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Semi-topological K-theory for certain projective varieties

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Abstract

In this paper we compute Lawson homology groups and semi-topological K-theory for certain threefolds and fourfolds. We consider smooth complex projective varieties whose zero cycles are supported on a proper subvariety. Rationally connected varieties are examples of such varieties. The computation makes use of different techniques of decomposition of the diagonal cycle, of the Bloch–Kato conjecture and of the spectral sequence relating morphic cohomology and semi-topological K-theory.

MSC: 19E20; 19E15; 14F43

1. Introduction

Eric Friedlander and Mark Walker introduced in [16] the (singular) semi-topological K-theory of a complex projective variety X. This is defined by

$$K_*^{sst}(X) = \pi_*(Mor(X, Grass)^+)$$

where $Grass = \amalg_{n,N} Grass_n(\mathbb{P}^N)$. By $Mor(X, Grass)^+$ we define the topological group given by the homotopy completion of the space of algebraic maps between X and Grass.

Semi-topological K-theory lies between algebraic and topological K-theory in the sense that the natural map from the algebraic K-theory $K_*(X)$ of a variety X to the connective (complex) topological K-theory $ku^*(X^{an})$ of its underlying analytic space X^{an} factors through the semi-topological K-theory of X, i.e.

$$K_q(X) \to K_q^{sst}(X) \to ku^{-q}(X^{an})$$

for any $q \geq 0$.

In [26] Blaine Lawson Jr. introduced the (Lawson) homology groups of a projective complex variety X, which are given by

$$L_r H_n(X) = \pi_{n-2r}(\mathscr{Z}_r(X))$$

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where $\mathscr{Z}_r(X)$ is the naive group completion of the topological monoid

$$\mathscr{C}_r(X) = \amalg_d \mathscr{C}_{r,d}(X)$$

with $\mathscr{C}_{r,d}(X)$ the Chow variety of subvarieties of X of dimension r and degree d.

In [15] Eric Friedlander and Blaine Lawson Jr. introduced the morphic cohomology, a cohomology theory dual to Lawson homology [14]. They defined

$$L^r H^n(X) = \pi_{2r-n}(\mathscr{Z}^r(X))$$

where $\mathscr{Z}^r(X)$ is the naive group completion of the following topological monoid $Mor(X, \mathscr{C}_0(\mathbb{P}^r))/Mor(X, \mathscr{C}_0(\mathbb{P}^{r-1}))$.

Morphic cohomology groups are related to the semi-topological K-theory by means of a semi-topological spectral sequence compatible with the motivic spectral sequence and the Atiyah–Hirzebruch spectral sequence [13].

In this paper we study the map

$$K_*^{sst}(X) \to ku^{-*}(X^{an})$$

for various complex projective varieties X.

We divide the paper in eight sections. In the second section we fix the notations and recall some essential results that we need in the paper.

In the third section we study the effects of the Bloch–Kato conjecture on the kernel and cokernel of the generalized cycle maps. We give a new proof of a theorem of Bloch about the torsion of the singular cohomology of a smooth projective variety. We also study the torsion of the Borel–Moore homology of a quasi-projective smooth variety. At the end of the section we construct a birational invariant using Lawson homology.

In the fourth section we study the action of an algebraic cycle on morphic cohomology groups. Our approach is slightly different than the one used by Peters [30] and our results include the results of [30].

In the fifth section we start comparing the Lawson homology and the singular homology of smooth projective varieties with zero cycles supported on a subvariety. We essentially use the results of the previous two sections and a technique introduced by Bloch and Srinivas [4].

The main goal of this section is to study the semi-topological K-theory of our "degenerate" varieties. One of the main results of the section is the following theorem which computes semi-topological K-theory of "degenerate" threefolds.

Theorem 1.1. *Let X be a smooth projective complex threefold such that there is a proper subvariety $V \subset X$ with $CH_0(X \setminus V) = 0$. Then:*

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an}), \quad i \geq 1,$$
$$K_0^{sst}(X) \hookrightarrow ku^0(X^{an}).$$

Moreover if X is a rationally connected threefold then

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an}), \quad i \geq 0.$$

This computation generalizes a result of [13] about the semi-topological K-theory of a rational threefold. In the last section of the paper we will analyze in more detail the Lawson homology of rational smooth threefolds and fourfolds.

The following result describes the semi-topological K-theory of some "degenerate" fourfolds.

Theorem 1.2. *Let X be a smooth projective fourfold such that there is a proper subvariety $V \subset X$ of $dim(V) \leq 2$ with $CH_0(X \setminus V) = 0$. Then:*

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an}), \quad i \geq 3,$$
$$K_2^{sst}(X) \hookrightarrow ku^{-2}(X^{an}),$$
$$K_i^{sst}(X)_{\mathbb{Q}} \simeq ku^{-i}(X^{an})_{\mathbb{Q}}, \quad i = 1, 2,$$
$$K_0^{sst}(X)_{\mathbb{Q}} \hookrightarrow ku^0(X^{an})_{\mathbb{Q}}.$$

We may contrast the above results with a result of Gillet [18] (see also Pedrini and Weibel [29]). He proved that the image of the map

$$K_n(X) \to ku^{-n}(X)$$

is finite for any $n > 0$ and for any smooth complex projective variety X.

In the sixth section of the paper we give some consequences of a theorem of Jannsen [22] and Laterveer [25] concerning a special decomposition of the diagonal for varieties with small Chow groups.

In the seventh section of the paper we study morphic cohomology of projective smooth linear varieties. The main idea is to use a *Künneth* formula for such varieties proved by Joshua [23] and by Totaro [36]. The results in this section were proved in [13] using other tools.

The last section of the paper is a short discussion of the morphic cohomology of a rational variety. The results here generalize and explain previous results of [13,20].

2. Notations and Recollection

Throughout this paper X will define a smooth projective irreducible variety over the complex numbers of dimension d (unless otherwise stated). By $H^{p,q}_{\mathbb{M}}(X)$, $L^qH^p(X)$, $L_pH_q(X)$ and $H_q(X)$ we define motivic cohomology, morphic cohomology, Lawson homology and singular homology with integer coefficients. For a field E we define $K^M_*(E)$ to be the Milnor K-theory of E. By $cyc^{p,q}$, respectively $cyc_{p,q}$ we define the generalized cycle maps

$$cyc^{p,q} : L^pH^q(X) \to H^q(X)$$

respectively

$$cyc_{p,q} : L_pH_q(X) \to H_q(X).$$

Let $K^{q,n} = Ker\{cyc^{q,n}\}$, $K_{p,q} = Ker(cyc_{p,q})$ and $C^{q,n} = Coker\{cyc^{q,n}\}$, $C_{q,n} = Coker\{cyc_{q,n}\}$. For an abelian group A we define ${}_mA = \{a \in A | ma = 0\}$.

If for a variety X there is a proper subvariety $V \subset X$ such that $CH_0(X \setminus V) = 0$ we say as in [4] that X is "degenerate" and also that its zero cycles "are supported on subvariety V".

For a complex variety X we let X^* denote a resolution of singularities for X.

We will start recalling the basics about the (co)niveau filtration of the singular (co)homology. Let

$$N_kH_n(X) = \sum_{dim(W)\le k} Im(H_n(W) \to H_n(X))$$

be a step in the niveau filtration of $H_n(X)$. This is an ascending filtration

$$0 \subset N_0H_n(X) \subset \cdots \subset N_kH_n(X) \subset \cdots \subset H_n(X)$$

which has the property that

$$N_kH_n(X) = H_n(X) \tag{1}$$

for any $k \ge min\{n, d\}$.

It is easy to see that $N_dH_n(X) = H_n(X)$ for any natural n. For $n < d$ the above equality follows from an induction argument using weak Lefshetz theorem. For a smooth projective variety X we know that the niveau filtration is isomorphic to the coniveau filtration of the cohomology of X, i.e.

$$N_kH_n(X) \simeq N^{d-k}H^{2d-n}(X) \tag{2}$$

where we define

$$N^kH^n(X) = \sum_{cd(W)\ge k} Im(H^n_W(X) \to H^n(X)).$$

From (1) and from (2) we conclude that

$$N_{d-1}H_{2d-n}(X) \simeq N^1H^n(X) \simeq H^n(X) \simeq H_{2d-n}(X)$$

for any n such that $2d - n \leq d - 1 (\Leftrightarrow n \geq d + 1)$. We also know [9,40] the following property of the generalized cycle maps

Proposition 2.1 (*[9,40]*). *For a smooth projective variety X*

$$Im(cyc^{q,n}) \subset N^{n-q} H^n(X)$$

with equality when $n = 2q$ *or* $n = 2q - 1$.

For a quasi-projective variety U, Deligne [6] and Gillet-Soule [19] defined a weight filtration on the Borel–Moore homology of U^{an} (written $H_*^{BM}(U^{an})$). We recall the definition of this filtration. Choose a compactification $U \subseteq X$ so that X is a projective complex variety and let Y be the reduced complement of U in X. Consider $\mathbb{Z}Sing_*()$ the functor taking a space Z to the complex associated to the simplicial set $Sing_* Z$. We may construct two hypercovers [19] $X_* \to X$ and $Y_* \to Y$ such that X_n and Y_n are smooth projective varieties and such that there is a map $Y_* \to X_*$ which covers the embedding $Y \subset X$. Denoting $U_n = X_n \sqcup Y_{n-1}$ we may construct a bicomplex

$$\cdots \to \mathbb{Z}Sing_*(U_1) \to \mathbb{Z}Sing_*(U_0). \tag{3}$$

The homology of the total complex of the bicomplex (3) gives the Borel–Moore homology [19]. The weight filtration for $H_*^{BM}(U^{an})$ is the increasing filtration

$$\cdots \subseteq W_t H_n^{BM}(U^{an}) \subseteq W_{t+1} H_n^{BM}(U^{an}) \subseteq \cdots$$

where

$$W_t H_n^{BM}(U^{an}) := image(h_n(\mathbb{Z}Sing_*(U_{n+t}) \to \cdots \to \mathbb{Z}Sing_*(U_0)) \to H_n^{BM}(U^{an})).$$

It can be proved [19] that

$$W_t H_n^{BM}(U^{an}) = 0$$

for any $t < -n$ and

$$W_t H_n^{BM}(U^{an}) = H_n^{BM}(U^{an})$$

for any $t \geq d - n$, where d is the dimension of the variety U.

