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Cardinal invariants and independence results in the poset of precompact group topologies[1]

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Abstract

We study the poset $\mathscr{B}(G)$ of all precompact Hausdorff group topologies on an infinite group G and its subposet $\mathscr{B}_\sigma(G)$ of topologies of weight σ, extending earlier results of Berhanu, Comfort, Reid, Remus, Ross, Dikranjan, and others. We show that if $\mathscr{B}_\sigma(G) \neq \emptyset$ and $2^{|G/G'|} = 2^{|G|}$ (in particular, if G is abelian) then the poset $[2^{|G|}]^\sigma$ of all subsets of $2^{|G|}$ of size σ can be embedded into $\mathscr{B}_\sigma(G)$ (and vice versa). So the study of many features (depth, height, width, size of chains, etc.) of the poset $\mathscr{B}_\sigma(G)$ is reduced to purely set-theoretical problems. We introduce a cardinal function $Ded^e(\sigma)$ to measure the length of chains in $[X]^\sigma$ for $|X| > \sigma$ generalizing the well-known cardinal function $Ded(\sigma)$. We prove that $Ded^e(\sigma) = Ded(\sigma)$ iff $cf\,Ded(\sigma) \neq \sigma^+$ and we use earlier results of Mitchell and Baumgartner to show that $Ded^e(\aleph_1) = Ded(\aleph_1)$ is independent of Zermelo–Fraenkel set theory (ZFC). We apply this result to show that it cannot be established in ZFC whether $\mathscr{B}_{\aleph_1}(\boldsymbol{Z})$ has chains of bigger size than those of the bounded chains.

We prove that the poset $\mathscr{H}_{\aleph_0}(G)$ of all Hausdorff metrizable group topologies on the group $G = \bigoplus_{\aleph_0} \boldsymbol{Z}_2$ has uncountable depth, hence cannot be embedded into $\mathscr{B}_{\aleph_0}(G)$. This is to be contrasted with the fact that for every infinite abelian group G the poset $\mathscr{H}(G)$ of all Hausdorff group topologies on G can be embedded into $\mathscr{B}(G)$. We also prove that it is independent of ZFC whether the poset $\mathscr{H}_{\aleph_0}(G)$ has the same height as the poset $\mathscr{B}_{\aleph_0}(G)$. © 1998 Elsevier Science B.V.

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[1] Part of the results of this paper were announced by the second named author in a survey-talk at the Colloquium on Topology, Szekszàrd (Hungary) 1993 (see [17]).

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1. Introduction

We will only consider infinite groups. A topological group G is *precompact* if for every non-empty open set U of G there exists a finite subset F of G such that $G = FU$, i.e. the underlying uniform space is precompact (every uniform cover has a finite subcover). If G is also Hausdorff this occurs iff G is (algebraically and topologically isomorphic to) a subgroup of a compact group, or equivalently, if the completion $\hat{G}$ of G is a compact group [38].

We denote by $\tilde{\mathscr{B}}(G)$ the poset of all precompact group topologies on G ordered by inclusion and by $\mathscr{B}(G)$ the subset of $\tilde{\mathscr{B}}(G)$ consisting of Hausdorff topologies. Actually, $\tilde{\mathscr{B}}(G)$ is a complete lattice, while $\mathscr{B}(G)$ is a complete upper semi-lattice (to see that suprema of precompact topologies are precompact it suffices to note that the Tychonoff product of precompact topologies is again a precompact topology).

For every cardinal σ denote by $\mathscr{B}_\sigma(G)$ the subset of $\mathscr{B}(G)$ consisting of topologies τ of weight $w(G,\tau) = \sigma$ (where the weight is the minimal cardinality of an open base), similarly for $\tilde{\mathscr{B}}$.

We will consider also the poset $\mathscr{H}(G) \supseteq \mathscr{B}(G)$ of all Hausdorff group topologies on G and the poset $\mathscr{H}_\sigma(G) \supseteq \mathscr{B}_\sigma(G)$ of all Hausdorff group topologies of weight σ. Note that $\mathscr{H}_\omega(G)$ consists entirely of metrizable group topologies.

All abelian groups have a precompact group topology, but a non-abelian group may or may not have such a topology: $SL_2(\boldsymbol{C})$ [36] and the infinite symmetric group S_ω [22] do not have precompact topologies, $SO_3(\boldsymbol{R})$ has only one such topology [37].

In this paper, we study the structure of the poset $\mathscr{B}(G)$ and of its subposets $\mathscr{B}_\sigma(G)$.

As a first step, one can measure the size of $\mathscr{B}(G)$. Our interest in this paper will be on groups with $|\mathscr{B}(G)| > 0$ known as *maximally almost periodic* (see [36]). Clearly, we always have $|\mathscr{B}(G)| \leq |\mathscr{H}(G)| \leq 2^{2^{|G|}}$.

Even in the abelian case, many features of the posets $\mathscr{B}(G)$ and $\mathscr{B}_\sigma(G)$ depend on the algebraic structure of G, such as minimal elements of $\mathscr{B}(G)$, complementary width, in the sense of Birkhoff [5], of $\mathscr{B}(G)$, etc. (see [16]). It is known, however (see Section 1.1 below), that many other invariants of these posets depend only on the cardinality of G, at least if G is abelian or it has large abelian quotients.

1.1. Some history

Comfort and Ross [13] proved that abelian groups are maximally almost periodic and even $|\mathscr{B}(G)| = 2^{2^{|G|}}$ for an infinite abelian group G (see [4] or [29]). This was extended by Remus [30] for maximally almost periodic group G admitting "large" abelian quotients (i.e. $|G/G'| = |G|$).

Berhanu et al. [4] computed the height, the width and the depth (see Section 2 for definitions) of the posets $\mathscr{B}_\sigma(G)$ for an abelian group G. These results were extended later by Remus [32] for maximally almost periodic groups G satisfying $|G/G'| = |G|$, in particular for free groups.

Chains in $\mathscr{B}(G)$ were considered for the first time by Comfort and Remus in [8]. It was shown that in case the group G is abelian or free the existence of a chain of length λ in $\mathscr{B}(G)$ is equivalent to the existence of such chains in the power set $\boldsymbol{P}(2^{|G|})$, so in particular does not depend on the algebraic properties of G but only on $|G|$ [8, Theorem 4.4]. Later, Comfort announced without proof that for a maximally almost periodic group G satisfying $|G/G'| = |G|$ the maximum length of chains in the power set $\mathscr{B}_\sigma(G)$ is the same as the maximum length of chains in the poset $[2^{|G|}]^\sigma$ consisting of all subsets of $2^{|G|}$ of size σ [6, Theorem II].

Lower bounds regarding the subset $\mathscr{P}(G)$ of pseudocompact topologies on some groups G (with large abelian quotients or relatively free[2]) are given in [17] and [15][3] (pseudocompact groups are precompact, so that $\mathscr{P}(G) \subseteq \mathscr{B}(G)$). It was shown also that the existence of a bounded (from above) chain of length λ in $\mathscr{B}_\sigma(G)$ is equivalent to the existence of a chain of length λ in $\boldsymbol{P}(\sigma)$. It was also pointed out that under GCH the condition on boundedness can be relaxed. However, it remained unclear whether the question of the existence of chains of length greater than the length of any bounded chain can be answered in Zermelo–Fraenkel set theory ZFC. We show here that this question cannot be answered in ZFC even in the case of precompact topologies (see Section 8 and Theorem 8.1). Chains of pseudocompact topologies are considered also by Comfort and Remus [10, 11]. We do not consider pseudocompact group topologies in this paper.

1.2. Main results

The above results suggest to look for a general theorem for the posets $\mathscr{B}(G)$ and $\mathscr{B}_\sigma(G)$.

To begin with we prove in Lemmas 6.5 and 7.3 the following (where $\log \alpha$, for a cardinal α, denotes the smallest cardinal β such that $2^\beta \geq \alpha$, for the definitions of $\gamma(G)$ and $\Gamma(G)$ see Section 7):

If G is any infinite group with $\mathscr{B}(G) \neq \emptyset$, there are two cardinals $\gamma(G) \leq \Gamma(G)$ such that $\mathscr{B}_\sigma(G) \neq \emptyset$ if and only if $\gamma(G) \leq \sigma \leq \Gamma(G)$. If G is abelian $\gamma(G) = \log(|G|)$ and $\Gamma(G) = 2^{|G|}$. If G is not abelian the interval may be smaller on both sides.

Under the extra assumption that G has large abelian quotients (it suffices $2^{|G|} = 2^{|G/G'|}$, actually a weaker condition works as well, see Definition 7.6) we can prove that, for any σ in the above interval, all the poset invariants of $\mathscr{B}(G)$ and $\mathscr{B}_\sigma(G)$ mentioned above depend only on σ and the cardinality $2^{|G|}$ (see Section 3). In other

[2] A group G is *relatively free* if G is a free group of the variety $\mathrm{Var}(G)$ generated by G (in the sense of Birkhoff, i.e. the smallest class of groups containing G and closed with respect to taking subgroups, quotient groups and direct products).

[3] More precisely, it was shown that for some groups G the poset $\mathscr{P}_\sigma(G)$ "contains a copy" of $\boldsymbol{P}(\sigma)$ whenever $\mathscr{P}_\sigma(G) \neq \emptyset$. It should be emphasized that establishing $\mathscr{P}_\sigma(G) \neq \emptyset$ may be highly non-trivial [19].

words, from this point of view the groups behave as *sets*. Our results are phrased in terms of poset embeddings.

Definition 1.1. Two partially ordered sets X and Y are *quasi-isomorphic* ($X \overset{\text{q.i.}}{\cong} Y$) if each one of them is isomorphic to a subset of the other.

Although quasi-isomorphic posets need not be isomorphic, they share a lot of common properties, such as monotone poset cardinal invariants (i.e. which do not increase strictly under the passage to suborders) e.g. height, depth, width, *Ded* (see Section 2).

We are interested in computing the monotone cardinal invariants of the poset $\mathscr{B}_\sigma(G)$.

Given an infinite set X we consider the posets $[X]^\sigma$, $[X]^{\leq\sigma}$, $[X]^{<\sigma}$ consisting of all subsets of X of size σ, $\leq \sigma$, $< \sigma$, respectively (ordered by inclusion). We denote by $\boldsymbol{P}(X)$ or 2^X the power set of X. It is easy to see that $[X]^{\leq\sigma} \overset{\text{q.i.}}{\cong} [X]^\sigma$.

We have obvious embeddings

$$\mathscr{B}(G) \hookrightarrow \mathscr{H}(G) \hookrightarrow \boldsymbol{P}(2^G).$$

If G is close to be abelian ($2^{|G/G'|} = 2^{|G|}$ suffices), then we can show that $\mathscr{B}(G)$ is very large, i.e. we have an embedding

$$\boldsymbol{P}(2^G) \hookrightarrow \mathscr{B}(G)$$

and, therefore,

$$\mathscr{B}(G) \overset{\text{q.i.}}{\cong} \mathscr{H}(G) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^G).$$

If we consider the subposets consisting of topologies of weight σ, then the situation changes.

Since a topology of weight σ has at most 2^σ open sets, we have an obvious poset embedding

$$\mathscr{H}_\sigma(G) \hookrightarrow [2^G]^{2^\sigma}$$

(and an injection $\mathscr{H}_\sigma(G) \hookrightarrow [2^G]^\sigma$ which need not be an embedding, see Section 8). Hence the monotone cardinal invariants of $\mathscr{H}_\sigma(G)$ are bounded by those of $[2^G]^{2^\sigma}$ and the latter are easy to compute (see Section 2).

For precompact group topologies, using characters (in the abelian case) or finite-dimensional unitary representations (in the non-abelian case) it is known (see Theorem 7.1) that there is an embedding

$$\mathscr{B}_\sigma(G) \hookrightarrow [2^G]^\sigma.$$

The precompactness is essential. We prove that $\mathscr{H}_\sigma(G)$ *does not embed in* $[2^G]^\sigma$ *even for* $\sigma = \omega$, $G = \bigoplus_{i\in\omega} \boldsymbol{Z}_2$ (Corollary 7.4). Thus for this choice of σ and G $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\ncong} \mathscr{H}_\sigma(G)$ in contrast with the fact that $\mathscr{B}(G) \overset{\text{q.i.}}{\cong} \mathscr{H}(G)$.

For a group with large abelian quotients and for σ in the appropriate interval, we characterize the structure of $\mathscr{B}(G)$ and $\mathscr{B}_\sigma(G)$ up to quasi-isomorphism: *$\mathscr{B}(G)$ is quasi-isomorphic to the power set $\mathbf{P}(2^{|G|})$ of $2^{|G|}$ and $\mathscr{B}_\sigma(G)$ is quasi isomorphic to the poset* $[2^G]^\sigma$ (Theorem 6.10 and Corollary 7.9):

$$\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^G]^\sigma.$$

This implies most of the results given in [4, 8, 29, 32].

The classification via the posets $[2^G]^\sigma$ is faithful since

$[\alpha]^\sigma$ *is quasi-isomorphic to* $[\beta]^\sigma$ *if and only if* $\alpha = \beta$ (Corollary 2.4).

Our methods are heavily based on unitary representations of groups. For example for an abelian group G the group of continuous characters G^* of G has cardinality $2^{|G|}$ and the poset $\mathscr{S}^{\mathrm{d}}(G^*)$ of all dense subgroups of G^* is isomorphic to $\mathscr{B}(G)$ and quasi-isomorphic to the power set of $2^{|G|}$. Hence $\mathscr{B}(G) \overset{\text{q.i.}}{\cong} \mathbf{P}(2^{|G|})$. Some more work shows that $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^{|G|}]^\sigma$ for σ in the interval $\log|G| \le \sigma \le 2^{|G|}$, and $\mathscr{B}_\sigma(G) = \emptyset$ for σ outside of this interval.

For non-abelian groups the situation is more complicated and the role of G^* is taken by the set $\Sigma(G)$ of equivalence classes of finite-dimensional irreducible unitary representations of G. If G is maximally almost periodic and G has "large abelian quotients" we obtain results similar to those of the abelian case with one important difference: the interval $\log|G| \le \sigma \le 2^{|G|}$ must be replaced by a smaller one.

The most difficult monotone cardinal invariants of $\mathscr{B}(G)$ and $\mathscr{B}_\sigma(G)$ are the ones related to the size of the chains and we will show that some of the relevant questions are independent of ZFC. It is convenient to introduce for any poset P a cardinal invariant $Ded^e(P)$ to measure the length of its chains and a cardinal invariant $Ded(P)$ to measure the lengths of its bounded chains (see Section 2 for definitions). For posets with a top element these two invariants coincide.

For the poset $\mathbf{P}(\sigma)$ ordered by inclusion, $Ded(\mathbf{P}(\sigma))$ coincides with the well-known cardinal function $Ded(\sigma)$. We introduce and study the cardinal function $Ded^e(\sigma) = Ded^e([X]^\sigma)$, where X is any set of cardinality $> \sigma$ ($|X| = \sigma^+$ suffices, see Lemma 4.3).

