Global Attractors for the Klein–Gordon–Schrödinger Equation in Unbounded Domains

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In this paper, we study the long time behavior of solutions for the Klein–Gordon–Schrödinger system in the whole space $\mathbb{R}^n$ with $n \geq 3$. We first prove the continuity of the solutions on initial data and then establish the asymptotic compactness of solutions. Finally, we show the existence of the global attractor for this model in the space $H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ for each integer $k \geq 1$.

1. INTRODUCTION

In this paper, we study the long time behavior of solutions for the following dissipative Klein–Gordon–Schrödinger system in the whole space $\mathbb{R}^n$ with $n \leq 3$,

\begin{align}
\imath \psi_t + A\psi + \imath v \psi + \phi \psi &= f, \\
\phi_t + \gamma \phi_t - 4\phi + \phi - |\psi|^2 &= g,
\end{align}

where $\psi$ is a complex valued function and $\phi$ is a real valued function, $v$ and $\gamma$ are positive constants, $f$ and $g$ are driving terms. System (1.1)-(1.2) describes the interaction of a nucleon field $\psi$ and a meson field $\phi$ through the Yukawa coupling. The dissipative mechanism of this model is introduced by the terms $\imath v \psi$ and $\gamma \phi_t$. 

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The well-posedness problem of system (1.1)-(1.2) was studied by Fukuda and Tsutsumi [1, 2], Bachelot [3], and Hayashi and von Wahl [4]. In [5], Biler studied the long time behavior of solutions for system (1.1)-(1.2) in a bounded domain and proved the existence of a weak global attractor in the Hilbert space $H^1 \times H^1$. Recently, Wang and Lange [6] improved this result and showed that the weak global attractor is, in fact, the strong one. In [7], Guo and Li considered system (1.1)-(1.2) in $\mathbb{R}^3$ and proved the existence of a global attractor in $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ which attracts bounded sets of $H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ in the topology of $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$. The lack of the compact embedding here causes the difficulty to show the attractor attracts the bounded sets in $H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$.

In present paper, we consider system (1.1)-(1.2) in $\mathbb{R}^n$ for $1 \leq n \leq 3$ and prove the existence of a global attractor in the space $H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ for $k \geq 1$ which attracts all bounded sets of $H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ in the norm topology. The difficulty caused by the non-compact imbedding is overcome.

In order to study the asymptotic behavior of solutions for system (1.1)-(1.2), we first prove the continuity of solutions with respect to initial data in $H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ for each integer $k \geq 1$. This can be done by the standard method when $k \geq 2$. However, the continuity of solutions with respect to the initial data in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ seems non-trivial and needs to be treated separately. For the space $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, we first establish an energy equation and then apply this energy equation to prove the continuity of solutions in the initial data.

Next, we establish the asymptotic compactness of solutions for system (1.1)-(1.2) in $\mathbb{R}^n$, which is a crucial step to get the global attractor. The method used by Wang and Lange [6] for system (1.1)-(1.2) in a bounded domain can not be directly employed to our case since the Sobolev embeddings are no longer compact. To overcome this difficulty, we approximate the space $\mathbb{R}^n$ by a large ball and then establish detailed estimates for solutions on the complement of the ball. A key observation is that the solutions of system (1.1)-(1.2) on the complement of a sufficiently large ball are uniformly small for large times. This together with the energy equation yields the asymptotic compactness of the solutions in $\mathbb{R}^n$.

We mention that the method of using energy equations to prove the asymptotic compactness is essentially due to Ball [8]. This idea was also exploited later by Temam [9], Wang [10], Ghidaglia [11], Wang [12] and the others.

We organize this paper as follows. In Section 2, we derive a priori estimates on the solutions of system (1.1)-(1.2) in $H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ for each integer $k \geq 1$. These estimates will be used to prove the existence of bounded absorbing sets and the asymptotic compactness. Section 3 is devoted to the existence of a continuous dynamical system associated with problem (1.1)-(1.2) in the space $H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$. We first establish an
energy equation for solutions in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ and then prove the continuity of solutions on initial data in that space. We shall also give the continuous dependence of the solutions on initial data in the space $H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ for $k \geq 2$. In Section 4, we show the asymptotic compactness of the dynamical system and prove the existence of the global attractor in the space $H^k(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ with $k \geq 1$ by an abstract result from [9, 13–16].

2. A PRIORI ESTIMATES

In this section, we derive a priori estimates on solutions of Klein–Gordon–Schrödinger equations which will be used when we prove the continuity of solutions and existence of global attractors.

We should mention that the method used by Wang and Lange [6] for system (1.1)–(1.2) in a bounded domain can not be directly employed to our case since the Sobolev embeddings are no longer compact. To overcome this difficulty, we approximate the space $\mathbb{R}^n$ by a large ball and then establish detailed estimates for solutions on the complement of the ball. A key observation is that the solutions of system (1.1)–(1.2) on the complement of a sufficiently large ball are uniformly small for large times.

We first introduce the transformation $\theta = \phi_i + \delta \phi$ with $\delta$ a small positive constant which will be specified later. Then, system (1.1)–(1.2) becomes

\[
\begin{align*}
    i\psi_t + A\psi + iv\psi + \phi \psi &= f & \text{in } \mathbb{R}^n \times \mathbb{R}^+ , & (2.1) \\
    \phi_t + (\gamma - \delta) \theta - A\phi + (1 - \delta(\gamma - \delta)) \phi - |\psi|^2 &= g & \text{in } \mathbb{R}^n \times \mathbb{R}^+ , & (2.2) \\
    \theta_t + (\gamma - \delta) \theta - (1 - \delta(\gamma - \delta)) \phi - |\psi|^2 &= g & \text{in } \mathbb{R}^n \times \mathbb{R}^+ , & (2.3)
\end{align*}
\]

where $n \leq 3$. Problem (2.1)–(2.3) is supplemented with the initial condition

\[
\psi(x, 0) = \psi_\delta(x), \quad \phi(x, 0) = \phi_\delta(x), \quad \theta(x, 0) = \theta_\delta(x), \quad x \in \mathbb{R}^n .
\]

In the sequel, we denote by $H^s(\mathbb{R}^n)$ both the standard real and complex Sobolev spaces and $H=L^2(\mathbb{R}^n)$. We also use $\| \cdot \|$ and $\langle \cdot , \cdot \rangle$ for the usual norm and inner product of $L^2(\mathbb{R}^n)$. For any $1 \leq p \leq \infty$, we denote by $\| \cdot \|_p$ the norm of $L^p(\mathbb{R}^n)$ ($\| \cdot \|_2 = \| \cdot \|$). In general, $\| \cdot \|_X$ denotes the norm of Banach space $X$.

We recall the following inequalities which will be used frequently later:

\[
\begin{align*}
    \| u \|_\infty &\leq C \| u \|_{H^4}^{n/4} \| u \|_1^{1-(n/4)}, \quad u \in H^2(\mathbb{R}^n), \quad n \leq 3 , & (2.5) \\
    \| u \|_3 &\leq C \| u \|_{H^6}^{n/6} \| u \|_1^{1-(n/6)}, \quad u \in H^1(\mathbb{R}^n), \quad n \leq 3 . & (2.6)
\end{align*}
\]
Hereafter, we denote by $C$ any positive constants which may change from line to line.

We are now in a position to derive the estimates on solutions of problem (2.1)-(2.4). We start with the estimates in $H$.

**Lemma 2.1.** Let $f$ belong to $H$. Then, every solution of problem (2.1)-(2.4) satisfies

$$
\| \psi(t) \| \leq C_1, \quad t \geq t_1,
$$

where $C_1$ is a constant depending only on $v$ and $\| f \|$; $t_1$ depending on $v$ and $\| f \|$ and $R$ when $\| \psi_0 \| \leq R$.

**Proof.** Taking the imaginary part of the inner product of (2.1) with $\psi$ in $H$, we get

$$
\frac{1}{2} \frac{d}{dt} \| \psi \|^2 + v \| \psi \|^2 = \text{Im} \int_{\Re^N} f \overline{\psi}.
$$

In the sequel, we denote by $\overline{\psi}$ the conjugate of $\psi$. Obviously, the right-hand side of (2.7) is bounded by

$$
\| f \| \| \psi \| \leq \frac{1}{2} v \| \psi \|^2 + \frac{1}{2v} \| f \|^2,
$$

so we have

$$
\frac{d}{dt} \| \psi \|^2 + v \| \psi \|^2 \leq \frac{1}{v} \| f \|^2.
$$

By Gronwall lemma we get (when $\| \psi(0) \| \leq R$)

$$
\| \psi(t) \|^2 \leq e^{-\alpha t} \| \psi(0) \|^2 + \frac{1}{v} \| f \|^2 \leq e^{-\alpha t} R^2 + \frac{1}{v} \| f \|^2 \leq \frac{2}{v} \| f \|^2,
$$

for $t \geq t_1 := (1/v) \ln(vR^2 \| f \|^2)$, which concludes the proof of Lemma 2.1.

The following lemma is concerned with the estimates in $H^1 \times H^1 \times H$.

**Lemma 2.2.** Assume that $f$ and $g$ belong to $H$. Then, there exists a constant $\delta_1$ such that when $\delta \leq \delta_1$, every solution $(\psi, \phi, \theta)$ of problem (2.1)-(2.4) satisfies

$$
\| \psi(t) \|_{H^1} + \| \phi(t) \|_{H^1} + \| \theta(t) \| \leq M, \quad t \geq t_2,
$$

where $M$ depends on $(v, \gamma, \delta, \| f \|, \| g \|)$; $t_2$ depends on $(v, \gamma, \delta, \| f \|, \| g \|)$ and $R$ when $\| (\psi_0, \phi_0, \theta_0) \|_{H^1 \times H^1 \times H} \leq R$. 

Proof. We give only the proof of the lemma for \( n = 3 \) since the case of \( n \leq 2 \) is simpler.