The generalized cycle maps of a quasi-projective variety have the following property:

Proposition 2.2 (*[13]*). *For any quasi-projective complex variety U the image of the canonical map*

$$cyc_{t,n} : L_t H_n(U) \to H_n^{BM}(U^{an})$$

lies in the part of weight at most $-2t$ *of Borel–Moore homology.*

We will recall now the following conjecture due to Bloch and Kato.

Theorem 2.1 (*Bloch–Kato Conjecture*). *For any* $n \geq 0$ *and any field E the norm residue homomorphism*

$$K_n^M(E)/l \to H_{et}^n(E, \mu_m^{\otimes q})$$

is an isomorphism.

This conjecture was proved by V. Voevodsky for any $m = 2^l$ and for any natural number $l > 0$ (this part is also called Milnor's conjecture). The general case appears to be proved from the work of Rost [31,32], Voevodsky [37] and Weibel [41]. Suslin and Voevodsky [34] (see also Geisser–Levine [17]) proved that the Bloch–Kato conjecture is equivalent to a conjecture due to Beilinson–Lichtenbaum.

Theorem 2.2 (*Beilinson–Lichtenbaum Conjecture*). *The map*

$$H_{\mathbb{M}}^n(X, \mathbb{Z}/m(q)) \to H_{et}^n(X, \mu_m^{\otimes q})$$

is an isomorphism for $n \leq q$ *and a monomorphism for* $n \leq q + 1$ *for any smooth quasi-projective variety X.*

A. Suslin proposed the following characterization for the morphic cohomology with integral coefficients (see [13,40]).

Conjecture 2.1 (*Suslin's Conjecture*). *The map*

$$L^q H^n(X,\mathbb{Z}) \to H^n(X,\mathbb{Z})$$

is an isomorphism for $n \le q$ and a monomorphism for $n \le q+1$ for any smooth quasi-projective variety X.

We notice that the last conjecture contains a conjecture due to Friedlander and Mazur [9].

Conjecture 2.2 (*Friedlander–Mazur Conjecture*). *For any complex smooth quasi-projective variety X*

$$L^q H^n(X) = 0$$

for any $n < 0$.

The Friedlander–Lawson duality theorem [14,11] between morphic cohomology and Lawson homology will be used throughout the paper.

Theorem 2.3. *(Friedlander–Lawson Jr. [14], Friedlander [11]) For any quasi-projective smooth complex variety X of dimension d*

$$L^s H^n(X) \simeq L_{d-s} H_{2d-n}(X)$$

for any $n \le 2s$, $n \in \mathbb{Z}$ and $0 \le s \le 2d$.

The relation between morphic cohomology and semi-topological K-theory is recalled in the following theorem.

Theorem 2.4 (*[13]*). *For any smooth, quasi-projective complex variety X and any abelian group A, there are natural maps of strongly convergent spectral sequences*

$$\begin{array}{ccc}
E_2^{p,q}(alg) = H_{\mathbb{M}}^{p-q}(X, A(-q)) & \Longrightarrow & K^{alg}_{-p-q}(X,A) \\
 & \downarrow & \\
E_2^{p,q}(sst) = L^{-q} H^{p-q}(X,A) & \Longrightarrow & K^{sst}_{-p-q}(X,A) \\
 & \downarrow & \\
E_2^{p,q}(top) = H^{p-q}(X^{an},A) & \Longrightarrow & ku^{p+q}(X^{an},A).
\end{array}$$

inducing the usual maps on both E_2-terms and abutments.

The following soft improvement of Theorem 3.7 in [13] will be used later in the text.

Theorem 2.5. *Let X be a smooth quasi-projective complex variety of dimension d. Let A be an abelian group and $k \le 0$. Then if*

$$cyc^{q,n} : L^q H^n(X,A) \to H^n(X^{an},A)$$

is an isomorphism for $n-2q \le k-1$ and a monomorphism for $n-2q \le k$, then the map

$$K_i^{sst}(X,A) \to ku^{-i}(X^{an},A)$$

is an isomorphism for $i \ge -k+1$ and a monomorphism for $i = -k$.

Proof. We will use the idea of proof used in Theorem 3.7 [13]. We will prove by induction the following statements:
(1) The map $E_r^{p,q}(sst) \to E_r^{p,q}(top)$ is an isomorphism provided $p+q \le k-1$.
(2) The map $E_r^{p,q}(sst) \to E_r^{p,q}(top)$ is a monomorphism provided $p+q \le k$.

We notice that $E_2^{p,q}(sst) \to E_2^{p,q}(top)$ is an isomorphism if and only if $cyc^{-q,p-q} : L^{-q}H^{p-q} \to H^{p-q}$ is an isomorphism. From the hypotheses we know that $cyc^{-q,p-q}$ is an isomorphism if $p-q-2(-q) = p+q \le k-1$ and a monomorphism if $p-q-2(-q) = p+q \le k$. This implies our assertions in case $r = 2$.

Let us suppose that both assertions are true for a fix $r \geq 2$ and prove them for $r+1$. Theorem 2.4 shows that there is a commutative diagram

$$\begin{array}{ccccc} E_r^{p-r,q+r-1}(sst) & \xrightarrow{d_r^{p-r,q+r-1}} & E_r^{p,q}(sst) & \xrightarrow{d_r^{p,q}} & E_r^{p+r,q-r+1}(sst) \\ \downarrow & & \downarrow & & \downarrow \\ E_r^{p-r,q+r-1}(top) & \xrightarrow{d_r^{p-r,q+r-1}} & E_r^{p,q}(top) & \xrightarrow{d_r^{p,q}} & E_r^{p+r,q-r+1}(top) \end{array}$$

with the middle homology groups of the rows given by $E_{r+1}^{p,q}(sst)$ and $E_{r+1}^{p,q}(top)$. If $p+q \leq k$ then the left vertical map in this diagram is an isomorphism and the middle vertical map is a monomorphism. It implies that the maps $E_{r+1}^{p,q}(sst) \to E_{r+1}^{p,q}(top)$ are monomorphisms for any $p+q \leq k$. If $p+q < k$ then the left and middle arrow in the above diagram are isomorphisms and the right one is a monomorphism. This implies that for $p+q < k$ the maps $E_{r+1}^{p,q}(sst) \to E_{r+1}^{p,q}(top)$ are surjective. This concludes our induction.

This induction shows that (1) and (2) hold also on E_∞-terms of the spectral sequences, and then the map $K_i^{sst}(X, A) \to ku^{-i}(X^{an}, A)$ has a finite filtration whose quotients are isomorphisms for $i \geq -k+1$ and monomorphisms for $i = -k$. $\square$

3. First results concerning generalized cycle maps

We start this section with some applications of the Bloch–Kato conjecture in the context of Lawson homology. The point (b) in Proposition 3.1 is known as Bloch's theorem [3]. In Proposition 3.2 we analyze the torsion of the Borel–Moore homology of a smooth quasi-projective variety.

Proposition 3.1. *Let X be a quasi-projective smooth variety. Assume that the Bloch–Kato conjecture is valid for all the primes. Then:*

(a) *Let $n \leq q+1$. Then $K^{q,n}$ is divisible and $C^{q,n}$ is torsion free.*

(b) *Suppose X is projective. Then the torsion of $H^n(X)$ is supported in codimension one for any $n > 0$.*

(c) *$L^q H^n(X)$ is uniquely divisible for $n < 0$ and $L^q H^0(X)$ is torsion free (for any $q \geq 0$).*

Proof. We write the diagram of universal coefficient sequences for both cohomologies:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^q H^n(X) \otimes \mathbb{Z}/m & \longrightarrow & L^q H^n(X, \mathbb{Z}/m) & \longrightarrow & {}_m L^q H^{n+1}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^n(X) \otimes \mathbb{Z}/m & \longrightarrow & H^n(X, \mathbb{Z}/m) & \longrightarrow & {}_m H^{n+1}(X) & \longrightarrow & 0. \end{array}$$

We recalled in the second section that Bloch–Kato conjecture implies that the map

$$H^n_{\mathbb{M}}(X, \mathbb{Z}/m(q)) \to H^n_{et}(X, \mu_m^{\otimes q})$$

is an isomorphism for $n \leq q$ and a monomorphism for $n \leq q+1$ for any smooth quasi-projective variety X. The above map factors through the cycle map from morphic cohomology to the singular cohomology [15]. In [35] it is proved that

$$H^n_{\mathbb{M}}(X, \mathbb{Z}/m(q)) \simeq L^q H^n(X, \mathbb{Z}/m)$$

for any allowed n, q and any complex projective variety X.