In this terminology, Comfort and Remus in [8] proved that $Ded(\mathscr{B}(G)) = Ded(\mathbf{P}(2^{|G|})) = Ded(2^{|G|})$ if G is abelian or free. This follows also from the quasi-isomorphism between $\mathscr{B}(G)$ and $\mathbf{P}(2^{|G|})$ proved in this paper.

For $\mathscr{B}_\sigma(G)$ the situation is more complicated because this poset has no top element. If G is abelian and $\log(|G|) \le \sigma \le 2^{|G|}$, the quasi-isomorphism $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^G]^\sigma$ gives us easily $Ded(\sigma) = Ded(\mathscr{B}_\sigma(G))$ and $Ded^e(\sigma) = Ded^e(\mathscr{B}_\sigma(G))$.

Under the generalized continuum hypothesis (GCH) $Ded(\sigma) = Ded^e(\sigma) = (2^\sigma)^+$ for every infinite σ. Without GCH we prove that

$$Ded(\sigma) \neq Ded^e(\sigma) \quad \textit{iff} \quad cf\, Ded(\sigma) = \sigma^+$$

(Theorem 4.11). Using earlier results of Mitchell and Baumgartner we deduce that the equality $Ded(\sigma) = Ded^e(\sigma)$ is independent of ZFC (Theorem 5.7).

We then apply this result to $\mathscr{B}_\sigma(G)$, together with the quasi isomorphism $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^{|G|}]^\sigma$, to show that *it cannot be established in ZFC whether $\mathscr{B}_{\aleph_1}(\boldsymbol{Z})$ has chains of size bigger than the size of every bounded chain* (Theorem 8.1). Finally, in Section 8.3 we prove that *for $\sigma = \aleph_0$ and $G = \bigoplus_{\aleph_0} \boldsymbol{Z}_2$, it is independent of ZFC whether the poset $\mathscr{H}_\sigma(G)$ has the same height as the poset $\mathscr{B}_\sigma(G)$*. This is proved using an embedding of the poset $\boldsymbol{P}(\omega)/\text{fin}$ (the power set of ω modulo the ideal of finite subsetes of ω) into the poset $\mathscr{H}_{\aleph_0}(G)$. Since the depth of $\boldsymbol{P}(\omega)/\text{fin}$ is uncountable, this proves (in ZFC) that the poset $\mathscr{H}_{\aleph_0}(G)$ cannot be embedded into $\mathscr{B}_{\aleph_0}(G)$.

2. Monotone cardinal invariants

We follow [26] for the set-theoretical notation.

A *monotone cardinal invariant* of a poset $(P, \leq)$ is a cardinal invariant which does not increase strictly under the passage to suborders. Examples are:

1. $|P|$ = the cardinality of P.
2. $\text{height}(P) = \sup\{|A| : A \subseteq P$ is well ordered$\}$.
3. $\text{height}^b(P) = \sup\{|A| : A \subseteq P$ is well ordered and bounded$\}$.
4. $\text{depth}(P) = \sup\{|A| : A$ is well ordered in the reverse order$\}$.
5. $\text{width}(P) = \sup\{|A| : A \subseteq P$ is an antichain$\}$.
6. $Ded(P) = \min\{\kappa :$ there is no totally ordered and bounded $A \subseteq P$ of cardinality $\kappa\}$.
7. $Ded^e(P) = \min\{\kappa :$ there is no totally ordered $A \subseteq P$ of cardinality $\kappa\}$.
8. $\Lambda(P) = \sup\{\kappa :$ there is a totally ordered and bounded $A \subseteq P$ of cardinality $\kappa\}$.
9. $\Lambda^e(P) = \sup\{\kappa :$ there is a totally ordered $A \subseteq P$ of cardinality $\kappa\}$.

Here "antichain" means a set of incomparable elements. Clearly, $Ded^e(P)$ coincides with $Ded(P)$ if P has a top element.

With these notations the well-known cardinal function $Ded(\sigma)$ is equal to $Ded(\boldsymbol{P}(\sigma))$ (and also to $Ded^e(\boldsymbol{P}(\sigma))$ since the poset $\boldsymbol{P}(\sigma)$ has a top element).

We define

$$Ded^e(\sigma) = Ded([\sigma^+]^\sigma)$$

and we will see that $Ded^e(\sigma)$ also coincides with $Ded([X]^\sigma)$ for all X of cardinality strictly greater than σ (Lemma 4.3).

Except for Ded and Ded^e, the above-mentioned cardinal invariants are easy to compute for the posets $[X]^\sigma$ ($\sigma \leq |X|$). Namely:

- height $[X]^\sigma = \min\{\sigma^+, |X|\}$,
- depth $[X]^\sigma = \sigma$, and
- width $[X]^\sigma = |X|^\sigma$.

To see the last fact the reader has to identity X with $\sigma \times X$ and note that for two functions $f, g : \sigma \to X$ considered as subsets of $\sigma \times X$ (under the identification with their graph) any inclusion $f \subseteq g$ or $g \subseteq f$ yields $f = g$. Therefore there is an antichain in $\boldsymbol{P}(\sigma \times X)$ of cardinality $|X|^\sigma$. It should be mentioned that also in the case of the height and the depth, the supremum is attained.

We can give an upper bound on $\Lambda(P)$ and $\Lambda^e(P)$ in terms of depth(P), height(P) and height$^b(P)$ using the Erdös Rado theorem $(2^\kappa)^+ \to (\kappa^+)^2_2$ (see [26, Theorem 6.9, p. 323], or [26, Exercise 29.1, p. 324]).

Proposition 2.1. (1) *If depth(P), height$(P) \leq \kappa$ then $\Lambda^e(P) \leq 2^\kappa$.*
(2) *If depth(P), height$^b(P) \leq \kappa$ then $\Lambda(P) \leq 2^\kappa$.*

Proof. Let $\leq$ be the order of the poset P and let $\prec$ be a well order on P. Suppose $\Lambda^e(P) > 2^\kappa$ and let $A \subseteq P$ be a chain in $(P, \leq)$ of size $|A| > 2^\kappa$. Define $f : [A]^2 \to \{0,1\}$ by $f(\{a,b\}) = 1$ iff $\leq$ and $\prec$ agree on $\{a,b\}$. By the Erdös Rado theorem there is $H \subseteq A$ of size κ^+ such that f restricted to $[H]^2$ is constant, i.e. H is well ordered or reverse well ordered by $\leq$. It then follows that either the depth or the height of P is $> \kappa$ – a contradiction. Hence point (1) follows. The proof of point (2) is similar, considering bounded chains $A \subseteq P$. □

Moreover, we have

Proposition 2.2. $\Lambda^e(P) \leq \Lambda(P)^+$.

Proof. Suppose $\Lambda^e(P) > \Lambda(P)^+$. So there is a chain $C \subseteq P$ of size $|C| = \kappa^{++}$, where $\kappa = \Lambda(P)$. Write $C = \bigcup_{\alpha < cf(C)} C_\alpha$, where $cf(C)$ is the cofinality of C and each C_α is a bounded chain, hence of size $|C_\alpha| \leq \kappa$. It follows that $\kappa^{++} = |C| \leq cf(C)\kappa$ and therefore $cf(C) = \kappa^{++}$. Hence C has a well-ordered subchain of size κ^{++} and therefore a bounded well-ordered subchain of size κ^+, contradiction. □

The above inequality is optimal: take P to be the ordinal σ^+, then $\Lambda^e(P) = \sigma^+$ and $\Lambda(P) = \sigma$ (as every bounded chain of σ^+ has size $\leq \sigma$).

For the poset $P = [X]^\sigma$ we shall prove $\Lambda^e(P) = \Lambda(P)$ (Lemma 4.2).

As explained in the introduction, under suitable hypotheses $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^G]^\sigma$, and therefore all the monotone cardinal invariants of the poset $\mathscr{B}_\sigma(G)$ coincide with those of the poset $[2^G]^\sigma$. This classification is faithful because of the following result.

Lemma 2.3. *If $\alpha' < \alpha$ and $\kappa > 1$ are cardinals, then there is no order-embedding $h : [\alpha]^{<\kappa} \to [\alpha']^{<\kappa}$ (so in particular $[\alpha]^{<\kappa}$ and $[\alpha']^{<\kappa}$ are not quasi-isomorphic).*

Proof. Suppose, toward a contradiction, that h were such an embedding. We consider two cases.

Case 1: $(\forall \beta \in \alpha)(\exists \xi \in h(\{\beta\}))(\forall \gamma \in \alpha \setminus \{\beta\})\, \xi \notin h(\{\gamma\})$.

Then the function assigning to each $\beta \in \alpha$ the smallest such ξ is a one-to-one map $\alpha \to \alpha'$. This contradicts $\alpha' < \alpha$.

Case 2: Otherwise: $(\exists \beta \in \alpha)(\forall \xi \in h(\{\beta\}))(\exists \gamma \in \alpha \setminus \{\beta\})\, \xi \in h(\{\gamma\})$.

Fix such a β. For each $\xi \in h(\{\beta\})$, fix $\beta_\xi \in \alpha \setminus \{\beta\}$ with $\xi \in h(\{\beta_\xi\})$. Let $b = \{\beta_\xi |\ \xi \in h(\{\beta\})\}$. Then $b \in [\alpha]^{<\kappa}$ because $|b| \leq |h(\{\beta\})| < \kappa$. For each $\xi \in h(\{\beta\})$, we have $\xi \in h(\{\beta_\xi\}) \subseteq h(b)$. So $h(\{\beta\}) \subseteq h(b)$. As h is an order-embedding, $\{\beta\} \subseteq b$, i.e. $\beta = \beta_\xi$ for some $\xi \in h(\{\beta\})$. This contradicts the choice of β_ξ. □

Corollary 2.4. *If $\alpha' < \alpha$ and $\sigma \geq 1$ are cardinals, then there is no order-embedding $h : [\alpha]^\sigma \to [\alpha']^\sigma$ (so in particular $[\alpha]^\sigma$ and $[\alpha']^\sigma$ are not quasi-isomorphic).*

Proof. It suffices to observe that $[\beta]^\sigma$ and $[\beta]^{<\sigma^+}$ are quasi isomorphic. □

Notice that, in general, $[\alpha]^\sigma$ and $[\alpha']^\sigma$ cannot be distinguished by the invariants width, depth, height, *Ded*, Ded^e.

For a poset P one can define a new monotone cardinal invariant Inv_σ by

$$\mathrm{Inv}_\sigma(P) = \sup\{|\alpha| \colon [\alpha]^\sigma \text{ can be embedded in } P\}.$$

By Corollary 2.4 $\mathrm{Inv}_\sigma([\alpha]^\sigma) = |\alpha|$.

3. Chains in $\boldsymbol{P}(\sigma)$

We have already computed height, width, depth for the posets $[X]^\sigma$ and in particular for the poset $\boldsymbol{P}(\sigma) = [\sigma]^{\leq\sigma} \overset{\text{q.i.}}{\cong} [\sigma]^\sigma$. In this section and in the next one, we consider the invariants *Ded* and Ded^e for these posets.

Since $\boldsymbol{P}(\sigma)$ has a top element, $Ded(\boldsymbol{P}(\sigma)) = Ded^e(\boldsymbol{P}(\sigma))$ and $\Lambda(\boldsymbol{P}(\sigma)) = \Lambda^e(\boldsymbol{P}(\sigma))$. We shall see below (Lemma 4.3) that both $Ded^e([X]^\sigma)$ and $\Lambda^e([X]^\sigma)$ do not depend on X, provided $|X| > \sigma$ (in particular, X can be taken of cardinality σ^+). These facts motivate the following:

Definition 3.1. For an infinite cardinal σ set:
- $Ded(\sigma) = Ded(\boldsymbol{P}(\sigma))$, $Ded^e(\sigma) = Ded^e([X]^\sigma)$, where $|X| > \sigma$,
- $\Lambda(\sigma) = \Lambda(\boldsymbol{P}(\sigma))$, $\Lambda^e(\sigma) = \Lambda^e([X]^\sigma)$, where $|X| > \sigma$.

Note that $Ded(\sigma)$ gives more information than $\Lambda(\sigma)$, in the sense that $\Lambda(\sigma)$ is uniquely determined by $Ded(\sigma)$ (if $Ded(\sigma)$ is a successor cardinal, then $\Lambda(\sigma)$ is its predecessor, otherwise $\Lambda(\sigma) = Ded(\sigma)$).

The cardinal function $Ded(\sigma) = Ded(\boldsymbol{P}(\sigma))$ is well known, and we state in this section some of the relevant results about it. For the posets $[X]^\sigma$ we prove that the invariants Λ and Λ^e coincide, as in the case of the the poset $\boldsymbol{P}(\sigma)$, and we show that the problem of whether *Ded* and Ded^e also coincide, is independent of ZFC. More specifically we show that $\Lambda(\sigma) = \Lambda^e(\sigma)$ for every σ, whilst $Ded(\aleph_1) = Ded^e(\aleph_1)$ is independent of ZFC (under GCH, $Ded(\sigma) = Ded^e(\sigma) = (2^\sigma)^+$). We also prove $Ded(\sigma) \neq Ded^e(\sigma)$ iff $cf\, Ded(\sigma) = \sigma^+$, where $cf(\kappa)$ is the cofinality of the cardinal κ.

A *chain* in $\boldsymbol{P}(\sigma)$ is a subset of $\boldsymbol{P}(\sigma)$ which is totally ordered by the inclusion relation. A *tree* is a partially ordered set $(T, \leq)$ such that for each $x \in T$ the set of predecessors of x, namely the set $\{y \in T \mid y < x\}$, is well ordered by $\leq$. One then defines the *height* of x as the order type of $\{y \in T \mid y < x\}$. The height of the tree is the supremum of the heights of its elements. $\text{Level}_\delta(T)$ is the set of all elements of T of height δ.

A *chain* in a tree T is a subset of T totally ordered by $\leq$. A *branch* is a maximal chain.

Definition 3.2. Let λ and σ be infinite cardinals and let

(a) $C(\sigma, \lambda)$ stay for the set-theoretic assumption: there is a chain of cardinality λ in $\boldsymbol{P}(\sigma)$ [8].

(b) $D(\sigma, \lambda)$ stay for the set-theoretic assumption: a set of cardinality λ can be totally ordered in a way to have a dense subset of cardinality σ [2].

(c) $T(\sigma, \lambda)$ stay for the set-theoretic assumption: there is a tree of height $\leq \sigma$ and cardinality $\leq \sigma$ with at least λ branches [2].

These three conditions are equivalent (see [2, Theorem 2.1(b)]).

Note that

$$\Lambda(\sigma) = \sup\{\lambda\colon C(\sigma, \lambda) \text{ holds}\}. \tag{1}$$

Since $Ded(\sigma) = \min\{\lambda : C(\sigma, \lambda) \text{ does not hold}\}$, obviously $Ded(\sigma)$ is either $\Lambda(\sigma)$ or $\Lambda(\sigma)^+$, and the latter holds if and only if $C(\sigma, \Lambda(\sigma))$ holds, i.e. there exists a chain of maximal cardinality in $\boldsymbol{P}(\sigma)$. Mitchell showed that the existence of a chain of cardinality 2^σ in $\boldsymbol{P}(\sigma)$ cannot be decided in *ZFC*.