Taking the real part of the inner product of (2.1) with \(-\psi_i\) in \( H\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \psi\|^2 + v \text{ Im} \int_{\mathbb{R}^n} \psi \bar{\psi}_i - \frac{1}{2} \int_{\mathbb{R}^n} \phi \frac{\partial}{\partial t} |\psi|^2 + \text{ Re} \int_{\mathbb{R}^n} f \bar{\psi}_i = 0. \tag{2.9}
\]

By (2.1) again, we have

\[
v \text{ Im} \int_{\mathbb{R}^n} \psi \bar{\psi}_i = v \|\nabla \psi\|^2 - \nu \int_{\mathbb{R}^n} \phi |\psi|^2 + v \text{ Re} \int_{\mathbb{R}^n} f \bar{\psi}_i. \tag{2.10}
\]

By (2.9) and (2.10) we get

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \psi\|^2 + v \|\nabla \psi\|^2 - v \int_{\mathbb{R}^n} \phi |\psi|^2 + v \text{ Re} \int_{\mathbb{R}^n} f \bar{\psi}_i
\]

\[
- \frac{1}{2} \int_{\mathbb{R}^n} \phi \frac{\partial}{\partial t} |\psi|^2 + \text{ Re} \int_{\mathbb{R}^n} f \bar{\psi}_i = 0. \tag{2.11}
\]

Using (2.2) we find

\[
\int_{\mathbb{R}^n} \phi \frac{\partial}{\partial t} |\psi|^2 = \frac{d}{dt} \int_{\mathbb{R}^n} \phi |\psi|^2 - \int_{\mathbb{R}^n} \phi |\psi|^2
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^n} \phi |\psi|^2 + \delta \int_{\mathbb{R}^n} \phi |\psi|^2 - \int_{\mathbb{R}^n} \theta |\psi|^2. \tag{2.12}
\]

By (2.11)–(2.12) we see

\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \psi\|^2 - \int_{\mathbb{R}^n} \phi |\psi|^2 + 2 \text{ Re} \int_{\mathbb{R}^n} f \bar{\psi}_i \right)
\]

\[
+ v \|\nabla \psi\|^2 - \left( v + \frac{1}{2} \delta \right) \int_{\mathbb{R}^n} \phi |\psi|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \theta |\psi|^2 + v \text{ Re} \int_{\mathbb{R}^n} f \bar{\psi}_i = 0. \tag{2.13}
\]

Taking the inner product of (2.3) with \( \theta \) in \( H \), we have

\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + (\gamma - \delta) \|\theta\|^2 - \int_{\mathbb{R}^n} \theta A\phi + (1 - \delta(\gamma - \delta)) \int_{\mathbb{R}^n} \phi \theta - \int_{\mathbb{R}^n} |\psi|^2 \theta
\]

\[
= \int_{\mathbb{R}^n} g \theta. \tag{2.14}
\]
Using (2.2), we get from (2.14) that
\[
\frac{1}{2} \frac{d}{dt} (\|\theta\|^2 + (1 - \delta(\gamma - \delta)) \|\phi\|^2 + \|\nabla \phi\|^2) + (\gamma - \delta) \|\theta\|^2 \\
+ \delta(1 - \delta(\gamma - \delta)) \|\phi\|^2 + \delta \|\nabla \phi\|^2 - \int_{\mathbb{R}^n} |\psi|^2 \theta = \int_{\mathbb{R}^n} g\theta.
\] (2.15)

Then, by 4 \times (2.13) + 2 \times (2.15), we get
\[
\frac{d}{dt} E(t) + \delta E(t) = F(t),
\] (2.16)
where
\[
E = 2 \|\nabla \phi\|^2 + (1 - \delta(\gamma - \delta)) \|\phi\|^2 + \|\nabla \phi\|^2 - 2 \int_{\mathbb{R}^n} |\psi|^2 \phi \\
+ 4 \text{Re} \int_{\mathbb{R}^n} f\tilde{\psi},
\]
and
\[
F = -2(2\gamma - \delta) \|\nabla \phi\|^2 - \delta(1 - \delta(\gamma - \delta)) \|\phi\|^2 - \delta \|\nabla \phi\|^2 \\
- (2\gamma - 3\delta) \|\theta\|^2 + 4\nu \int_{\mathbb{R}^n} \phi \|\psi\|^2 + 4(\delta - \nu) \text{Re} \int_{\mathbb{R}^n} f\tilde{\psi} + 2 \int_{\mathbb{R}^n} g\theta.
\] (2.17)

In order to get the estimates on solutions in $H^1 \times H^1 \times H$, we need to handle each term in (2.17). We first choose a positive constant $\delta_1$ such that when $\delta \leq \delta_1$, the following holds:
\[
2\gamma - \delta > 0, \quad 1 - \delta(\gamma - \delta) > 0, \quad 2\gamma - 3\delta > 0.
\] (2.18)

Then, we majorize the last three terms in (2.17) as follows. First we have
\[
\left| 4\nu \int_{\mathbb{R}^n} \phi \|\psi\|^2 \right| \leq 4\nu \|\phi\|_4 \|\psi\|_3 \|\psi\| \\
\leq C \|\phi\|_{H^1} \|\psi\|_2 \|\psi\|_3^{3/2} \quad \text{(by (2.6))}
\leq C \|\phi\|_{H^1} \|\psi\|_2^{3/2} \quad \text{(by Lemma 2.1)}
\leq \delta(1 - \delta(\gamma - \delta)) \|\phi\|^2 + \delta \|\nabla \phi\|^2 + 2(2\gamma - \delta) \|\nabla \phi\|^2 + C.
\] (2.19)
By Lemma 2.1 again, we find
\[
4(\delta - \nu) \operatorname{Re} \int_{\mathbb{R}^n} f \overline{\psi} \leq C \|f\| \|\psi\| \leq C, \tag{2.20}
\]
and
\[
2 \int_{\mathbb{R}^n} g \theta \leq 2 \|g\| \|\theta\| \leq (2\nu - 3\delta) \|\theta\|^2 + C. \tag{2.21}
\]
From (2.16)–(2.21) it follows that
\[
\frac{d}{dt} E(t) + \delta E(t) \leq C, \quad t \geq t_1,
\]
where \(t_1\) is the constant in Lemma 2.1. Then, Gronwall’s lemma gives Lemma 2.2.

Next, we improve the estimates in the previous lemma for the space \(H^{k+2} \times H^{k+2} \times H^{k+1}\) when \(f\) and \(g\) belong to \(H^k\) with \(k \geq 0\).

**Lemma 2.3.** Assume that \(f\) and \(g\) belong to \(H^k(\mathbb{R}^n)\) with \(k \geq 0\). Then, every solution \((\psi, \phi, \theta)\) of problem (2.1)–(2.4) satisfies
\[
\|\psi(t)\|_{H^{k+2}} + \|\phi(t)\|_{H^{k+2}} + \|\theta(t)\|_{H^{k+1}} \leq M_k, \quad t \geq t_k,
\]
where \(M_k\) depends on \((\nu, \gamma, \delta, \|f\|_{H^k}, \|g\|_{H^k})\) and \(k\); \(t_k\) depends on \((\nu, \gamma, \delta, \|f\|_{H^k}, \|g\|_{H^k})\) and \(k\) and \(R\) when \(\|(\phi_0, \psi_0, \theta_0)\|_{H^{k+2} \times H^{k+2} \times H^{k+1}} \leq R\).

**Proof.** Taking the real part of the inner product of (2.1) with \((-1)^k (A^{k+1} \psi + \nu A^{k+1} \psi)\) in \(H\), we have that
\[
\frac{1}{2} \frac{d}{dt} \|\nabla^{k+2} \psi\|^2 + \nu \|\nabla^{k+2} \psi\|^2 + \nu \operatorname{Re} \int_{\mathbb{R}^n} \nabla^{k+2} \psi \cdot \nabla^{k+2} \phi \overline{\psi}
- \nu \operatorname{Re} \int_{\mathbb{R}^n} \nabla^k f \cdot \nabla^{k+2} \psi + (-1)^k \operatorname{Re} \int_{\mathbb{R}^n} \phi \psi A^{k+1} \overline{\psi}_t
- \operatorname{Re} \int_{\mathbb{R}^n} \nabla^k f \cdot \nabla^{k+2} \overline{\psi}_t = 0. \tag{2.22}
\]
Note that

\[
(-1)^k \text{Re} \int_{\mathbb{R}^n} \phi \psi \cdot A^k \psi = \text{Re} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi,
\]
\[
= \frac{d}{dt} \text{Re} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi
\]
\[
- \text{Re} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi
\]
\[
- \text{Re} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi. \quad (2.23)
\]

By (2.1) and (2.2), we first substitute \(t\) and \(\psi\) in (2.23), and then from (2.22) we find that

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla^k \phi \psi \|^2 + 2 \text{Re} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi - 2 \text{Re} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi \right)
\]
\[
+ \nu \| \nabla^k \phi \psi \|^2 + (2\nu + \delta) \text{Re} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi - \nu \text{Re} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi
\]
\[
- \text{Re} \int_{\mathbb{R}^n} \nabla^k (\theta \psi) \cdot \nabla^k \psi \text{Im} \int_{\mathbb{R}^n} \nabla^k (\phi A \psi) \cdot \nabla^k \psi
\]
\[
- \text{Im} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi + \text{Im} \int_{\mathbb{R}^n} \nabla^k (\phi \psi) \cdot \nabla^k \psi = 0. \quad (2.24)
\]

Now, we derive an energy equation for \(\phi\) and \(\theta\). Taking the inner product of (2.3) with \((-1)^{k+1} A^{k+1} \theta\) in \(H\), we find

\[
\frac{1}{2} \frac{d}{dt} \| \nabla^k \phi \|_2^2 + (\gamma - \Delta) \| \nabla^k \psi \|_2^2 - (-1)^{k+1} \int_{\mathbb{R}^n} A \phi \cdot A^{k+1} \theta
\]
\[
+ (-1)^{k+1} (1 - \Delta (\gamma - \Delta)) \int_{\mathbb{R}^n} \phi \cdot A^{k+1} \theta - (-1)^{k+1} \int_{\mathbb{R}^n} |\psi|^2 \cdot A^{k+1} \theta
\]
\[
= (-1)^{k+1} \int_{\mathbb{R}^n} \Re A^{k+1} \theta. \quad (2.25)
\]
By (2.2) we have

\[
\begin{align*}
-\int_{\mathbb{R}^n} A \phi A^{k+1} \theta + (-1)^{k+1} (1 - \delta(\gamma - \delta)) \int_{\mathbb{R}^n} \phi A^{k+1} \theta \\
= \frac{1}{2} (1 - \delta(\gamma - \delta)) \frac{d}{dt} \|V^{k+1} \phi \|^2 + \delta(1 - \delta(\gamma - \delta)) \|V^{k+1} \phi \|^2 \\
+ \frac{1}{2} \frac{d}{dt} \|V^{k+2} \phi \|^2 + \delta \|V^{k+2} \phi \|^2,
\end{align*}
\]

and

\[
(-1)^{k+1} \int_{\mathbb{R}^n} g \phi A^{k+1} \theta = -\frac{d}{dt} \int_{\mathbb{R}^n} V^k g \cdot V^{k+2} \phi - \delta \int_{\mathbb{R}^n} V^k g \cdot V^{k+2} \phi.
\]

