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Orthogonal rational functions and rational modifications of a measure on the unit circle☆

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Abstract

In this paper we present formulas expressing the orthogonal rational functions associated with a rational modification of a positive bounded Borel measure on the unit circle, in terms of the orthogonal rational functions associated with the initial measure. These orthogonal rational functions are assumed to be analytic inside the closed unit disc, but the extension to the case of orthogonal rational functions analytic outside the open unit disc is easily made. As an application we obtain explicit expressions for the orthogonal rational functions associated with a rational modification of the Lebesgue measure.

Keywords: Orthogonal rational functions; Rational modifications of a measure; Unit circle

1. Introduction

Given a positive bounded Borel measure μ on the unit circle, Godoy et al. [3] derived formulas, in determinant form, expressing the orthogonal polynomials (OPs) associated with

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the polynomial modification $\tilde{\mu}$ of the measure μ on the unit circle in terms of the OPs associated with μ. Suppose $d\tilde{\mu}$ is given by

$$d\tilde{\mu} = |p_m(z)|^2 d\mu, \quad z = e^{\mathbf{i}\theta},$$

where $p_m(z)$ is a polynomial of degree m, and let $\phi_n(z; \tilde{\mu})$ denote the OP of degree n associated with $\tilde{\mu}$. Then these derivations are based on the fact that $p_m(z)\phi_n(z; \tilde{\mu})$ is again a polynomial, but now of degree $m + n$, so that a relation exists between $\phi_n(z; \tilde{\mu})$ and $\phi_{m+n}(z; \mu)$, the OP of degree $m + n$ associated with μ.

Later on, Godoy et al. [4] derived formulas for OPs and rational modifications of a positive bounded Borel measure on a compact set of the complex plane (including the unit circle). Suppose now that $d\tilde{\mu}$ is given by

$$d\tilde{\mu} = \frac{d\mu}{|p_m(z)|^2}, \quad z = e^{\mathbf{i}\theta},$$

where $p_m(z)$ has no zeros on the unit circle. Then clearly $\phi_n(z; \tilde{\mu})/p_m(z)$ is not necessarily a polynomial of degree $n - m$. Hence, the derivations in [4] are different from those in [3]. Instead, they are based on the so-called functions of the second kind.

Orthogonal rational functions (ORFs) analytic inside the closed unit disc are a generalisation of OPs on the unit circle in such a way that the OPs return if all the poles are at infinity (see e.g. [2, p. 1]). The aim of this paper is to generalise the results for OPs and polynomial and rational modifications of a measure on the unit circle to the case of ORFs. The main difference between OPs and ORFs is that for the latter, polynomial and rational modification of a measure on the unit circle can be treated simultaneously, in a similar way as has been done in [3]. This due to the fact that a rational function multiplied with, or divided by, a polynomial is obviously again a rational function.

Although the ORFs are assumed to be analytic inside the closed unit disc, the results obtained in this paper can easily be extended to the case of ORFs analytic outside the open unit disc with the aid of [8].

The outline of this paper is as follows. After giving the necessary theoretical preliminaries in Section 2, Section 3 contains the main result with respect to ORFs and rational modifications of a measure on the unit circle. This result is highly elegant, mathematically, but it is hardly useful for computational purposes. Hence, Section 4 deals with computing the monic orthogonal rational functions (MORFs) associated with the rational modifications of a measure on the unit circle through the MORFs associated with the initial measure. Finally, in Section 5 we derive expressions for the ORFs and MORFs associated with the rational modification $\tilde{\mu}$ of the Lebesgue measure μ on the unit circle, given by

$$d\tilde{\mu}(z) = \left|\frac{z - \gamma}{1 - \overline{\beta} z}\right|^2 d\mu(z), \quad |\gamma| \leq 1 \text{ and } |\beta| < 1,$$

or equivalently

$$d\tilde{\mu}(\theta) = \frac{(1 - r)^2 + 4r \sin^2\left(\frac{\theta - t}{2}\right)}{\left|1 - \overline{\beta} e^{\mathbf{i}\theta}\right|^2} d\theta, \quad r \in [0, 1], t \in \mathbb{R} \text{ and } |\beta| < 1,$$

where $\gamma = r e^{\mathbf{i}t}$ and β are parameters that can be chosen freely.

2. Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$. For the real line we use the symbol $\mathbb{R}$ and for the positive real line $\mathbb{R}^+ = \{z \in \mathbb{R} : z \geq 0\}$. Furthermore, we will use the blackboard $\mathbb{N}$ to represent the set of natural numbers. The unit circle, the open and closed unit disc will be denoted respectively by

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}, \qquad \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad \mathbb{O} = \mathbb{D} \cup \mathbb{T}.$$

If a subset Y of X is omitted from the set X, this will be represented by X_Y, e.g. $\mathbb{C}_{\mathbb{R}} = \mathbb{C} \setminus \mathbb{R}$ and $\mathbb{D}_0 = \mathbb{D} \setminus \{0\}$.

Let $\mathcal{P}_n$ denote the space of polynomials of degree less than or equal to n. Then we define the substar conjugate and the super-c conjugate of a function $p_n(z) \in \mathcal{P}_n$ respectively as

$$p_{n*}(z) = \overline{p_n(1/\overline{z})} \quad \text{and} \quad p_n^c(z) = \overline{p_n(\overline{z})},$$

and the superstar transformation as

$$p_n^*(z) = z^n p_{n*}(z).$$

Note that $p_{n*}(z) = \overline{p_n(z)}$ if $z \in \mathbb{T}$, and that $p_n^c(z) = \overline{p_n(z)}$ if $z \in \mathbb{R}$.

Suppose a sequence of complex numbers $\mathcal{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \mathbb{D}$ is given and define the Blaschke factors

$$\zeta_k(z) := \zeta_{\alpha_k}(z) = \eta_{\alpha_k} \frac{z - \alpha_k}{1 - \overline{\alpha}_k z}, \qquad \eta_{\alpha_k} = \begin{cases} \dfrac{\overline{\alpha}_k}{|\alpha_k|}, & \alpha_k \neq 0 \\ 1, & \alpha_k = 0, \end{cases} \qquad k = 1, 2, \ldots, \tag{1}$$

and the Blaschke products[1]

$$B_0(z) \equiv 1, \qquad B_k(z) = B_{k-1}(z)\zeta_k(z), \quad k = 1, 2, \ldots; \tag{2}$$

see e.g. [2, pp. 42–43]. Then the space of rational functions with poles in $\{1/\overline{\alpha}_1, \ldots, 1/\overline{\alpha}_n\}$ is defined as

$$\mathcal{L}_n = \text{span}\{B_0(z), \ldots, B_n(z)\}.$$

In the special case of all $\alpha_k = 0$, the factor in (1) becomes $\zeta_k(z) = z$ and the products in (2) become $B_k(z) = z^k$. Define

$$\pi_0(z) \equiv 1, \qquad \pi_k(z) = \prod_{j=1}^{k} (1 - \overline{\alpha}_j z), \quad k = 1, 2, \ldots,$$

then we may write equivalently

$$B_k(z) = \nu_k \frac{\pi_k^*(z)}{\pi_k(z)}, \quad \nu_k = \prod_{j=1}^{k} \eta_{\alpha_j} \in \mathbb{T}$$

and

$$\mathcal{L}_n = \{p_n/\pi_n : p_n \in \mathcal{P}_n\}.$$

[1] The factors and products are named after Wilhelm Blaschke, who introduced these for the first in [1].

Further, we let $\mathcal{L}_{-1} = \{0\}$ and $\mathcal{L}_0 = \mathbb{C}$ to be the trivial subspaces.

The definitions for the substar conjugate and the super-c conjugate of a function $f_n(z) \in \mathcal{L}_n$ are the same as before, for the polynomial case, but the superstar transformation is now defined as

$$f_n^*(z) = B_n(z) f_{n*}(z).$$

Let μ be a positive bounded Borel measure on the unit circle. Orthonormalising the basis $\{B_0(z), \ldots, B_n(z)\}$ with respect to this measure μ and inner product

$$\langle f, g\rangle = \frac{1}{2\pi} \oint_{\mathbb{T}} f(z) g_*(z) \mathrm{d}\mu(z)$$

we obtain the orthonormal rational functions (ORFs) $\{\varphi_0(z;\mu), \ldots, \varphi_n(z;\mu)\}$. Explicit expressions for the so-called Takenaka–Malmquist basis (see [5,6]), which is a basis of ORFs on the unit circle with respect to the Lebesgue measure ($\mathrm{d}\mu(z) = \frac{\mathrm{d}z}{\mathbf{i}z}$), are well known. These are given by

$$\varphi_0(z;\mu) \equiv 1 \quad \text{and} \quad \varphi_n(z;\mu) = \eta_{\alpha_n} \sqrt{1 - |\alpha_n|^2} \frac{z B_{n-1}(z)}{1 - \overline{\alpha}_n z}, \quad n \geq 1.$$

