both $[X]$ and $[Y]$ are disconnected and therefore a union of $p$ components each of which is a vertex symmetric graph of order $p$. It is then not difficult to see that there exist automorphisms $\alpha_1, \beta_1 \in \text{Aut}[X]$ and $\alpha_2, \beta_2 \in \text{Aut}[Y]$ of order $p$ such that $\alpha_1(X_i) = X_i$, $\beta_1(X_i) = X_{i+1}$, $\alpha_2(Y_i) = Y_i$, $\beta_2(Y_i) = Y_{i+1}$ ($i \in Z_p$) and $\alpha_1 \beta_1 = \beta_1 \alpha_1$, $\alpha_2 \beta_2 = \beta_2 \alpha_2$. Define automorphisms $\alpha', \beta'$ by

$$\alpha'(v) = \begin{cases} \alpha_1(v), & \text{if } v \in X, \\ \alpha_2(v), & \text{if } v \in Y, \end{cases} \quad \beta'(v) = \begin{cases} \beta_1(v), & \text{if } v \in X, \\ \beta_2(v), & \text{if } v \in Y. \end{cases}$$

Then $\langle \alpha', \beta' \rangle$ is a semiregular subgroup of $\Pi$ with orbits $X$ and $Y$, a contradiction.

We are therefore left (since $G$ is connected) with the case when either $[Y]/\mathcal{W}$ and all $[X_i]$ are connected or $[X]/\mathcal{V}$ and all $[Y_i]$ are connected. We lose no generality in assuming that the latter is the case. Choose distinct vertices $x_i, x'_i \in X_i$ such that $x_i \sim x_{i+1}$ for each $i \in Z_p$. Then there exists a Hamiltonian path $Q_i$ in $[X_i \cup Y_i]$ with origin $x_i$ and terminus $x'_i$ for each $i \in Z_p$. Thus
$Q_0Q_1\ldots Q_{p-1}x_0$ is a Hamiltonian cycle in $G$. This completes the proof of Theorem 2.5. □

References