Fact 3.3. *Let σ be a cardinal.*

(1) *$C(\sigma, \lambda)$ implies $\lambda \leq 2^\sigma$. In other words $\Lambda(\sigma) \leq 2^\sigma$* [2, Theorem 2.2(e)].

(2) *Let $\sigma \leq \sigma'$ and $\lambda \geq \lambda'$, then $C(\sigma, \lambda)$ implies $C(\sigma', \lambda')$* [2, Theorem 2.2(b)].

(3) *If ρ is the least cardinal such that $\lambda \leq \sigma < \lambda^\rho$, then $C(\sigma, \lambda^\rho)$ holds. In particular $C(\sigma, \sigma^+)$ holds. Hence $Ded(\sigma) > \sigma^+$ and $\Lambda(\sigma) \geq \sigma^+$* [2, Corollary 2.4].

(4) *$C(\sigma, \lambda)$ implies $C(\sigma^\rho, \lambda^\rho)$ for all ρ.* [2, Corollary 4.2(a)].

(5) *If $\omega_\alpha^\rho = \omega_\alpha$ and $2^{\omega_\alpha} \geq \omega_{\alpha+\rho^+}$, then $C(\omega_\alpha, \omega_{\alpha+\rho^+})$ holds. Hence $Ded(\omega^\alpha) > \omega_{\alpha+\rho^+}$* [33; 2, Theorem 4.5].

(6) *$C(\omega, 2^\omega)$ is true in ZFC.*

(7) *Mitchell* [28] *found models of ZFC where $C(\omega_1, 2^{\omega_1})$ fails.*

(8) *If $\forall i < \sigma$ $C(\sigma, \lambda_i)$, then $C(\sigma, \sup_{i<\sigma} \lambda_i)$* [2, Theorem 2.2(c)].

By Fact 3.3, $\sigma^+ \leq \Lambda(\sigma) \leq 2^\sigma$. Under GCH there is a chain of cardinality 2^σ in $\boldsymbol{P}(\sigma)$, which is the maximum possible value.

Note that point (5) yields $C(\omega_\alpha, \omega_{\alpha+n})$ for every $n < \omega$ such that $\omega_{\alpha+n} \leq 2^{\omega_\alpha}$.

An immediate corollary of Fact 3.3(8) is the following:

$$\text{If } cf(\kappa) \leq \sigma \quad \text{and} \quad \forall \lambda < \kappa\, C(\sigma, \lambda), \text{ then } C(\sigma, \kappa). \tag{2}$$

In particular taking $\kappa = Ded(\sigma)$ we get

$$cf(Ded(\sigma)) > \sigma. \tag{3}$$

On the other hand, taking $\kappa = \Lambda(\sigma)$ we obtain

$$\text{If } cf(\Lambda(\sigma)) \leq \sigma, \text{ then } C(\sigma, \Lambda(\sigma)). \tag{4}$$

This implication cannot be reversed: for $\sigma = \omega$ we have $\Lambda(\sigma) = 2^\sigma$, $cf(\Lambda(\sigma)) > \sigma$ and $C(\sigma, \Lambda(\sigma))$ holds.

4. Chains in $[\sigma^+]^\sigma$

If $\gamma \geq \sigma$, the poset $[\gamma]^{\leq\sigma}$ (consisting of all the subsets of γ of cardinality $\leq \sigma$) contains the poset $\boldsymbol{P}(\sigma) = [\sigma]^{\leq\sigma}$, so every chain in $\boldsymbol{P}(\sigma)$ is also a chain in $[\gamma]^{\leq\sigma}$. We are interested in the question whether for some $\gamma > \sigma$, $[\gamma]^{\leq\sigma}$ contains chains of cardinality bigger than the cardinality of any chain in $\boldsymbol{P}(\sigma)$.

Definition 4.1. Let λ and σ be cardinals and let $C^e(\sigma, \lambda)$ stay for the set-theoretic assumption "there is a chain $\boldsymbol{C}$ of cardinality λ consisting of sets of cardinality $\leq \sigma$". (Note that we are not assuming that $\boldsymbol{C}$ is a chain in $\boldsymbol{P}(\sigma)$, it could be a chain in $[\gamma]^{\leq\sigma}$ for some $\gamma > \sigma$).

Note that:

$$\Lambda^e(\sigma) = \sup\{\lambda\colon C^e(\sigma, \lambda) \text{ holds}\}.$$

$$Ded^e(\sigma) = \min\{\lambda\colon C^e(\sigma, \lambda) \text{ does not hold}\}.$$

So our problem is whether $Ded^e(\sigma)$ can be strictly greater than $Ded(\sigma)$ (clearly it is greater or equal). The question is interesting only if GCH fails, in fact under GCH, $Ded^e(\sigma) = Ded(\sigma) = (2^\sigma)^+$ (see below). We will prove that a necessary and sufficient condition for $Ded^e(\sigma) > Ded(\sigma)$ is that $cf(Ded(\sigma)) = \sigma^+$. So our problem about chains in $[\gamma]^{\leq\sigma}$ can be entirely reduced to a question about $Ded(\sigma)$.

By Proposition 2.2 we know that $\Lambda^e(P) \leq \Lambda(P)^+$ for any poset, so in particular $\Lambda^e(\sigma) \leq \Lambda(\sigma)^+$. We strengthen this by:

Lemma 4.2. $\Lambda^e(\sigma) = \Lambda(\sigma)$.

Proof. Let $\boldsymbol{C}$ be a chain of sets with $\forall X \in \boldsymbol{C}\, |X| \leq \sigma$. We must show that $|\boldsymbol{C}| \leq \Lambda(\sigma)$. By $cf(\boldsymbol{C})$ we mean the cofinality of $\boldsymbol{C}$ considered as a totally ordered set (ordered by inclusion). This is not to be confused with $cf(|\boldsymbol{C}|)$, the cofinality of the cardinality of $\boldsymbol{C}$. We will give upper bounds on $cf(\boldsymbol{C})$, $|\bigcup \boldsymbol{C}|$, and $|\boldsymbol{C}|$.

Claim. If γ is an ordinal and $f : \gamma \to \boldsymbol{C}$ is a strictly increasing function (in the sense that $\alpha < \beta < \gamma$ entails $f(\alpha) \subset f(\beta)$), then for each $\alpha < \alpha + 1 < \gamma$, $f(\alpha + 1)$ has at least $|\alpha|$ elements.

Proof. For each $\alpha < \alpha + 1 < \gamma$ fix some $x_\alpha \in f(\alpha+1) \setminus f(\alpha)$ and consider the set $\{x_\alpha \mid \beta < \alpha\} \subseteq f(\alpha+1)$.

An immediate consequence is

$$cf(\boldsymbol{C}) \leq \sigma^+.$$

Since $\bigcup \boldsymbol{C}$ can be seen as a union of a family of cardinality $\leq cf(\boldsymbol{C})$ of sets of cardinality $\leq \sigma$, we have: $|\bigcup \boldsymbol{C}| \leq cf(\boldsymbol{C}) \cdot \sup\{|X| : X \in \boldsymbol{C}\} \leq \sigma^+\sigma = \sigma^+$, hence

$$|\bigcup \boldsymbol{C}| \leq \sigma^+.$$

So in the definition of $C^e(\sigma, \lambda)$ we can assume that $\boldsymbol{C}$ is a chain in $[\sigma^+]^{\leq\sigma}$. To finish the proof we must show that

$$|\boldsymbol{C}| \leq \Lambda(\sigma).$$

To see this let $\gamma = cf(\boldsymbol{C})$ and let $f : \gamma \to \boldsymbol{C}$ be a cofinal map. We can write $\boldsymbol{C} = \bigcup_{\delta<\gamma} \boldsymbol{C}_\delta$ where $\boldsymbol{C}_\delta = \{X \in \boldsymbol{C} : X \subseteq f(\delta)\}$. Since $f(\delta)$ has cardinality $\leq \sigma$, $\boldsymbol{C}_\delta$ has cardinality at most $\Lambda(\sigma)$. Hence $|\boldsymbol{C}| \leq \gamma\Lambda(\sigma) = cf(\boldsymbol{C})\Lambda(\sigma)$. But since $cf(\boldsymbol{C}) \leq \sigma^+ \leq \Lambda(\sigma)$ (the last inequality holding by fact 3.3(3)), $|\boldsymbol{C}| \leq \Lambda(\sigma)$ as desired. □

A corollary of the proof is the following:

Lemma 4.3. *In the definition of $C^e(\sigma, \lambda)$ we can assume that $\boldsymbol{C}$ is a chain in $[\sigma^+]^{\leq\sigma}$.*

Since $\Lambda^e(\sigma) = \Lambda(\sigma)$, the only case in which $Ded^e(\sigma)$ can be greater than $Ded(\sigma)$ is when there is a chain of maximal cardinality in $[\sigma^+]^{\leq\sigma}$, but there is no chain of maximal cardinality in $\boldsymbol{P}(\sigma)$. In this case this maximal cardinality must be $\Lambda(\sigma)$ and $Ded^e(\sigma) = \Lambda(\sigma)^+ = Ded(\sigma)^+$. So in any case $Ded^e(\sigma)$ is either $Ded(\sigma)$ or $Ded(\sigma)^+$, and the latter holds iff $C^e(\sigma, Ded(\sigma))$ holds.

Remark 4.4. (1) Since $C(\omega, 2^\omega)$ holds, $\Lambda(\omega) = \Lambda^e(\omega) = 2^\omega$ and $Ded(\omega) = Ded^e(\omega) = (2^\omega)^+$.

(2) Under GCH $\Lambda(\sigma) = \Lambda^e(\sigma) = 2^\sigma$ and $Ded(\sigma) = Ded^e(\sigma) = (2^\sigma)^+$.

Notice that always $\sigma^+ \leq \Lambda(\sigma) = \Lambda^e(\sigma) \leq Ded(\sigma) \leq Ded^e(\sigma) \leq \Lambda^e(\sigma)^+ \leq (2^\sigma)^+$.

Lemma 4.5. *If $cf(Ded(\sigma)) \neq \sigma^+$, then $Ded^e(\sigma) = Ded(\sigma)$.*

Proof. If $Ded^e(\sigma) \neq Ded(\sigma)$, then $C^e(\sigma, Ded(\sigma))$ holds. Let $\boldsymbol{C}$ be a chain of cardinality $Ded(\sigma)$ witnessing $C^e(\sigma, Ded(\sigma))$. We can write $\boldsymbol{C} = \bigcup_{\delta<\gamma} \boldsymbol{C}_\delta$ where $\gamma = cf(\boldsymbol{C})$ and $|\boldsymbol{C}_\delta| < Ded(\sigma) = |\boldsymbol{C}|$. So $|\boldsymbol{C}|$ is a supremum of $cf(\boldsymbol{C})$ sets of cardinality $< |\boldsymbol{C}|$, and therefore $cf(Ded(\sigma)) = cf(|\boldsymbol{C}|) \leq cf(\boldsymbol{C}) \leq \sigma^+$. On the other hand, we have already seen in a previous section (see (3)) that $cf(Ded(\sigma)) > \sigma$, hence $cf(Ded(\sigma)) = \sigma^+$. □

We will see that the converse of the above lemma also holds. To see this, it is convenient to give an equivalent formulation of $C^e(\sigma,\lambda)$ in terms of dense linear orders.

Definition 4.6. Let $D^e(\sigma,\lambda)$ be the following statement: a set S of cardinality λ can be totally ordered so as to have a dense subset Q of cardinality $\leq \sigma^+$ with the property that each subset of Q bounded from above in Q has cardinality $\leq \sigma$.

Note that the assumption that Q is dense in S can be replaced with the apparently weaker assumption that Q is *weakly dense*, i.e. for each $s_1 < s_2$ in S, there is $x \in Q$ with $s_1 \leq x \leq s_2$. To see this, note that we can reduce to the case in which Q is dense by replacing each element $x \in Q$ by a copy of the rationals.

Proposition 4.7. *$C^e(\sigma,\lambda)$ is equivalent to $D^e(\sigma,\lambda)$.*

Proof. We can assume without loss of generality that $\sigma^+ < \lambda$.

Assume $D^e(\sigma,\lambda)$. We must prove $C^e(\sigma,\lambda)$. Let $(S,\leq)$ be a total order of cardinality λ with a dense subset Q of cardinality σ^+ witnessing $D^e(\sigma,\lambda)$. We can assume that S has no maximum element. So a subset of Q has an upper bound in Q iff it has an upper bound in S. Consider the chain $\boldsymbol{C}$ in $\boldsymbol{P}(Q)$ consisting of the downward closed bounded subsets of Q. So each element of $\boldsymbol{C}$ has cardinality $\leq \sigma$. $\boldsymbol{C}$ has cardinality at least λ since there is an injection of S into $\boldsymbol{C}$ sending each $s \in S$ into the set $X_s = \{x \in Q \mid x \leq s\}$. Thus $\boldsymbol{C}$ witnesses $C^e(\sigma,\lambda)$.

Conversely assume $C^e(\sigma,\lambda)$. Then there exists a chain $\boldsymbol{C}$ in $[\sigma^+]^{\leq\sigma}$ of cardinality $\geq \lambda$. Without loss of generality we can assume that $\boldsymbol{C}$ is a maximal chain in $[\sigma^+]^{\leq\sigma}$. It follows that $Q = \bigcup \boldsymbol{C}$ is a subset of σ^+ of cardinality σ^+. $\boldsymbol{C}$ induces a linear order $\leq_{\mathrm{C}}$ on Q by letting $a \leq_{\mathrm{C}} b$ iff every set of the chain containing b contains a. To verify the antisymmetry, note that if for a contradiction a and b are distinct elements with $a \leq_{\mathrm{C}} b$ and $b \leq_{\mathrm{C}} a$, then a and b belong to exactly the same elements of the chain. Let $X \subseteq \sigma^+$ be the set obtained by removing a from the intersection of all the sets of the chain containing both a and b. Then X is not in the chain but it is comparable with every set in the chain, contradicting the maximality of the chain. So $\leq_{\mathrm{C}}$ is indeed a partial order on Q. It is easy to see that $\leq_{\mathrm{C}}$ is linear. Note that the sets belonging to the chain are downward closed with respect to $\leq_{\mathrm{C}}$. It follows that for each $y \in Q$ the set $\{x \in Q \mid x \leq_{\mathrm{C}} y\}$ is contained in some set belonging to the chain (namely in any set $X \in \boldsymbol{C}$ containing y) and therefore it has cardinality $\leq \sigma$. For $y \in Q$ let $X_y = \{x \in Q \mid x \leq_{\mathrm{C}} y\}$. Note that if $X \in \boldsymbol{C}$ and $X \nsubseteq X_y$, then X contains some $z >_{\mathrm{C}} y$, and therefore $X \supseteq X_y$. Thus X_y is comparable with every element of the chain. But then by maximality of $\boldsymbol{C}$, $X_y \in \boldsymbol{C}$. Consider $\boldsymbol{C}$ as a total order ordered by inclusion. $\boldsymbol{C}$ contains the subset Q^* of cardinality σ^+ consisting of the sets of the form X_y with $y \in Q$. This subset might not be dense in $\boldsymbol{C}$ but it is certainly weakly dense. In fact, if $U \subset V$ are in $\boldsymbol{C}$, there exists $y \in Q$ with $y \in V - U$, hence $U \subseteq X_y \subseteq V$. Since $|\boldsymbol{C}| \geq \lambda$ we can conclude that $\boldsymbol{C}$ (ordered by inclusion) together with the weakly dense suborder Q^* witness $D^e(\sigma,\lambda')$ for some $\lambda' \geq \lambda$, hence by monotony $D^e(\sigma,\lambda)$ holds. □

Using the equivalence between $C^e(,)$ and $D^e(,)$ and the equivalence between $C(,)$ and $D(,)$ we can now prove:

Lemma 4.8. *If* $\forall i < \sigma^+ \ C(\sigma, \lambda_i)$, *then* $C^e(\sigma, \sup_{i<\sigma^+} \lambda_i)$.