Then, it follows from (2.25)–(2.27) that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \|V^{k+1} \theta \|^2 + (1 - \delta(\gamma - \delta)) \|V^{k+1} \phi \|^2 + \|V^{k+2} \phi \|^2 + 2 \int_{\mathbb{R}^n} V^k g \cdot V^{k+2} \phi \right) \\
+ \delta (\gamma - \delta) \|V^{k+1} \phi \|^2 + \delta(1 - \delta(\gamma - \delta)) \|V^{k+1} \phi \|^2 + \delta \|V^{k+2} \phi \|^2 \\
+ \delta \int_{\mathbb{R}^n} V^k g \cdot V^{k+2} \phi - \int_{\mathbb{R}^n} V^{k+1} |\phi|^2 \cdot \nabla \theta = 0.
\end{align*}
\]

Summing up (2.24) and (2.28), we get that

\[
\frac{d}{dt} E_k(\psi(t), \phi(t), \theta(t)) + \delta E_k(\psi(t), \phi(t), \theta(t)) = F_k(\psi(t), \phi(t), \theta(t)),
\]

where

\[
E_k(\psi, \phi, \theta) = \|V^{k+2} \psi \|^2 + (1 - \delta(\gamma - \delta)) \|V^{k+1} \phi \|^2 + \|V^{k+2} \phi \|^2 \\
+ \|V^{k+1} \theta \|^2 + 2 \text{Re} \int_{\mathbb{R}^n} V^k(\phi \psi) \cdot V^{k+2} \phi - 2 \text{Re} \int_{\mathbb{R}^n} V^k f \cdot V^{k+2} \psi \\
+ 2 \int_{\mathbb{R}^n} V^k g \cdot V^{k+2} \phi.
\]
Thus, this lemma follows from (2.29) and Gronwall’s lemma. The details are omitted here.

In the following, we denote by $B$ the ball

$$B = \{ (\psi, \phi, \theta) \in H^1 \times H^1 \times H : \| \psi \|_{H^1} + \| \phi \|_{H^1} + \| \theta \| \leq M \},$$

and for each $k = 0, \ldots$,

$$B_k = \{ (\psi, \phi, \theta) \in H^{k+2} \times H^{k+2} \times H^{k+1} : \| \psi \|_{H^{k+2}} + \| \phi \|_{H^{k+2}} + \| \theta \|_{H^{k+1}} \leq M_k \},$$

where $M$ and $M_k$ are the constants in (2.8) and Lemma 2.3, respectively. Then, Lemma 2.2 and Lemma 2.3 show that $B$ and $B_k$ are bounded absorbing sets for problem (2.1)–(2.4) in $H^1 \times H^1 \times H$ and $H^{k+2} \times H^{k+2} \times H^{k+1}$ with $k \geq 0$, respectively. Since $B$ is bounded, by Lemma 2.2 again, we see that there exists a constant $T(B)$ depending on $B$ such that

$$S(t) B \subset B, \quad \forall t \geq T(B).$$

We note that to prove the existence of global attractors, a key is to establish the asymptotic compactness of solutions. For unbounded domains, it is difficult to get such asymptotic compactness because the Sobolev embeddings are no long compact. Here, we first approximate the whole space $\mathbb{R}^n$ by a bounded ball $\Omega_m = \{ x \in \mathbb{R}^n : |x| \leq m \}$ for each $m > 0$. Then, we establish an estimate for the solutions on the complement of $\Omega_m$ when $m$ is large.
enough. In fact, we have the following estimate which states that all solutions on the complement of $Q_m$ are uniformly small for large times.

**Lemma 2.4.** Let $u_0 \in B$, the bounded absorbing set in (2.32). Then for every $\varepsilon > 0$, there exist $T(\varepsilon)$ and $M(\varepsilon)$ such that every solution $(\psi, \phi, 0)$ of problem (2.1)-(2.4) satisfies

$$
\int_{|x| \geq m} \left( |\psi(t)|^2 + |\phi(t)|^2 + |\nabla\psi(t)|^2 + |\theta(t)|^2 \right) \, dx \leq \varepsilon^2
$$

$$
\forall t \geq T(\varepsilon), \quad m \geq M(\varepsilon),
$$

(2.35)

where $T(\varepsilon)$ and $M(\varepsilon)$ depend on $\varepsilon$.

**Proof.** Choose a smooth function $\beta$ such that $0 \leq \beta(s) \leq 1$ for $s \in \mathbb{R}^+$, and $\beta(s) = 0$ for $0 \leq s \leq 1$; $\beta(s) = 1$ for $s \geq 2$.

Then, there exists a constant $C$ such that $|\beta'(s)| \leq C$ for $s \in \mathbb{R}^+$.

Taking the imaginary part of the inner product of (2.1) with $\beta(|x|^2/m^2) \psi$ in $H$, we get

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \beta\left(\frac{|x|^2}{m^2}\right) |\psi|^2 + v \int_{\mathbb{R}^n} \beta\left(\frac{|x|^2}{m^2}\right) |\psi|^2 = -\text{Im} \left[ \int_{\mathbb{R}^n} \beta\left(\frac{|x|^2}{m^2}\right) \bar{\psi} \phi + \text{Im} \left[ \int_{\mathbb{R}^n} \beta\left(\frac{|x|^2}{m^2}\right) \bar{\psi} f \right] \right].
$$

(2.36)

Note that

$$
-\text{Im} \left[ \int_{\mathbb{R}^n} \beta\left(\frac{|x|^2}{m^2}\right) \bar{\psi} \phi = \text{Im} \left[ \int_{\mathbb{R}^n} \beta\left(\frac{|x|^2}{m^2}\right) |\nabla\psi|^2 + \text{Im} \left[ \int_{\mathbb{R}^n} \beta\left(\frac{|x|^2}{m^2}\right) \bar{\psi} \frac{2v}{m^2} \nabla\psi \right] \right] \right.
$$

$$
\leq \frac{C}{m} \int_{m < |x| < \sqrt{2}m} |\psi| |\nabla\psi|
$$

$$
\leq \frac{C}{m} \left[ |\psi| |\nabla\psi| \leq \frac{C}{m} \right], \quad t \geq T(B) \quad \text{(by (2.34))},
$$

(2.37)
where $C$ is independent of $m$. For the second term on the right hand side of (2.36), we have

$$\left| \Im \int_{|x| \geq m} \beta \left( \frac{|x|^2}{m^2} \right) \bar{\psi} f \right| = \left| \Im \int_{|x| \geq m} \beta \left( \frac{|x|^2}{m^2} \right) \bar{\psi} f \right|$$

$$\leq \left( \int_{|x| \geq m} |f|^2 \right)^{1/2} \left( \int_{|x| \geq m} \beta^2 \left( \frac{|x|^2}{m^2} \right) |\psi|^2 \right)^{1/2}$$

$$\leq \left( \int_{|x| \geq m} |f|^2 \right)^{1/2} \left( \int_{\mathbb{R}^n} \beta \left( \frac{|x|^2}{m^2} \right) |\psi|^2 \right)^{1/2}$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^n} \beta \left( \frac{|x|^2}{m^2} \right) |\psi|^2 + \frac{1}{2} \int_{|x| \geq m} |f|^2.$$  (2.38)

By (2.36)–(2.38) we get

$$\frac{1}{2} \left. \frac{d}{dt} \right|_{|x| \geq m} \beta \left( \frac{|x|^2}{m^2} \right) |\psi|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \beta \left( \frac{|x|^2}{m^2} \right) |\psi|^2 \leq C + \frac{1}{2} \int_{|x| \geq m} |f|^2.$$  (2.39)

Since $f \in H$, for given $\epsilon > 0$, we find that there exists $M_1(\epsilon) > 0$ such that when $m \geq M_1(\epsilon)$, the right-hand side of (2.39) is bounded by $\frac{\epsilon}{2}$. Therefore, we get for $t \geq T(B)$ and $m \geq M_1(\epsilon)$,

$$\frac{d}{dt} \int_{\mathbb{R}^n} \beta \left( \frac{|x|^2}{m^2} \right) |\psi|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \beta \left( \frac{|x|^2}{m^2} \right) |\psi|^2 \leq \epsilon.$$  (2.40)

By (2.40), it follows from Gronwall’s lemma that

$$\int_{\mathbb{R}^n} \beta \left( \frac{|x|^2}{m^2} \right) |\psi(t)|^2 \leq e^{-\nu(t-T(B))} \int_{\mathbb{R}^n} \beta \left( \frac{|x|^2}{m^2} \right) |\psi(T(B))|^2 + \frac{E}{\nu}$$

$$\leq e^{-\nu(t-T(B)) \|\psi(T(B))\|^2} + \frac{E}{\nu} \leq M^2 e^{-\nu(t-T(B))} + \frac{E}{\nu}.$$  (2.41)

Letting $T_1(\epsilon) = \frac{1}{2} \ln \left( \frac{1}{\epsilon} \right) + T(B)$ and $T_2(\epsilon) = \max \{ T_1(\epsilon), T(B) + 1 \}$, then for $t \geq T_2(\epsilon)$ and $m \geq M_1(\epsilon)$ we have

$$\int_{|x| \geq 2m} |\psi(t)|^2 \leq \int_{\mathbb{R}^n} \beta \left( \frac{|x|^2}{m^2} \right) |\psi(t)|^2 \leq \frac{2\epsilon}{\nu}.$$  (2.41)
We now derive the estimates for $\phi$ and $\theta$. Taking the inner product of (2.3) with $\beta(|x|^2/m^2) \theta$, we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) \theta^2 + (\gamma - \delta) \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) |\theta|^2 - \int_{\mathbb{R}^m} A\phi \beta \left( \frac{|x|^2}{m^2} \right) \theta \\
+ (1 - \delta(\gamma - \delta)) \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) \phi \theta - \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) \theta |\psi|^2 = \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) g \theta.
\]
(2.42)