We denote the leading coefficient of $\varphi_n(z;\mu)$, i.e. the coefficient of $B_n(z)$ in the expansion of $\varphi_n(z;\mu)$ with respect to the basis $\{B_0(z), \ldots, B_n(z)\}$, by $\kappa_n = \overline{\varphi_n^*(\alpha_n;\mu)}$. In the remainder of this paper, we will assume that $\kappa_n \in \mathbb{R}_0^+$. With this leading coefficient, we define the monic orthogonal rational functions (MORFs) associated with the sequence $\mathcal{A}$ and the measure μ as $\phi_n(z;\mu) = \kappa_n^{-1} \varphi_n(z;\mu)$, e.g. for the Lebesgue measure this gives

$$\phi_0(z;\mu) \equiv 1 \quad \text{and} \quad \phi_n(z;\mu) = \eta_{\alpha_n} (1 - |\alpha_n|^2) \frac{z B_{n-1}(z)}{1 - \overline{\alpha}_n z}, \quad n \geq 1. \tag{3}$$

The reproducing kernel for $\mathcal{L}_n$ associated with the measure μ is given by

$$k_n(z, w; \mu) = \sum_{k=0}^{n} \varphi_k(z;\mu) \overline{\varphi_k(w;\mu)}. \tag{4}$$

The following Christoffel–Darboux relation has been proved in [2, Thm. 3.1.3] for $n \geq 1$:

$$\begin{aligned} k_n(z, w; \mu) &= \frac{\varphi_{n+1}^*(z;\mu) \overline{\varphi_{n+1}^*(w;\mu)} - \varphi_{n+1}(z;\mu) \overline{\varphi_{n+1}(w;\mu)}}{1 - \zeta_{n+1}(z) \overline{\zeta_{n+1}(w)}} \\ &= \kappa_{n+1}^2 \frac{\phi_{n+1}^*(z;\mu) \overline{\phi_{n+1}^*(w;\mu)} - \phi_{n+1}(z;\mu) \overline{\phi_{n+1}(w;\mu)}}{1 - \zeta_{n+1}(z) \overline{\zeta_{n+1}(w)}}. \end{aligned} \tag{5}$$

The earlier definitions with respect to the sequence of complex numbers $\mathcal{A}$, can be repeated for other sequences. Suppose a sequence $\mathcal{B}_m = \{\beta_1, \ldots, \beta_m\} \subset \mathbb{D}$ is given and let $\mathcal{C} = \mathcal{B}_m \cup \mathcal{A} = \{\delta_1, \delta_2, \ldots\} \subset \mathbb{D}$ with

$$\delta_k = \begin{cases} \beta_k, & k \leq m \\ \alpha_{k-m}, & k > m. \end{cases}$$

Then we denote the Blaschke factors and Blaschke products associated with the sequence $\mathcal{C}$ respectively as $\tilde{\zeta}_k(z) := \zeta_{\delta_k}(z)$ and $\tilde{B}_k(z)$. The space of rational functions with poles in

$\{1/\overline{\delta}_1, \ldots, 1/\overline{\delta}_{m+n}\}$ is then defined as

$$\tilde{\mathcal{L}}_{m+n} = \text{span}\{\tilde{B}_0(z), \ldots, \tilde{B}_{m+n}(z)\}$$

and orthonormalising this basis with respect to μ on the unit circle, we obtain the ORFs $\{\tilde{\varphi}_0(z;\mu), \ldots, \tilde{\varphi}_{m+n}(z;\mu)\}$ and MORFs $\{\tilde{\phi}_0(z;\mu), \ldots, \tilde{\phi}_{m+n}(z;\mu)\}$ with $\tilde{\phi}_k(z;\mu) = \tilde{\kappa}_k^{-1}\tilde{\varphi}_k(z;\mu)$ for $k = 0, \ldots, m+n$. Finally, the reproducing kernel for $\tilde{\mathcal{L}}_{m+n}$ associated with the measure μ will be denoted by $\tilde{k}_{m+n}(z, w; \mu)$.

3. Rational modifications of a measure on the unit circle

Let μ_m be a positive bounded Borel measure on the unit circle, given by

$$\mathrm{d}\mu_m = |A_m(z)|^2 \mathrm{d}\mu, \quad m \in \mathbb{N}_0, \tag{6}$$

where $A_m(z) \in \tilde{\mathcal{L}}_m \setminus \tilde{\mathcal{L}}_{m-1}$. For $z \in \mathbb{T}$ it holds that

$$|z - a|^2 = |1 - \overline{a}z|^2, \quad \forall a \in \mathbb{C},$$

and hence, without loss of generality we can assume $A_m(z)$ has all its zeros γ_k, for $k = 1, \ldots, m$, in $\mathbb{O}$.

Next, let $\varphi_n(z;\mu_m)$ (respectively $\phi_n(z;\mu_m)$) represent the ORF (respectively MORF) associated with the sequence $\mathcal{A}$ and the measure μ_m on the unit circle. Define χ_n as

$$\chi_0 = \overline{A_m^*(\beta_m)}, \qquad \chi_n = \overline{A_m^*(\alpha_n)}, \quad n \geq 1. \tag{7}$$

Then it holds that

$$\tilde{\phi}_{m+n}(z;\mu) - \frac{A_m(z)}{\chi_n}\phi_n(z;\mu_m) \in \tilde{\mathcal{L}}_{m+n-1}. \tag{8}$$

Let us now consider the orthogonal decomposition

$$\tilde{\mathcal{L}}_{m+n-1} = A_m\mathcal{L}_{n-1} \oplus \left[A_m\mathcal{L}_{n-1}\right]_{\mu}^{\perp m+n-1}, \tag{9}$$

where $\{\tilde{k}_{m+n-1}(z, \gamma_i; \mu)\}_{i=1}^{m}$ is a basis for the space $\left[A_m\mathcal{L}_{n-1}\right]_{\mu}^{\perp m+n-1}$ if $\gamma_i \neq \gamma_j$ for $i \neq j$ (we give a proof of this statement in the Appendix). We then have the following theorem.

Theorem 1. *Let μ_m be a rational modification of the positive bounded Borel measure μ on the unit circle, given by (6), where $A_m(z) \in \tilde{\mathcal{L}}_m$ is a rational function with simple zeros $\{\gamma_1, \ldots, \gamma_m\} \subset \mathbb{O}$. Let $\phi_n(z;\mu_m)$ denote the MORF associated with the sequence $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ and the measure μ_m. Similarly, let $\tilde{\phi}_{m+n}(z;\mu)$ denote the MORF associated with the sequence $\mathcal{C} = \{\beta_1, \ldots, \beta_m, \alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ and the measure μ. Then it holds that*

$$\frac{A_m(z)}{\chi_n}\phi_n(z;\mu_m) = \frac{1}{\det K}\begin{vmatrix} \tilde{\phi}_{m+n}(z;\mu) & \tilde{k}_{m+n-1}(z,\gamma_1;\mu) & \ldots & \tilde{k}_{m+n-1}(z,\gamma_m;\mu) \\ \tilde{\phi}_{m+n}(\gamma_1;\mu) & \tilde{k}_{m+n-1}(\gamma_1,\gamma_1;\mu) & \ldots & \tilde{k}_{m+n-1}(\gamma_1,\gamma_m;\mu) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\phi}_{m+n}(\gamma_m;\mu) & \tilde{k}_{m+n-1}(\gamma_m,\gamma_1;\mu) & \ldots & \tilde{k}_{m+n-1}(\gamma_m,\gamma_m;\mu) \end{vmatrix}, \tag{10}$$

where χ_n is given by (7), $\tilde{k}_{m+n-1}(z, w; \mu)$ denotes the reproducing kernel for $\tilde{\mathcal{L}}_{m+n-1}$, and the matrix K is given by

$$K = [\tilde{k}_{m+n-1}(\gamma_i, \gamma_j; \mu)]_{i,j=1}^{m}. \tag{11}$$

Proof. With the decomposition given by (9) it follows from (8) that there exists a unique sequence of complex numbers $\{\Lambda_{m+n-1,0}, \ldots, \Lambda_{m+n-1,n-1}, \lambda_{m+n-1,1}, \ldots, \lambda_{m+n-1,m}\}$ so that

$$\tilde{\phi}_{m+n}(z; \mu) - \frac{A_m(z)}{\chi_n}\phi_n(z; \mu_m)$$
$$= \sum_{k=0}^{n-1} \Lambda_{m+n-1,k} A_m(z)\phi_k(z; \mu_m) + \sum_{j=1}^{m} \lambda_{m+n-1,j}\tilde{k}_{m+n-1}(z, \gamma_j; \mu).$$