In conclusion the middle vertical map from the above diagram is injective for $n \leq q+1$ and isomorphism for $n \leq q$. Using the snake lemma we conclude that we have the following exact sequence:

$$0 \to L^q H^n(X) \otimes \mathbb{Z}/m \to H^n(X) \otimes \mathbb{Z}/m \to Ker({}_m L^q\, H^{n+1}(X) \to_m H^{n+1}(X)) \to 0$$

for any $n \leq q$ and that the map $L^q H^n(X) \otimes \mathbb{Z}/m \to H^n(X) \otimes \mathbb{Z}/m$ is an injection for $n \leq q+1$. Moreover we conclude that for $n \leq q$ we have that the map

$${}_m L^q\, H^{n+1}(X) \to_m H^{n+1}(X)$$

is surjective. This means actually that

$$ {}_mL^q\, H^n(X) \to_m H^n(X) $$

is surjective for any $n \leq q+1$ and any $m > 1$. It implies that

$$ \text{torsion}(Im(cyc^{q,n})) = \text{torsion}(H^n(X)) \tag{4} $$

for any $n \leq q+1$. As each image of a generalized cycle map is included in a step of the coniveau filtration, we have

$$ \text{torsion}(N^{n-q} H^n(X)) = \text{torsion}(H^n(X)) $$

for any $n \leq q+1$. The only case when we conclude something nontrivial from the equality above is when $n = q+1$. In this case for any $0 < n \leq d+1$ we have

$$ \text{torsion}(N^1 H^n(X)) = \text{torsion}(H^n(X)) $$

which implies our point (b). Consider now the composition

$$ L^q H^n(X) \otimes \mathbb{Z}/m \to Im(cyc^{q,n}) \otimes \mathbb{Z}/m \to H^n(X) \otimes \mathbb{Z}/m. $$

The first map is still surjective because $- \otimes \mathbb{Z}/m$ is a right exact functor. For $n \leq q+1$ the composition is injective. This implies that

$$ L^q H^n(X) \otimes \mathbb{Z}/m \simeq Im(cyc^{q,n}) \otimes \mathbb{Z}/m $$

and that

$$ Im(cyc^{q,n}) \otimes \mathbb{Z}/m \hookrightarrow H^n(X) \otimes \mathbb{Z}/m $$

for any $n \leq q+1$ and $m > 1$. Consider now the following short exact sequence

$$ 0 \to K^{q,n} \to L^q H^n(X) \to Im(cyc^{q,n}) \to 0. $$

Tensoring with $\mathbb{Z}/m$ we obtain the following exact sequence:

$$ \begin{aligned} 0 \to_m K^{q,n} &\to_m L^q H^n(X) \xrightarrow{a_1}{}_m Im(cyc^{q,n}) \to K^{q,n} \otimes \mathbb{Z}/m \\ &\to L^q H^n(X) \otimes \mathbb{Z}/m \xrightarrow{a_2} Im(cyc^{q,n}) \otimes \mathbb{Z}/m \to 0. \end{aligned} $$

For $n \leq q+1$ the map a_2 is an isomorphism and the map a_1 is a surjection. From the exactness of the sequence we get

$$ K^{q,n} \otimes \mathbb{Z}/m = 0 $$

for any $n \leq q+1$ and $m > 1$. This implies that $K^{q,n}$ is divisible for any $n \leq q+1$.

Consider now the following exact sequence

$$ 0 \to Im(cyc^{q,n}) \to H^n(X) \to C^{q,n} \to 0. $$

Tensoring with $\mathbb{Z}/m$ we obtain the following long exact sequence:

$$ 0 \to_m Im(cyc^{q,n}) \xrightarrow{a_3}{}_m H^n(X) \to_m C^{q,n} \to Im(cyc^{q,n}) \otimes \mathbb{Z}/m \xrightarrow{a_4} H^n(X) \otimes \mathbb{Z}/m \to C^{q,n} \otimes \mathbb{Z}/m \to 0. $$

For $n \leq q+1$ the map a_3 is bijective and the map a_4 is injective. From the exactness of the sequence we get

$$ {}_mC^{q,n} = 0 $$

for any $n \leq q+1$ and any $m > 1$. This implies that $C^{q,n}$ is torsion free for any $n \leq q+1$.

Suppose now that $n < 0$. Because $0 \leq q \leq d = dim(X)$, we have $n < q$. We have the following short exact sequence

$$ 0 \to L^q H^n(X) \otimes \mathbb{Z}/m \to L^q H^n(X, \mathbb{Z}/m) \to_m L^q H^{n+1}(X) \to 0. $$

Because $L^q H^n(X, \mathbb{Z}/m) = 0$ for any $n < 0$ and for any $m > 1$, we conclude that $L^q H^n(X) \otimes \mathbb{Z}/m = 0$ for any $n < 0, m > 1$ (i.e. $L^q H^n(X)$ is divisible for $n < 0$) and that ${}_mL^q\, H^{n+1}(X) = 0$ (i.e. $L^q H^n(X)$ is torsion free for any $n \leq 0$). $\square$

Corollary 3.1. *Let $n \le q+1$. Then:*
(a) *If there is a natural nonzero number M such that $MK^{q,n} = 0$, then $cyc^{q,n}$ is injective.*
(b) *Suppose that $cyc^{q,n} \otimes \mathbb{Q}$ is surjective. Then $cyc^{q,n}$ is surjective.*

Point (b) in Proposition 3.1 has the following formulation in the quasi-projective case:

Proposition 3.2. *Let X be a smooth quasi-projective variety of dimension d and let $n \ge d$. Fix $s = n-d+1$. Then*

$$\mathrm{torsion}(W_{-2s}H_n(X)) = \mathrm{torsion}(H_n^{BM}(X))$$

where $W_{-2s}H_n(X)$ is a step in the weight filtration of the Borel–Moore homology $H_n^{BM}(X)$.

Proof. From (4) we have that the groups $Im(cyc^{q,q+1})$ and $H^{q+1}(X)$ have the same torsion for any q with $0 \le q \le d$. Using the Friedlander–Lawson duality theorem [11] we have that

$$tors(Im(L_{n-d+1}H_n(X) \to H_n(X))) = tors(H_n(X))$$

for any $n \ge d$.

We recalled in the second section that the cycle map from Lawson homology to the Borel–Moore homology of a smooth quasi-projective variety factors through steps in the weight filtration [13], i.e.

$$L_sH_n(X) \to W_{-2s}H_n(X) \hookrightarrow H_n(X)$$

for any $0 \le s \le d$ and $n \ge 2s$. This implies the statement of the theorem. □

The above discussion gives us the following reformulation of the Friedlander–Mazur conjecture.

Proposition 3.3. *Let X be a smooth quasi-projective variety. Then the Friedlander–Mazur conjecture is valid for X if and only if $L^qH^n(X)_{\mathbb{Q}} \simeq 0$ for any $n < 0$.*

Proof. The point (c) in the Proposition 3.1 shows that these groups are torsion free. □

For a smooth projective variety X of dimension d with

$$H^2_{Zar}(X, O_X) = H^1_{Zar}(X, O_X) = 0$$

we know (see for example [5]) that the cohomological Brauer group of X has the following characterization

$$Br(X) \simeq tors(H^3(X)).$$

Suslin's conjecture predicts that the cycle map

$$L^qH^n(X) \to H^n(X)$$

is an isomorphism for any $n \le q$ and a monomorphism for $n = q+1$. Assuming Suslin's conjecture for X we obtain that

$$\mathrm{torsion}(L^3H^3(X)) \simeq \mathrm{torsion}(L^4H^3(X)) \simeq \cdots \simeq \mathrm{torsion}(H^3(X)).$$

But (4) shows that

$$\mathrm{torsion}(Im(cyc^{2,3})) = \mathrm{torsion}(H^3(X))$$

and, because of our assumption, we get that the cycle map $cyc^{2,3}$ is injective. We obtain that

$$\mathrm{torsion}(L^2H^3(X)) \simeq \mathrm{torsion}(L^3H^3(X)) \simeq \cdots \simeq \mathrm{torsion}(H^3(X))$$

giving a characterization of the cohomological Brauer group of X by means of morphic cohomology. We will show in sections five, six and seven that Suslin's conjecture can be verified for certain projective varieties.

A natural question to ask is whether $tors(L^2H^3)$ is a birational invariant in general as Suslin's conjecture predicts. We will prove below that this is indeed the case. We will use a blow-up formula for Lawson homology proved by Hu [20] and the fact that birational maps between projective smooth varieties factor as a composition of blow-ups with centers of codimension greater than two [1].

Proposition 3.4. *$tors(L^2H^3)$ is a birational invariant.*

That is, for any birational equivalent smooth projective varieties X, X' we have:

$$tors(L^2H^3(X)) \simeq tors(L^2H^3(X')).$$

Proof. Let $X_Y \to X$ be a blow-up of a smooth center Y of codimension greater than or equal with 2. From [20] we know that

$$L_{n-2}H_{2n-3}(X_Y) = \bigoplus_{1\le j\le r-1} L_{n-2-j}H_{2n-3-2j}(Y) \bigoplus L_{n-2}H_{2n-3}(X)$$

where r is the codimension of Y in X.