Using (2.2) we find that
\[
- \int_{\mathbb{R}^m} A\phi \beta \left( \frac{|x|^2}{m^2} \right) \theta = \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) \nabla \phi \cdot \nabla \theta + \int_{\mathbb{R}^m} \theta \nabla \phi \cdot \nabla \beta \left( \frac{|x|^2}{m^2} \right)
\]
\[
= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) |\nabla \phi|^2
\]
\[
+ \delta \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) |\nabla \phi|^2 + \int_{\mathbb{R}^m} \theta \beta' \left( \frac{|x|^2}{m^2} \right) \nabla \phi \cdot \frac{2x}{m^2}.
\]
(2.43)

On the other hand, by (2.2) again, we have
\[
(1 - \delta(\gamma - \delta)) \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) \phi \theta = \frac{1}{2} (1 - \delta(\gamma - \delta)) \frac{d}{dt} \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) |\phi|^2
\]
\[
+ \delta(1 - \delta(\gamma - \delta)) \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) |\phi|^2.
\]
(2.44)

Then, it follows from (2.42)–(2.44) that
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) ((1 - \delta(\gamma - \delta)) |\phi|^2 + |\nabla \phi|^2 + |\theta|^2)
\]
\[
+ \delta \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) ((1 - \delta(\gamma - \delta)) |\phi|^2
\]
\[
+ |\nabla \phi|^2 + |\theta|^2) + (\gamma - 2\delta) \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) |\theta|^2
\]
\[
= \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) g \theta + \int_{\mathbb{R}^m} \beta \left( \frac{|x|^2}{m^2} \right) \theta |\psi|^2 - \int_{\mathbb{R}^m} \theta \beta' \left( \frac{|x|^2}{m^2} \right) \nabla \phi \cdot \frac{2x}{m^2}.
\]
(2.45)
Using (2.34) and (2.41) and proceeding as (2.37) and (2.38), we see that for given, there exists $M_2(e)$ such that for $m \geq M_2(e)$, the right-hand side of (2.45) is bounded by $\varepsilon$. Choosing $\varepsilon > 0$ small enough such that (2.18) holds and $\gamma - 2\delta > 0$, then, it follows from (2.45) that for $m \geq M_2(e)$ and $t \geq T(B)$,

$$
\frac{1}{2} \frac{d}{dt} \int_{|x|} \beta \left( \frac{|x|^2}{m^2} \right) \left( (1 - \delta(\gamma - \delta)) |\phi|^2 + |V\phi|^2 + |\theta|^2 \right) + \delta \int_{|x|} \beta \left( \frac{|x|^2}{m^2} \right) \left( (1 - \delta(\gamma - \delta)) |\phi|^2 + |V\phi|^2 + |\theta|^2 \right) \leq \varepsilon. \tag{2.46}
$$

Similar to (2.41), by (2.46) and Gronwall's inequality we can obtain that there exists $T_3(e)$ such that for $m \geq M_2(e)$ and $t \geq T_3(e)$,

$$
\int_{|x| \geq 2m} \left( (1 - \delta(\gamma - \delta)) |\phi|^2 + |V\phi|^2 + |\theta|^2 \right) \leq \frac{2\varepsilon}{\delta}. \tag{2.47}
$$

Thus, (2.41) and (2.47) imply this lemma. The proof is complete.

Next, we state some estimates on the solutions on a finite time interval which will be used when we establish the existence and uniqueness and continuity of solutions. Repeating the procedure similar to the proof of Lemmas 2.1–2.3, we can get the following result.

**Lemma 2.5.** Assume that $f$ and $g$ belong to $H$. Then, every solution $(\psi, \phi, \theta)$ of problem (2.1)(2.4) satisfies

$$
\|\psi(t)\|_{H^1} + \|\phi(t)\|_{H^1} + \|\theta(t)\| \leq L, \quad 0 \leq t \leq T,
$$

where $L$ depends on $(v, \gamma, \delta, \|f\|, \|g\|, T)$ and $\|((\psi_0, \phi_0, \theta_0)\|_{H^{k+2} \times H^{k+1} \times H^k}$.

The following fact is the analogue of Lemma 2.3.

**Lemma 2.6.** Assume that $f$ and $g$ belong to $H^k (\mathbb{R}^n)$ with $k \geq 0$. Then every solution $(\psi, \phi, \theta)$ of problem (2.1)(2.4) satisfies

$$
\|\psi(t)\|_{H^{k+2}} + \|\phi(t)\|_{H^{k+2}} + \|\theta(t)\|_{H^{k+1}} \leq L_k, \quad 0 \leq t \leq T,
$$

where $L_k$ depends on $(v, \gamma, \delta, \|f\|_{H^k}, \|g\|_{H^k})$ and $T$ and $k$ and $\|((\psi_0, \phi_0, \theta_0)\|_{H^{k+2} \times H^{k+2} \times H^{k+1}}$. 

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In this section, we establish the existence of the dynamical system associated with problem (2.1)-(2.4). What is more, we shall show the existence, uniqueness, and continuity of solutions in the space $H^{k+1}(\mathbb{R}^n) \times H^{k+1}(\mathbb{R}^n) \times H^k(\mathbb{R}^n)$ for each integer $k \geq 0$. The existence and uniqueness of solutions follow from standard methods and estimates in Lemmas 2.5-2.6, see [15] for more details. But the continuity property of solutions on initial data in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H$ seems a difficult question and was left open in [5, 7].

In this section, we shall first derive an energy equation for solutions in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H$, and then apply this energy equation to show the continuity of solutions on initial data. We shall also present such continuous property for solutions in $H^{k+\frac{1}{2}}(\mathbb{R}^n) \times H^{k+\frac{1}{2}}(\mathbb{R}^n) \times H^{k+1}(\mathbb{R}^n)$ for each integer $k \geq 0$.

By Lemma 2.5, it is easy to show that problem (2.1)-(2.4) is well-posed in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H$, more precisely, if $f$ and $g$ belong to $H$, then for every $(\psi_0, \phi_0, \theta_0) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H$, problem (2.1)-(2.4) has a unique solution $(\psi, \phi, \theta)$ such that $(\psi, \phi, \theta) \in C([0, \infty), H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H)$. This establishes the existence of a dynamical system $\{S(t)\}_{t \geq 0}$ which maps $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H$ to $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H$ such that $S(t)(\psi_0, \phi_0, \theta_0) = (\psi(t), \phi(t), \theta(t))$, the solution of problem (2.1)-(2.4). On the other hand, if $f$ and $g$ belong to $H^k(\mathbb{R}^n)$ with $k \geq 0$, by Lemma 2.6 we can easily prove that the solution $(\psi(t), \phi(t), \theta(t)) \in C([0, \infty), H^{k+\frac{1}{2}}(\mathbb{R}^n) \times H^{k+\frac{1}{2}}(\mathbb{R}^n) \times H^{k+1}(\mathbb{R}^n))$ provided $(\psi_0, \phi_0, \theta_0)$ is in $H^{k+\frac{1}{2}}(\mathbb{R}^n) \times H^{k+\frac{1}{2}}(\mathbb{R}^n) \times H^{k+1}(\mathbb{R}^n)$, which means, in this case, $\{S(t)\}_{t \geq 0}$ is also a dynamical system in $H^{k+\frac{1}{2}}(\mathbb{R}^n) \times H^{k+\frac{1}{2}}(\mathbb{R}^n) \times H^{k+1}(\mathbb{R}^n)$.

Next, we shall prove the continuity of $S(t)$ on initial data. To this end, we first have to establish the weak continuity on initial data in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H$ which can be stated as follows.

**Lemma 3.1.** Assume that $f$ and $g$ belong to $H$. If $(\psi_n, \phi_n, \theta_n) \rightarrow (\psi, \phi, \theta)$ weakly in $H^1 \times H^1 \times H$, then for every $T > 0$, we have

$$S(t)(\psi_n, \phi_n, \theta_n) \rightarrow S(t)(\psi, \phi, \theta) \quad \text{weakly in } L^2(0, T; H^1 \times H^1 \times H)$$

(3.1)

and

$$S(t)(\psi_n, \phi_n, \theta_n) \rightarrow S(t)(\psi, \phi, \theta) \quad \text{weakly in } H^1 \times H^1 \times H, \quad 0 < t < T.$$  