Because $\{A_m(z)\phi_k(z; \mu_m)\}_{k=0}^{n-1}$ forms an orthogonal basis for the space $A_m\mathcal{L}_{n-1}$ with respect to the measure μ, it holds that $\Lambda_{m+n-1,k} = 0$ for $k = 0, \ldots, n-1$. Furthermore, for $z = \gamma_i$ we get that

$$\tilde{\phi}_{m+n}(\gamma_i; \mu) = \sum_{j=1}^{m} \lambda_{m+n-1,j}\tilde{k}_{m+n-1}(\gamma_i, \gamma_j; \mu), \quad i = 1, \ldots, m,$$

so that

$$\begin{bmatrix} \lambda_{m+n-1,1} \\ \vdots \\ \lambda_{m+n-1,m} \end{bmatrix} = K^{-1} \begin{bmatrix} \tilde{\phi}_{m+n}(\gamma_1; \mu) \\ \vdots \\ \tilde{\phi}_{m+n}(\gamma_m; \mu) \end{bmatrix},$$

with K given by (11). Consequently, we have that

$$\frac{A_m(z)}{\chi_n}\phi_n(z; \mu_m)$$
$$= \tilde{\phi}_{m+n}(z; \mu) - [\tilde{k}_{m+n-1}(z, \gamma_1; \mu) \ldots \tilde{k}_{m+n-1}(z, \gamma_m; \mu)]K^{-1} \begin{bmatrix} \tilde{\phi}_{m+n}(\gamma_1; \mu) \\ \vdots \\ \tilde{\phi}_{m+n}(\gamma_m; \mu) \end{bmatrix},$$

which can also be written in determinant form, as in (10). □

Remark 2. From the previous theorem, together with Eq. (5), it follows that $\phi_n(z; \mu_m)$ can be computed only by means of $\tilde{\phi}_{m+n}(z; \mu)$.

Remark 3. If the zeros of $A_m(z)$ are not simple, i.e. if for $i = 1, \ldots, j$ it holds that γ_i has multiplicity m_i with $\sum_{i=1}^{j} m_i = m$, then there exists a new basis for $\left[A_m\mathcal{L}_{n-1}\right]_{\mu}^{\perp m+n-1}$ given by

$$\left[\left[\left.\frac{\partial^k \tilde{k}_{m+n-1}(z, w; \mu)}{\partial \overline{w}^k}\right|_{w=\gamma_i}\right]_{k=0}^{m_i-1}\right]_{i=1}^{j}. \tag{12}$$

This way, Theorem 1 can be generalised to the case of multiple zeros.

4. Computing the MORFs for the rational modifications

Eq. (10) is highly elegant, mathematically, but it is hardly useful for computational purposes. Therefore, it will be more interesting to compute $\phi_n(z;\mu_m)$ by means of computing intermediate MORFs, using the minimum possible number of terms from the sequence $\{\tilde{\phi}_n(z;\mu)\}$. As a consequence of Theorem 1 we have the following corollary.

Corollary 4. *Let the sequences of complex numbers* $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ *and* $\mathcal{C} = \{\beta, \alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ *be given. Suppose* $\tilde{\mu}$ *is given by*

$$\mathrm{d}\tilde{\mu} = \left|\frac{z-\gamma}{1-\overline{\beta}z}\right|^2 \mathrm{d}\mu, \quad \gamma \in \mathbb{O}.$$

Let $\phi_n(z;\tilde{\mu})$ *denote the MORF associated with the sequence* $\mathcal{A}$ *and the measure* $\tilde{\mu}$. *Similarly, let* $\tilde{\phi}_{n+1}(z;\mu)$ *denote the MORF associated with the sequence* $\mathcal{C}$ *and the measure* μ. *Then it holds for* $n \geq 1$ *that*

$$\left(\frac{z-\gamma}{1-\overline{\beta}z}\right)\phi_n(z;\tilde{\mu}) = \overline{\eta}_\beta \frac{1-\gamma\overline{\alpha}_n}{1-\beta\overline{\alpha}_n}\left[\tilde{\phi}_{n+1}(z;\mu) - \frac{\tilde{\phi}_{n+1}(\gamma;\mu)}{\tilde{k}_n(\gamma,\gamma;\mu)}\tilde{k}_n(z,\gamma;\mu)\right],$$

where $\tilde{k}_n(z,w;\mu)$ *denotes the reproducing kernel for the space of rational functions* $\tilde{\mathcal{L}}_n$, *associated with the measure* μ.

Remark 5. If $n = 0$, it is easily verified with the aid of (4) and Theorem 1, with $m = 1$, that

$$\phi_0(z;\tilde{\mu}) = \overline{\eta}_\beta\left(\frac{1-\gamma\overline{\beta}}{1-|\beta|^2}\right)\left(\frac{1-\overline{\beta}z}{z-\gamma}\right)\left(\tilde{\phi}_1(z;\mu) - \tilde{\phi}_1(\gamma;\mu)\right) \equiv 1.$$

So, in remainder we will restrict ourselves to the case in which $n \geq 1$.

From Eq. (5) it follows that

$$\begin{aligned}(1-\overline{\gamma}z)\tilde{k}_n(z,\gamma;\mu) = {} & \tilde{\kappa}_{n+1}^2\frac{1-\alpha_n\overline{\gamma}}{1-|\alpha_n|^2}(1-\overline{\alpha}_n z) \\ & \times\left[\tilde{\phi}_{n+1}^*(z;\mu)\overline{\tilde{\phi}_{n+1}^*(\gamma;\mu)} - \tilde{\phi}_{n+1}(z;\mu)\overline{\tilde{\phi}_{n+1}(\gamma;\mu)}\right].\end{aligned}$$

Consequently, we have that

$$\begin{aligned}&\frac{(1-\overline{\gamma}z)(z-\gamma)}{1-\overline{\beta}z}\phi_n(z;\tilde{\mu}) = \chi_n(1-\overline{\gamma}z)\tilde{\phi}_{n+1}(z;\mu) \\ &\quad + \chi_n\upsilon_n(1-\overline{\alpha}_n z)\left[\tilde{\phi}_{n+1}(z;\mu)\overline{\tilde{\phi}_{n+1}(\gamma;\mu)} - \tilde{\phi}_{n+1}^*(z;\mu)\overline{\tilde{\phi}_{n+1}^*(\gamma;\mu)}\right],\end{aligned} \tag{13}$$

where

$$\chi_n = \overline{\eta}_\beta\frac{1-\gamma\overline{\alpha}_n}{1-\beta\overline{\alpha}_n} \quad \text{and} \quad \upsilon_n = \tilde{\kappa}_{n+1}^2\left(\frac{1-\alpha_n\overline{\gamma}}{1-|\alpha_n|^2}\right)\frac{\tilde{\phi}_{n+1}(\gamma;\mu)}{\tilde{k}_n(\gamma,\gamma;\mu)}. \tag{14}$$

With this, we can prove the following theorems.

Theorem 6. *Let a sequence of complex numbers* $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ *and* $\mathcal{C} = \{\beta, \alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ *be given. Suppose* $\tilde{\mu}$ *is given by*

$$\mathrm{d}\tilde{\mu} = \left|\frac{z-\gamma}{1-\overline{\beta}z}\right|^2 \mathrm{d}\mu, \quad \gamma \in \mathbb{T}.$$

Let $\phi_n(z;\tilde{\mu})$ *denote the MORF associated with the sequence* $\mathcal{A}$ *and the measure* $\tilde{\mu}$*. Similarly, let* $\tilde{\phi}_{n+1}(z;\mu)$ *denote the MORF associated with the sequence* $\mathcal{C}$ *and the measure* μ*. Then it holds that*

$$\frac{(z-\gamma)^2}{1-\overline{\beta}z}\phi_n(z;\tilde{\mu}) = \frac{\overline{\eta}_\beta \frac{1-\gamma\overline{\alpha}_n}{1-\beta\overline{\alpha}_n}}{\begin{vmatrix} \tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu) \\ \tilde{\phi}'_{n+1}(\gamma;\mu) & \tilde{\phi}^{*\prime}_{n+1}(\gamma;\mu) \end{vmatrix}} \times \begin{vmatrix} (z-\gamma)\tilde{\phi}_{n+1}(z;\mu) & \tilde{\phi}_{n+1}(z;\mu) & \tilde{\phi}^*_{n+1}(z;\mu) \\ 0 & \tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu) \\ \left(\frac{1-\overline{\alpha}_n z}{1-\overline{\alpha}_n\gamma}\right)\tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}'_{n+1}(\gamma;\mu) & \tilde{\phi}^{*\prime}_{n+1}(\gamma;\mu) \end{vmatrix}, \tag{15}$$

where $\tilde{\phi}'$ *represents the derivative of* $\tilde{\phi}$*.*

Proof. If $\gamma \in \mathbb{T}$, it holds that

$$\overline{\tilde{\phi}_{n+1}(\gamma;\mu)} = \tilde{B}_{n+1*}(\gamma)\tilde{\phi}^*_{n+1}(\gamma;\mu) \quad \text{and} \quad \overline{\tilde{\phi}^*_{n+1}(\gamma;\mu)} = \tilde{B}_{n+1*}(\gamma)\tilde{\phi}_{n+1}(\gamma;\mu).$$