It suffices to show that $tors(L_{n-2-j}H_{2n-3-2j}(Y)) = 0$ for any $1 \le j \le r-1$. We notice that

$$dim(Y) - 1 \le n - 2 - j \le n - 3$$

and

$$2dim(Y) - 1 \le 2n - 3 - 2j \le 2n - 5.$$

If $n - 2 - j \ge dim(Y)$, it is obvious that

$$tors(L_{n-2-j}H_{2n-3-2j}(Y)) = 0.$$

We also have

$$tors(L_{dim\,Y-1}H_{2dim\,Y-1}(Y)) \simeq tors(H_{2dim\,Y-1}(Y)).$$

From the universal coefficient sequence one can obtain that

$$tors(H_{2dim\,Y-1}(Y)) = tors(H_0(Y)) = 0.$$

We can conclude now that

$$tors(L^2H^3(X)) \simeq tors(L^2H^3(X'))$$

for any birational equivalent smooth projective varieties X, X'. □

4. Cycle action on morphic cohomology

Let α be a dimension $d = dim(X)$ irreducible algebraic cycle in $X \times X$ with the support contained in $V \times W$, where $V \subset X$ and $W \subset X$ are irreducible subvarieties. Let $v = dim(V)$ and $w = dim(W)$. Consider the compositions $i : V^* \to V \hookrightarrow X$ and $j : W^* \to W \hookrightarrow X$, with V^* and W^* resolutions of singularities of V, respectively W. We may suppose that $pr_1(\alpha) = V$ and that $pr_2(\alpha) = W$. Let $\alpha' = (i \times j)^{-1}(\alpha)$. Based on our assumption, the cycle α is not entirely included in the singular locus of $V \times W$. This implies that

$$(i \times j)_*\alpha' = \alpha$$

where $(i \times j)_* : CH_d(V^* \times W^*) \to CH_d(X \times X)$. The cycle α gives the following action

$$\alpha_* : L^mH^l(X) \to L_{d-m}H_{2d-l}(X)$$

defined as

$$\alpha_*(x) = pr_{2*}(pr_1^*(x) \cap \alpha).$$

The above map depends only on the algebraic equivalence class of α because of the properties of cap product [15]. A similar action in the context of Lawson homology was considered by Peters in [30] (see also [15]).

The above map decomposes in the following way

$$\alpha_*(x) = pr_{2*}(pr_1^*(x) \cap (i \times j)_*(\alpha')) = pr_{2*}(i \times j)_*((i \times j)^* pr_1^*(x) \cap (\alpha')) = j_* pr_{W^**}(pr_{V^*}^* i^*(x) \cap \alpha')$$

by using projection formula (see [12] for a proof of the projection formula) in the morphic cohomology setting. If we consider, for example, the action of the diagonal cycle of $X \times X$ we will obtain the Friedlander–Lawson duality isomorphism between morphic cohomology and Lawson homology.

The above action commutes with the similarly defined action of the algebraic cycle α on the singular cohomology. This can be seen from the fact that the cycle maps from morphic cohomology (resp. Lawson homology) to singular cohomology (resp. singular homology) are natural [12] and commute with the cap product with an algebraic cycle [12].

The above discussion is summarized in the following sequence of commutative diagrams (the horizontal maps are given by the decomposition of the action of the algebraic cycle α and the vertical maps are given by the cycle maps)

$$\begin{tikzcd} L^{d-m}H^{2d-l}(X) \arrow[r, "i^*"] \arrow[d, "c_1"'] & L^{d-m}H^{2d-l}(V^*) \arrow[r, "pr_{V^*}^*"] \arrow[d, "c_2"'] & L^{d-m}H^{2d-l}(V^*\times W^*) \arrow[r, "\cap\alpha'"] \arrow[d] & {} \\ H^{2d-l}(X) \arrow[r, "i^*"] & H^{2d-l}(V^*) \arrow[r, "pr_{V^*}^*"] & H^{2d-l}(V^*\times W^*) \arrow[r, "\cap\alpha'"] & {} \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r] & L_mH_l(V^*\times W^*) \arrow[r, "(pr_{W^*})_*"] \arrow[d] & L_mH_l(W^*) \arrow[r, "j_*"] \arrow[d, "c_3"'] & L_mH_l(X) \arrow[d, "c_4"'] \\ {} \arrow[r] & H_l(V^*\times W^*) \arrow[r, "(pr_{W^*})_*"] & H_l(W^*) \arrow[r, "j_*"] & H_l(X) \end{tikzcd} \tag{5}$$

for any $0 \le m \le d$ and any $l \ge 2m$.

The map c_2 is an isomorphism for any $m \le d - v$, where $v = dim(V)$. To see this we divide it in several cases depending on the value of $2d - l$. If $2d - l > 2v \ge 0$ then

$$L^{d-m}H^{2d-l}(V^*) = H^{2d-l}(V^*) = 0$$

by [15]. If $0 \le 2d - l \le 2v$ then we can consider the following morphic cohomology group $L^vH^{2d-l}(V^*)$ which is isomorphic with $H^{2d-l}(V^*)$ from the Poincare duality and the Dold–Thom theorem (see [12]). At the same time the composition of s-maps

$$L^vH^{2d-l}(V^*) \to L^{d-m}H^{2d-l}(V^*)$$

is an isomorphism in this range and commutes with the cycle maps [14]. This implies that

$$L^{d-m}H^{2d-l}(V^*) \simeq H^{2d-l}(V^*).$$

Consider now $2d - l < 0$. Then by the Friedlander–Lawson duality theorem and the isomorphism of s-maps in this range we obtain

$$0 = L^vH^{2d-l}(V^*) = L^{d-m}H^{2d-l}(V^*).$$

The map c_3 is an isomorphism for any $m \ge w$. In the case $m = w$ we have

$$L_mH_{2m}(W^*) \simeq H_{2m}(W^*)$$

because W^* is irreducible. For $m > w$ we obviously have $L_mH_l(W^*) = H_l(W^*) = 0$ since $l \ge 2m > 2w$.

The above discussion proves the following proposition:

Proposition 4.1. *Let α be an irreducible algebraic cycle in $CH^d(X \times X)$ with the support contained in $V \times W$, where $V \subset X$ and $W \subset X$ are irreducible subvarieties of dimension v, respectively w. The action of the cycle α on the kernel and the cokernel of the map c_1 is zero for $m \ge w = dim(W)$ or for $m \le d - v = codim(V)$.*

Proof. From the above discussion we conclude that if $m \ge w$ then c_3 is an isomorphism and that if $m \le d - v$ then c_2 is an isomorphism. These imply the conclusion of our proposition. $\square$

Corollary 4.1. *Suppose α as in Proposition 4.1 and suppose $dim(X) = dim(V) + dim(W)$. Then the action of the cycle α on the kernel and on the cokernel of the map c_1 is zero for any $0 \le m \le d$.*

Proof. Direct consequence of Proposition 4.1. □

Remark 4.1. We remark that to study the action of a cycle $\alpha = \sum n_i \alpha_i \in CH^d(X \times X)$ with $supp(\alpha) \subset V \times W$ it is enough to study the action of each irreducible cycle α_i. It is obvious that

$$supp(\alpha_i) \subset supp(\alpha) \subset V \times W$$

and that because $supp(\alpha_i)$ is irreducible there are $V_i \subset V$, $W_i \subset W$ irreducible components such that

$$supp(\alpha_i) \subset V_i \times W_i$$

By using the Friedlander–Lawson duality theorem [14] we will identify the cycle map c_1 with the cycle map $L_m H_l(X) \to H_l(X)$.

This will identify the action of the diagonal cycle with the identity map.

Convention 4.1. *From now on by "the action of α is zero for m in some certain range" we will understand that the action of the cycle α on the kernel and cokernel of the cycle map $L_m H_* \to H_*$ is zero for m in the respective range.*

5. Comparing Lawson homology with singular homology

In this section we study the cycle maps $cyc^{q,n} : L^q H^n(X) \to H^n(X)$ for a smooth projective complex variety X with the property that its zero cycles are supported on a proper subvariety. We prove that these cycle maps behave nicely for threefolds and fourfolds with this property (being injective or bijective most of the time). We expect that there are cycle maps $cyc^{q,n}$ with nontrivial kernel for varieties of large dimension with zero cycles supported on a proper subvariety. A support for our expectation is a theorem of Albano and Collino [2] proving that for a generic smooth cubic hypersurface $X \subset \mathbb{P}^8$ the Griffiths group $Griff^4(X) \otimes \mathbb{Q}$ is infinitely generated.

We start the section by recalling a result of Friedlander. He proved [10] that for any smooth connected complex projective variety X of dimension d we have

$$L_{d-1} H_{2d-2}(X) \hookrightarrow H_{2d-2}(X),$$
$$L_{d-1} H_{2d-1}(X) \simeq H_{2d-1}(X),$$
$$L_{d-1} H_{2d}(X) \simeq H_{2d}(X) \simeq \mathbb{Z}$$

and that

$$L_{d-1} H_k(X) = 0$$

for any $k > 2d$.

We recall that Bloch and Srinivas [4] proved that if a smooth projective variety X has its zero cycles supported on a not necessary irreducible subvariety V, i.e. $CH_0(X \setminus V) = 0$, then the diagonal cycle decomposes as

$$N\Delta = \alpha + \beta$$

for some natural nonzero number N and some cycles $\alpha, \beta \in CH^d(X \times X)$ with the support of α included in $V \times X$ and the support of β included in $X \times D$, where D is a divisor of X. We will also use the transpose of this decomposition, i.e.

$$N\Delta = \alpha^t + \beta^t$$

where $\alpha^t, \beta^t \in CH^d(X \times X)$ are supported on $X \times V$, respectively $D \times X$.

In case $X = \mathbb{P}^n$, we can choose $N = 1$ in the equality above. Moreover, because the minimal N in the above decomposition is a birational invariant, we can choose $N = 1$ for any smooth projective rational variety X. This remark is a consequence of the proof of the above-mentioned result. We include its proof below because of the lack of a good reference.

Proposition 5.1. *Let X be a rational projective variety of dimension d over $\mathbb{C}$ and let $\Delta \subset X \times X$ be the diagonal cycle. Then there exist a divisor $D \subset X$ and d-cycles Γ_1, Γ_2 on $X \times X$ such that $supp(\Gamma_1) \subset Spec(\mathbb{C}) \times X$, $supp(\Gamma_2) \subset X \times D$ and*

$$[\Delta] = [\Gamma_1] + [\Gamma_2]$$

in $CH^d(X \times X)$.