(3.2)
Proof. By the weak convergence of \((\psi_n, \phi_n, \theta_n)\) to \((\psi, \phi, \theta)\), we know \((\psi_n, \phi_n, \theta_n)\) is bounded in \(H^1 \times H^1 \times H\). So, it follows from Lemma 2.5 that
\[
S(t)(\psi_n, \phi_n, \theta_n) \quad \text{is bounded in} \quad L^\infty(0, T; H^1 \times H^1 \times H). \tag{3.3}
\]
By (3.3) and (2.1)–(2.3), it is easy to see that
\[
\frac{\partial}{\partial t} S(t) \psi_n \quad \text{and} \quad \frac{\partial}{\partial t} S(t) \phi_n \quad \text{are bounded in} \quad L^\infty(0, T; H^{-1}(\mathbb{R}^n)), \tag{3.4}
\]
and
\[
\frac{\partial}{\partial t} S(t) \phi_n \quad \text{is bounded in} \quad L^\infty(0, T; H). \tag{3.5}
\]
Hereafter, we denote by \((\psi_n(t), \phi_n(t), \theta_n(t)) = S(t)(\psi_n, \phi_n, \theta_n) = (S(t) \psi_n, S(t) \phi_n, S(t) \theta_n)\) when no confusion arises. By (3.3)–(3.5) we infer that there exist a subsequence \((\psi_{n_j}, \phi_{n_j}, \theta_{n_j})\) of \((\psi_n, \phi_n, \theta_n)\) and \((\psi_\infty, \phi_\infty, \theta_\infty)\) \(\in L^\infty(0, T; H^1 \times H^1 \times H)\) such that
\[
S(t)(\psi_{n_j}, \phi_{n_j}, \theta_{n_j}) \rightharpoonup S(t)(\psi_\infty, \phi_\infty, \theta_\infty) \quad \text{weakly in} \quad L^2(0, T; H^1 \times H^1 \times H), \tag{3.6}
\]
\[
\frac{\partial}{\partial t} S(t) \psi_{n_j} \rightharpoonup \frac{\partial}{\partial t} \psi_\infty \quad \text{weakly in} \quad L^2(0, T; H^{-1}(\mathbb{R}^n)), \tag{3.7}
\]
\[
\frac{\partial}{\partial t} S(t) \theta_{n_j} \rightharpoonup \frac{\partial}{\partial t} \theta_\infty \quad \text{weakly in} \quad L^2(0, T; H^{-1}(\mathbb{R}^n)), \tag{3.8}
\]
and
\[
\frac{\partial}{\partial t} S(t) \phi_{n_j} \rightharpoonup \frac{\partial}{\partial t} \phi_\infty \quad \text{weakly in} \quad L^2(0, T; H). \tag{3.9}
\]
Similar to the proof of the existence of solutions (see, e.g., [2–5]), (3.6)–(3.9) enable us to show that \((\psi_\infty, \phi_\infty, \theta_\infty)\) is a solution of problem (2.1)–(2.4) with \((\psi_\infty(0), \phi_\infty(0), \theta_\infty(0)) = (\psi, \phi, \theta)\). By the uniqueness of solutions, we have \((\psi_\infty, \phi_\infty, \theta_\infty) = S(t)(\psi, \phi, \theta)\). This, together with (3.6)–(3.9), shows any sequence \(S(t)(\psi_n, \phi_n, \theta_n)\) has a weakly convergent subsequence in \(L^2(0, T; H^1 \times H^1 \times H)\) and the limit of any such a subsequence is equal to \(S(t)(\psi, \phi, \theta)\). Therefore, by contradiction arguments, we conclude (3.1).
We now prove (3.2). For a fixed $t \in [0, T]$, it follows from Lemma 2.5 that $S(t)(\psi_n, \phi_n, \theta_n)$ is bounded in $H^1 \times H^1 \times H$, so there exist a subsequence $(\psi_{n_k}, \phi_{n_k}, \theta_{n_k})$ and $(\psi, \phi, \theta) \in H^1 \times H^1 \times H$ such that $S(t)(\psi_{n_k}, \phi_{n_k}, \theta_{n_k})$ weakly converges to $(\psi, \phi, \theta)$ in $H^1 \times H^1 \times H$. In this case, (3.6)–(3.9) enable us to show that $(\psi_{n_k}, \phi_{n_k}, \theta_{n_k})$ is a solution of problem (2.1)–(2.3) not only with $(\psi(0), \phi(0), \theta(0)) = (\psi, \phi, \theta)$, but also with $(\psi_{n_k}(t), \phi_{n_k}(t), \theta_{n_k}(t)) = (\psi, \phi, \theta)$. Again, by the uniqueness of solutions, we have $(\psi_{n_k}(t), \phi_{n_k}(t), \theta_{n_k}(t)) = S(t)(\psi_{n_k}, \phi_{n_k}, \theta_{n_k})$ weakly convergent subsequence in $H^1 \times H^1 \times H$ and the limit of any such a subsequence is the same. Again, by contradiction arguments, (3.2) follows.

Clearly, similar to (3.1), we also have that if $(\psi_{n_k}, \phi_{n_k}, \theta_{n_k})$ weakly in $H^1 \times H^1 \times H$, then for $0 \leq s \leq T$,

$$S(t)(\psi_{n_k}, \phi_{n_k}, \theta_{n_k}) \to S(t)(\psi, \phi, \theta)$$ weakly in $L^2(s, T; H^1 \times H^1 \times H)$,

(3.10)

which will be needed for the proof of the asymptotic compactness of solutions in the next section.

We are now in a position to prove the strong continuity of solutions on initial data in $H^1 \times H^1 \times H$.

**Theorem 3.1.** Assume that $f$ and $g$ belong to $H$. Then the solution $(\psi, \phi, \theta)$ of problem (2.1)–(2.4) depends continuously on initial data in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times H$, and satisfies the energy equation

$$\frac{d}{dt} E(\psi(t), \phi(t), \theta(t)) + 2 \Delta E(\psi(t), \phi(t), \theta(t)) = F(\psi(t), \phi(t), \theta(t)), \quad \forall t > 0,$$

(3.11)

where

$$E(\psi, \phi, \theta) = \frac{1}{2} \|\psi\|^2 + \frac{1}{2} \|\nabla \psi\|^2 + (1 - \delta)(\gamma - \delta) \|\phi\|^2 + \|\nabla \phi\|^2 + \|\theta\|^2 - 2 \int_{\mathbb{R}^n} |\psi|^2 \phi + 4 \text{Re} \int_{\mathbb{R}^n} f \bar{\psi},$$

(3.12)

and

$$F(\psi, \phi, \theta) = -4(\nu - \delta)(\|\psi\|^2 + \|\nabla \psi\|^2) - 2(\gamma - 2\delta) \|\theta\|^2 + 2(2\nu - \delta) \int_{\mathbb{R}^n} |\phi|^2 \phi + 4 \text{Im} \int_{\mathbb{R}^n} f \bar{\psi} + 4(2\delta - \nu) \text{Re} \int_{\mathbb{R}^n} f \bar{\psi} + 2 \int_{\mathbb{R}^n} g \theta.$$
Proof. We first prove energy equation (3.11). To this end, for \((\psi_0, \theta_0, \theta_0) \in H^1 \times H^1 \times H\) given, we take a sequence \((\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \in H^2 \times H^2 \times H^1\) such that
\[
(\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \to (\psi_0, \phi_0, \theta_0) \quad \text{in} \quad H^1 \times H^1 \times H.
\] (3.14)
Note that (3.14) implies that \((\psi_{0,n}, \phi_{0,n}, \theta_{0,n})\) is bounded in \(H^1 \times H^1 \times H\). Therefore, by Lemma 2.5 we know that for every \(T > 0\), the solution \((\psi_{n}(t), \phi_{n}(t), \theta_{n}(t)) = S(t)(\psi_{0,n}, \phi_{0,n}, \theta_{0,n})\) satisfies
\[
\|(\psi_{n}(t), \phi_{n}(t), \theta_{n}(t))\|_{H^1 \times H^1 \times H} \leq C, \quad \forall 0 \leq t \leq T, \quad n = 1, \ldots \quad \text{(3.15)}
\]
By (3.2) and (3.14), we have
\[
(\psi_{n}(t), \phi_{n}(t), \theta_{n}(t)) \to (\psi(t), \phi(t), \theta(t)) \quad \text{weakly in} \quad H^1 \times H^1 \times H,
\] \(\forall 0 \leq t \leq T. \quad \text{(3.16)}
In what follows, we show
\[
\psi_{n}(t) \to \psi(t) \quad \text{in} \quad H. \quad \text{(3.17)}
\]
Since (3.16) implies \(\psi_{n}(t) \to \psi(t)\) weakly in \(H\), then (3.17) will be proved if we can show
\[
\|\psi_{n}(t)\|^2 \to \|\psi(t)\|^2. \quad \text{(3.18)}
\]
We now prove (3.18) by an energy equation in \(H\). Taking the imaginary part of the inner product of (2.1) with \(\psi_{n}\) in \(H\), we get
\[
\frac{d}{dt} \|\psi_{n}\|^2 + 2 \nu \|\psi_{n}\|^2 = 2 \text{Im} \int_{B^n} f \psi_n.
\]
Then, from Gronwall’s lemma it follows that
\[
\|\psi_{n}(t)\|^2 = e^{-2\nu t} \|\psi_{n}(0)\|^2 + 2 \text{Im} \int_{0}^{t} e^{-2\nu(t-s)}(f, \psi_{n}(s)) \, ds. \quad \text{(3.19)}
\]
On the other hand, we also have
\[
\|\psi(t)\|^2 = e^{-2\nu t} \|\psi(0)\|^2 + 2 \text{Im} \int_{0}^{t} e^{-2\nu(t-s)}(f, \psi(s)) \, ds. \quad \text{(3.20)}
\]
Taking the limit of (3.19), then from (3.14) and (3.16) we get
\[
\lim_{n \to \infty} \| \psi_n(t) \|^2 = e^{-2\alpha t} \| \psi(0) \|^2 + 2 \text{ Im} \int_0^t e^{-2\alpha(s-t)} (f, \psi(s)) \, ds. \tag{3.21}
\]

By (3.20)–(3.21) we obtain (3.18) and hence (3.17) holds. By (3.15) and (3.17) we find that
\[
\| \psi_n(t) - \psi(t) \|_3 \leq C \| \psi_n(t) - \psi(t) \|^{1/2} \| \psi_n(t) - \psi(t) \|^{1/2} \leq C \| \psi_n(t) - \psi(t) \|^{1/2} \to 0. \tag{3.22}
\]

Next, we prove \((\phi_n(t), \theta_n(t)) \to (\phi(t), \theta(t))\) in \(H^1 \times H\). We set
\[
\psi_n(t) = \psi_n(t) - \psi(t), \quad \psi_n(t) = \phi_n(t) - \psi(t), \quad \psi_n(t) = \theta_n(t) - \theta(t). \tag{3.23}
\]

Then, it follows from (2.2)–(2.3) that
\[
\frac{d}{dt} \psi_n + \delta \psi_n = w_n, \tag{3.24}
\]
\[
\frac{d}{dt} w_n + (\gamma - \delta) w_n - \Delta w_n + (1 - \delta(\gamma - \delta)) v_n + u_n \psi_n + \psi \bar{u}_n = 0. \tag{3.25}
\]

Taking the inner product of (3.25) with \(w_n\) in \(H\), we find
\[
\frac{1}{2} \frac{d}{dt} \| w_n \|^2 + (\gamma - \delta) \| w_n \|^2 - (\Delta v_n, w_n) + (1 - \delta(\gamma - \delta))(v_n, w_n) + (u_n \psi_n + \psi \bar{u}_n, w_n) = 0.
\]