Proof. For each $i < \sigma^+$ let S_i be a total order of size λ_i with a dense subset of cardinality σ. To construct an order witnessing $D^e(\sigma, \sup_{i<\sigma^+} \lambda_i)$ we can take the disjoint union of the S_i's ordered according to the order of their indexes. □

Using a diagonal argument we can strengthen this as follows:

Lemma 4.9. *If* $\forall i < \sigma^+ \ C^e(\sigma, \lambda_i)$, *then* $C^e(\sigma, \sup_{i<\sigma^+} \lambda_i)$.

Proof. Consider for each $i < \sigma^+$ a total order S_i of size λ_i with a dense subset Q_i of cardinality $\leq \sigma^+$ and such that each subset of Q_i bounded from above has cardinality $\leq \sigma$. We can assume that all the orders we consider have no end-points. Since each S_i has a dense subset of cardinality σ^+, the cofinality of each S_i is at most σ^+, so we can assume that S_i is the union of a family of downward closed proper subsets $S_{i,j} \subset S_i$ where $i, j < \sigma^+$ and for $a < b < \sigma^+$, $S_{i,a} \subseteq S_{i,b}$. The intersection of Q_i with $S_{i,j}$ is dense in $S_{i,j}$ and has cardinality $\leq \sigma$ because $S_{i,j}$ is bounded in S_i. Let $\lambda_{i,j} = |S_{i,j}|$. Then we have $C(\sigma, \lambda_{i,j})$ and by the previous lemma $C^e(\sigma, \sup_{i,j<\sigma^+} \lambda_{i,j})$. But $\sup_{i,j<\sigma^+} \lambda_{i,j} = \sup_{i<\sigma^+} \lambda_i$ and we are done. □

Since $C^e(\sigma, Ded^e(\sigma))$ fails, we have:

Corollary 4.10. $cf(Ded^e(\sigma)) > \sigma^+$.

We can now prove the converse of Lemma 4.5.

Theorem 4.11. (1) *If* $cf(Ded(\sigma)) \neq \sigma^+$, *then* $Ded^e(\sigma) = Ded(\sigma)$.
(2) *If* $cf(Ded(\sigma)) = \sigma^+$, *then* $Ded^e(\sigma) = Ded(\sigma)^+$.

Proof. Part (1) is Lemma 4.5. By Corollary 4.10 if $cf(Ded(\sigma)) = \sigma^+$, then $Ded(\sigma) \neq Ded^e(\sigma)$, hence $Ded^e(\sigma) = Ded(\sigma)^+$. □

Remark 4.12. If $Ded^e(\sigma) \neq Ded(\sigma)$, then $cf(Ded(\sigma)) = \sigma^+$ and $Ded(\sigma)$ is a singular cardinal. So in particular $Ded(\sigma) = \Lambda(\sigma)$ in this case.

It will be shown in the next section that $cf(Ded(\sigma)) = \sigma^+$ is consistent with ZFC.

5. Computation of $Ded(\sigma)$ and $Ded^e(\sigma)$ in Mitchell's models

We recall the following theorems of Mitchell:

Theorem 5.1 (Mitchell [28, Theorem 4.4]). *Let $\mathcal{M}$ be a model of Zermelo Fraenkel set theory plus GCH. Suppose that in $\mathcal{M}$, σ and θ are regular and $\theta < \sigma$. Then there is a forcing extension $\mathcal{N}$ of $\mathcal{M}$ with the same cardinals in which $2^\theta = \sigma^+$, $2^\sigma = \aleph_{\sigma^+}$, and $C(\sigma, 2^\sigma)$ fails.*

Theorem 5.2 (Mitchell [28, Corollary 4.3]). *There is a model of Zermelo Fraenkel set theory where $2^{\aleph_0} = \aleph_{\omega_1}$, $2^{\aleph_1} = \aleph_{\omega_1}^+$ and $C(\aleph_1, 2^{\aleph_1})$ fails.*

We also need a result of Baumgartner:

Theorem 5.3 (Baumgartner [2, Theorem 3.5]). *Let κ, λ, λ' and μ be cardinals (λ may be finite), and let $\kappa = \aleph_\alpha$. If $\lambda^{<\mu} < \aleph_{\alpha + cf\,\mu}$, $\lambda^{<\mu} < \lambda^\mu$, $\lambda' \le \lambda^\mu$ and either $cf\,\lambda' \le \kappa$ or $cf\,\lambda' > \lambda^{<\mu}$, then $C(\kappa, \lambda')$ holds.*

The next corollary shows that if $2^{<\sigma}$ is "not too big", then we can find very long chains in $\boldsymbol{P}(\sigma)$, namely $\Lambda(\sigma) = 2^\sigma$.

Corollary 5.4. *Suppose that σ is regular, $2^{<\sigma} < \aleph_\sigma$, and $2^{<\sigma} < 2^\sigma$. Then:*

(1) $\Lambda(\sigma) = 2^\sigma$;

(2) *moreover, if $2^{<\sigma} < cf(2^\sigma)$ (in particular if 2^σ is regular), then $Ded(\sigma) = (2^\sigma)^+$ (i.e. $C(\sigma, 2^\sigma)$ holds).*

Proof. Let σ be regular, $2^{<\sigma} < \aleph_\sigma$ and $2^{<\sigma} < 2^\sigma$. We apply Theorem 5.3 with $\kappa = \mu = \sigma$, $\lambda = 2$. Let $\lambda' \le 2^\sigma$, $cf(\lambda') > 2^{<\sigma}$. By Baumgartner's result $C(\kappa, \lambda')$ holds. If $2^{<\sigma} < cf(2^\sigma)$, then we can take $\lambda' = 2^\sigma$ and we have $C(\sigma, 2^\sigma)$, i.e. $Ded(\sigma) = (2^\sigma)^+$ (and $\Lambda(\sigma) = 2^\sigma$).

If $2^{<\sigma} \ge cf(2^\sigma)$, then in particular 2^σ is singular (as $2^{<\sigma} < 2^\sigma$), and therefore it is the supremum of regular cardinals λ' which we can take as well of cardinality $> 2^{<\sigma}$. Hence $C(\sigma, \lambda')$ holds for such cardinals and taking their sup we obtain $\Lambda(\sigma) = 2^\sigma$. □

Alternatively, we can argue as follows.

Direct proof of Corollary 5.4. Let $T^*(\sigma, \lambda, \mu)$ mean that there is a tree of height σ, whose levels have size not exceeding μ and with at least λ branches of height σ. Clearly $T^*(\sigma, \lambda, \mu)$ implies $T(\sigma\mu, \lambda)$, which in turn implies $C(\sigma\mu, \lambda)$. In particular if $\mu \le \sigma$, then $T^*(\sigma, \lambda, \mu)$ implies $C(\sigma, \lambda)$. We need the following three claims.

Claim. $T^*(\sigma, 2^\sigma, 2^{<\sigma})$.

Proof. The tree T of all binary sequences of length $< \sigma$ (usually denoted by $2^{<\sigma}$) witnesses $T^*(\sigma, 2^\sigma, 2^{<\sigma})$.

Claim. *If $\sigma \leq \mu$, $\mu^+ < cf(\lambda)$ and $T^*(\sigma, \lambda, \mu^+)$, then $T^*(\sigma, \lambda, \mu)$.*

Proof. Let $T = (T, \prec)$ be a tree of height σ witnessing $T^*(\sigma, \lambda, \mu^+)$ and such that $T^*(\sigma, \lambda, \mu)$ fails. We can assume that $T \subseteq \sigma \times \mu^+$, and that the αth level of the tree is contained in $\{\alpha\} \times \mu^+$. Let f be a branch of T of height σ. Since $\mu^+ = cf(\mu^+) > \sigma$, there is $\eta_f < \mu^+$, such that $f \subseteq \sigma \times \eta_f$. Since $cf(\lambda) > \mu^+$, there is $\eta < \mu^+$ such that $\mathscr{F} = \{f \mid \eta_f = \eta\}$ has cardinality λ. Let $T' = (T', \prec)$ be the subtree of T consisting of all the nodes appearing in the branches $f \in \mathscr{F}$. Since $|\eta| \leq \mu$, T' witnesses $T^*(\sigma, \lambda, \mu)$.

Claim. *If $cf(\lambda) > \mu \geq \sigma > cf(\mu)$ and $T^*(\sigma, \lambda, \mu)$, then $\exists \eta < \mu\ T^*(\sigma, \lambda, \sigma|\eta|)$.*

Proof. Let $T = (T, \prec)$ be a tree witnessing $T^*(\sigma, \lambda, \mu)$, and assume that $T \subseteq \sigma \times \mu$ and the α's level of the tree is contained in $\{\alpha\} \times \mu$. Note that a branch f of T of height σ can be thought of as a function $f : \sigma \to \mu$ (where $f(\xi) = \beta$ iff the node (ξ, β) belongs to the branch f). Since $cf(\mu) < \sigma$ and σ is regular, for every branch f of T of height σ, there is $\eta_f < \mu$ such that $f < \eta_f$ on a subset of σ of size $\sigma|\eta|$. Since $cf(\lambda) > \mu$, there is $\eta < \mu$ such that the set $\mathscr{G} = \{f \mid \eta_f = \eta\}$ has size λ. Take the subtree S of T having exactly the nodes belonging to some of the branches in $\mathscr{G}$. (Note that S is not necessarily contained in $\sigma \times \eta$.) So S is a tree with λ branches of height σ. To finish the proof it suffices to show that each level S_α of S has size $\sigma|\eta|$. To this purpose we define an injective map $h : S_\alpha \to \sigma \times \eta$ as follows. Given $x \in S_\alpha$ choose the least $f \in \mathscr{G}$, with respect to a fixed well ordering of $\mathscr{G}$, such that x is on the branch f. Then choose ξ minimal such that $\alpha < \xi < \sigma$ and $f(\xi) < \eta$. Note that ξ exists since $f < \eta$ on a set of cardinality σ. Now define $h(x) = (\xi, f(\xi))$. To prove that h is injective suppose $(\xi, f(\xi)) = h(x) = h(y) = (\tau, g(\tau))$ (hence $y = g(\alpha)$). Then $\xi = \tau$ and $f(\xi) = g(\tau) = g(\xi)$. But $\xi > \alpha$ and $\mathscr{G}$ is a tree, so $f(\alpha) = g(\alpha)$, i.e. $x = y$.

To prove the corollary fix λ such that $\lambda \leq 2^\sigma$ and $cf(\lambda) > 2^{<\sigma}$. Since $\lambda \leq 2^\sigma$, by the first claim we obtain $T^*(\sigma, \lambda, 2^{<\sigma})$. Let $v \leq 2^{<\sigma}$ be minimal such that $T^*(\sigma, \lambda, v)$ holds. We will show that $v \leq \sigma$. Suppose for a contradiction that $v > \sigma$. Since $cf(\lambda) > 2^{<\sigma} \geq v$, by the second claim v cannot be a successor cardinal. So v is a limit cardinal. Now since $v \leq 2^{<\sigma} < \aleph_\sigma$, the cofinality of v must be $< \sigma$. But then the hypothesis of the third claim are satisfied and v is not minimal, which is absurd.

We have thus proved that the least v such that $T^*(\sigma, \lambda, v)$ holds is $\leq \sigma$. Thus $T(\sigma, \lambda)$ holds, hence $\Lambda(\sigma) \geq \lambda$. It follows that $\Lambda(\sigma) \geq \sup\{\lambda \mid \lambda \leq 2^\sigma,\ cf(\lambda) > 2^{<\sigma}\}$. Since $2^{<\sigma} < 2^\sigma$, this supremum is 2^σ, and the supremum is achieved if $cf(2^\sigma) > 2^{<\sigma}$. The thesis of the lemma follows. □

We have thus proved that if $2^{<\sigma}$ is "small", then $\boldsymbol{P}(\sigma)$ contains long chains. Baumgartner proved that there are models of ZFC where 2^σ is arbitrarily large, and yet $C(\sigma, 2^\sigma)$ holds, i.e. $\boldsymbol{P}(\sigma)$ contains chains of the maximal possible cardinality. In fact, it is possible to have a model where for every regular σ $C(\sigma, 2^\sigma)$ holds and yet the continuum function $\sigma \mapsto 2^\sigma$ has an arbitrary (reasonable) behavior on the regular cardinals. The following result is a particular case of Baumgartner's theorem (obtained by taking X = the class of all regular cardinals, in his Theorem 5.7).

Theorem 5.5 (Baumgartner [2, Theorem 5.7]). *Let $\mathcal{M}$ be a countable transitive model of Gödel–Bernays set theory + GCH. Let F be a function in $\mathcal{M}$ mapping the class of regular cardinals of $\mathcal{M}$ into the class of all cardinals of $\mathcal{M}$ with the properties:*

(1) *For every regular σ, $cf(F(\sigma)) > \sigma$.*

(2) *$F(\sigma) \geq F(\lambda)$ for every $\lambda < \sigma$.*

Then there is a forcing extension $\mathcal{N}$ of $\mathcal{M}$ preserving cofinalities and such that for every regular σ, $2^\sigma = F(\sigma)$ and $C(\sigma, 2^\sigma)$.

Corollary 5.4 is optimal in the sense that the hypotheses $2^{<\sigma} < \aleph_\sigma$ and $2^{<\sigma} < 2^\sigma$ do not determine, in ZFC, whether $C(\sigma, 2^\sigma)$ holds. Indeed, using Mitchell's models (Theorem 5.1) with $\sigma = \theta^+$ we obtain $2^{<\sigma} = \sigma^+$, $2^\sigma = \aleph_{\sigma^+}$ whereas $C(\sigma, 2^\sigma)$ fails. On the other hand, using Baumgartner's models (Theorem 5.5) we obtain that $C(\sigma, 2^\sigma)$ holds with the same values of $2^{<\sigma}, 2^\sigma$.