Proof. Consider $\mathbb{Q} \subset k$ to be an extension of finite transcendence degree which contains all the coefficients of the polynomials that cut out X in some projective space, all the coefficients of the birational morphism between X and $\mathbb{P}^d_{\mathbb{C}}$ and of its inverse. Let μ be the generic point of $X := X/k$, $L = k(X) = O_{X,\mu}$ and choose an embedding $L \hookrightarrow \mathbb{C}$. We can see μ as a zero cycle on X_L. Owing to the way we choose k, we obtain that X is a rational variety over k. This implies that $CH_0(X_F) = \mathbb{Z}$ for any finitely generated field extension F/k [27]. In particular $CH_0(X_L) = \mathbb{Z}$. Let $pt := Spec(k)$ be a k-rational point of X. We know that such a point exists because X is rational over an infinite field k. We have that

$$\mu \in CH^d(X_L) \to CH^d((X \setminus \{pt\})_L) = 0$$

so we can conclude that $\mu = 0$ in $CH^d((X \setminus \{pt\})_L)$, i.e. there is γ a cycle supported on $Spec(k) \times Spec(L)$ such that $\mu = \gamma$. Let Γ_1 be the closure of γ in $Spec(k) \times X$. Then

$$\Delta - \Gamma_1 \in Ker(CH^d(X \times X) \to CH^d(X_L)).$$

As $CH^d(X_L) = lim_D CH^d(X \times (X \setminus D))$, where the limit runs over all divisors D of X, we have that

$$[\Delta] = [\Gamma_1] + [\Gamma_2]$$

with Γ_2 supported on $X \times D$. A fortiori, this decomposition can be seen over $\mathbb{C}$. □

The following theorem computes K^{sst} for "degenerate" threefolds.

Theorem 5.1. *Let X be a smooth projective complex threefold such that there is a proper subvariety $V \subset X$ with $CH_0(X \setminus V) = 0$. Then:*

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an}), \quad i \geq 1,$$
$$K_0^{sst}(X) \hookrightarrow ku^0(X^{an}).$$

Moreover if X is a rationally connected threefold then

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an})$$

for any $i \geq 0$.

Some examples which fulfill the conditions in Theorem 5.1 are: rationally connected threefolds (e.g. smooth Fano threefolds [24]), Kummer threefolds [4], certain quotient varieties such

$$(X \times E)/(\mathbb{Z}/2)$$

with X a K3 covering of an Enriques surface and E an elliptic curve [4].

The above result on rationally connected threefolds generalizes the same result on rational threefolds proved in [13] with other tools.

Proof. The proof of the above theorem is based on the spectral sequence relating morphic cohomology and semi-topological K-theory [13] and on the following two propositions which compute the Lawson homology groups of a threefold X with zero cycles supported on a subvariety.

Proposition 5.2. *Let X be a smooth projective complex threefold such that there is a proper subvariety $V \subset X$ with $CH_0(X \setminus V) = 0$ and with $dim(V) \leq 1$. Then:*

(a) *$L_1H_2(X) \hookrightarrow H_2(X)$ is injective and a rational isomorphism.*

(b) *$L_1H_3(X) \simeq H_3(X)$.*

(c) $L_2H_4(X) \simeq L_1H_4(X) \simeq H_4(X)$.
(d) $L_2H_5(X) \simeq L_1H_5(X) \simeq H_5(X)$.
(e) $L_3H_6(X) \simeq L_2H_6(X) \simeq L_1H_6(X) \simeq H_6(X)$.
(f) $L_kH_n(X) = 0$ *for any* $n \geq 7$ *and any* $k \geq 0$.
In particular any such a threefold fulfills Suslin's conjecture.
Moreover if X *is a rationally connected threefold then*

$$L_*H_*(X) = H_*(X)$$

for all possible indices.

Proof. It is enough to prove the case $dim(V) = 1$. Consider the above decomposition

$$N\Delta = \alpha^t + \beta^t$$

with α^t supported on $X \times V$ and β^t supported on $D \times X$, with D a divisor on X. Remark 4.1 shows that it is enough to consider the case when V and D are irreducible. We recall that we let D^*, respectively by V^*, to be the resolution of singularities of D, respectively V.

Proposition 4.1 gives us that the action of β^t is zero on $Ker(L_mH_*(X) \to H_*(X))$ for $m \leq codim(D) = 1$ and the action of α^t on the same kernel is zero for $m \geq v = dim(V) = 1$. This implies that for $m = 1$ we have

$$N(Ker(L_1H_k(X) \to H_k(X))) = 0$$

for any $2 \leq k \leq 6$ and that

$$NL_1H_k(X) = 0$$

for any $k \geq 7$. Proposition 3.1 implies that $L_1H_k(X) = 0$ for any $k \geq 7$ and Corollary 3.1 implies that the cycle maps $L_1H_k(X) \to H_k(X)$ are injective for any $3 \leq k \leq 6$.

Let $x \in H_k(X) \simeq H^{6-k}(X)$, where $3 \leq k \leq 6$. Then, from the Diagram (5), we can see that

$$\beta^t_* x \in Im(L_1H_k(X) \to H_k(X))$$

because $L^2H^{6-k}(D^*) \simeq H^{6-k}(D^*)$.

For $k \geq 3$, we have $\alpha^t_* x = 0$ because the action of α^t on $H_k(X)$ factors through $H_k(V^*)$ and $dim(V) = 1$. This implies that

$$Nx = \beta^t_* x \in Im(L_1H_k(X) \to H_k(X))$$

for any $3 \leq k \leq 6$. For $k = 2$, we can see that

$$\alpha^t_* x \in Im(L_1H_2(X) \to H_2(X))$$

because $L_1H_2(V^*) \simeq H_2(V^*)$.

Because $x \in H_k(X)$ is arbitrary chosen, we conclude that the rational cycle maps

$$L_1H_k(X) \otimes \mathbb{Q} \to H_k(X) \otimes \mathbb{Q}$$

are surjective for any $3 \leq k \leq 6$. Corollary 3.1 shows that the cycle maps $L_1H_k(X) \to H_k(X)$ are surjective for any $3 \leq k \leq 6$. For $k = 2$ we have

$$Nx = \alpha^t_* x + \beta^t_* x \in Im(L_1H_2(X) \to H_2(X))$$

This means that $L_1H_2(X) \otimes \mathbb{Q} \to H_2(X) \otimes \mathbb{Q}$ is a surjective map. For $k = 2$ we use a result of Bloch–Srinivas. They prove that varieties with our hypothesis have the property that algebraic equivalence and homological equivalence coincide for codimension 2 cycles [4]. This means that the cycle map

$$L_1H_2(X) \to H_2(X)$$

is injective.

Consider now the decomposition

$$N\Delta = \alpha + \beta$$

with α supported on $V \times X$ and β supported on $X \times D$. Proposition 4.1 gives that the action of α is zero on $Ker(L_m H_*(X) \to H_*(X))$ for $m \leq d - v = codim(V) = 2$ and that the action of β is zero on $Ker(L_m H_*(X) \to H_*(X))$ for $m \geq dim(D) = 2$. This implies that for $m = 2$ we have

$$N(Ker(L_2 H_k(X) \to H_k(X))) = 0$$

for $4 \leq k \leq 6$ and that

$$N L_2 H_k(X) = 0$$

for $k \geq 7$. Proposition 3.1 implies that $L_2 H_k(X) = 0$ for any $k \geq 7$ and Corollary 3.1 implies that the cycle maps $L_2 H_k(X) \to H_k(X)$ are injective for any $4 \leq k \leq 6$.

Let $x \in H_k(X)$ with $4 \leq k \leq 6$. Then, as the Diagram (5) shows,

$$\alpha_*(x) \in Im(L_2 H_k(X) \to H_k(X))$$

because $L^1 H^{6-k}(V^*) \simeq H^{6-k}(V^*)$. The action of β on $H_k(X)$ is zero for $5 \leq k \leq 6$ because this action factors through $H_k(D^*)$ and $dim(D^*) = 2$. If $k = 4$ then

$$\beta_*(x) \in Im(L_2 H_4(X) \to H_4(X)),$$

because $L_2 H_4(D^*) \simeq H_4(D^*)$ (see Diagram (5)). As

$$Nx = \alpha_*(x) + \beta_*(x)$$

we conclude that for $4 \leq k \leq 6$ the cycle maps

$$L_2 H_k(X) \otimes \mathbb{Q} \to H_k(X) \otimes \mathbb{Q}$$

are surjective. Applying Corollary 3.1 we conclude that these maps are surjections with integer coefficients.

Let us consider now the case of a smooth rationally connected threefold X. We want to prove that, in this case, $L_1 H_2(X) \simeq H_2(X, \mathbb{Z})$. For a smooth rationally connected threefold X we have that $H^4(X, \mathbb{C}) = H^{2,2}(X)$ because $h^{1,3} = h^{3,1} = h^{2,0} = 0$ (see [39]). This implies that

$$H_2(X, \mathbb{Z}) \simeq H^4(X, \mathbb{Z}) \simeq H^{2,2}(X, \mathbb{Z})$$

where we defined $H^{2,2}(X, \mathbb{Z}) = \{\eta \in H^4(X, \mathbb{Z})$ such that $coef_*(\eta) \in H^{2,2}(X)$ with $coef_* : H^4(X, \mathbb{Z}) \to H^4(X, \mathbb{C})$ being the coefficient map$\}$.

C. Voisin proved the following theorem:

Theorem 5.2 (*Voisin [38]*). *The Hodge conjecture with integral coefficients is valid for any smooth uniruled threefold X. That is, the map*

$$CH_1(X) \to H^{2,2}(X, \mathbb{Z})$$

is surjective for any such X. In particular all torsion cycles of $H_2(X, \mathbb{Z})$ are algebraic.