Letting \(\delta\) small enough such that \(\gamma > \delta\), then using (3.15) and (3.24) we get
\[
\frac{1}{2} \frac{d}{dt} \left( \| w_n \|^2 + (1 - \delta(\gamma - \delta)) \| v_n \|^2 + \| \nabla w_n \|^2 \right) + (\gamma - \delta) \| w_n \|^2 + \delta(1 - \delta(\gamma - \delta)) \| v_n \|^2 + \delta \| \nabla w_n \|^2
\]
\[
\leq \int_{\mathbb{R}^n} (u_n \psi_n w_n + \psi \bar{u}_n w_n)
\]
\[
\leq C \| \psi_n \|_6 \| u_n \|_3 \| w_n \| + C \| \psi \|_6 \| u_n \|_3 \| w_n \|
\]
\[
\leq C \| \psi \|_{H^1} \| u_n \|_3 \| w_n \| + C \| \psi \|_{H^1} \| u_n \|_3 \| w_n \|
\]
\[
\leq C \| u_n \|_3 \| w_n \| \leq \frac{1}{2} (\gamma - \delta) \| w_n \|^2 + C \| u_n \|_3^2.
\]
Hence, we have
\[
\frac{d}{dt} \left( \|w_n\|^2 + (1 - \delta(\gamma - \delta)) \|v_n\|^2 + \|\nabla v_n\|^2 \right) \leq C \|u_n\|^2.
\]
By the Gronwall lemma, we obtain
\[
\|w_n(t)\|^2 + (1 - \delta(\gamma - \delta)) \|v_n(t)\|^2 + \|\nabla v_n(t)\|^2 \\
\leq \|w_n(0)\|^2 + (1 - \delta(\gamma - \delta)) \|v_n(0)\|^2 + \|\nabla v_n(0)\|^2 + C \int_0^t \|u_n\|^2 dt.
\]
Taking the limit of the above and using (3.22), we infer that
\[
\| (v_n(t), w_n(t)) \|_{H^1 \times H} \to 0, \quad \text{as} \quad n \to \infty,
\]
that is,
\[
(\phi_n(t), \theta_n(t)) \to (\phi(t), \theta(t)) \quad \text{in} \quad H^1 \times H. \tag{3.26}
\]
Since \((\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \in H^2 \times H^2 \times H^1\), by Lemma 2.6 we know that for every fixed \(n = 1, \ldots\), \((\psi_n, \phi_n, \theta_n) \in L^\infty(0, T; H^2 \times H^2 \times H^1)\). This justifies the derivation of (2.7) and (2.13). Therefore, we have
\[
\frac{d}{dt} G(\psi_n(t), \phi_n(t), \theta_n(t)) + 2 \nu G(\psi_n(t), \phi_n(t), \theta_n(t)) = H(\psi_n(t), \phi_n(t), \theta_n(t)),
\]
(3.27)
where
\[
G(\psi_n, \phi_n, \theta_n) = \|\psi_n\|^2 + \|\nabla \psi_n\|^2 - \int_{\mathbb{R}^n} \phi_n |\psi_n|^2 + 2 \nu \int_{\mathbb{R}^n} f \bar{\psi}_n, \tag{3.28}
\]
and
\[
H(\psi_n, \phi_n, \theta_n) = \delta \int_{\mathbb{R}^n} \phi_n |\psi_n|^2 - \int_{\mathbb{R}^n} \theta_n |\psi_n|^2 + 2 \nu \int_{\mathbb{R}^n} f \bar{\psi}_n + 2 \text{Im} \int_{\mathbb{R}^n} f \bar{\psi}_n. \tag{3.29}
\]
By (3.27), we get
\[
G(\psi_n(t), \phi_n(t), \theta_n(t)) = e^{-2\nu t} G(\psi_n(0), \phi_n(0), \theta_n(0)) \\
+ \int_0^t e^{-2\nu (t-s)} H(\psi_n(s), \phi_n(s), \theta_n(s)) ds. \tag{3.30}
\]
By (3.14) we find
\[
G(\psi_n(0), \phi_n(0), \theta_n(0)) \to G(\psi(0), \phi(0), \theta(0)).
\] (3.31)

We now handle each term on the right-hand side of (3.29) as follows. We first have
\[
\left| \int_{\mathbb{R}^d} \phi_n |\psi_n|^2 - \int_{\mathbb{R}^d} \phi |\psi|^2 \right|
= \left| \int_{\mathbb{R}^d} (\phi_n - \phi) |\psi_n|^2 + \int_{\mathbb{R}^d} \phi (|\psi_n|^2 - |\psi|^2) \right|
\leq \|\phi_n - \phi\|_A \|\psi_n\|_A^2 + \|\phi\|_3 \|\psi_n - \psi\| (\|\psi_n\|_6 + \|\psi\|_6)
\leq \|\phi_n - \phi\|_A \|\psi_n\|_H^2 + \|\phi\|_3 \|\psi_n - \psi\| (\|\psi_n\|_H + \|\psi\|_H)
\leq C \|\phi_n - \phi\| + C \|\psi_n - \psi\| \to 0 \quad \text{(by (3.17) and (3.26)).} \] (3.32)

For the second term on the right-hand side of (3.29), we find
\[
\left| \int_{\mathbb{R}^d} \theta_n |\psi_n|^2 - \int_{\mathbb{R}^d} \theta |\psi|^2 \right|
= \left| \int_{\mathbb{R}^d} (\theta_n - \theta) |\psi_n|^2 + \int_{\mathbb{R}^d} \theta (|\psi_n|^2 - |\psi|^2) \right|
\leq \|\theta_n - \theta\|_A \|\psi_n\|_A^2 + \|\theta\|_3 \|\psi_n - \psi\| (\|\psi_n\|_6 + \|\psi\|_6)
\leq \|\theta_n - \theta\|_A \|\psi_n\|_H^2 + \|\theta\|_3 \|\psi_n - \psi\| (\|\psi_n\|_H + \|\psi\|_H)
\leq C \|\theta_n - \phi\| + C \|\psi_n - \psi\|_3 \to 0 \quad \text{(by (3.22) and (3.26)).} \] (3.33)

Then, from (3.32)-(3.33) and (3.17) it follows that
\[
H(\psi_n(t), \phi_n(t), \theta_n(t)) \to H(\psi(t), \phi(t), \theta(t)), \quad \forall 0 \leq t \leq T. \] (3.34)

By (3.15) and (3.34) and the Lebesgue dominated convergence theorem, we obtain
\[
\int_0^T e^{-2i(t-s)} H(\psi_n(s), \phi_n(s), \theta_n(s)) \, ds \to \int_0^T e^{-2i(t-s)} H(\psi(s), \phi(s), \theta(s)) \, ds.
\] (3.35)
Taking the limit of (3.30), by (3.28)--(3.29), (3.31)--(3.32), and (3.35) we get
\[
\lim_{n \to \infty} (\|\psi_n\|^2 + \|\nabla \psi_n\|^2) - \int_{\mathbb{R}^n} \phi |\psi|^2 + 2 \Re \int_{\mathbb{R}^n} \psi \bar{\psi} = e^{-2\sigma t} G(\psi(0), \phi(0), \theta(0)) + \int_0^t e^{-2\sigma(t-s)} H(\psi(s), \phi(s), \theta(s)) \, ds.
\]
(3.36)

On the other hand, the weak convergence (3.16) implies that
\[
\|\bar{\psi}\|^2 + \|\nabla \bar{\psi}\|^2 \leq \liminf_{n \to \infty} (\|\psi_n\|^2 + \|\nabla \psi_n\|^2).
\]
(3.37)

Therefore, by (3.28) and (3.36)--(3.37) we obtain
\[
G(\psi(t), \phi(t), \theta(t)) \leq e^{-2\sigma t} G(\psi(0), \phi(0), \theta(0)) + \int_0^t e^{-2\sigma(t-s)} H(\psi(s), \phi(s), \theta(s)) \, ds.
\]
(3.38)

In what follows, we prove the reverse inequality of (3.38):
\[
G(\psi(t), \phi(t), \theta(t)) \geq e^{-2\sigma t} G(\psi(0), \phi(0), \theta(0)) + \int_0^t e^{-2\sigma(t-s)} H(\psi(s), \phi(s), \theta(s)) \, ds.
\]
(3.39)

To this end, for fixed \(t_0 > 0\), we set
\[
(x(t), y(t), z(t)) = (\psi(t_0 - t), \phi(t_0 - t), \theta(t_0 - t)), \quad \forall 0 \leq t \leq t_0.
\]
(3.40)

Then we find \((x, y, z)\) also satisfies system (2.1)--(2.3) with \(\psi, \phi, \theta\) replaced by \(-x, -y, -z\) and \(-\sigma\), respectively. We now take a sequence \((x_{0,n}, y_{0,n}, z_{0,n}) \in H^1 \times H^2 \times H^3\) such that \((x_{0,n}, y_{0,n}, z_{0,n}) \to (x_0, y_0, z_0)\) in \(H^1 \times H^2 \times H^3\). Then, repeating the procedure above we can get the following inequality which is similar to (3.38):
\[
G(x(t), y(t), z(t)) \leq e^{2\sigma t} G(x(0), y(0), z(0)) - \int_0^t e^{2\sigma(t-s)} H(x(s), y(s), z(s)) \, ds.
\]

Taking \(t = t_0\) in the above and using (3.40) we get
\[
G(\psi(0), \phi(0), \theta(0)) \leq e^{2\sigma t_0} G(\psi(t_0), \phi(t_0), \theta(t_0)) - \int_0^{t_0} e^{2\sigma s} H(\psi(s), \phi(s), \theta(s)) \, ds.
\]
(3.41)
Then, multiplying (3.41) by $e^{-2it_0}$ and noting $t_0$ is arbitrary, we obtain (3.39). Hence, by (3.38)–(3.39) we have

$$G(\psi(t), \phi(t), \theta(t)) = e^{-2it}G(\psi(0), \phi(0), \theta(0))$$

$$+ \int_0^t e^{-2i(t-s)}H(\psi(s), \phi(s), \theta(s)) \, ds,$$  \hspace{1cm} (3.42)

that is,

$$\frac{d}{dt} G(\psi, \phi, \theta) + 2\sigma G(\psi, \phi, \theta) = H(\psi, \phi, \theta), \quad \forall 0 \leq t \leq T. \hspace{1cm} (3.43)$$

On the other hand, since $(\psi_n, \phi_n, \theta_n) \in L^\infty(0, T; H^2 \times H^2 \times H^1)$ for every fixed $n = 1, \ldots$, we see the derivation of (2.15) is justified, and hence we get

$$\frac{d}{dt} \left( \|\phi_n\|^2 + (1 - \delta(\gamma - \delta)) \|\phi_n\|^2 + \|\nabla \phi_n\|^2 \right) + 2(\gamma - \delta) \|\theta_n\|^2$$

$$+ 2\delta(1 - \delta(\gamma - \delta)) \|\phi_n\|^2 + 2\delta \|\nabla \phi_n\|^2 - 2 \int_{\mathbb{R}^1} |\psi_n|^2 \theta_n = 2 \int_{\mathbb{R}^1} g \theta_n,$$  \hspace{1cm} (3.44)

Taking the limit of (3.44) and using (3.26) and (3.33) we obtain that

$$\frac{d}{dt} \left( \|\theta\|^2 + (1 - \delta(\gamma - \delta)) \|\phi\|^2 + \|\nabla \phi\|^2 \right) + 2(\gamma - \delta) \|\theta\|^2$$

$$+ 2\delta(1 - \delta(\gamma - \delta)) \|\phi\|^2 + 2\delta \|\nabla \phi\|^2 - \int_{\mathbb{R}^1} |\psi|^2 \theta = \int_{\mathbb{R}^1} g \theta,$$  \hspace{1cm} (3.45)

Then, by (3.45) + $2 \times (3.43)$, we get the energy equation (3.11) with $E$ and $F$ given in (3.12) and (3.13), respectively.