Using Theorem 4.11 we can now compute $Ded(\sigma)$ and $Ded^e(\sigma)$ in Mitchell's models.

Theorem 5.6. (1) *In the model of Theorem 5.2 (where $2^{\aleph_0} = \aleph_{\omega_1}$, $2^{\aleph_1} = \aleph_{\omega_1}^+$ and $C(\aleph_1, 2^{\aleph_1})$ fails), we have: $\Lambda(\aleph_1)^+ = 2^{\aleph_1} = Ded(\aleph_1) = Ded^e(\aleph_1)$.*

(2) *In the model of Theorem 5.1, taking $\theta = \omega$ and $\sigma = \aleph_1$, we have: $2^{\omega_1} = \aleph_{\omega_2}$, $C(\omega_1, 2^{\aleph_1})$ fails and $\Lambda(\aleph_1) = 2^{\aleph_1} = Ded(\aleph_1) < Ded^e(\aleph_1) = (2^{\aleph_1})^+$.*

Proof. (1) We show that $C(\aleph_1, \aleph_{\omega_1})$ holds. Apply Theorem 5.3 with $\kappa = \aleph_1$, $\mu = \aleph_0$, $\lambda = 2$, $\lambda' = \aleph_{\omega_1}$. The hypotheses are verified because $2^{<\mu} = \aleph_0 < \aleph_\omega = \aleph_{1+\omega}$, $2^{<\mu} = \aleph_0 < 2^{\aleph_0} = 2^\mu$, $\lambda' = \aleph_{\omega_1} = 2^{\aleph_0} = 2^\mu$ and $cf\,\lambda' = \omega_1$. Hence $C(\aleph_1, \aleph_{\omega_1})$ holds, and we obtain $\Lambda(\aleph_1)^+ = 2^{\aleph_1} = Ded(\aleph_1)$. By Theorem 4.11 $Ded^e(\aleph_1) = Ded(\aleph_1)$.

(2) By Corollary 5.4 $\Lambda(\aleph_1) = 2^{\aleph_1}$ and again $Ded(\aleph_1) = 2^{\aleph_1}$. Now Theorem 4.11 applies. □

Corollary 5.7. *$Ded^e(\aleph_1) = Ded(\aleph_1)$ cannot be decided in ZFC.*

In both models of Mitchell $Ded(\aleph_1) = 2^{\aleph_1}$. We do not know whether it is consistent to have $Ded(\aleph_1) < 2^{\aleph_1}$ (i.e. there exists $\alpha < 2^{\aleph_1}$ such that $C(\aleph_1, \alpha)$ fails). Notice that $\Lambda^+(\aleph_1) < 2^{\aleph_1}$ implies $Ded(\aleph_1) < 2^{\aleph_1}$.

6. The structure of $\mathscr{B}_\sigma(G)$ for abelian groups

6.1. The lattice $\mathscr{S}(G)$ of all subgroups of an abelian group G

Now we show that "uncountable abelian groups have many subgroups". More precisely, the complete lattice $\mathscr{S}(G)$ of all subgroups of such a group G is quasi isomorphic to $\boldsymbol{P}(G)$. Actually, for each infinite σ the subset $\mathscr{S}_\sigma(G) = \mathscr{S}(G) \cap [G]^\sigma$ of $\mathscr{S}(G)$ consisting of all subgroups of cardinality σ is quasi isomorphic to $[G]^\sigma$.

Lemma 6.1. *Let G be an uncountable abelian group. Then there exists an embedding $f : \boldsymbol{P}(G) \to \mathscr{S}(G)$ which sends $[G]^\sigma$ into $\mathscr{S}_\sigma(G)$ for each infinite cardinal $\sigma \leq |G|$. Consequently $\boldsymbol{P}(G) \overset{\text{q.i.}}{\cong} \mathscr{S}(G)$ and $[G]^\sigma \overset{\text{q.i.}}{\cong} \mathscr{S}_\sigma(G)$ for such σ.*

Proof. First we prove the assertion in case $G = \bigoplus\{C_i\colon i \in I\}$ is a direct sum of non-trivial cyclic subgroups C_i. Obviously $|I| = |G|$. Now for each $A \in \boldsymbol{P}(I)$ set $G_A = \bigoplus\{C_i\colon i \in A\}$. Since $|G_A| = |A|$ for infinite A and $G_A \subseteq G_B$ iff $A \subseteq B$, the assignment $A \mapsto G_A$ defines the desired embedding f. In the general case it suffices to observe that the group G contains a subgroup $G_1 = \bigoplus\{C_i\colon i \in I\}$ where each C_i is a non-trivial cyclic subgroup and $|G_1| = |G|$ (for a proof see Theorem 1.1 in [4]). Then by the above argument there exists an embedding $f_1 : \boldsymbol{P}(|G_1|) \to \mathscr{S}(G_1)$ which sends $[G_1]^\sigma$ into $\mathscr{S}_\sigma(G_1)$ for all infinite σ. Since $\mathscr{S}_\sigma(G_1)$ is a subset of $\mathscr{S}_\sigma(G)$ and $|G_1| = |G|$ we are through.

To finish the proof it suffices to note that the inclusion $\mathscr{S}(G) \to \boldsymbol{P}(G)$ is an embedding sending each $\mathscr{S}_\sigma(G)$ into $[G]^\sigma$. □

6.2. Characters and Pontryagin duality

We show in the next subsection that Lemma 6.1 can be used for a more detailed study of the poset $\mathscr{B}(G)$. To this end we need a link between this poset and the poset $\mathscr{S}(G^*)$ for an appropriate group G^*. Such a group is available through the following description of precompact topologies on abelian groups based on Peter–Weyl's theorem.

For a topological group G denote by G^* the group of all *continuous characters* of G, i.e. continuous homomorphisms $\chi : G \to \boldsymbol{T}$ into the circle group $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ equipped with the usual euclidean topology. The group G^* is always equipped with the compact-open topology. For a discrete group G the group G^* coincides with the (compact) group $Hom(G, \boldsymbol{T})$ of all characters of G and its topology coincides with the topology of pointwise convergence of $Hom(G, \boldsymbol{T})$.

For a subgroup H of G^* denote by T_H the initial topology of the family of homomorphisms $\{\chi\colon G \to \boldsymbol{T} : \chi \in H\}$. Then for $K = \bigcap\{\ker\chi\colon \chi \in H\}$ the quotient group $(G, T_H)/K$ is topologically isomorphic to a subgroup of $\boldsymbol{T}^H$. In fact, it suffices to factorize the diagonal homomorphism $\varphi_H : G \to \boldsymbol{T}^H$ produced by the family H, so

T_H is precompact. Clearly $\ker \varphi_H = 0$ iff H *separates the points of* G, i.e. for each $x \in G\backslash\{0\}$ there exists $h \in H$ such that $h(x) \neq 0$.

On the other hand, for any group topology T on G one can consider the subgroup $H = (G,T)^*$ of G_{d}^*, where G_{d} denotes the group G equipped with the discrete topology. In this way we obtain a correspondence between the complete lattice $\tilde{\mathscr{B}}(G)$ and the complete lattice $\mathscr{S}(G^*)$ for a discrete abelian group G:

Theorem 6.2. *Let G be an infinite abelian group. Then:*

(a) *the correspondence* $H \mapsto T_H$ $(H \in \mathscr{S}(G_{\mathrm{d}}^*))$ *is a poset isomorphism between* $\mathscr{S}(G_{\mathrm{d}}^*)$ *and* $\tilde{\mathscr{B}}(G)$ *with inverse* $T \mapsto (G,T)^*$ $(T \in \tilde{\mathscr{B}}(G))$;

(b) $w(G,T_H) = |H|$, *i.e. the correspondence* (a) *sends* $\mathscr{S}_\sigma(G_{\mathrm{d}}^*)$ *onto* $\tilde{\mathscr{B}}_\sigma(G)$;

(c) *for a subgroup* H *of* G_{d}^* *the following are equivalent*:

(c_1) T_H *is Hausdorff*;

(c_2) H *separates the points of* G;

(c_3) H *is dense in* G_{d}^*.

A proof of this theorem can be found in [13] (see also [18, Theorem 2.2.3]). Point (c) was essentially checked above, since the homomorphism $\varphi_H : G \to \boldsymbol{T}^H$ is injective iff T_H is Hausdorff.

Denote by $\mathscr{S}^{\mathrm{d}}(G_{\mathrm{d}}^*)$ the family of subgroups H satisfying the equivalent conditions from point (c) of Theorem 6.2 and set $\mathscr{S}_\sigma^{\mathrm{d}}(G_{\mathrm{d}}^*) = \mathscr{S}_\sigma(G_{\mathrm{d}}^*) \cap \mathscr{S}^{\mathrm{d}}(G_{\mathrm{d}}^*)$ Note that the subset $\mathscr{S}^{\mathrm{d}}(G_{\mathrm{d}}^*)$ of $\mathscr{S}(G_{\mathrm{d}}^*)$ is upward-closed.

In the next subsection, we prove that $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^G]^\sigma$ whenever $\mathscr{B}_\sigma(G) \neq \emptyset$ making use of the following immediate corollary of Theorem 6.2 which we give separately for reader's convenience.

Corollary 6.3. *Let G be an infinite abelian group and let σ be an infinite cardinal. Then* $\mathscr{B}_\sigma(G) \cong \mathscr{S}_\sigma^{\mathrm{d}}(G_{\mathrm{d}}^*)$ *and* $\mathscr{B}(G) \cong \mathscr{S}^{\mathrm{d}}(G_{\mathrm{d}}^*)$.

6.3. The poset $\mathscr{B}_\sigma(G)$ in the abelian case

The next fact is usually attributed to Kakutani [23].

Lemma 6.4. $|G_{\mathrm{d}}^*| = 2^{|G|}$ *for every infinite abelian group.*

Now we are in a position to describe the structure of $\mathscr{B}_\sigma(G)$ up to quasi-isomorphism. First we see which σ should be taken into consideration. In the next two lemmas, we recall some folklore facts (see [4]), for reader's convenience we give a proof here.

Lemma 6.5. *Let G be an infinite group and let σ be an infinite cardinal such that* $\mathscr{B}_\sigma(G) \neq \emptyset$. *Then*

$$\log|G| \leq \sigma \leq 2^{|G|}. \tag{5}$$

Proof. We shall prove (5) for an arbitrary topological space X. Let $\boldsymbol{B}$ be a base of X with $|\boldsymbol{B}| = \sigma$. Then $\boldsymbol{B} \subseteq \boldsymbol{P}(X)$ so that the second inequality is obvious. In case X is Hausdorff the assignment $x \mapsto \{B \in \boldsymbol{B}: x \in B\}$ defines an injection $X \to \boldsymbol{P}(\boldsymbol{B})$. This proves the first inequality in (5). □

In view of Corollary 6.3 the above lemma says that also $\mathscr{S}^{\mathrm{d}}_{\sigma}(G^*_{\mathrm{d}}) \neq \emptyset$ yields (5). In order to prove that the lower bound in (5) is attained for G abelian we need also the following lemma.

For an abelian group G we denote by $r_0(G)$ the free-rank of G, by $r_p(G)$ the p-rank of G for a prime number p and set $r(G) = \sup\{r_\alpha(G): \alpha \in \{0\} \cup P\}$ where P is the set of the primes [21].

Lemma 6.6. *Let G be an infinite abelian group. There is a dense subgroup H_0 of G^*_{d} with $|H_0| = \log|G|$.*

Proof. Let $\sigma_0 = \log|G|$. Clearly, $r_0(\boldsymbol{T}^{\sigma_0}) = 2^{\sigma_0} \geq |G|$, and for each prime p, $r_p(\boldsymbol{T}^{\sigma_0}) = 2^{\sigma_0} \geq |G|$. Since $\boldsymbol{T}^{\sigma_0}$ is divisible this implies that there exists a monomorphism i: $G \to \boldsymbol{T}^{\sigma_0}$. Let T_0 be the Hausdorff precompact topology on G induced by i. Then $w(G, T_0) \leq \sigma_0$, since $w(\boldsymbol{T}^{\sigma_0}) = \sigma_0$. On the other hand, by (5) $w(G, T_0) \geq \sigma_0$. Hence $w(G, T_0) = \sigma_0$. Let $H_0 = (G, T_0)^*$. Then by Theorem 6.2 $|H_0| = \sigma_0$ and H_0 is dense. □

For a poset P and $a \in P$ we denote by $\uparrow a$ the upward closed subset $\{x \in P : x \geq a\}$ of the poset P. Hence for an abelian group G and $N \in \mathscr{S}(G^*_{\mathrm{d}})$ $\uparrow N$ is the family of those subgroups of G^*_{d} which contain N.

Lemma 6.7. *If G is infinite and $N \in \mathscr{S}^{\mathrm{d}}(G^*_{\mathrm{d}})$ with $|N| < 2^{|G|}$, then there exists an embedding of $\boldsymbol{P}(2^G)$ into $\mathscr{S}^{\mathrm{d}}(G^*_{\mathrm{d}}) \cap (\uparrow N)$ which sends $[2^G]^\sigma$ into $\mathscr{S}^{\mathrm{d}}_\sigma(G^*_{\mathrm{d}})$ for each $\sigma \geq |N|$.*

Proof. By Lemma 6.4 $|G^*_{\mathrm{d}}| = 2^{|G|}$; hence $|G^*_{\mathrm{d}}/N| = |G^*_{\mathrm{d}}| = 2^{|G|} > \omega$. Now by Lemma 6.1 there exists an embedding $j: \boldsymbol{P}(2^G) \to \mathscr{S}(G^*_{\mathrm{d}}/N)$ which sends $[2^G]^\sigma$ into $\mathscr{S}_\sigma(G^*_{\mathrm{d}}/N)$ for each infinite cardinal σ. Let $\varphi: G^*_{\mathrm{d}} \to G^*_{\mathrm{d}}/N$ be the canonical homomorphism. Now to each subgroup $H \in \mathscr{S}(G^*_{\mathrm{d}}/N)$ the subgroup $H' = \varphi^{-1}(H)$ is a subgroup of G^*_{d} containing N. Since the topology $T_0 = T_N$ is Hausdorff,

$$N \in \mathscr{S}^{\mathrm{d}}(G^*_{\mathrm{d}}). \tag{6}$$

Thus also $H' \in \mathscr{S}^{\mathrm{d}}(G^*_{\mathrm{d}})$ for each subgroup $H \in \mathscr{S}(G^*_{\mathrm{d}}/N)$. Hence the correspondence $H \mapsto H' = \varphi^{-1}(H)$ defines an embedding $i : \mathscr{S}(G^*_{\mathrm{d}}/N) \to (\uparrow N) \cap \mathscr{S}^{\mathrm{d}}(G^*_{\mathrm{d}})$. Moreover, for each $\sigma \geq |N|$ and $H \in \mathscr{S}_\sigma(G^*_{\mathrm{d}}/N)$ we have $|H'| = |H| = \sigma$ since $|\ker \varphi| = |N| \leq \sigma$. Now the restriction of the composition $i \circ j : \boldsymbol{P}(2^G) \to (\uparrow N) \cap \mathscr{S}^{\mathrm{d}}(G^*_{\mathrm{d}})$ sends $[2^G]^\sigma$ into $\mathscr{S}^{\mathrm{d}}_\sigma(G^*_{\mathrm{d}}) \cap (\uparrow N)$, hence $i \circ j$ is the desired embedding. □

The previous two results yield:

Theorem 6.8. *Let G be an infinite abelian group and let σ be an infinite cardinal satisfying* (5). *Then there exists an embedding $f : \boldsymbol{P}(2^G) \to \mathscr{S}^{\mathrm{d}}(G_{\mathrm{d}}^*)$ which sends $[2^G]^\sigma$ into $\mathscr{S}_\sigma^{\mathrm{d}}(G_{\mathrm{d}}^*)$. Consequently $\mathscr{S}^{\mathrm{d}}(G_{\mathrm{d}}^*) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^G)$ and $\mathscr{S}_\sigma^{\mathrm{d}}(G_{\mathrm{d}}^*) \overset{\text{q.i.}}{\cong} [2^G]^\sigma$.*

Now we can finally describe the structure of the poset of precompact group topologies of weight σ up to quasi-isomorphism. First we give a more precise (local) form corresponding to the more precise (local) result given Lemma 6.7. We need this stronger form in the proof of the non-abelian version of Theorem 6.8 in the next subsection.