We recall that a smooth projective complex variety is uniruled if there is a rational curve through every point of the variety. As any smooth rationally connected variety is uniruled, we conclude using Voisin's result that for a smooth rationally connected threefold X the cycle map $CH_1(X) \to H_2(X, \mathbb{Z}) = H^{2,2}(X, \mathbb{Z})$ is surjective. This implies that $L_1 H_2(X) \to H_2(X, \mathbb{Z})$ is a surjective map. $\square$

Proposition 5.3. *Let X be a smooth projective complex threefold such that there is a proper subvariety $V \subset X$ with $CH_0(X \setminus V) = 0$ and with $dim(V) = 2$. Then:*

(a) $L_1 H_2(X) \hookrightarrow H_2(X)$.
(b) $L_1 H_3(X) \simeq H_3(X)$.
(c) $L_2 H_4(X) \simeq H^{1,1}(X, \mathbb{Z}) \hookrightarrow L_1 H_4(X) \simeq H_4(X)$.
(d) $L_2 H_5(X) \simeq L_1 H_5(X) \simeq H_5(X)$.
(e) $L_3 H_6(X) \simeq L_2 H_6(X) \simeq L_1 H_6(X) \simeq H_6(X)$.
(f) $L_k H_n(X) = 0$ *for any* $n \geq 7$ *and any* $k \geq 0$.

In particular any such a threefold fulfills Suslin's conjecture.

Proof. Consider the decomposition

$$N\Delta = \alpha + \beta$$

with α and β being supported on $V \times X$, respectively on $X \times D$. It is enough to consider the case when V and D are irreducible varieties (see Remark 4.1). The action of α is zero for $m \leq d - v = codim(V) = 1$ and the action of β is zero for $m \geq dim(D) = 2$ (see Convention 4.1).

Suppose that $m = 1$. As D^* is an irreducible surface, we have

$$L_1 H_l(D^*) \simeq H_l(D^*)$$

for any $l \geq 3$. This implies that the action of β is zero for $m = 1$ and $l \geq 3$ (see Diagram (5)). As we already know that the action of α is zero for $m = 1$, it implies that

$$L_1 H_l(X)_{\mathbb{Q}} \simeq H_l(X)_{\mathbb{Q}}$$

for any $l \geq 3$. Applying Corollary 3.1 we obtain

$$L_1 H_l(X) \simeq H_l(X)$$

for any $l \geq 3$.

Suppose that $m = 2$. As V^* is an irreducible surface, we have

$$L^1 H^{6-l}(V^*) \simeq H^{6-l}(V^*)$$

for any $l \geq 5$, which means that the action of α is zero on $L_2 H_l(X)$ for any $l \geq 5$. As we know that for $m = 2$ the action of β is zero we can conclude as above that

$$L_2 H_l(X) \simeq H_l(X)$$

for any $l \geq 5$.

The injectivity with integer coefficients in (a) and (c) follows from the result of Bloch–Srinivas used in the Proposition 5.2 and from the fact that algebraic equivalence coincides with homological equivalence for divisors. The isomorphism $L_2 H_4(X) \simeq H^{1,1}(X, \mathbb{Z})$ comes from Lefschetz (1, 1) theorem. □

Using now Theorem 2.5 for $k = 0$, together with Propositions 5.2 and 5.3 we can conclude Theorem 5.1. □

Remark 5.1. For X and V as in the Theorem 5.1 and $dim(V) \leq 1$ the injection

$$K_0^{sst}(X) \hookrightarrow ku^0(X^{an})$$

is moreover a rational isomorphism.

Convention 5.1. *Remark 4.1 shows that it is enough to study the action of an irreducible cycle. In the rest of the paper, without reducing the generality, we will understand that a decomposition of the form $N\Delta = \alpha + \beta$, with α supported on $V \times X$ and β supported on $X \times D$, has V and D irreducible varieties.*

The next theorem computes K^{sst} for certain "degenerate" smooth fourfolds.

Theorem 5.3. *Let X be a smooth projective fourfold such that there is a proper subvariety $V \subset X$ of $dim(V) \leq 2$ with $CH_0(X \setminus V) = 0$. Then:*

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an}), \quad i \geq 3,$$
$$K_2^{sst}(X) \hookrightarrow ku^{-2}(X^{an}),$$
$$K_i^{sst}(X)_{\mathbb{Q}} \simeq ku^{-i}(X^{an})_{\mathbb{Q}}, \quad i = 1, 2,$$
$$K_0^{sst}(X)_{\mathbb{Q}} \hookrightarrow ku^0(X^{an})_{\mathbb{Q}}.$$

Some examples of varieties which fulfill the conditions of the theorem are: rationally connected fourfolds, certain quotient varieties as in [4] etc.

Proof. The proof is similar to the proof of the Theorem 5.1. It is a corollary of the spectral sequence relating morphic cohomology groups and K^{sst} and of the computation of some Lawson groups given in the propositions below.

Proposition 5.4. *Let X be a smooth projective fourfold such that there is a proper subvariety $V \subset X$ of $dim(V) \leq 1$ with $CH_0(X \setminus V) = 0$. Then:*

(a) $L_1H_2(X)_{\mathbb{Q}} \simeq H_2(X)_{\mathbb{Q}}$.
(b) $L_1H_3(X)_{\mathbb{Q}} \simeq H_3(X)_{\mathbb{Q}}$.
(c) $L_2H_4(X) \hookrightarrow L_1H_4(X) \simeq H_4(X)$.
(d) $L_2H_5(X) \simeq L_1H_5(X) \simeq H_5(X)$.
(e) $L_3H_6(X) \simeq L_2H_6(X) \simeq L_1H_6(X) \simeq H_6(X)$.
(f) $L_3H_7(X) \simeq L_2H_7(X) \simeq L_1H_7(X) \simeq H_7(X)$.
(g) $L_4H_8(X) \simeq L_3H_8(X) \simeq L_2H_8(X) \simeq L_1H_8(X) \simeq H_8(X)$.
(h) $L_kH_n(X) = 0$ *for any* $n \geq 9$ *and any* $k \geq 0$.

In particular any such a fourfold fulfills Suslin's conjecture.

Proof. Consider the decomposition

$$N\Delta = \alpha + \beta$$

with α supported on $V \times X$ and β supported on $X \times D$. The action of α is zero for $m \leq 3 = codim(V)$ and the action of β is zero for $m \geq 3 = dim(D)$. This implies that

$$L_3H_*(X) \otimes \mathbb{Q} \simeq H_*(X) \otimes \mathbb{Q}$$

and because of Corollary 3.1, we obtain

$$L_3H_*(X) \simeq H_*(X).$$

As D^* is a smooth irreducible threefold, we have that the cycle map $L_2H_l(D^*) \to H_l(D^*)$ is an isomorphism for $l \geq 5$ and a monomorphism for $l = 4$. This implies that the action of β on the kernel and cokernel of the cycle map

$$L_2H_l(X) \to H_l(X)$$

is zero for $l \geq 5$. As we already know that the action of α is zero for $m = 2$, we conclude using Corollary 3.1 that

$$L_2H_l(X) \simeq H_l(X)$$

for any $l \geq 5$. The injection from the point (c) comes from the fact that for such varieties algebraic equivalence and homological equivalence coincide on codimension 2 cycles.

Consider now the decomposition

$$N\Delta = \alpha^t + \beta^t$$

with α^t supported on $X \times V$ and β^t supported on $D \times X$. The action of α^t is zero for $m \geq dim(V) = 1$ and the action of β^t is zero for $m \leq codim(D) = 1$ (see Convention 4.1). This implies that

$$L_1H_l(X) \otimes \mathbb{Q} \simeq H_l(X) \otimes \mathbb{Q}$$

for any $l \geq 2$ and from Corollary 3.1 we obtain

$$L_1H_l(X) \simeq H_l(X)$$

for any $l \geq 4$. □

Proposition 5.5. *Let X be a smooth projective fourfold such that there is a proper subvariety $V \subset X$ of $dim(V) = 2$ with $CH_0(X \setminus V) = 0$. Then:*

(a) $L_1H_2(X)_{\mathbb{Q}} \hookrightarrow H_2(X)_{\mathbb{Q}}$.
(b) $L_1H_3(X)_{\mathbb{Q}} \simeq H_2(X)_{\mathbb{Q}}$.
(c) $L_2H_4(X) \hookrightarrow L_1H_4(X) \simeq H_4(X)$.
(d) $L_2H_5(X) \simeq L_1H_5(X) \simeq H_5(X)$.
(e) $L_3H_6(X) \hookrightarrow L_2H_6(X) \simeq L_1H_6(X) \simeq H_6(X)$.

(f) $L_3H_7(X) \simeq L_2H_7(X) \simeq L_1H_7(X) \simeq H_7(X)$.
(g) $L_4H_8(X) \simeq L_3H_8(X) \simeq L_2H_8(X) \simeq L_1H_8(X) \simeq H_8(X)$.
(h) $L_kH_n(X) = 0$ *for any* $n \geq 9$ *and any* $k \geq 0$.
In particular any such a fourfold fulfills Suslin's conjecture.

Proof. Consider the decomposition

$$N\Delta = \alpha + \beta$$

with α supported on $V \times X$ and β supported on $X \times D$. The action of α is zero for $m \leq codim(V) = 2$ and the action of β is zero for $m \geq dim(D) = 3$ (see Convention 4.1).

As D^* is a smooth threefold, we have that the cycle map

$$L_2H_l(D^*) \to H_l(D^*)$$

is an isomorphism for $l \geq 5$ and a monomorphism for $l = 4$. This implies that the action of β on the kernel and cokernel of the cycle map

$$L_2H_l(X) \to H_l(X)$$

is zero for $l \geq 5$. As we already know that the action of α is zero for $m = 2$, we conclude that

$$L_2H_l(X) \simeq H_l(X)$$

for any $l \geq 5$. The injection from the point (c) comes from the fact that for such varieties algebraic equivalence and homological equivalence coincide on codimension 2 cycles [4].