Hence, it follows that

$$\frac{(z-\gamma)^2}{1-\overline{\beta}z}\phi_n(z;\tilde{\mu}) = \chi_n(z-\gamma)\tilde{\phi}_{n+1}(z;\mu) - \gamma\chi_n\upsilon_n(1-\overline{\alpha}_n z)\tilde{B}_{n+1*}(\gamma)\begin{vmatrix} \tilde{\phi}_{n+1}(z;\mu) & \tilde{\phi}^*_{n+1}(z;\mu) \\ \tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu) \end{vmatrix}.$$

Dividing by $(z-\gamma)$ we get that

$$\frac{z-\gamma}{1-\overline{\beta}z}\phi_n(z;\tilde{\mu}) = \chi_n\tilde{\phi}_{n+1}(z;\mu) - \gamma\chi_n\upsilon_n(1-\overline{\alpha}_n z)\tilde{B}_{n+1*}(\gamma)\begin{vmatrix} \frac{\tilde{\phi}_{n+1}(z;\mu)-\tilde{\phi}_{n+1}(\gamma;\mu)}{z-\gamma} & \frac{\tilde{\phi}^*_{n+1}(z;\mu)-\tilde{\phi}^*_{n+1}(\gamma;\mu)}{z-\gamma} \\ \tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu) \end{vmatrix}.$$

When z tends to γ, we find that

$$\gamma\upsilon_n\tilde{B}_{n+1*}(\gamma) = \frac{\tilde{\phi}_{n+1}(\gamma;\mu)}{(1-\overline{\alpha}_n\gamma)\begin{vmatrix} \tilde{\phi}'_{n+1}(\gamma;\mu) & \tilde{\phi}^{*\prime}_{n+1}(\gamma;\mu) \\ \tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu) \end{vmatrix}},$$

so that

$$\frac{(z-\gamma)^2}{1-\overline{\beta}z}\phi_n(z;\tilde{\mu}) = \chi_n(z-\gamma)\tilde{\phi}_{n+1}(z;\mu)$$

$$+\frac{\chi_n(1-\overline{\alpha}_n z)\tilde{\phi}_{n+1}(\gamma;\mu)}{(1-\overline{\alpha}_n\gamma)\begin{vmatrix}\tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu)\\ \tilde{\phi}'_{n+1}(\gamma;\mu) & \tilde{\phi}^{*\prime}_{n+1}(\gamma;\mu)\end{vmatrix}}\begin{vmatrix}\tilde{\phi}_{n+1}(z;\mu) & \tilde{\phi}^*_{n+1}(z;\mu)\\ \tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu)\end{vmatrix}. \tag{16}$$

Finally, note that (16) is equivalently with (15), which ends the proof. □

Theorem 7. *Let a sequence of complex numbers* $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ *and* $\mathcal{C} = \{\beta, \alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ *be given. Suppose* $\tilde{\mu}$ *is given by*

$$\mathrm{d}\tilde{\mu} = \left|\frac{z-\gamma}{1-\overline{\beta}z}\right|^2 \mathrm{d}\mu, \quad \gamma \in \mathbb{D}.$$

Let $\phi_n(z;\tilde{\mu})$ *denote the MORF associated with the sequence* $\mathcal{A}$ *and the measure* $\tilde{\mu}$. *Similarly, let* $\tilde{\phi}_{n+1}(z;\mu)$ *denote the MORF associated with the sequence* $\mathcal{C}$ *and the measure* μ. *Then it holds that*

$$\frac{(1-\overline{\gamma}z)(z-\gamma)}{1-\overline{\beta}z}\phi_n(z;\tilde{\mu}) = \frac{\overline{\eta}_\beta \frac{1-\gamma\overline{\alpha}_n}{1-\beta\overline{\alpha}_n}}{\begin{vmatrix}\tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu)\\ \overline{\tilde{\phi}^*_{n+1}(\gamma;\mu)} & \overline{\tilde{\phi}_{n+1}(\gamma;\mu)}\end{vmatrix}}$$

$$\times\begin{vmatrix}(1-\overline{\gamma}z)\tilde{\phi}_{n+1}(z;\mu) & (1-\overline{\alpha}_n z)\tilde{\phi}_{n+1}(z;\mu) & (1-\overline{\alpha}_n z)\tilde{\phi}^*_{n+1}(z;\mu)\\ \left(\frac{1-|\gamma|^2}{1-\overline{\alpha}_n\gamma}\right)\tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu)\\ 0 & \overline{\tilde{\phi}^*_{n+1}(\gamma;\mu)} & \overline{\tilde{\phi}_{n+1}(\gamma;\mu)}\end{vmatrix}. \tag{17}$$

Proof. Define a_n and b_n respectively as

$$a_n = \upsilon_n\overline{\tilde{\phi}_{n+1}(\gamma;\mu)} \quad \text{and} \quad b_n = -\upsilon_n\overline{\tilde{\phi}^*_{n+1}(\gamma;\mu)}, \tag{18}$$

where υ_n is given by (14). Eq. (13) can then be rewritten as

$$\frac{(1-\overline{\gamma}z)(z-\gamma)}{1-\overline{\beta}z}\frac{\phi_n(z;\tilde{\mu})}{\chi_n} = (1-\overline{\gamma}z)\tilde{\phi}_{n+1}(z;\mu)$$
$$+ a_n(1-\overline{\alpha}_n z)\tilde{\phi}_{n+1}(z;\mu) + b_n(1-\overline{\alpha}_n z)\tilde{\phi}^*_{n+1}(z;\mu). \tag{19}$$

For $z = \gamma \in \mathbb{D}$ we get that

$$0 = (1-|\gamma|^2)\tilde{\phi}_{n+1}(\gamma;\mu) + a_n(1-\overline{\alpha}_n\gamma)\tilde{\phi}_{n+1}(\gamma;\mu) + b_n(1-\overline{\alpha}_n\gamma)\tilde{\phi}^*_{n+1}(\gamma;\mu).$$

On the other hand, because $|\upsilon_n| < \infty$, it follows from (18) that

$$0 = a_n\overline{\tilde{\phi}^*_{n+1}(\gamma;\mu)} + b_n\overline{\tilde{\phi}_{n+1}(\gamma;\mu)}. \tag{20}$$

Consequently, it holds that

$$\begin{vmatrix}R_n(z) & (1-\overline{\alpha}_n z)\tilde{\phi}_{n+1}(z;\mu) & (1-\overline{\alpha}_n z)\tilde{\phi}^*_{n+1}(z;\mu)\\ -\left(\frac{1-|\gamma|^2}{1-\overline{\alpha}_n\gamma}\right)\tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu)\\ 0 & \overline{\tilde{\phi}^*_{n+1}(\gamma;\mu)} & \overline{\tilde{\phi}_{n+1}(\gamma;\mu)}\end{vmatrix} = 0, \tag{21}$$

where

$$R_n(z) = \frac{(1-\overline{\gamma}z)(z-\gamma)}{1-\overline{\beta}z}\frac{\phi_n(z;\tilde{\mu})}{\chi_n} - (1-\overline{\gamma}z)\tilde{\phi}_{n+1}(z;\mu).$$

Finally, note that (21) is equivalent with (17), which ends the proof. □

Note that

$$\overline{\tilde{\phi}^*_{n+1}(\gamma;\mu)} = \frac{\tilde{\phi}_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right)}{\tilde{B}_{n+1}\left(\frac{1}{\overline{\gamma}}\right)} \quad \text{and} \quad \overline{\tilde{\phi}_{n+1}(\gamma;\mu)} = \frac{\tilde{\phi}^*_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right)}{\tilde{B}_{n+1}\left(\frac{1}{\overline{\gamma}}\right)},$$

so that Eq. (20) becomes

$$0 = \frac{1}{\tilde{B}_{n+1}\left(\frac{1}{\overline{\gamma}}\right)}\left(a_n\tilde{\phi}_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right) + b_n\tilde{\phi}^*_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right)\right).$$

Furthermore, if $\gamma \notin \mathcal{C}$, this is equivalent with

$$0 = a_n\tilde{\phi}_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right) + b_n\tilde{\phi}^*_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right), \tag{22}$$

which is the same equation we would get when evaluating Eq. (19) in $z = 1/\overline{\gamma}$ for $\gamma \notin \mathcal{C}$. Thus, basically Eq. (20) means that the multiplicity of $1/\overline{\gamma}$ as a zero or pole of the right hand side of (19) equals the multiplicity of $1/\overline{\gamma}$ as a zero or pole of the left hand side of (19). Hence, we have proved the following corollary.