Corollary 6.9. *Let G be an infinite abelian group and let $T \in \mathscr{B}_{\sigma_0}(G)$ for some $\sigma_0 < 2^{|G|}$. Then there exists an embedding of $\boldsymbol{P}(2^G)$ into $(\uparrow T) \cap \mathscr{B}(G)$ which sends $[2^G]^\sigma$ into $(\uparrow T) \cap \mathscr{B}_\sigma(G)$ (so that $[2^G]^\sigma \overset{\text{q.i.}}{\cong} (\uparrow T) \cap \mathscr{B}_\sigma(G)$) for each σ satisfying $\sigma_0 \le \sigma \le 2^{|G|}$.*

Theorem 6.10. *Let G be an infinite abelian group and let σ be an infinite cardinal satisfying* (5). *Then $\mathscr{B}(G) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^G)$ and $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^G]^\sigma$.*

Proof. Apply the above theorem and Corollary 6.3. □

In particular, we obtain the following converse of Lemma 6.5:

Corollary 6.11 (Berhanu et al. [4]). *Let G be an infinite abelian group and let σ be an infinite cardinal. Then $\mathscr{B}_\sigma(G) \neq \emptyset$ iff* (5) *holds.*

7. The poset $\mathscr{B}_\sigma(G)$ in the non-abelian case

We shall see here (Corollary 7.9) that Theorem 6.10 can be extended for a group G satisfying $2^{|G|} = 2^{|G/G'|}$ (residually finite relatively free groups satisfy the even stronger condition $|G| = |G/G'|$, for more details see Corollary 7.11 below or [15]).

In the non-abelian case the role of the continuous characters $G \to \boldsymbol{T}$ of a topological group G is played by finite-dimensional irreducible unitary representation of G, i.e. continuous homomorphism $h : G \to U(n)$ into the group $U(n)$ of unitary $n \times n$ matrices such that $h(G)$ acts transitively on $\boldsymbol{C}^n$. Now the set $\Sigma(G)$ of all such representations of G has no natural group structure even if G is compact. Nevertheless, when G is precompact, $\Sigma(G)$ still determines uniquely the original precompact topology T of G as the weak topology of all $h \in \Sigma(G)$ and $w(G,T) = |\Sigma(G)|$ as in the abelian case (see Theorem 7.1).

In the case when G is discrete, one can define a correspondence between $\tilde{\mathscr{B}}(G)$ and $\boldsymbol{P}(\Sigma(G))$ by putting $\varphi(T) = \Sigma(G,T)$ for every $T \in \tilde{\mathscr{B}}(G)$. In the opposite direction, for every subset Σ of $\Sigma(G)$ the weak topology $T(\Sigma)$ of all $h \in \Sigma$ is precompact, hence gives a map $\psi : \boldsymbol{P}(\Sigma(G)) \rightarrow \tilde{\mathscr{B}}(G)$ defined by $\psi(\Sigma) = T(\Sigma)$. By what we said above it follows that φ is monotone and $\psi(\varphi(T)) = T$ for every $T \in \tilde{\mathscr{B}}(G)$. Consequently, φ is an embedding.

Theorem 7.1. *Let G be an infinite maximally almost periodic group. Then*:
(1) *the embedding* $\varphi : \tilde{\mathscr{B}}(G) \hookrightarrow \boldsymbol{P}(\Sigma(G))$ *sends each* $\tilde{\mathscr{B}}_\sigma(G)$ *into* $[\Sigma(G)]^\sigma$;
(2) $\psi : \boldsymbol{P}(\Sigma(G)) \rightarrow \tilde{\mathscr{B}}(G)$ *satisfies* $\varphi(\psi(\Sigma)) \supseteq \Sigma$ *for every* $\Sigma \subseteq \Sigma(G)$ *and sends each* $[\Sigma(G)]^\sigma$ *into* $\tilde{\mathscr{B}}_\sigma(G)$.

Proof. (1) Follows easily from the well-known basic properties of Tanaka–Krein duality applied to the Bohr compactification of G [24, (28.2), (28.9), (28.10)]. More details can be found in [32, Theorem 2.3].

(2) Although this is a direct consequence of Part (1) we give a proof in full detail for reader's convenience. Let $\Sigma \subseteq \Sigma(G)$. Since every $h \in \Sigma$ is continuous, we immediately get $\Sigma \subseteq \varphi(\psi(\Sigma))$. To prove the last assertion note that obviously $w(G, T(\Sigma)) \leq |\Sigma|$ by the definition of $T(\Sigma)$. This fact, along with Part (1) and $\Sigma \subseteq \varphi(\psi(\Sigma)) = \Sigma(G, T(\Sigma))$, gives

$$|\Sigma| \geq w(G, T(\Sigma)) = |\Sigma(G, T(\Sigma))| \geq |\Sigma|.$$

Therefore all these cardinals coincide. □

We begin with the first important step: in the non-abelian case the interval (5) in Lemma 6.5 should be replaced by a smaller one defined as follows.

Consider the minimal weight of a Hausdorff precompact topology on a given maximally almost periodic group G defined obviously by $\gamma(G) = \min\{\kappa \mid \mathscr{B}_\kappa(G) \neq \emptyset\}$. In analogy with the minimal weight $\gamma(G)$ one can introduce also the upper bound $\Gamma(G) = \sup\{\kappa \mid \mathscr{B}_\kappa(G) \neq \emptyset\}$. Note that $\Gamma(G)$ is attained since $\mathscr{B}(G)$ has a top element (being a complete upper semi-lattice). By Lemma 6.5 $\log|G| \leq \gamma(G) \leq \Gamma(G) \leq 2^{|G|}$, so that $\mathscr{B}_\sigma(G) \neq \emptyset$ now will imply

$$\log|G| \leq \gamma(G) \leq \sigma \leq \Gamma(G) \leq 2^{|G|}.$$

For relatively free groups or abelian groups $\gamma(G) = \log|G|$ and $\Gamma(G) = 2^{|G|}$ (Corollary 6.11 and [17] resp.). Now we give an example with $G = G'$ and $\log|G| = \gamma(G) = \Gamma(G)$.

Remark 7.2. In general, $\mathscr{B}_\sigma(G)$ may be empty even for all $\log|G| < \sigma \leq 2^{|G|}$, while $\mathscr{B}_{\log|G|}(G) \neq \emptyset$. In fact, it follows from a classical result of van der Waerden that $G = SO_3(\boldsymbol{R})$ has a unique precompact topology, namely the usual compact metrizable one (see [37]). Now $\sigma = \omega = \log|G| = \gamma(G) = \Gamma(G)$ and $G = G'$, so that $|G/G'| = 1$.

In the following lemma, we give an easy upper bound of the minimal weight $\gamma(G)$ and the natural substitute of Corollary 6.11 in the non-abelian case.

Lemma 7.3. *Let G be an infinite maximally almost periodic group. Then*:

(1) $\gamma(G) \leq |G|$ *and* $\mathscr{B}_\sigma(G) \neq \emptyset$ *precisely for* $\gamma(G) \leq \sigma \leq \Gamma(G)$;

(2) $\gamma(H) \leq \gamma(G)$ *for any subgroup H of G; if H is a direct summand of G, then also* $\Gamma(H) \leq \Gamma(G)$;

(3) *if* $G = G_1 \oplus G_2$, *then* $\gamma(G) = \gamma(G_1)\gamma(G_2)$ *and* $\Gamma(G) = \Gamma(G_1)\Gamma(G_2)$.

Proof. (1) The inequality $\gamma(G) \leq |G|$ is well known (see [32, Lemma 2.9]). Since $\mathscr{B}_{\Gamma(G)}(G) \neq \emptyset$, it suffices to show that if $\sigma \leq \kappa$, $\mathscr{B}_\sigma(G) \neq \emptyset$ and $\mathscr{B}_\kappa(G) \neq \emptyset$, then also $\mathscr{B}_\lambda(G) \neq \emptyset$ for all λ satisfying $\sigma \leq \lambda \leq \kappa$. Let now $T \in \mathscr{B}_\sigma(G)$ and $T' \in \mathscr{B}_\kappa(G)$. It is not restrictive to assume that $T \leq T'$ (otherwise take the join of T and T' instead of T'). Let $\Sigma = \Sigma(G,T)$ and $\Sigma' = \Sigma(G,T')$ be the sets of irreducible finite-dimensional unitary representations corresponding to T and T', respectively. Then $|\Sigma| = \sigma$ and $|\Sigma'| = \kappa$ by Theorem 7.1. Now for any Σ'' between Σ and Σ' with $|\Sigma''| = \lambda$ the precompact group topology $T'' = T(\Sigma)$ on G determined by Σ'' has weight λ by Theorem 7.1.

(2) The first inequality is trivial. To prove the second inequality assume $G = H \oplus K$ and consider any topology $T_1 \in \mathscr{B}_{\Gamma(H)}(H)$ and an arbitrary $T_2 \in \mathscr{B}(K)$. Then the Tychonoff topology of $(H,T_1) \oplus (K,T_2)$ has weight $\Gamma(H)w(K,T_2) \geq \Gamma(H)$. Hence $\Gamma(H) \leq \Gamma(G)$.

(3) Take any topology $T_1 \in \mathscr{B}_{\gamma(G_1)}(G_1)$ and analogously $T_2 \in \mathscr{B}_{\gamma(G_2)}(G_2)$. Then the Tychonoff topology of $(G_1,T_1) \oplus (G_2,T_2)$ has weight $\gamma(G_1)\gamma(G_2)$. Hence $\gamma(G_1)\gamma(G_2) \geq \gamma(G)$. To prove the inequality $\gamma(G_1)\gamma(G_2) \leq \gamma(G)$ apply item (2) to the subgroups G_1 and G_2 of G.

The inequality $\Gamma(G_1)\Gamma(G_2) \leq \Gamma(G)$ is proved in the same way with recourse to the second part of item (2). To prove the opposite inequality fix $T \in \mathscr{B}_{\Gamma(G)}(G)$ and denote by T_1 and T_2 the topologies induced by T on G_1 and G_2, respectively. Since they are precompact, we have $w(G,T_1) \leq \Gamma(G_1)$ and $w(G,T_2) \leq \Gamma(G_2)$. Let K, K_1 and K_2 denote the compact completions of G, G_1 and G_2, respectively. Then $w(K) = \Gamma(G)$, $w(K_1) \leq \Gamma(G_1)$ and $w(K_2) \leq \Gamma(G_2)$. The inclusion $G_1 \hookrightarrow K$ admits a unique extension $\imath_1 : K_1 \to K$ which is a continuous homomorphism (actually, a topological embedding). Analogously $\imath_2 : K_2 \to K$ is defined. Since the elements of G_1 and G_2 commute, it follows by continuity that $\imath_1(x)$ and $\imath_2(y)$ commute for every $x \in G_1$ and every $y \in G_2$. Hence the map $f : K_1 \times K_2 \to K$ defined by $f(x,y) = \imath_1(x)\imath_2(y)$ is a continuous homomorphism. Moreover, f is surjective since the image of f is compact and contains the dense subgroup G of K. Since the weight may only decrease under continuous surjective homomorphisms of compact groups (see [24], or [32, Theorem 2.2(a) and (c)]), we conclude $\Gamma(G) = w(K) \leq w(K_1 \times K_2) \leq \Gamma(G_1)\Gamma(G_2)$. □

Remark 7.4. (1) Example 7.13 below shows that the estimate of item (1) of Lemma 7.3 cannot be improved. It shows actually that for a maximally almost periodic group G the cardinal invariant $\gamma(G)$ may take all possible values between $\log |G|$ and $|G|$.

(2) The group G of Remark 7.2 shows that in item (2) of the above lemma one can have actually $\Gamma(H) = 2^{\Gamma(G)} > \Gamma(G)$ (take as H any infinite cyclic subgroup of G).

(3) The proof of the equality $\Gamma(G)=\Gamma(G_1)\Gamma(G_2)$ shows a more precise property. Namely, *the Bohr compactification commutes with finite cartesian products.* A categorical proof of this fact in the case of arbitrary cartesian products can be found in [25].

Combining Lemma 7.3 with Theorem 7.1 we get:

Corollary 7.5. *Let G be an infinite maximally almost periodic group. Then there exists an embedding $\tilde{\mathscr{B}}(G) \to \mathbf{P}(\Gamma(G))$ which sends each $\mathscr{B}_\sigma(G)$ into $[\Gamma(G)]^\sigma$.*

Definition 7.6. A maximally almost periodic infinite group G is *weakly abelian* if $\gamma(G) < 2^{|G/G'|}$.

Note that any maximally almost periodic group G satisfying $2^{|G/G'|}=2^{|G|}$ (or, a fortiori, $|G/G'|=|G|$) is weakly abelian. In particular, maximally almost periodic relatively free groups and abelian groups are weakly abelian. An example of a non-weakly-abelian group G with $\log|G| = \gamma(G) = \Gamma(G)$ was given in Remark 7.2 above.

Now we show that a weakly abelian group G satisfies $2^{|G/G'|} \leq \Gamma(G)$ and admits embeddings $[2^{|G/G'|}]^\sigma \hookrightarrow \mathscr{B}_\sigma(G) \hookrightarrow [\Gamma(G)]^\sigma$.

Theorem 7.7. *A weakly abelian group G satisfies $2^{|G/G'|} \leq \Gamma(G)$. Moreover, for any infinite cardinal σ satisfying*

$$\gamma(G) \leq \sigma \leq 2^{|G/G'|}, \tag{7}$$

there exists an embedding $[2^{|G/G'|}]^\sigma \hookrightarrow \mathscr{B}_\sigma(G)$.