Consider the action of α on the kernel and the cokernel of the cycle maps

$$L_3H_l(X) \to H_l(X).$$

This action factors through $L^1H^{8-l}(V^*) \simeq L_1H_{l-4}(V^*)$ (see Diagram (5)). As V^* is an irreducible surface, we have that the cycle map

$$L_1H_{l-4}(V^*) \to H_{l-4}(V^*)$$

is an isomorphism for any $l \geq 7$ and injective for $l = 6$. As the action of β is zero for $m = 3$, we have that

$$L_3H_l(X) \otimes \mathbb{Q} \simeq H_l(X) \otimes \mathbb{Q}$$

for any $l \geq 7$. From Corollary 3.1 we conclude that

$$L_3H_l(X) \simeq H_l(X)$$

for any $l \geq 7$. The injectivity in point (e) comes from the fact that on divisors algebraic equivalence coincides with homological equivalence.

Consider now the decomposition

$$N\Delta = \alpha^t + \beta^t$$

with α^t and β^t being supported on $X \times V$, respectively $D \times X$. The action of α^t is zero for $m \geq dim(V) = 2$ and the action of β^t is zero for $m \leq codim(D) = 1$. As V^* is a surface, we have that

$$L_1H_l(V^*) \to H_l(V^*)$$

is an isomorphism for any $l \geq 3$ and a monomorphism for $l = 2$. This implies that

$$L_1H_l(X) \otimes \mathbb{Q} \to H_l(X) \otimes \mathbb{Q}$$

is an isomorphism for any $l \geq 3$ and a monomorphism for $l = 2$. Using Corollary 3.1 we can conclude that

$$L_1H_l(X) \simeq H_l(X)$$

for any $l \geq 4$. □

Using now Theorem 2.5 and our Propositions 5.4 and 5.5 we can conclude our Theorem 5.3. □

Remark 5.2. As smooth cubic fourfolds show, the injectivity

$$K_0^{sst}(X)_{\mathbb{Q}} \hookrightarrow ku^0(X^{an})_{\mathbb{Q}}$$

is the best we can obtain. This is because a generic cubic fourfold has nontrivial nondiagonal Hodge numbers.

The following proposition was previously known only in the case of generic cubic hypersurfaces [21] which are known to be rationally connected varieties.

Proposition 5.6. *Let X be a projective smooth variety of dimension $d \geq 3$ and suppose that there is a subvariety $V \subset X$ of $dim(V) \leq 2$ with $CH_0(X \setminus V) = 0$. Then*

$$N^1H^d(X, \mathbb{Z}) = H^d(X, \mathbb{Z}).$$

Proof. Without restricting the generality we may suppose that $dim(V) = 2$. We consider the following decomposition of the diagonal

$$N\Delta = \alpha^t + \beta^t$$

with α^t supported on $X \times V$ and β^t supported on $D \times X$. As before we obtain that the action of α^t is zero for $m \geq 2 = dim(V)$ and the action of β^t is zero for $m \leq codim(D) = 1$ (see Convention 4.1). Let V^* be a desingularization of V. As V^* is a surface, we have

$$L_1H_l(V^*) \otimes \mathbb{Q} \simeq H_l(V^*) \otimes \mathbb{Q}$$

for any $l \geq 3$. This implies that

$$L_1H_l(X)_{\mathbb{Q}} \simeq H_l(X)_{\mathbb{Q}}$$

for any $l \geq 3$. From Corollary 3.1 we get

$$L_1H_d(X) \simeq H_d(X)$$

and because the image of this cycle map is included in $N_{d-1}H_d(X) = N^1H^d(X)$, we obtain our conclusion. □

6. About varieties with small Chow group

It is proved by Jannsen [22] and independently by Laterveer [25] that if the following cycle maps are injective

$$CH_k(X) \otimes \mathbb{Q} \hookrightarrow H_{2k}(X) \otimes \mathbb{Q}$$

for any $0 \leq k \leq r$ (in which case, we say that X has small Chow group of rank r [8,25]), then we have the following decomposition of the diagonal

$$N\Delta = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_r + \beta \tag{6}$$

where α_i are supported on $V_i \times W_{d-i}$ and β is supported on $X \times \Gamma^{r+1}$ and N is a nonzero natural number (the lower indices represent the dimension of the subvariety and the upper indices represent the codimension of the subvariety). We let d to be the dimension of the variety X.

The following theorem and corollary are extensions of the main results of Peters [30].

Theorem 6.1. *Let X be a smooth projective variety with small Chow group of rank r. Then there is a natural nonzero number N such that*

(a) $NK_{s,*} = 0$ *for any* $s \in \{0, 1, \ldots, r+1\}$ [30].

(b) $NK^{s,*} = 0$ *for any* $s \in \{0, 1, \ldots, r+2\}$.

Proof. As X has small Chow group of rank r, the diagonal cycle decomposes as in (6). As the cycles α_i are supported on $V_i \times W_{d-i}$ with $dim(V_i) + dim(W_{d-i}) = dim(X)$, we know from Corollary 4.1 that the action of α_i is zero for any m (see Convention 4.1). This implies that

$$N\Delta_* = \beta_*$$

on the kernel and the cokernel of the cycle map

$$L_m H_l(X) \to H_l(X).$$

But the action of β is zero for $m \geq d-r-1$ because it factors through $L_m H_l(\Gamma^{r+1})$ (see Diagram (5)). We know [10] that the cycle map

$$L_{d-r-2} H_*(\Gamma^{r+1*}) \to H_*(\Gamma^{r+1*})$$

is injective (where $\Gamma^{r+1*} \to \Gamma^{r+1}$ is a resolution of singularities). This implies that the action of β on the kernel of the cycle map

$$L_m H_l(X) \to H_l(X)$$

is zero for any $m \geq d-r-2$. This means that

$$N(Ker(L^m H^*(X) \to H^*(X))) = 0$$

for $0 \leq m \leq r+2$

Point (a) was proved in [30]. □

Corollary 6.1. *Let X be a smooth projective variety such that rational equivalence coincides with homological equivalence in $CH_*(X) \otimes \mathbb{Q}$ in degrees less than or equal to r. Then the algebraic equivalence coincides with homological equivalence in $CH_*(X) \otimes \mathbb{Q}$ in degrees less than or equal to $r+1$* [30] *and in degrees greater than or equal to $d-r-2$.*

Proof. This is just a reformulation of Theorem 6.1. □

Remark 6.1. It is conjectured [28] that for a smooth complete intersection X in $\mathbb{P}^{n+1}$ of multi-degree $d_1 \geq d_2 \geq \cdots \geq d_s \geq 2$ and of dimension d we have

$$CH_l(X)_{\mathbb{Q}} \simeq \mathbb{Q}$$

for any $l \leq k-1$, where $k = [\frac{n+1-\sum_{i=2,s} d_i}{d_1}]$, the integer part of the rational number $\frac{n+1-\sum_{i=2,s} d_i}{d_1}$.

In particular this would imply that X has small Chow group of rank $k-1$. Supposing this conjecture and using Theorem 6.1 we conclude that in our case we have

$$Griff^r(X)_{\mathbb{Q}} = 0$$

for any $r \geq d-k-1$ and $r \leq k$ and moreover

$$K^{r,*}_{\mathbb{Q}} = 0$$

for the same range of indices.

H. Esnault, M. Levine and E. Viehweg made in [7] an attempt to prove this conjecture on a class of complete intersections. We will analyze below some applications of their results in the context of morphic cohomology.

It is known that the conjecture from Remark 6.1 is valid for generic cubic fivefold and sixfold [28]. The next proposition studies the Lawson homology groups of such cubics.

Proposition 6.1. (a) *Let X be a smooth generic cubic of dimension $d = 5$. Then*

$$L_* H_*(X) \otimes \mathbb{Q} \simeq H_*(X) \otimes \mathbb{Q}$$

for all defined indices.

(b) *Let X be a smooth generic cubic of dimension $d = 6$. Then*

(1) *$L_m H_l(X) \otimes \mathbb{Q} \simeq H_{l+2m}(X) \otimes \mathbb{Q}$ for any $m \neq 3$ and $l \geq 2m$ or for $m = 3$ and any $l \geq 7$.*

(2) *$L_3 H_6(X) \hookrightarrow H_6(X)$ is an injective map.*

In particular Suslin's conjecture is valid for both the generic cubics X.

Proof. In [28] it is proved that a generic smooth cubic of dimension $d \geq 5$ has

$$CH_0(X) \simeq CH_1(X) \simeq \mathbb{Z}.$$

This implies that there is a decomposition

$$N\Delta = \alpha_0 + \alpha_1 + \beta$$

with α_i supported on $V_i \times W_{d-i}$ and β supported on $X \times \Gamma^2$, cycles of codimension d in $X \times X$. As in Theorem 6.1, we get the equality $N\Delta_* = \beta_*$ on the kernel and the cokernel of the cycle map $L_m H_*(X) \to H_*(X)$ for any m.

Suppose that $d = 5$. As the action of β is zero for $m \geq dim(\Gamma^2) = 3$ and the action of β^t is zero for $m \leq codim(\Gamma^2) = 2$, we obtain that the Lawson homology of a generic smooth cubic fivefold is isomorphic with singular homology up to torsion, i.e.

$$L_* H_*(X) \otimes \mathbb{Q} \simeq H_*(X) \otimes \mathbb{Q}.$$

Using Corollary 3.1, we obtain Suslin's conjecture for generic smooth cubic fivefold.