We are now in a position to prove the continuity of solutions on initial data. Clearly, from (3.42) and (3.46) we have

$$\lim_{n \to \infty} (\|\psi_n\|^2 + \|\nabla \psi_n\|^2) = \|\psi\|^2 + \|\nabla \psi\|^2,$$

which together with the weak convergence (3.16) implies

$$\psi_n(t) \to \psi(t) \quad \text{in} \ H^1.$$  \hspace{1cm} (3.46)

Therefore, it follows from (3.26) and (3.46) that

$$S(t) (\psi_{0,n}, \phi_{0,n}, \theta_{0,n}) \to S(t) (\psi_0, \phi_0, \theta_0) \quad \text{in} \ H^1 \times H^1 \times H, \quad \forall 0 \leq t \leq T, \hspace{1cm} (3.47)$$
when \((\psi_{0_n}, \phi_{0_n}, \theta_{0_n}) \to (\psi_0, \phi_0, \theta_0)\) in \(H^1 \times H^1 \times H\) and \((\psi_{0_n}, \phi_{0_n}, \theta_{0_n}) \in H^2 \times H^2 \times H^1\).

We now want to prove that (3.47) also holds when \((\psi_{0_n}, \phi_{0_n}, \theta_{0_n}) \in H^1 \times H^1 \times H\).

Assume \((\tilde{\psi}_{0_n}, \tilde{\phi}_{0_n}, \tilde{\theta}_{0_n})\) and \((\psi_0, \phi_0, \theta_0)\) are in \(H^1 \times H^1 \times H\) such that

\[
(\tilde{\psi}_{0_n}, \tilde{\phi}_{0_n}, \tilde{\theta}_{0_n}) \to (\psi_0, \phi_0, \theta_0) \quad \text{in} \quad H^1 \times H^1 \times H, \quad \text{as} \quad n \to \infty.
\]

(3.48)

Then, for each \(n = 1, \ldots\), we take a sequence \((\psi_{0_n}^{(k)}, \phi_{0_n}^{(k)}, \theta_{0_n}^{(k)}) \in H^2 \times H^2 \times H^1\) such that

\[
(\psi_{0_n}^{(k)}, \phi_{0_n}^{(k)}, \theta_{0_n}^{(k)}) \to (\tilde{\psi}_{0_n}, \tilde{\phi}_{0_n}, \tilde{\theta}_{0_n}) \quad \text{in} \quad H^1 \times H^1 \times H, \quad \text{as} \quad k \to \infty.
\]

(3.49)

By (3.47) and (3.49) we find

\[
S(t)(\psi_{0_n}^{(k)}, \phi_{0_n}^{(k)}, \theta_{0_n}^{(k)}) \to S(t)(\tilde{\psi}_{0_n}, \tilde{\phi}_{0_n}, \tilde{\theta}_{0_n}) \quad \text{in} \quad H^1 \times H^1 \times H, \quad \text{as} \quad k \to \infty.
\]

(3.50)

And hence, for each \(n = 1, \ldots\), we can choose \((\psi_{0_n}^{(n)}, \phi_{0_n}^{(n)}, \theta_{0_n}^{(n)}) \in H^2 \times H^2 \times H^1\) such that

\[
\| (\psi_{0_n}^{(n)}, \phi_{0_n}^{(n)}, \theta_{0_n}^{(n)}) - (\tilde{\psi}_{0_n}, \tilde{\phi}_{0_n}, \tilde{\theta}_{0_n}) \|_{H^1 \times H^1 \times H} \leq \frac{1}{n}.
\]

(3.51)

and

\[
\| S(t)(\psi_{0_n}^{(n)}, \phi_{0_n}^{(n)}, \theta_{0_n}^{(n)}) - S(t)(\tilde{\psi}_{0_n}, \tilde{\phi}_{0_n}, \tilde{\theta}_{0_n}) \|_{H^1 \times H^1 \times H} \leq \frac{1}{n}.
\]

(3.52)

By (3.48) and (3.51) we get

\[
(\psi_{0_n}^{(n)}, \phi_{0_n}^{(n)}, \theta_{0_n}^{(n)}) \to (\psi_0, \phi_0, \theta_0) \quad \text{in} \quad H^1 \times H^1 \times H, \quad \text{as} \quad n \to \infty.
\]

(3.53)

Since \((\psi_{0_n}^{(n)}, \phi_{0_n}^{(n)}, \theta_{0_n}^{(n)}) \in H^2 \times H^2 \times H^1\), by (3.47) and (3.53) we infer that

\[
S(t)(\psi_{0_n}^{(n)}, \phi_{0_n}^{(n)}, \theta_{0_n}^{(n)}) \to S(t)(\psi_0, \phi_0, \theta_0) \quad \text{in} \quad H^1 \times H^1 \times H, \quad \text{as} \quad n \to \infty.
\]

(3.54)

Therefore, from (3.52) and (3.54) it follows that

\[
S(t)(\tilde{\psi}_{0_n}, \tilde{\phi}_{0_n}, \tilde{\theta}_{0_n}) \to S(t)(\psi_0, \phi_0, \theta_0) \quad \text{in} \quad H^1 \times H^1 \times H, \quad \text{as} \quad n \to \infty.
\]
Since \((\tilde{\psi}_{0,n}, \tilde{\phi}_{0,n}, \tilde{\theta}_{0,n})\) is an arbitrary sequence in \(H^1 \times H^1 \times H\) which satisfies (3.48), then (3.55) means the continuity of solutions on initial data in \(H^1 \times H^1 \times H\). The proof is complete.

The following statement is concerned with the continuity of solutions on initial data in \(H^{k+\frac{3}{2}} \times H^{k+\frac{3}{2}} \times H^{k+1}\) for each \(k \geq 0\).

**Theorem 3.2.** Assume that \(f\) and \(g\) belong to \(H^k\) with \(k \geq 0\). Then, the solution \((\tilde{\psi}, \tilde{\phi}, \tilde{\theta})\) of problem (2.1)–(2.4) is continuous with respect to initial data in \(H^{k+\frac{3}{2}} \times H^{k+\frac{3}{2}} \times H^{k+1}\), and satisfies the energy equation

\[
\frac{d}{dt} E_k(\tilde{\psi}(t), \tilde{\phi}(t), \tilde{\theta}(t)) + \delta E_k(\tilde{\psi}(t), \tilde{\phi}(t), \tilde{\theta}(t)) = F_k(\tilde{\psi}(t), \tilde{\phi}(t), \tilde{\theta}(t)),
\]

(3.56)

where \(E_k\) and \(F_k\) are given by (2.30) and (2.31), respectively.

**Proof.** The proof of this theorem is much easier than Theorem 3.1, and we only sketch it here.

Consider two solutions \((\psi_1, \phi_1, \theta_1)\) and \((\psi_2, \phi_2, \theta_2)\) of problem (2.1)–(2.4). Then, the difference \((\psi, \phi, \theta) = (\psi_1 - \psi_2, \phi_1 - \phi_2, \theta_1 - \theta_2)\) satisfies

\[
\dot{\psi} + A\psi + iv\psi + \phi_1 + \phi_2 \psi = 0,
\]

(3.57)

\[
\phi + \delta\phi = \theta,
\]

(3.58)

\[
\theta + (\gamma - \delta) \theta - A\phi + (1 - \delta(\gamma - \delta)) \phi + \psi\phi_1 - \psi\phi_2 = 0.
\]

(3.59)

Taking the real part of the inner product of (3.57) with \((-1)^k (A^{k+1} \psi_1 + v A^{k+1} \phi)\) in \(H\), and then using Lemma 2.6 to estimate the terms in the resulting identity, by a computation we can get

\[
\frac{d}{dt} \|\psi\|^2_{\dot{H}^{k+\frac{3}{2}}} \leq C(\|\psi\|^2_{\dot{H}^{k+\frac{3}{2}}} + \|\phi\|^2_{\dot{H}^{k+\frac{3}{2}}}).
\]

(3.60)

Similarly, taking the inner product of (3.59) with \((-1)^{k+1} A^{k+1} \theta\) in \(H\), then using (3.58) and Lemma 2.6, we can derive the inequality

\[
\frac{d}{dt} ((1 - \delta(\gamma - \delta)) \|\phi\|^2_{\dot{H}^{k+1}} + \|\phi\|^2_{\dot{H}^{k+2}} + \|\theta\|^2_{\dot{H}^{k+1}})
\]

\[
\leq C((1 - \delta(\gamma - \delta)) \|\phi\|^2_{\dot{H}^{k+1}} + \|\phi\|^2_{\dot{H}^{k+2}} + \|\theta\|^2_{\dot{H}^{k+1}}).
\]

(3.61)

It follows from (3.60) and (3.61) that

\[
\frac{d}{dt} (\|\psi\|^2_{\dot{H}^{k+\frac{3}{2}}} + (1 - \delta(\gamma - \delta)) \|\phi\|^2_{\dot{H}^{k+1}} + \|\phi\|^2_{\dot{H}^{k+2}} + \|\theta\|^2_{\dot{H}^{k+1}})
\]

\[
\leq C(\|\psi\|^2_{\dot{H}^{k+\frac{3}{2}}} + (1 - \delta(\gamma - \delta)) \|\phi\|^2_{\dot{H}^{k+1}} + \|\phi\|^2_{\dot{H}^{k+2}} + \|\theta\|^2_{\dot{H}^{k+1}}).
\]
This and Gronwall’s lemma give the Lipschitz continuity of solutions in $H^{k+2} \times H^{k+2} \times H^{k+1}$.