Proof. Fix an infinite cardinal σ satisfying (7) and a Hausdorff precompact topology T of G of weight $\sigma_0 = \gamma(G) < 2^{|G/G'|}$. The existence of such a topology follows from the definition of $\gamma(G)$ and our hypothesis.

Following the idea of the proof of Theorem 2.13 of [32], consider the canonical homomorphism $f: G \to G/G'$ and let $\overline{T}$ be the quotient topology of G/G'. For $\mu \in \mathscr{B}(G/G')$ denote by $\tilde{\mu}$ the initial topology on G with respect to μ and f. Then the assignment $\mu \mapsto \tilde{\mu} \vee T$ defines a monotone map $p : \mathscr{B}(G/G') \to \mathscr{B}(G)$ which sends $\mathscr{B}_\sigma(G/G')$ into $\mathscr{B}_\sigma(G)$ since $w(G, \tilde{\mu} \vee T) = \sigma\sigma_0 = \sigma$. By Corollary 6.9 there exists an embedding q of $[2^{G/G'}]^\sigma$ into the subset M of $\mathscr{B}_\sigma(G/G')$ consisting of topologies finer than $\overline{T}$. Our next aim will be to show that the restriction of p to M is injective. To this end it suffices to show that for any $\mu \in M$ the quotient topology of $p(\mu)$ on G/G' is precisely μ. Actually, for any $\mu \in \mathscr{B}(G/G')$ the quotient topology μ' of $p(\mu)$ on G/G' is precisely $\mu \vee \overline{T}$. Indeed, a typical $p(\mu)$-neighbourhood of the neutral element e_G of G is of the form $f^{-1}(U) \cap W$, where W is a T-neighbourhood of e_G and U is a μ-neighbourhood of $e_{G/G'}$. Since obviously $f(f^{-1}(U) \cap W) = U \cap f(W)$, we can conclude that the topology μ' is contained in $\mu \vee \overline{T}$. On the other hand, the continuity of $f : (G, p(\mu)) \to (G/G', \mu \vee \overline{T})$ and the categorical property of the quotient topology yield that μ' is finer than $\mu \vee \overline{T}$. Thus $\mu' = \mu \vee \overline{T}$. To get an embedding of $[2^{G/G'}]^\sigma$ into $\mathscr{B}_\sigma(G)$ take the composition of p and q. With $\sigma = 2^{|G/G'|}$ this proves the inequality $2^{|G/G'|} \leq \Gamma(G)$. □

For a weakly abelian group G with $\Gamma(G) = 2^{|G/G'|}$ we get now a quasi-isomorphism for the full interval $\gamma(G) \leq \sigma \leq \Gamma(G)$.

Corollary 7.8. *Let G be a weakly abelian group with* $\Gamma(G) = 2^{|G/G'|}$.

(1) $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [\Gamma(G)]^\sigma$ *for all* σ *satisfying* $\gamma(G) \leq \sigma \leq \Gamma(G)$.

(2) *A necessary and sufficient condition to have* $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^G]^\sigma$ *for all* σ *satisfying* $\gamma(G) \leq \sigma \leq \Gamma(G)$, *is that* $\Gamma(G) = 2^{|G|}$.

Proof. For part (1) apply Theorem 7.7 with $\Gamma(G) = 2^{|G/G'|}$ and Corollary 7.5. For part (2) observe that by Lemma 2.4, if $[2^G]^\sigma \hookrightarrow [\Gamma(G)]^\sigma$ then $2^G \leq \Gamma(G)$. □

If $2^{|G|} = 2^{|G/G'|}$ we obtain a similar result as in the abelian case except that the lower bound of the interval is $\gamma(G)$ instead of $\log |G|$.

Corollary 7.9. *Let G be a maximally almost periodic group with* $2^{|G|} = 2^{|G/G'|}$. *Then* $\mathscr{B}(G) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^G)$ *and* $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^G]^\sigma$ *for all* σ *satisfying* $\gamma(G) \leq \sigma \leq 2^{|G|}$.

Proof. Note that $2^{|G|} = 2^{|G/G'|}$ implies that G is weakly abelian with $2^{|G|} = \Gamma(G) = 2^{|G/G'|}$ so that now Corollary 7.8 applies. □

We do not know whether $[\Gamma(G)]^\sigma \hookrightarrow \mathscr{B}_\sigma(G)$ when $\Gamma(G) > 2^{|G/G'|} > \gamma(G)$ (compare with Corollary 7.8).

Corollary 7.10. *Let G be an infinite maximally almost periodic group with* $2^{|G|} = 2^{|G/G'|}$ *and* σ *be a cardinal satisfying* $\gamma(G) \leq \sigma \leq 2^{|G|}$. *Then* $\text{height}(\mathscr{B}_\sigma(G)) = \min\{2^{|G|}, \sigma^+\}$, $\text{depth}(\mathscr{B}_\sigma(G)) = \sigma$, $\text{width}(\mathscr{B}_\sigma(G)) = 2^{\sigma|G|}$. *All these values are attained.*

The particular case $|G| = |G/G'|$ was proved in [32, Theorem 2.13].

In the following corollary we show that relatively free groups behave as abelian groups from the point of view of the semilattice of precompact group topologies. A group is *residually finite* if the intersection of the normal subgroups of finite index is trivial. Residually finite groups are obviously maximally almost periodic, just take the pro-finite topology (generated by the normal subgroups of finite index).

Corollary 7.11. *Let G be a relatively free group. Then G is maximally almost periodic iff G is residually finite. In such a case* $\log |G| \leq \sigma \leq 2^{|G|}$ *is equivalent to* $\mathscr{B}_\sigma(G) \neq \emptyset$.

Proof. The first assertion was proved in [15]. Since $|G| = |G/G'|$ obviously holds in this case, we have $\Gamma(G) = 2^{|G|}$. Finally, $\gamma(G) = \log |G|$ by Lemma 2.2 in [17], so that Lemma 7.3 applies. □

Remark 7.12. Residual finiteness of the maximally almost periodic relatively free groups is deeply related to Burnside problems (see [15] or [19] for more details).

The next example shows that $\gamma(G)$ can assume all the possible values between $\log |G|$ and $|G|$ both for weakly abelian groups and for groups satisfying $H = H'$.

Example 7.13. For each uncountable cardinal α which is not a strong limit (i.e., $\alpha \neq \log \alpha$) and for each cardinal β satisfying $\log \alpha < \beta \leq \alpha$ there exists a group $G = G_{\alpha,\beta}$ such that $|G| = |G/G'| = \alpha$, and $\gamma(G) = \beta$.

Proof. In view of item (3) of Lemma 7.3, to prove our lemma it suffices to find groups N and Z such that $N = N'$, $\gamma(N) = |N| = \beta$, $|Z| = |Z/Z'| = \alpha$ and $\gamma(Z) = \log \alpha$. Then $G_{\alpha,\beta} = N \oplus Z$ will satisfy the required conditions since $\log \alpha \leq \beta \leq \alpha$.

Let A be the alternate group of degree 5 and let $N = \bigoplus_\beta A$ be the direct sum of β copies of A. Then $N = N'$ and $\gamma(N) \leq \beta = |N|$ by Lemma 7.3. Let $h : N \to U(n)$ be a finite-dimensional unitary representation of N. Since N, as well as A, is of exponent 60, also the image $h(N)$, as well as its closure $\overline{h(N)}$ in the group of $U(n)$, is of exponent 60. Every compact group of finite exponent is totally disconnected, hence $\overline{h(N)}$ is totally disconnected. On the other hand, $\overline{h(N)}$ is a compact Lie group as a closed subgroup of $U(n)$. Since totally disconnected compact Lie groups are finite, we conclude that $\overline{h(N)}$ is finite. The kernel K of h is a normal subgroup of N, hence by the simplicity of A also K is a direct sum of copies of A. Since N/K is finite it follows that K contains almost all coordinate subgroups isomorphic to A. Hence every finite-dimensional unitary representation of N is trivial on almost all summands of N. Therefore, to separate the points of N one needs at least β such representations. This proves that $\mathscr{B}_\kappa(N) = \emptyset$ for each $\kappa < \beta$, i.e. $\gamma(N) \geq \beta$.

Now let $Z = \bigoplus_\alpha \boldsymbol{Z}$, i.e. the direct sum of α copies of $\boldsymbol{Z}$. Then $|Z| = |Z/Z'| = \alpha$ while $\gamma(Z) = \log \alpha$ by Corollary 6.11. □

Remark 7.14. The three cardinal invariants $\gamma(G)$, $|G|$ and $\Gamma(G)$ of a maximally almost periodic group G are subject to the relations (2) and $\gamma(G) \leq |G|$. In Remark 7.2 we saw a group G with $\gamma(G) = \Gamma(G) < |G|$. Now we compute $\gamma(G)$, $|G|$ and $\Gamma(G)$ for the group $G = G_{\alpha,\beta}$ defined as in the above example, i.e. $G = \bigoplus_\alpha \boldsymbol{Z} \oplus (\bigoplus_\beta A)$, but now without any relation between α and β. The argument in the above proof shows actually that the Tychonoff topology is the unique precompact topology of $\bigoplus_\beta A$, thus $|\mathscr{B}(\bigoplus_\beta A)| = 1$. In particular, $\gamma(\bigoplus_\beta A) = \Gamma(\bigoplus_\beta A) = |\bigoplus_\beta A| = \beta$. Since $\log \alpha = \gamma(\bigoplus_\alpha \boldsymbol{Z}) < \alpha = |\bigoplus_\alpha \boldsymbol{Z}| < \Gamma(\bigoplus_\alpha \boldsymbol{Z}) = 2^\alpha$, we get from Lemma 7.3(3)

$$\gamma(G) = \beta \log \alpha \leq |G| = \beta\alpha \leq \Gamma(G) = \beta 2^\alpha.$$

Therefore, $\Gamma(G) < 2^{|G|}$ iff $2^\alpha < 2^\beta$. In such a case we have two possibilities:

- $\alpha < \beta < 2^\alpha < 2^\beta$, then $\gamma(G) = |G| = \beta < \Gamma(G) = 2^\alpha < 2^{|G|}$;
- $\alpha < 2^\alpha \leq \beta < 2^\beta$, then $\beta = \gamma(G) = |G| = \Gamma(G) = 2^\alpha < 2^{|G|}$.

In case $2^\alpha \geq 2^\beta$ (i.e. $\Gamma(G) = 2^{|G|}$) we have $\Gamma(G) > |G|$. Now we have three cases:

- $\alpha \leq \beta < 2^\beta = 2^\alpha$, then $\gamma(G) = |G| = \beta < \Gamma(G) = 2^{|G|}$;
- $\beta < \alpha \leq 2^\beta \leq 2^\alpha$, then $\beta = \gamma(G) < \alpha = |G| < \Gamma(G) = 2^{|G|}$;
- $\beta < 2^\beta < \alpha < 2^\alpha$, then $\log \alpha = \gamma(G) \leq \alpha = |G| < \Gamma(G) = 2^{|G|}$.

The group $G = G_{\alpha,\beta}$ obviously has the additional constraint $|G| \leq \Gamma(G)$. Moreover, $\Gamma(G) < 2^{|G|}$ yields $\gamma(G) = |G|$. Hence we have no witness for the relations $\gamma(G) < \Gamma(G) < |G|$ and $\gamma(G) < |G| \leq \Gamma(G) < 2^{|G|}$.

8. Independence results in the poset of precompact group topologies

8.1. Bounded versus unbounded chains in $\mathscr{B}_\sigma(G)$

Theorem 8.1. *It is independent of ZFC whether $\mathscr{B}_{\aleph_1}(\mathbf{Z})$ has chains of cardinality greater than the cardinality of any bounded chain.*

Proof. Set for brevity $B = \mathscr{B}_{\aleph_1}(\mathbf{Z})$. We claim that the equality $Ded(B) = Ded^e(B)$ cannot be decided in ZFC. To this end we prove the following statement: ZFC+CH implies $Ded(B) = Ded^e(B)$, *while* ZFC+¬CH implies $Ded^e(B) = Ded^e(\aleph_1)$ and $Ded(B) = Ded(\aleph_1)$. In Mitchell's model of Theorem 5.1, with $\theta = \omega$ and $\sigma = \aleph_1$, one has $2^\omega = \aleph_2$ and $Ded(\aleph_1) < Ded^e(\aleph_1)$ (see item (2) of Theorem 5.6). Hence in that model $Ded(B) < Ded^e(B)$.

By Theorem 6.10 $B \overset{\text{q.i.}}{\cong} [2^\omega]^{\aleph_1}$, hence $Ded^e(B) = Ded^e([2^\omega]^{\aleph_1})$, while $Ded(B) = Ded([2^\omega]^{\aleph_1}) = Ded(\aleph_1)$. Assuming CH, i.e. $2^\omega = \aleph_1$, we have $Ded^e([\aleph_1]^{\aleph_1}) = Ded([\aleph_1]^{\aleph_1}) = Ded(\aleph_1)$. Whence $Ded^e(B) = Ded(B)$. Assuming ¬CH, i.e. $2^\omega \geq \aleph_2$ we have $Ded^e([2^\omega]^{\aleph_1}) = Ded^e(\aleph_1)$. □

The reader may take in the above theorem any free group G of cardinality $\leq 2^{\aleph_1}$, or any maximally almost periodic uncountable group G such that $|G| = |G/G'| \leq 2^{\aleph_1}$.

In the next two subsections we discuss the natural question as to when our results of the form $\mathscr{B}(G) \overset{\text{q.i.}}{\cong} \mathbf{P}(2^G)$ and $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^G]^\sigma$ can be extended to the posets $\mathscr{H}(G)$ and $\mathscr{H}_\sigma(G)$ of Hausdorff group topologies on G. We begin with the bigger poset $\mathscr{H}(G)$ by showing that such a quasi-isomorphism is available in ZFC. In the last subsection we show that $\mathscr{B}_\sigma(G) \overset{\text{q.i.}}{\cong} \mathscr{H}_\sigma(G)$ cannot be proved in ZFC.