Corollary 8. *Let a sequence of complex numbers* $\mathcal{A} = \{\alpha_1,\ldots,\alpha_n\} \subset \mathbb{D}$ *and* $\mathcal{C} = \{\beta,\alpha_1,\ldots,\alpha_n\} \subset \mathbb{D}$ *be given. Suppose* $\tilde{\mu}$ *is given by*

$$\mathrm{d}\tilde{\mu} = \left|\frac{z-\gamma}{1-\overline{\beta}z}\right|^2 \mathrm{d}\mu, \quad \gamma \in \mathbb{D}_{\mathcal{C}}.$$

Let $\phi_n(z;\tilde{\mu})$ *denote the MORF associated with the sequence* $\mathcal{A}$ *and the measure* $\tilde{\mu}$*. Similarly, let* $\tilde{\phi}_{n+1}(z;\mu)$ *denote the MORF associated with the sequence* $\mathcal{C}$ *and the measure* μ*. Then it holds that*

$$\frac{(1-\overline{\gamma}z)(z-\gamma)}{1-\overline{\beta}z}\phi_n(z;\tilde{\mu}) = \frac{\overline{\eta}_\beta\frac{1-\gamma\overline{\alpha}_n}{1-\beta\overline{\alpha}_n}}{\begin{vmatrix}\tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu)\\ \tilde{\phi}_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right) & \tilde{\phi}^*_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right)\end{vmatrix}}$$

$$\times\begin{vmatrix}(1-\overline{\gamma}z)\tilde{\phi}_{n+1}(z;\mu) & (1-\overline{\alpha}_n z)\tilde{\phi}_{n+1}(z;\mu) & (1-\overline{\alpha}_n z)\tilde{\phi}^*_{n+1}(z;\mu)\\ \left(\frac{1-|\gamma|^2}{1-\overline{\alpha}_n\gamma}\right)\tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}_{n+1}(\gamma;\mu) & \tilde{\phi}^*_{n+1}(\gamma;\mu)\\ 0 & \tilde{\phi}_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right) & \tilde{\phi}^*_{n+1}\left(\frac{1}{\overline{\gamma}};\mu\right)\end{vmatrix}. \tag{23}$$

Note that although Eq. (22) does not hold for $\gamma \in \mathcal{C}$, Eq. (23) does in a limiting sense. Nevertheless, it is clear that Eq. (17) is much more interesting for $\gamma \in \mathbb{D}$, due to the fact that each value in the determinants is finite at any time.

Finally, the following corollary is easily verified with the aid of Theorem 7.

Corollary 9. *Let a sequence of complex numbers* $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ *and* $\mathcal{C} = \{\beta, \alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$ *be given. Suppose* $\tilde{\mu}$ *is given by*

$$\mathrm{d}\tilde{\mu} = \left|\frac{z-\gamma}{1-\overline{\beta}z}\right|^2 \mathrm{d}\mu, \quad \gamma \in \mathbb{D}.$$

Let $\phi_n(z; \tilde{\mu})$ *denote the MORF associated with the sequence* $\mathcal{A}$ *and the measure* $\tilde{\mu}$. *Similarly, let* $\tilde{\phi}_{n+1}(z; \mu)$ *denote the MORF associated with the sequence* $\mathcal{C}$ *and the measure* μ.

(a) *If* γ *is a zero of* $\tilde{\phi}_{n+1}(z; \mu)$, *i.e. if* $\tilde{\phi}_{n+1}(\gamma; \mu) = 0$, *then it holds that*

$$\phi_n(z; \tilde{\mu}) = \overline{\eta}_\beta \frac{1-\gamma\overline{\alpha}_n}{1-\beta\overline{\alpha}_n}\left(\frac{1-\overline{\beta}z}{z-\gamma}\right)\tilde{\phi}_{n+1}(z; \mu). \tag{24}$$

(b) *If* $\gamma = \alpha_n$, *it holds that*

$$\phi_n(z; \tilde{\mu}) = \frac{\overline{\eta}_\beta \frac{1-|\alpha_n|^2}{1-\beta\overline{\alpha}_n}}{1-|\tilde{\phi}_{n+1}(\alpha_n; \mu)|^2}\left(\frac{1-\overline{\beta}z}{z-\alpha_n}\right) \times \left[\tilde{\phi}_{n+1}(z; \mu) - \tilde{\phi}_{n+1}(\alpha_n; \mu)\tilde{\phi}^*_{n+1}(z; \mu)\right]. \tag{25}$$

5. An application

If the measure μ on the unit circle is absolutely continuous, then we have with $z = \mathrm{e}^{\mathbf{i}\theta}$ that

$$\mathrm{d}\mu(z) = \mu'(z)\mathrm{d}z = w(\theta)\mathrm{d}\theta.$$

Hence,

$$\frac{1}{2\pi}\oint_{\mathbb{T}} f(z)\mathrm{d}\mu(z) = \frac{1}{2\pi}\oint_{\mathbb{T}} f(z)\mu'(z)\mathrm{d}z = \frac{1}{2\pi}\int_0^{2\pi} f\left(\mathrm{e}^{\mathbf{i}\theta}\right) w(\theta)\mathrm{d}\theta.$$

So, as an application, let us consider the Lebesgue measure $w(\theta) = 1 = \mathbf{i}z\mu'(z)$ on the unit circle and the rational modification given by

$$w(\theta, \beta, r\mathrm{e}^{\mathbf{i}t}) = \frac{(1-r)^2 + 4r\sin^2\left(\frac{\theta-t}{2}\right)}{\left|1-\overline{\beta}\mathrm{e}^{\mathbf{i}\theta}\right|^2} = \frac{(1+r^2) - 2r\cos(\theta - t)}{\left|1-\overline{\beta}\mathrm{e}^{\mathbf{i}\theta}\right|^2},$$

where $r \in [0, 1]$, $t \in \mathbb{R}$ and $|\beta| < 1$. Or equivalently,

$$\tilde{\mu}'(z, \beta, \gamma) = \left|\frac{z-\gamma}{1-\overline{\beta}z}\right|^2 \mu'(z), \quad \gamma = r\mathrm{e}^{\mathbf{i}t}.$$

Example 10. First, let us consider the case where $t = 0$ and $r = 1$, i.e.

$$w(\theta, \beta) = w(\theta, \beta, 1) \quad \text{and} \quad \tilde{\mu}'(z, \beta) = \tilde{\mu}'(z, \beta, 1).$$

Then we have that

$$\frac{1}{2\pi}\int_0^{2\pi} w(\theta, \beta)\mathrm{d}\theta = \frac{1}{2\mathbf{i}\pi}\oint_{\mathbb{T}}\left|\frac{z-1}{1-\overline{\beta}z}\right|^2\frac{\mathrm{d}z}{z} = \frac{-1}{2\mathbf{i}\pi}\oint_{\mathbb{T}}\frac{(z-1)^2}{z(z-\beta)(1-\overline{\beta}z)}\mathrm{d}z$$

$$= -\left[\text{Res}\left(\frac{(z-1)^2}{z(z-\beta)(1-\overline{\beta}z)}, 0\right) + \text{Res}\left(\frac{(z-1)^2}{z(z-\beta)(1-\overline{\beta}z)}, \beta\right)\right]$$
$$= \frac{2(1-\Re\{\beta\})}{1-|\beta|^2} \equiv \frac{1}{\varphi_0^2(z;\tilde{\mu})}.$$

For $n > 0$ it follows from (3) that

$$\tilde{\phi}_{n+1}(z;\mu) = \eta_{\alpha_n}(1-|\alpha_n|^2)\frac{z\tilde{B}_n(z)}{1-\overline{\alpha}_n z}, \qquad \tilde{B}_n(z) = \zeta_\beta(z)B_{n-1}(z),$$
$$\tilde{\phi}^*_{n+1}(z;\mu) = \frac{1-|\alpha_n|^2}{1-\overline{\alpha}_n z},$$
$$\tilde{\phi}_{n+1}(1;\mu) = \eta_{\alpha_n}(1-|\alpha_n|^2)\frac{\tilde{B}_n(1)}{1-\overline{\alpha}_n},$$
$$\tilde{\phi}^*_{n+1}(1;\mu) = \frac{1-|\alpha_n|^2}{1-\overline{\alpha}_n},$$
$$\tilde{\phi}'_{n+1}(1;\mu) = \eta_{\alpha_n}(1-|\alpha_n|^2)\frac{\tilde{B}_n(1)}{1-\overline{\alpha}_n}\left[Q_{n-1} + \frac{\overline{\alpha}_n}{1-\overline{\alpha}_n}\right],$$
$$\tilde{\phi}^{*'}_{n+1}(1;\mu) = \frac{\overline{\alpha}_n(1-|\alpha_n|^2)}{(1-\overline{\alpha}_n)^2},$$

where

$$Q_n = 1 + \frac{\tilde{B}'_{n+1}(1)}{\tilde{B}_{n+1}(1)} = 1 + \frac{1-|\beta|^2}{|1-\beta|^2} + \sum_{k=1}^{n}\frac{1-|\alpha_k|^2}{|1-\alpha_k|^2} \in \mathbb{R}_0^+.$$