Suppose $d = 6$. Then the action of β is zero for $m \geq 4 = dim(\Gamma_{d-2})$ and the action of β^t is zero for $m \leq 6 - 4 = 2 = codim(\Gamma_{d-2})$. We remark that the action of α_i is zero for any $m \geq 1$. As above, we conclude that for any generic smooth cubic sixfold

$$L_m H_*(X) \otimes \mathbb{Q} \simeq H_*(X) \otimes \mathbb{Q}$$

for any $m \geq 4$ and any $m \leq 2$. The action of β on $L_3 H_l(X)$ factors through $L_3 H_l(\Gamma_4) \to H_l(\Gamma_4)$ which is an isomorphism for any $l \geq 7$ and a monomorphism for $l = 6$. It implies that the cycle map $L_3 H_6(X) \otimes \mathbb{Q} \to H_6(X) \otimes Q$ is injective and that $L_3 H_l(X) \otimes \mathbb{Q} \simeq H_l(X) \otimes \mathbb{Q}$ for any $l \geq 7$. Using now Corollary 3.1 we conclude that Suslin's conjecture holds for any generic smooth cubic sixfold. □

Corollary 6.2. *Let X be a smooth generic cubic fivefold. Then*

$$K_*^{sst}(X) \otimes \mathbb{Q} \simeq ku^{-*}(X) \otimes \mathbb{Q}$$

for any $* \geq 0$.

Corollary 6.3. *Let X be a smooth generic cubic sixfold. Then*

$$K_i^{sst}(X) \otimes \mathbb{Q} \simeq ku^{-i}(X) \otimes \mathbb{Q}$$

for any $i \geq 1$ *and*

$$K_0^{sst}(X) \otimes \mathbb{Q} \hookrightarrow ku^{-0}(X) \otimes \mathbb{Q}.$$

Theorem 6.2. *Let X be a smooth projective variety. If all the cycle maps $CH^*_{\mathbb{Q}}(X) \to H^*_{\mathbb{Q}}(X)$ are injective, then $L_* H_*(X)_{\mathbb{Q}} \simeq H_*(X)_{\mathbb{Q}}$ for any possible indices.*

In particular Suslin's conjecture is valid for such an X.

Proof. Our condition on X gives us the following decomposition of the diagonal

$$N\Delta = \alpha_0 + \alpha_1 + \cdots + \alpha_d$$

where each α_i is supported on $V_i \times W_{d-i}$. We remark that $dim(V_i) + dim(W_{d-i}) = dim\, X = d$. Using Corollary 4.1 we can conclude that

$$Nx = N\Delta_*(x) = 0$$

for any x in the kernel and in the cokernel of the cycle map

$$L_* H_*(X) \to H_*(X).$$

This implies that

$$L_m H_l(X) \otimes \mathbb{Q} \simeq H_l(X) \otimes \mathbb{Q}.$$

We conclude now Suslin's conjecture for X by using Corollary 3.1. □

We notice that the same techniques used in Theorem 6.1 and in Theorem 6.2 give us the following proposition, initially proved by J. Lewis and, independently, by C. Schoen.

Proposition 6.2. *Let X be a smooth projective complex variety such that the cycle class maps*

$$cl : CH_l(X)_{\mathbb{Q}} \to H_{2l}(X, \mathbb{Q})$$

are injective for $0 \le l \le k$. Then $H^{p,q}(X) = 0$ for

(a) *$p \neq q$, $p + q$ even and $q \le k$.*

(b) *$| p - q |> 1$, $p + q$ odd and $q \le k$.*

Proof. The vanishing of the above Hodge numbers comes from the equalities in the coniveau filtration implied by the decomposition of the diagonal and from the fact that coniveau filtration with rational coefficients is included in the Hodge filtration. □

Let $X \subset \mathbb{P}^{n+1}$ be a complete intersection complex variety defined by r equations of degree $d_1 \ge d_2 \ge \cdots \ge d_r \ge 2$ and $k = [\frac{n+1-\sum_{i=2}^{r} d_i}{d_1}]$. H. Esnault, M. Levine and E. Viehweg proved the following result [7]:

Theorem 6.3 ([7]). *Let $X \subset \mathbb{P}^{n+1}$ be the union of some of the irreducible components of the intersection of r hyperplanes of degree $d_1 \ge d_2 \ge \cdots \ge d_r \ge 2$ and either $d_1 \ge 3$ or $r \ge l + 1$. If*

$$\sum_{i=1}^{r} \binom{d_i + l}{l + 1} \le n + 1$$

then $CH_s(X)_{\mathbb{Q}} \simeq \mathbb{Q}$ for any $0 \le s \le l$.

If $d_1 = d_2 = \cdots = d_r = 2$ and $r \le l$ we have the same conclusion assuming the inequality

$$r(l + 1) \le n - l + r$$

Corollary 6.4. *Assume the conditions from the above theorem. Then*

$$L_s H_*(X)_{\mathbb{Q}} \simeq H_*(X)_{\mathbb{Q}}$$

for any $s \ge n - l - 1$ and $s \le l + 1$.

Proof. This corollary follows from Theorem 6.3 using the same techniques used in the proof of Proposition 6.1. □

7. The case of projective linear varieties

Totaro [36] and Jannsen [22] gave the definition of a linear variety (see also [23]).

Definition. A complex variety is called 0-linear if it is either empty set or isomorphic to any affine space $\mathbb{A}^n_{\mathbb{C}}$. Let $n > 0$. A complex variety Z is n-linear if there is a triple (U, X, Y) of complex varieties so that $Y \subset X$ is a closed immersion with U its complement; Y and one of the varieties U or X is $(n - 1)$-linear and Z is the other member in U, X. We say that Z is linear if it is n-linear for some $n \ge 0$.

Among examples of linear varieties are toric varieties, flag varieties [36,22,23]. Joshua [23] and Totaro [36] proved the following *Künneth* formula for projective linear varieties:

Theorem 7.1 ([36,23]). *Let X be a projective smooth linear variety of dimension d. Then*

$$CH^*(X) \otimes CH^*(X) \simeq CH^*(X \times X).$$

In particular there is a decomposition of the diagonal cycle $\Delta \in CH^d(X \times X)$ of the form

$$\Delta = \sum \alpha_i \times \beta_i$$

with $\alpha_i, \beta_i \in CH^*(X)$ algebraic cycles with $dim(\alpha_i) + dim(\beta_i) = d$.

Using Corollary 4.1 and Theorem 7.1 we can conclude that the action of Δ is zero on the kernel and the cokernel of the cycle map $L_m H_*(X) \to H_*(X)$. This implies the following proposition:

Proposition 7.1. *Let X be a projective smooth linear variety. Then*

$$L_* H_*(X) \simeq H_*(X)$$

for any possible indices. In particular we have

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an})$$

for any $i \geq 0$.

The above proposition was first proved in [13] by other methods.

8. The case of projective rational varieties

Let X be a smooth projective rational variety over complex numbers. In this section, we compute some Lawson homology groups of X. These computations generalize and explain some the results of [13,20].

Theorem 8.1. *Let X be a smooth projective rational variety of dimension d. Then:*
(a) $L_1 H_{*+2}(X) \simeq H_{*+2}(X)$ *for any* $* \geq 0$.
(b) $L_{d-2} H_*(X) \simeq H_*(X)$ *for any* $* \geq 2d-3$ *and* $L_{d-2} H_{2d-4}(X) \hookrightarrow H_{2d-4}(X)$.

Proof. As X is a smooth rational variety, we can choose $N = 1$ in Bloch–Srinivas decomposition of the diagonal (see Proposition 5.1). Now applying the same idea of proof as in Proposition 5.4 we obtain the result. $\square$

Corollary 8.1. *Let X be a smooth rational fourfold. Then:*
(a) $L_1 H_2(X) \simeq H_2(X)$.
(b) $L_1 H_3(X) \simeq H_2(X)$.
(c) $L_2 H_4(X) \simeq H^{2,2}(X, \mathbb{Z}) \hookrightarrow L_1 H_4(X) \simeq H_4(X)$.
(d) $L_2 H_5(X) \simeq L_1 H_5(X) \simeq H_5(X)$.
(e) $L_3 H_6(X) \simeq L_2 H_6(X) \simeq L_1 H_6(X) \simeq H_6(X)$.
(f) $L_3 H_7(X) \simeq L_2 H_7(X) \simeq L_1 H_7(X) \simeq H_7(X)$.
(g) $L_4 H_8(X) \simeq L_3 H_8(X) \simeq L_2 H_8(X) \simeq L_1 H_8(X) \simeq H_8(X)$.
(h) $L_k H_n(X) = 0$ *for any* $n \geq 9$ *and any* $k \geq 0$.

Proof. It follows directly from Theorem 8.1. As integral Hodge conjecture holds for codimension 2 cycles on a rational variety [33], we obtain

$$L_2 H_4(X) \simeq H^{2,2}(X, \mathbb{Z}). \quad \square$$

The next corollary was proved in [13] by other methods.

Corollary 8.2. *Let X be a smooth rational fourfold. Then:*

$$K_*^{sst}(X) \simeq ku^{-*}(X^{an})$$

for any $* \geq 1$ and

$$K_0^{sst}(X) \hookrightarrow ku^0(X^{an}).$$

Proof. It follows directly from Corollary 8.1 using the spectral sequence relating morphic cohomology and semi-topological K-theory. $\square$

Remark 8.1. For a smooth rational fourfold, Corollary 8.2 is the best that can be obtained. This is because there are rational varieties with nonzero nondiagonal Hodge numbers.

Lawson homology of a smooth rational threefold can be obtained without the use of Bloch–Kato conjecture, as a corollary of Theorem 8.1.

Corollary 8.3. *Let X be a smooth rational threefold. Then*

$$L_*H_*(X) \simeq H_*(X)$$

for all defined indices.

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