8.2. ZFC proves that $\mathscr{B}(G)$ and $\mathscr{H}(G)$ are quasi-isomorphic

Our results from Section 7 immediately imply

Theorem 8.2. *Let G be an infinite maximally almost periodic group with $2^{|G|} = 2^{|G/G'|}$. Then $\mathscr{H}(G) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^{|G|}) \overset{\text{q.i.}}{\cong} \mathscr{B}(G)$.*

Proof. By Corollary 7.9 $\mathscr{B}(G) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^{|G|})$. In view of the inclusion $\mathscr{B}(G) \hookrightarrow \mathscr{H}(G)$ it suffices to note that obviously $\mathscr{H}(G) \hookrightarrow \boldsymbol{P}(2^{|G|})$. □

Let us note that the quasi-isomorphism $\mathscr{H}(G) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^{|G|})$ was established in the above proof making use of the full strength of Section 7. We give below an alternative proof for abelian groups G satisfying $r(G) \geq \omega$ without any recourse to characters (i.e. compact representations). It should be noted that an abelian group G with $r(G) < \omega$ is of very special type, namely: G is isomorphic to a subgroup of some of the groups $\boldsymbol{Q}^n \oplus \boldsymbol{Z}(p_1^\infty) \oplus \boldsymbol{Z}(p_2^\infty) \oplus \cdots \oplus \boldsymbol{Z}(p_n^\infty)$, where $n \in \boldsymbol{N}$, $p_1, p_2, \ldots, p_n$ are prime numbers and $\boldsymbol{Z}(p_i^\infty)$ is the Prüfer group ([4]). Hence an infinite abelian group G with $r(G) < \omega$ is countable and contains either a copy of $\boldsymbol{Z}$ or of some of the Prüfer's groups $\boldsymbol{Z}(p_i^\infty)$. In both cases one can easily establish the quasi isomorphism by noting that $\mathscr{H}(\boldsymbol{Z}) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^\omega)$ and $\mathscr{H}(\boldsymbol{Z}(p_i^\infty)) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^\omega)$.

Let G be an abelian group with infinite $r(G)$. Then G contains a subgroup of the form $G_1 = \bigoplus_{i \in A} C_i$ where $|A| = |G|$ and each C_i is a non-trivial cyclic group [4, Theorem 1.1]. Since $\mathscr{H}(G_1) \hookrightarrow \mathscr{H}(G)$, in order to prove the nontrivial part $\boldsymbol{P}(2^{|G|}) \hookrightarrow \mathscr{H}(G)$ of the quasi isomorphism $\mathscr{H}(G) \overset{\text{q.i.}}{\cong} \boldsymbol{P}(2^{|G|})$ we can assume, without loss of generality, that $G = G_1$ is a direct sum of cyclic subgroups C_i.

With $G = \bigoplus_{i \in A} C_i$ and $|A| = |G|$ as above let $\mathscr{I}(A)$ denote the poset of all ideals of the poset 2^A, containing the ideal $[A]^{<\omega}$, ordered by inclusion. We define first an embedding

$$\lambda : \mathscr{I}(A) \hookrightarrow \mathscr{H}(G). \tag{8}$$

For a subset $B \subseteq A$ set $U_A := \bigoplus_{i \in A \setminus B} C_i$. Note that the correspondence $B \mapsto U_B$ is order reversing (actually, an anti-isomorphic embedding of the poset 2^A into $\mathscr{S}(G)$), so that in particular

$$\bigcap U_{B_\alpha} = U_{\cup B_\alpha}. \tag{9}$$

Now for an ideal $I \in \mathscr{I}(A)$ let $\lambda(I)$ be the group topology on G having as a base of neighbourhoods of 0 the filterbase $\{U_B\}_{B \in I}$. By (9) $\lambda(I)$ is a Hausdorff topology, so that (8) is established.

To finish the proof it suffices to show that $\boldsymbol{P}(2^{|A|})$ embeds into $\mathscr{I}(A)$. Fix an independent family $\mathscr{F} = \{B_\kappa\}_{\kappa \in 2^A}$ of subsets of A (i.e. for $\kappa_1, \ldots, \kappa_n, \rho_1, \ldots, \rho_m \in 2^A$ pairwise distinct the intersection $B_{\kappa_1} \cap \cdots \cap B_{\kappa_n} \cap (A \setminus B_{\rho_1}) \cap \cdots \cap (A \setminus B_{\rho_m})$ is infinite; see [27] for the existence of such a family). Take distinct subsets $\mathscr{F}_1$ and $\mathscr{F}_2$ of $\mathscr{F}$, say $\mathscr{F}_1 \not\subseteq \mathscr{F}_2$. Let I_ν be the ideal of 2^A generated by $\mathscr{F}_\nu$ and the ideal $[A]^{<\omega}$ $(\nu = 1, 2)$. The independence of $\mathscr{F}$ easily entails $I_1 \not\subseteq I_2$. Thus we obtain the desired embedding $\boldsymbol{P}(2^{|A|}) \hookrightarrow \mathscr{I}(A)$.

8.3. When are $\mathscr{B}_\sigma(G)$ and $\mathscr{H}_\sigma(G)$ quasi-isomorphic?

Here we answer negatively to the following natural question which arises in view of Theorem 8.2:

Is it possible to establish $\mathscr{H}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^{|G|}]^\sigma$ for abelian groups G?

To construct our counterexample we make use of the inclusion modulo finite ($\subseteq^*$) preorder on $\boldsymbol{P}(\omega)$. To obtain a *poset* we have to take the quotient $\boldsymbol{P}(\omega)/\text{fin}$ with respect to the equivalence relation defined by the symmetric closure of the preorder $\subseteq^*$, i.e. $A \sim B$ iff the symmetric difference of A and B is finite.

In the sequel $\boldsymbol{Z}_2$ is the cyclic group of order 2.

Theorem 8.3. *There exists an embedding* $j : \boldsymbol{P}(\omega)/\text{fin} \hookrightarrow \mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)$.

Proof. For an infinite subset $B \subseteq \omega$ denote by (B) the ideal of 2^ω generated by B and $[\omega]^{<\omega}$, i.e. $(B) = \{C \in 2^\omega : C \subseteq^* B\}$. Note that $(B) = (B')$ iff $B \sim B'$, so that the correspondence $B \mapsto (B)$ defines an injection $\mu : \boldsymbol{P}(\omega)/\text{fin} \hookrightarrow \mathscr{I}(\omega)$. It is easy to see that μ is an embedding. Let j be the composite of μ with the embedding $\lambda : \mathscr{I}(\omega) \hookrightarrow \mathscr{H}(\bigoplus_\omega \boldsymbol{Z}_2)$ defined in Section 8.2. Note that the topology $\lambda((B))$ for $B \in 2^\omega$ has the subgroup $U_B = \bigoplus_{\omega \setminus B} \boldsymbol{Z}_2$ as an open subgroup and the induced topology of $\lambda((B))$ on U_B is the product topology. Hence $\lambda((B))$ is metrizable. This proves that j is the desired embedding $\boldsymbol{P}(\omega)/\text{fin} \hookrightarrow \mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)$. □

The height and the depth of $\boldsymbol{P}(\omega)/\text{fin}$ coincide since this poset is obviously "symmetric" (the involutive anti-isomorphism defined by the complement defines the symmetry under question). A *tower* in $\boldsymbol{P}(\omega)$ is an anti-well $\subseteq^*$-ordered subset $\mathscr{T} \subseteq \boldsymbol{P}(\omega)$ such that for every infinite $A \subseteq \omega$ there exists $B \in \mathscr{T}$ with $A \not\subseteq^* B$. Set $\boldsymbol{t} = \min\{|\mathscr{T}| : \mathscr{T} \text{ is a tower in } \boldsymbol{P}(\omega)\}$. It is known that $\omega_1 \le \boldsymbol{t} \le 2^\omega = 2^{<\boldsymbol{t}}$ [35, Theorem 3.1]. Moreover, $\boldsymbol{t} > \omega_1$ under the conjunction of the ω_1-Martin axiom and the negation of the Continuum Hypothesis ($MA_{\omega_1} + \neg CH$). While $\boldsymbol{t} = 2^\omega$ under Martin axiom (MA) [3, 35]. Obviously, $\text{depth}(\boldsymbol{P}(\omega)/\text{fin}) \ge \boldsymbol{t}$, so that Theorem 8.3 gives:

Corollary 8.4. $\text{height}(\mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)) \ge \boldsymbol{t}$ *and* $\text{depth}(\mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)) \ge \boldsymbol{t}$. *In particular,*
1. $\text{height}(\mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)) > \omega_1$ *and* $\text{depth}(\mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)) > \omega_1$ *under* $MA_{\omega_1} + \neg CH$;
2. $\text{depth}(\mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)) = 2^\omega$ *under MA.*

Proof. The inequalities $\text{height}(\mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)) \ge \boldsymbol{t}$ and $\text{depth}(\mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)) \ge \boldsymbol{t}$ follow directly from Theorem 8.3 and obviously imply item 1. To get the equality of item 2. one should note that $\text{depht}(\mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)) \le |\mathscr{H}_\omega(\bigoplus_\omega \boldsymbol{Z}_2)| \le 2^\omega$ and $\boldsymbol{t} = 2^\omega$ *MA*. □

Corollary 8.5. *For* $G = \bigoplus_\omega \boldsymbol{Z}_2$ *the question whether* $\text{height}(\mathscr{H}_\omega(G)) = \text{height}(\mathscr{B}_\omega(G))$ *cannot be answered in ZFC.*

Proof. Since $\mathscr{B}_\omega(G) \overset{\text{q.i.}}{\cong} [2^\omega]^\omega$ (by Theorem 6.10) we have $\text{height}(\mathscr{B}_\omega(G)) = \omega_1$. Hence, under $MA_{\omega_1} + \neg CH$ Corollary 8.4 gives $\text{height}(\mathscr{H}_\omega(G)) \ne \text{height}(\mathscr{B}_\omega(G))$. On the other hand, under CH height$(\mathscr{H}_\omega(G)) \leq |\mathscr{H}_\omega(G)| = 2^\omega = \omega_1 =$ height$(\mathscr{B}_\omega(G))$ (for the first equality it suffices to note that G has 2^ω (group) topologies of countable weight). Now the first part of Corollary 8.4 gives height$(\mathscr{H}_\omega(G)) =$ height$(\mathscr{B}_\omega(G))$ under CH. □

Theorem 8.6. *There exists no embedding* $\mathscr{H}_\omega(\bigoplus_\omega \mathbf{Z}_2) \hookrightarrow \mathscr{B}_\omega(\bigoplus_\omega \mathbf{Z}_2)$. *Under* $MA + \neg CH$ *there exists no embedding* $\mathscr{H}_\omega(\bigoplus_\omega \mathbf{Z}_2) \hookrightarrow \mathscr{B}_\sigma(\bigoplus_\omega \mathbf{Z}_2)$ *when* $\sigma < 2^\omega$.

Proof. According to Corollary 8.4 depth$(\mathscr{H}_\omega(\bigoplus_\omega \mathbf{Z}_2)) \geq \mathbf{t} > \omega =$ depth$(\mathscr{B}_\omega(\bigoplus_\omega \mathbf{Z}_2))$. This proves the first part of the theorem. To prove the second part assume $MA + \neg CH$ and take $\sigma < 2^\omega$. Then by Corollary 8.4 depth$(\mathscr{H}_\omega(\bigoplus_\omega \mathbf{Z}_2)) = 2^\omega > \sigma =$ depth$(\mathscr{B}_\sigma(\bigoplus_\omega \mathbf{Z}_2))$. □

The restraint $\sigma < 2^\omega$ is essential in the above theorem in view of the quasi-isomorphism $\mathscr{H}_{2^{|G|}}(G) \overset{\text{q.i.}}{\cong} \mathscr{B}_{2^{|G|}}(G)$ valid for every abelian group G. In fact, by Theorem 6.10 $\mathscr{H}_{2^{|G|}}(G) \overset{\text{q.i.}}{\cong} \mathbf{P}(2^{|G|}) \overset{\text{q.i.}}{\cong} \mathscr{B}(G)$, so that Theorem 8.2 yields $\mathscr{B}_{2^{|G|}}(G) \overset{\text{q.i.}}{\cong} \mathscr{H}(G)$; the rest is obvious in view of the inclusion $\mathscr{B}_{2^{|G|}}(G) \hookrightarrow \mathscr{H}_{2^{|G|}}(G)$.

In 8.4–8.7 the group $G = \bigoplus_\omega \mathbf{Z}_2$ can be replaced by any abelian group with $r(G) = \omega$.

Remark 8.7. For a *set* G let $\mathscr{T}(G)$ denote the poset of all topologies on G and for an infinite cardinal σ let $\mathscr{T}_\sigma(G)$ denote the poset of all topologies of weight σ on G.

(1) For a maximally almost periodic uncountable group G with $|G| = |G/G'| \leq 2^{\aleph_1}$ one can easily extend the quasi-isomorphism of Theorem 8.2 to $\mathscr{T}(G) \overset{\text{q.i.}}{\cong} \mathbf{P}(2^{|G|}) \overset{\text{q.i.}}{\cong} \mathscr{B}(G)$ – it suffices to note that (obviously) $\mathscr{T}(G) \hookrightarrow \mathbf{P}(2^{|G|})$.

(2) Note that the embedding $\mathscr{T}(G) \hookrightarrow P(2^{|G|})$ *does not* send $\mathscr{T}_\sigma(G)$ into $[2^{|G|}]^\sigma$. Of course, one can easily find an *injection* $i : \mathscr{T}_\sigma(G) \hookrightarrow [2^{|G|}]^\sigma$ by just choosing a base $\mathscr{B} = i(\tau)$ of cardinality σ for each $\tau \in \mathscr{T}_\sigma(G)$. This obviously yields $\Lambda^e(\mathscr{T}_\sigma(G)) \leq |\mathscr{T}_\sigma(G)| \leq 2^{|G|\sigma}$ (compare with Question 9.1(4).). It is clear from Theorem 8.6 that in general one cannot arrange in ZFC to have i also *monotone*, i.e. it is impossible to establish $\mathscr{T}_\sigma(G) \overset{\text{q.i.}}{\cong} [2^{|G|}]^\sigma$ (or $\mathscr{T}_\sigma(G) \overset{\text{q.i.}}{\cong} \mathscr{B}_\sigma(G)$).

9. Questions

We do not know the answer to the following questions regarding mainly the length of chains:

Question 9.1. (1) Is it consistent with ZFC to have $Ded(\aleph_1) < 2^{\aleph_1}$?

(2) Is there a maximally almost periodic group G with $\gamma(G) < \Gamma(G) < |G|$ or $\gamma(G) < |G| \leq \Gamma(G) < 2^{|G|}$?

(3) Is it possible to have a chain of strictly more than $2^{\aleph_0}$ Hausdorff group topologies on $\mathbf{R}$ of countable weight? More generally, is there a group G and an infinite cardinal σ such that $\mathscr{H}_\sigma(G)$ has chains of cardinality $> 2^\sigma$ (or $\geq Ded^e(\sigma)$). (We know that $\mathscr{B}_\sigma(G) \hookrightarrow [2^G]^\sigma$, hence $\mathscr{B}_\sigma(G)$ does not have chains of size $\geq Ded^e(\sigma)$.)

(4) Let I be a totally ordered set and let $\{\tau_i \mid i \in I\}$ be a chain of Hausdorff topologies of weight σ on an infinite *set* G. Can one give a good upper bound on $|I|$? (in the case of precompact topologies on an abelian group G we have seen that $|I| < Ded^e(\sigma)$ if $\log|G| \leq \sigma \leq 2^{|G|}$).

We do not know the answer to (3) even for arbitrary (not necessarily group) topologies.

Remark added (July, 1996). In a recent manuscript [12] Comfort and Remus prove the particular case of our Corollary 7.9 (announced in [6, Theorem II]) regarding only chain lengths (see the precise formulation in Section 1.1).

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