The last equality here follows from [7, Lem. 3.3]. So we get that

$$\frac{\tilde{\phi}_{n+1}(1;\mu)}{1-\overline{\alpha}_n}\frac{1}{\begin{vmatrix}\tilde{\phi}_{n+1}(1;\mu) & \tilde{\phi}^*_{n+1}(1;\mu)\\ \tilde{\phi}'_{n+1}(1;\mu) & \tilde{\phi}^{*'}_{n+1}(1;\mu)\end{vmatrix}} = -\frac{1}{(1-|\alpha_n|^2)Q_{n-1}},$$

and

$$(1-\overline{\alpha}_n z)\begin{vmatrix}\tilde{\phi}_{n+1}(z;\mu) & \tilde{\phi}^*_{n+1}(z;\mu)\\ \tilde{\phi}_{n+1}(1;\mu) & \tilde{\phi}^*_{n+1}(1;\mu)\end{vmatrix} = \frac{\eta_{\alpha_n}(1-|\alpha_n|^2)^2}{1-\overline{\alpha}_n}\left[z\tilde{B}_n(z) - \tilde{B}_n(1)\right].$$

Substitute this in (15) and simplify to find that

$$\phi_n(z;\tilde{\mu}) = \frac{c_n\left(a_n(1-\overline{\beta}z) + \frac{z(z-b_n)(z-\beta)B_{n-1}(z)}{1-\overline{\alpha}_n z}\right)}{(z-1)^2}, \tag{26}$$

where

$$c_n = \eta_{\alpha_n}(1-|\alpha_n|^2)\frac{(1-\overline{\alpha}_n)}{(1-\beta\overline{\alpha}_n)}\left[\frac{Q_{n-1}+\frac{\overline{\alpha}_n}{1-\overline{\alpha}_n}}{Q_{n-1}}\right],$$
$$a_n = \frac{(1-\beta)}{(1-\overline{\beta})}\left[\frac{\frac{B_{n-1}(1)}{1-\overline{\alpha}_n}}{Q_{n-1}+\frac{\overline{\alpha}_n}{1-\overline{\alpha}_n}}\right],$$

$$b_n = 1 + \frac{1}{Q_{n-1} + \frac{\overline{\alpha}_n}{1-\overline{\alpha}_n}}.$$

For $\varphi_n(z;\tilde{\mu})$, the constant c_n in (26) has to be replaced with $d_n = \kappa_n c_n$. It holds that

$$1 = \langle \varphi_n, \varphi_n \rangle = \langle \kappa_n \phi_n, \kappa_n B_n \rangle = \kappa_n^2 \langle \phi_n, B_n \rangle.$$

Furthermore, we have that

$$\begin{aligned}
\langle \phi_n, B_n \rangle &= -\frac{c_n}{2\pi \mathbf{i}} \left(a_n \oint_{\mathbb{T}} \frac{B_{n*}(z)}{z(z-\beta)} \mathrm{d}z + \overline{\eta}_{\alpha_n} \oint_{\mathbb{T}} \frac{(z-b_n)}{(z-\alpha_n)(1-\overline{\beta}z)} \mathrm{d}z \right) \\
&= \frac{c_n a_n}{2\pi \mathbf{i}} \oint_{\mathbb{T}} \frac{B_{n*}(\overline{z})}{\overline{z}(\overline{z}-\beta)} \mathrm{d}\overline{z} - c_n \overline{\eta}_{\alpha_n} \mathrm{Res} \left(\frac{(z-b_n)}{(z-\alpha_n)(1-\overline{\beta}z)}, \alpha_n \right) \\
&= -\frac{c_n a_n}{2\pi \mathbf{i}} \oint_{\mathbb{T}} \frac{B_n^c(z)}{(1-\beta z)} \mathrm{d}z + c_n \overline{\eta}_{\alpha_n} \frac{b_n - \alpha_n}{1-\overline{\beta}\alpha_n} \\
&= 0 + \frac{|1-\alpha_n|^2}{|1-\overline{\beta}\alpha_n|^2} (1-|\alpha_n|^2) \frac{Q_{n-1} + \frac{\overline{\alpha}_n}{1-\overline{\alpha}_n} + \frac{1}{1-\alpha_n}}{Q_{n-1}} \\
&= \frac{|1-\alpha_n|^2}{|1-\overline{\beta}\alpha_n|^2} (1-|\alpha_n|^2) \frac{Q_n}{Q_{n-1}},
\end{aligned}$$

so that

$$\kappa_n = \frac{|1-\overline{\beta}\alpha_n|}{|1-\alpha_n|} \sqrt{\frac{Q_{n-1}}{(1-|\alpha_n|^2) Q_n}}.$$

Let ρ_n and σ_n be defined respectively as

$$\rho_n = \eta_{\alpha_n} \frac{(1-\overline{\alpha}_n)(1-\overline{\beta}\alpha_n)}{|(1-\overline{\alpha}_n)(1-\overline{\beta}\alpha_n)|} \in \mathbb{T}$$

and

$$\sigma_n = B_{n-1}(1) \frac{(1-\beta)^2}{|1-\beta|^2} \in \mathbb{T}.$$

Then we find that the coefficients for $\varphi_n(z;\mu)$ are given by

$$\begin{aligned}
d_n &= \rho_n \sqrt{\frac{1-|\alpha_n|^2}{Q_{n-1}Q_n}} \left[Q_{n-1} + \frac{\overline{\alpha}_n}{1-\overline{\alpha}_n} \right] \\
d_n a_n &= \rho_n \sqrt{\frac{1-|\alpha_n|^2}{Q_{n-1}Q_n}} \left[\frac{1}{1-\overline{\alpha}_n} \right] \sigma_n \\
d_n b_n &= \rho_n \sqrt{\frac{1-|\alpha_n|^2}{Q_{n-1}Q_n}} \left[Q_{n-1} + \frac{1}{1-\overline{\alpha}_n} \right].
\end{aligned}$$

Example 11. Next, let us consider the case where $t = 0$ and $r \in [0, 1)$, i.e.

$$w(\theta, \beta) = w(\theta, \beta, r) \quad \text{and} \quad \tilde{\mu}'(z, \beta) = \tilde{\mu}'(z, \beta, r).$$

Then we have that

$$\frac{1}{2\pi}\int_0^{2\pi} w(\theta,\beta)\mathrm{d}\theta = \frac{1}{2\mathbf{i}\pi}\oint_{\mathbb{T}}\left|\frac{z-r}{1-\overline{\beta}z}\right|^2\frac{\mathrm{d}z}{z} = \frac{1}{2\mathbf{i}\pi}\oint_{\mathbb{T}}\frac{(z-r)(1-rz)}{z(z-\beta)(1-\overline{\beta}z)}\mathrm{d}z$$
$$= \left[\operatorname{Res}\left(\frac{(z-r)(1-rz)}{z(z-\beta)(1-\overline{\beta}z)},0\right)+\operatorname{Res}\left(\frac{(z-r)(1-rz)}{z(z-\beta)(1-\overline{\beta}z)},\beta\right)\right]$$
$$= \frac{(1+r^2)-2r\Re\{\beta\}}{1-|\beta|^2} \equiv \frac{1}{\varphi_0^2(z;\tilde{\mu})}.$$

For $n > 0$ it follows from (3) that

$$\begin{aligned}
\tilde{\phi}_{n+1}(z;\mu) &= \eta_{\alpha_n}(1-|\alpha_n|^2)\frac{z\tilde{B}_n(z)}{1-\overline{\alpha}_n z}, \qquad \tilde{B}_n(z)=\zeta_\beta(z)B_{n-1}(z),\\
\tilde{\phi}^*_{n+1}(z;\mu) &= \frac{1-|\alpha_n|^2}{1-\overline{\alpha}_n z},\\
\tilde{\phi}_{n+1}(r;\mu) &= \eta_{\alpha_n}(1-|\alpha_n|^2)\frac{r\tilde{B}_n(r)}{1-\overline{\alpha}_n r},\\
\tilde{\phi}^*_{n+1}(r;\mu) &= \frac{1-|\alpha_n|^2}{1-\overline{\alpha}_n r}.
\end{aligned} \tag{27}$$

So we get that

$$-\frac{1-r^2}{1-\overline{\alpha}_n r}\frac{\tilde{\phi}_{n+1}(r;\mu)}{\begin{vmatrix}\tilde{\phi}_{n+1}(r;\mu) & \tilde{\phi}^*_{n+1}(r;\mu)\\ \tilde{\phi}^*_{n+1}(r;\mu) & \tilde{\phi}_{n+1}(r;\mu)\end{vmatrix}} = \frac{\eta_{\alpha_n}(1-\alpha_n r)r\tilde{B}_n(r)}{(1-|\alpha_n|^2)(1-\overline{\alpha}_n r)q_{n-1}(r)},$$

where $q_{n-1}(r)$ is given by

$$q_{n-1}(r) = \frac{1-r^2|\tilde{B}_n(r)|^2}{1-r^2} \in \mathbb{R}_0^+,$$

and

$$(1-\overline{\alpha}_n z)\begin{vmatrix}\tilde{\phi}_{n+1}(z;\mu) & \tilde{\phi}^*_{n+1}(z;\mu)\\ \tilde{\phi}^*_{n+1}(r;\mu) & \tilde{\phi}_{n+1}(r;\mu)\end{vmatrix} = \frac{(1-|\alpha_n|^2)^2\left[z\tilde{B}_n(z)r\overline{\tilde{B}_n(r)}-1\right]}{1-\alpha_n r}.$$

Substituting this in (17) and simplifying, we find that

$$\phi_n(z;\tilde{\mu}) = \frac{c_n\left(a_n(1-\overline{\beta}z)+\frac{z(z-b_n)(z-\beta)B_{n-1}(z)}{1-\overline{\alpha}_n z}\right)}{(z-r)(rz-1)}, \tag{28}$$

where

$$\begin{aligned}
c_n &= \eta_{\alpha_n}(1-|\alpha_n|^2)\frac{r-\overline{\alpha}_n}{(1-\beta\overline{\alpha}_n)}\left[\frac{q_{n-1}(r)+\frac{\overline{\alpha}_n}{r-\overline{\alpha}_n}}{q_{n-1}(r)}\right],\\
a_n &= \frac{(r-\beta)}{(1-\overline{\beta}r)}\left[\frac{\frac{rB_{n-1}(r)}{r-\overline{\alpha}_n}}{q_{n-1}(r)+\frac{\overline{\alpha}_n}{r-\overline{\alpha}_n}}\right],
\end{aligned}$$

$$b_n = 1 + \frac{\frac{1-\overline{\alpha}_n}{r-\overline{\alpha}_n} - (1-r)q_{n-1}(r)}{q_{n-1}(r) + \frac{\overline{\alpha}_n}{r-\overline{\alpha}_n}}.$$

Note that $|\tilde{B}_n(1)| = 1$, so that

$$\begin{aligned}\lim_{r\to 1} q_{n-1}(r) &= \frac{1}{2}\lim_{r\to 1}\frac{1-r^2|\tilde{B}_n(r)|^2}{1-r} = 1 + \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}r}|\tilde{B}_n(r)|^2\right]_{r=1} \\ &= 1 + \Re\{\tilde{B}_n'(1)\overline{\tilde{B}_n(1)}\} = 1 + \Re\left\{\frac{\tilde{B}_n'(1)}{\tilde{B}_n(1)}\right\} = Q_{n-1}.\end{aligned}$$

Hence, for r tending to 1, the expression for the MORFs in Example 10 are recovered.

Again, for $\varphi_n(z; \tilde{\mu})$, the constant c_n in (28) has to be replaced with $d_n = \kappa_n c_n$. In a similar way as in Example 10, we now find that

$$\kappa_n = \frac{|1-\overline{\beta}\alpha_n|}{|r-\alpha_n|}\sqrt{\frac{q_{n-1}(r)}{(1-|\alpha_n|^2)\left(q_{n-1}(r) + \frac{1-|\alpha_n|^2}{|r-\alpha_n|^2}\right)}}.$$

So, let ρ_n and σ_n be defined respectively as

$$\rho_n = \eta_{\alpha_n}\frac{(r-\overline{\alpha}_n)(1-\overline{\beta}\alpha_n)}{|(r-\overline{\alpha}_n)(1-\overline{\beta}\alpha_n)|} \in \mathbb{T}$$

and

$$\sigma_n = rB_{n-1}(r)\left(\frac{r-\beta}{1-\overline{\beta}r}\right) \in \mathbb{D}.$$

Then we find that the coefficients for $\varphi_n(z; \mu)$ are given by

$$\begin{aligned} d_n &= \rho_n\sqrt{\frac{1-|\alpha_n|^2}{q_{n-1}(r)\left(q_{n-1}(r) + \frac{1-|\alpha_n|^2}{|r-\alpha_n|^2}\right)}}\left[q_{n-1}(r) + \frac{\overline{\alpha}_n}{r-\overline{\alpha}_n}\right] \\ d_n a_n &= \rho_n\sqrt{\frac{1-|\alpha_n|^2}{q_{n-1}(r)\left(q_{n-1}(r) + \frac{1-|\alpha_n|^2}{|r-\alpha_n|^2}\right)}}\left[\frac{1}{r-\overline{\alpha}_n}\right]\sigma_n \\ d_n b_n &= \rho_n\sqrt{\frac{1-|\alpha_n|^2}{q_{n-1}(r)\left(q_{n-1}(r) + \frac{1-|\alpha_n|^2}{|r-\alpha_n|^2}\right)}}\left[rq_{n-1}(r) + \frac{1}{r-\overline{\alpha}_n}\right].\end{aligned}$$

Remark 12. The case in which $r \in [0, 1)$ and $r \in \mathcal{C} \cup \{0\}$, requires some extra attention. If $r = r_1 \in \{0, \beta, \alpha_1, \ldots, \alpha_{n-1}\}$, it follows from (27) that $\tilde{\phi}_{n+1}(r_1; \mu) = 0$. Hence, using Eq. (24) we find that

$$\phi_n(z; \tilde{\mu}) = \hat{a}_n\frac{z(z-\beta)B_{n-1}(z)}{(z-r_1)(1-\overline{\alpha}_n z)},$$

where

$$\hat{a}_n = \eta_{\alpha_n}(1-|\alpha_n|^2)\frac{1-r_1\overline{\alpha}_n}{1-\beta\overline{\alpha}_n}.$$

Note that $q_{n-1}(r_1) = (1 - r_1^2)^{-1}$, so that it is not difficult to verify that

$$\lim_{r\to r_1} c_n = r_1 \hat{a}_n, \qquad \lim_{r\to r_1} c_n a_n = 0, \qquad \lim_{r\to r_1} c_n b_n = \hat{a}_n$$

and

$$\lim_{r\to r_1} \kappa_n = \frac{|1 - \overline{\beta}\alpha_n|}{|1 - r_1\alpha_n|} \frac{1}{\sqrt{1 - |\alpha_n|^2}}.$$

On the other hand, if $r = \alpha_n \notin \{0, \beta, \alpha_1, \ldots, \alpha_{n-1}\}$, it follows from (25) that

$$\phi_n(z; \tilde{\mu}) = \hat{b}_n \frac{\hat{c}_n(1 - \overline{\beta}z) - z(z - \beta)B_{n-1}(z)}{(z - \alpha_n)(\alpha_n z - 1)},$$

where

$$\hat{b}_n = \frac{\eta_{\alpha_n}(1 - \alpha_n^2)^2}{(1 - \beta\alpha_n)(1 - \alpha_n^2|\tilde{B}_n(\alpha_n)|^2)} \quad \text{and} \quad \hat{c}_n = \frac{(\alpha_n - \beta)\alpha_n B_{n-1}(\alpha_n)}{(1 - \overline{\beta}\alpha_n)}.$$

Note that $q_{n-1}(\alpha_n) = \frac{1-\alpha_n^2|\tilde{B}_n(\alpha_n)|^2}{1-\alpha_n^2}$, so that now it is not difficult either to verify that

$$\lim_{r\to \alpha_n} c_n = \alpha_n \hat{b}_n, \qquad \lim_{r\to \alpha_n} c_n a_n = \hat{b}_n \hat{c}_n, \qquad \lim_{r\to \alpha_n} c_n b_n = \hat{b}_n$$

and

$$\lim_{r\to \alpha_n} \kappa_n = \frac{|1 - \overline{\beta}\alpha_n|}{(1 - \alpha_n^2)^{3/2}} \sqrt{1 - \alpha_n^2|\tilde{B}_n(\alpha_n)|^2}.$$

Finally, the expression of $\varphi_n(z; \tilde{\mu})$ for the more general case of $\gamma \in \mathbb{O}$ can be found, using the following theorem.

Theorem 13 (*Rotated Weight Function*). *Let the weight function $\tilde{\mu}'$ be given by*

$$\tilde{\mu}'(z, \beta, r\mathrm{e}^{\mathbf{i}t}) = \frac{1}{\mathbf{i}z} \left| \frac{z - r\mathrm{e}^{\mathbf{i}t}}{1 - \overline{\beta}z} \right|^2, \quad r \in [0, 1],$$

and denote $\tilde{\mu}'(z, \beta, r)$ by $\tilde{\mu}'(z, \beta)$. Furthermore, let $\tilde{\phi}_{\mathcal{A},n}(z, \beta)$ represent the rational function with poles in $\mathcal{A} = \{1/\overline{\alpha}_1, \ldots, 1/\overline{\alpha}_n\}$, orthonormal on the unit circle with respect to the weight function $\tilde{\mu}'(z, \beta, r\mathrm{e}^{\mathbf{i}t})$, and let $\phi_{\mathcal{B},n}(u, \omega)$ represent the rational function with poles in $\mathcal{B} = \{1/\overline{\beta}_1, \ldots, 1/\overline{\beta}_n\}$, orthonormal on the unit circle with respect to the weight function $\tilde{\mu}'(u, \omega)$. Then it holds that

$$\tilde{\phi}_{\mathcal{A},n}(z, \beta) = \phi_{\mathcal{B},n}(u, \omega)$$

iff $u = \mathrm{e}^{-\mathbf{i}t}z$, $\omega = \mathrm{e}^{-\mathbf{i}t}\beta$ and $\mathcal{B} = \mathrm{e}^{-\mathbf{i}t}\mathcal{A}$.

Proof. We have that

$$\tilde{\mu}'(z, \beta, r\mathrm{e}^{\mathbf{i}t})\mathrm{d}z = \left| \frac{z - r\mathrm{e}^{\mathbf{i}t}}{1 - \overline{\beta}z} \right|^2 \frac{\mathrm{d}z}{\mathbf{i}z}$$

$$= \left| \frac{\mathrm{e}^{\mathbf{i}t}u - r\mathrm{e}^{\mathbf{i}t}}{1 - \overline{\mathrm{e}^{\mathbf{i}t}\omega}\mathrm{e}^{\mathbf{i}t}u} \right|^2 \frac{\mathrm{d}(\mathrm{e}^{\mathbf{i}t}u)}{\mathbf{i}\mathrm{e}^{\mathbf{i}t}u}, \quad u = \mathrm{e}^{-\mathbf{i}t}z, \omega = \mathrm{e}^{-\mathbf{i}t}\beta,$$

$$= \left| \frac{u - r}{1 - \overline{\omega} u} \right|^2 \frac{\mathrm{d}u}{\mathbf{i}u} = \tilde{\mu}'(u, \omega)\mathrm{d}u.$$

Hence, it holds that $\phi_{\mathcal{B},n}(u, \omega)$ is a rational function with poles $u = 1/\overline{\beta}_k$, for $k = 1, \ldots, n$, orthonormal on the unit circle with respect to the weight function $\tilde{\mu}'(z, \beta, r\mathrm{e}^{\mathbf{i}t})$. Moreover, if $\mathcal{B} = \mathrm{e}^{-\mathbf{i}t}\mathcal{A}$, it follows that $\phi_{\mathcal{B},n}(u, \omega)$ has poles in $z = 1/\overline{\alpha}_k$, for $k = 1, \ldots, n$. Hence, there exists a unimodular constant γ_n so that

$$\phi_{\mathcal{A},n}(z, \beta) = \gamma_n \phi_{\mathcal{B},n}(u, \omega).$$

Finally, if the leading coefficients of $\phi_{\mathcal{A},n}(z, \beta)$ and $\phi_{\mathcal{B},n}(u, \omega)$ are supposed to be positive real, then it holds that $\gamma_n = 1$ due to the fact that $\zeta_{\beta_k}(u) = \zeta_{\alpha_k}(z)$ for $k = 1, \ldots, n$. □

Appendix

Theorem 14. *The sequence of rational functions given by (12) forms a basis for the space $\left[A_m \mathcal{L}_{n-1}\right]_\mu^{\perp m+n-1}$. Let j and m_i be as defined before in Remark 3. Then, the sequence $\{\tilde{k}_{m+n-1}(z, \gamma_i; \mu)\}_{i=1}^m$ (for the special case in which all the zeros are simple) is recovered by setting $j = m$ and $m_i = 1$.*

Proof. First, notice that for a fixed k and i we have that

$$\begin{aligned} g_{i,k}(z) &:= \left. \frac{\partial^k \tilde{k}_{m+n-1}(z, w; \mu)}{\partial \overline{w}^k} \right|_{w=\gamma_i} = \sum_{l=0}^{m+n-1} \tilde{\varphi}_l(z; \mu) \left. \frac{\mathrm{d}^k \overline{\tilde{\varphi}_l(w; \mu)}}{\mathrm{d}\overline{w}^k} \right|_{w=\gamma_i} \\ &= \sum_{l=0}^{m+n-1} \tilde{\varphi}_l(z; \mu) \left. \overline{\left(\frac{\mathrm{d}^k \tilde{\varphi}_l(w; \mu)}{\mathrm{d}w^k} \right)} \right|_{w=\gamma_i} = \sum_{l=0}^{m+n-1} \tilde{\varphi}_l(z; \mu) \overline{\tilde{\varphi}_l^{(k)}(\gamma_i; \mu)}, \end{aligned}$$

where $\tilde{\varphi}_l^{(k)}$ represents the kth derivative of $\tilde{\varphi}_l$. Hence, $g_{i,k}(z) \in \tilde{\mathcal{L}}_{m+n-1}$ for every $i = 1, \ldots, j$ and $k = 0, \ldots, m_i - 1$.

Next, suppose $f(z)$ is an arbitrary function in $\tilde{\mathcal{L}}_{m+n-1} \cap A_m \mathcal{L}_{n-1}$, and of the form

$$f(z) = \sum_{l=0}^{m+n-1} f_l \tilde{\varphi}_l(z; \mu).$$

We then have for every $i = 1, \ldots, j$ and $k = 0, \ldots, m_i - 1$ that

$$\begin{aligned} \langle g_{i,k}(z), f(z) \rangle &= \left\langle \sum_{l=0}^{m+n-1} \tilde{\varphi}_l(z; \mu) \overline{\tilde{\varphi}_l^{(k)}(\gamma_i; \mu)}, \sum_{l=0}^{m+n-1} f_l \tilde{\varphi}_l(z; \mu) \right\rangle \\ &= \sum_{l=0}^{m+n-1} \overline{f_l \tilde{\varphi}_l^{(k)}(\gamma_i; \mu)} = \overline{f^{(k)}(\gamma_i)} = 0. \end{aligned}$$

Consequently, $g_{i,k}(z) \in \left[A_m \mathcal{L}_{n-1}\right]_\mu^{\perp m+n-1}$ for every $i = 1, \ldots, j$ and $k = 0, \ldots, m_i - 1$.

Finally, assume the rational functions $g_{i,k}(z)$, with $i = 1, \ldots, j$ and $k = 0, \ldots, m_i - 1$, are linear dependent, i.e. suppose there exist constants $a_{i,k}$, not all equal to zero, so that

$$\sum_{i=1}^{j} \sum_{k=0}^{m_i - 1} a_{i,k} g_{i,k}(z) \equiv 0.$$

It then follows that

$$0 \equiv \sum_{i=1}^{j} \sum_{k=0}^{m_i-1} a_{i,k} \left(\sum_{l=0}^{m+n-1} \tilde{\varphi}_l(z;\mu) \overline{\tilde{\varphi}_l^{(k)}(\gamma_i;\mu)} \right)$$
$$= \sum_{l=0}^{m+n-1} \tilde{\varphi}_l(z;\mu) \left(\sum_{i=1}^{j} \sum_{k=0}^{m_i-1} a_{i,k} \overline{\tilde{\varphi}_l^{(k)}(\gamma_i;\mu)} \right) = \sum_{l=0}^{m+n-1} \overline{b}_l \tilde{\varphi}_l(z;\mu),$$

where

$$b_l = \sum_{i=1}^{j} \sum_{k=0}^{m_i-1} \overline{a}_{i,k} \tilde{\varphi}_l^{(k)}(\gamma_i;\mu). \tag{29}$$

Since the ORFs $\tilde{\varphi}_l$ are linear independent, it must hold that

$$b_l = 0 \quad \text{for every } l \in \{0, \ldots, m+n-1\}. \tag{30}$$

Consider now the $m \times m$ square matrix

$$B = \begin{pmatrix} \tilde{\varphi}_0(\gamma_1;\mu) & \cdots & \tilde{\varphi}_0^{(m_1-1)}(\gamma_1;\mu) & \cdots & \tilde{\varphi}_0(\gamma_j;\mu) & \cdots & \tilde{\varphi}_0^{(m_j-1)}(\gamma_j;\mu) \\ \tilde{\varphi}_1(\gamma_1;\mu) & \cdots & \tilde{\varphi}_1^{(m_1-1)}(\gamma_1;\mu) & \cdots & \tilde{\varphi}_1(\gamma_j;\mu) & \cdots & \tilde{\varphi}_1^{(m_j-1)}(\gamma_j;\mu) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \tilde{\varphi}_{m-1}(\gamma_1;\mu) & \cdots & \tilde{\varphi}_{m-1}^{(m_1-1)}(\gamma_1;\mu) & \cdots & \tilde{\varphi}_{m-1}(\gamma_j;\mu) & \cdots & \tilde{\varphi}_{m-1}^{(m_j-1)}(\gamma_j;\mu) \end{pmatrix}.$$

From (29)–(30) and the fact that not every $a_{i,k}$ equals zero, we should have that $\det B = 0$. This implies that there exist constants c_l, not all equal to zero for $l = 0, \ldots, m-1$, and a rational function

$$h(z) = \sum_{l=0}^{m-1} c_l \tilde{\varphi}_l(z;\mu) \in \tilde{\mathcal{L}}_{m-1},$$

so that $h^{(k)}(\gamma_i) = 0$ for every $i = 1, \ldots, j$ and $k = 0, \ldots, m_i - 1$. But this can only be the case if $h(z) \equiv 0$. Due to the fact that the ORFs $\tilde{\varphi}_l$ are linear independent, we have that $h(z) \equiv 0$ iff $c_l = 0$ for $l = 0, \ldots, m-1$. Clearly, this is a contradiction, which means that the rational functions $g_{i,k}(z)$ are linear independent. □

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