Note that the continuity of solutions on initial data justifies the derivation of (2.29), and Therefore, the energy equation (3.56) holds. The proof is complete.

4. GLOBAL ATTRACTORS

In this section, we establish the existence of the global attractor for the dynamical system $S(t)$ in the space $H^{k+1} \subset H^{k+2} \subset H^{k+1}$. To this end, we first need to prove the asymptotic compactness of solutions which will be achieved by an energy equation method and the estimates in Lemma 2.4. And then we conclude our result by an abstract theorem which can be stated as follows. (see, e.g., [9, 13–16]).

**Proposition 4.1.** Assume that $X$ is a metric space and $\{S(t)\}_{t \geq 0}$ is a semigroup of continuous operators in $X$. If $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set and is asymptotically compact, then $\{S(t)\}_{t \geq 0}$ possesses a global attractor which is a compact invariant set and attracts every bounded set in $X$.

The next theorem gives the asymptotic compactness of $S(t)$.

**Theorem 4.1.** Assume that $f$ and $g$ belong to $H^{k+1}$. Then, the dynamical system $S(t)$ is asymptotically compact in $H^1 \times H^1 \times H$, that is, if $(\psi_n, \theta_n, \phi_n)$ is bounded in $H^1 \times H^1 \times H$ and $t_n \to \infty$, then $S(t_n)(\psi_n, \phi_n, \theta_n)$ is precompact in the same space.

**Proof.** Since $(\psi_n, \phi_n, \theta_n)$ is bounded, we can assume that $\|(\psi_n, \phi_n, \theta_n)\|_{H^1 \times H} \leq R$ for a suitable constant $R$. Then, by Lemma 2.2, we have that there exists a constant $T(R)$ depending on $R$ such that

$$S(t)\{(\psi_n, \phi_n, \theta_n)\} \subset B \quad \forall t \geq T(R),$$

where $B$ is the absorbing set in (2.32). Since $t_n \to \infty$, there exists $N_1(R)$ such that if $n \geq N_1(R)$, then $t_n \geq T(R)$, and hence

$$S(t_n)(\psi_n, \phi_n, \theta_n) \in B \quad \forall n \geq N_1(R).$$

By (4.2) we know that there exists $(\psi, \phi, \theta) \in B$ such that, up to a subsequence,

$$S(t_n)(\psi_n, \phi_n, \theta_n) \to (\psi, \phi, \theta) \quad \text{weakly in } H^1 \times H^1 \times H. (4.3)$$

For every $T > 0$, again by $t_n \to \infty$, there exists $N_2(R, T)$ such that for $n \geq N_2(R, T)$, we have $t_n - T \geq T(R)$. So, by (4.1) we get

$$S(t_n - T)(\psi_n, \phi_n, \theta_n) \in B \quad \forall n \geq N_2(R, T).$$
By (4.4) we have that there exists \((\psi_T, \theta_T) \in \mathcal{B}\) such that
\[
S(t_n - T)(\psi_n, \theta_n) \rightharpoonup (\psi_T, \theta_T) \quad \text{weakly in} \quad H^1 \times H^1 \times H.
\] (4.5)

By (4.3) and (4.5) and the weak continuity (3.2), it follows that
\[
(\psi, \theta) = S(T)(\psi_T, \theta_T).
\] (4.6)

By the weak convergence (4.3) we get
\[
\lim_{n \to \infty} \|S(t_n)(\psi_n, \theta_n)\|_{H^1 \times H^1 \times H} \geq \|(\psi, \theta)\|_{H^1 \times H^1 \times H}.
\] (4.7)

If we can also prove
\[
\limsup_{n \to \infty} \|S(t_n)(\psi_n, \theta_n)\|_{H^1 \times H^1 \times H} \leq \|(\psi, \theta)\|_{H^1 \times H^1 \times H},
\] (4.8)
then (4.3) and (4.7) and (4.8) will imply
\[
S(t_n)(\psi_n, \theta_n) \to (\psi, \theta) \quad \text{strongly in} \quad H^1 \times H^1 \times H.
\]

So, the proof will be finished once (4.8) is verified. In what follows, we apply the energy equation (3.11) to prove (4.8).

By (3.11) we find that for every \(t \geq 0\), any solution \((\psi_n, \theta_n) = S(t)(\psi_0, \theta_0)\) satisfies
\[
E(S(t)(\psi_0, \phi_0, \theta_0)) = e^{-2\varphi(t-s)}E(S(s)(\psi_0, \phi_0, \theta_0))
\]
\[
+ \int_s^t e^{-2\varphi(t-\tau)}F(S(\tau)(\psi_0, \phi_0, \theta_0)) \, d\tau.
\] (4.9)

where \(E\) and \(F\) is given by (3.12) and (3.13), respectively.

In the following, \(T(B)\) is the constant in (2.34), and for \(\varepsilon > 0\) given, \(T(\varepsilon)\) is the constant in (2.35). Let \(T_0(\varepsilon)\) be a fixed constant such that \(T_0(\varepsilon) \geq \max\{T(\varepsilon), T(B)\}\). Then, taking \(T \geq T_0(\varepsilon)\), and applying (4.9) to the solution \(S(t)(S(t_n - T)(\psi_n, \phi_n, \theta_n))\) with \(s = T_0\) and \(t = T\), we get that, for \(n \geq N_0(R, T)\),
\[
E(S(t_n)(\psi_n, \phi_n, \theta_n)) = E(S(T)(S(t_n - T)(\psi_n, \phi_n, \theta_n)))
\]
\[
= e^{-2\varphi(T-T_0)}E(S(T_0)(S(t_n - T)(\psi_n, \phi_n, \theta_n)))
\]
\[
+ \int_{T_0}^T e^{-2\varphi(T-\tau)}F(S(\tau)(S(t_n - T)(\psi_n, \phi_n, \theta_n))) \, d\tau.
\] (4.10)
Next, we pass to the limit of (4.10). We treat each term there. We start with the first term on the right-hand side. Since $T_0 \geq T(B)$, by (2.34) and (4.4) we have $S(T_0)(S(t_n - T)(\psi_n, \phi_n, \theta_n)) \in B$ for $n \geq N_2(R, T)$. Therefore, by the definition of $E$ in (3.12) we find that

$$
e^{-2(T - T_0)}E(S(T_0)(S(t_n - T)(\psi_n, \phi_n, \theta_n))) \leq Ce^{-2(T - T_0)}, \quad \forall n \geq N_2(R, T),$$

(4.11)

where $C$ is independent of $T$. We now consider the second term on the right-hand side of (4.10). Note that, by (3.13) we have

$$|T - T_0| e^{-2(T - T_0)} \geq 2(T - T_0) \geq T - T_0,$$

(4.12)

We now handle the first two terms on the right-hand side of (4.12). By (4.5) and (3.10) we get

$$e^{-\delta(T - \tau)} S(\tau)(S(t_n - T) \psi_n) \rightarrow e^{-\delta(T - \tau)} S(\tau) \psi_T \quad \text{weakly in } L^2(T_0, T; H^1),$$

and

$$e^{-\delta(T - \tau)} S(\tau)(S(t_n - T) \theta_n) \rightarrow e^{-\delta(T - \tau)} S(\tau) \theta_T \quad \text{weakly in } L^2(T_0, T; H).$$
So we find that
\[
\liminf_{n \to \infty} \|e^{-i(T-t_n)}S(t_n - T) \psi_n\|_{L^2(T_0; H')}
\geq \|e^{-i(T-t)}S(t) \psi_T\|_{L^2(T_0; H')},
\] (4.13)
and
\[
\liminf_{n \to \infty} \|e^{-i(T-t_n)}S(t_n - T) \theta_n\|_{L^1(T_0; H')}
\geq \|e^{-i(T-t)}S(t) \theta_T\|_{L^1(T_0; H')}. \] (4.14)

Then, choosing \(\delta\) small enough such that \(\nu - \delta > 0\), and \(\gamma - 2\delta > 0\), we have the following estimates for the first two terms on the right-hand side of (4.12),
\[
\limsup_{n \to \infty} -4(\nu - \delta) \int_{T_0}^T e^{-2i(T-t_n)}(\|S(t)(S(t_n - T) \psi_n\|_2^2
+ \|\nabla S(t)(S(t_n - T) \psi_n\|_2^2)
+ \limsup_{n \to \infty} -2(\gamma - 2\delta) \int_{T_0}^T e^{-2i(T-t_n)}(\|S(t)(S(t_n - T) \theta_n\|_2^2)
\leq -2 \int_{T_0}^T e^{-2i(T-t_n)}(2(\nu - \delta)(\|S(t) \psi_T\|_2^2 + \|\nabla S(t) \psi_T\|_2^2)
+ (\gamma - 2\delta) \|S(t) \theta_T\|_2^2). \] (4.15)

By (4.5) and (3.10) we also have
\[
4 \int_{T_0}^T e^{-2i(T-t_n)} \text{Im} \int_{\mathbb{R}^n} \overline{S(t)(S(t_n - T) \psi_n)}
+ 2 \int_{T_0}^T e^{-2i(T-t_n)} \left(2(\nu - \delta) \text{Re} \int_{\mathbb{R}^n} S(t)(S(t_n - T) \psi_n) \right)
+ \left( \int_{\mathbb{R}^n} gS(t)(S(t_n - T) \theta_n) \right)
\to 2 \int_{T_0}^T e^{-2i(T-t_n)} \left(2 \text{Im} \int_{\mathbb{R}^n} S(t) \psi_T \right)
+ 2(\nu - \delta) \text{Re} \int_{\mathbb{R}^n} S(t) \psi_T \right. \right) \] (4.16)

We now consider the nonlinear term, the last term on the right-hand side of (4.12). To pass to the limit of this term, we have to use some kind of
compactness. In order to overcome the difficulty of noncompactness of Sobolev imbeddings in $\mathbb{R}^n$, we here approach the whole space $\mathbb{R}^n$ by a bounded domain. Then the compactness of Sobolev embeddings in bounded domains and the estimates in Lemma 2.4 will yield the result.

We write the nonlinear term as

$$
\int_{T_0}^T e^{-2d(T-t)} \int_{\mathbb{R}^n} S(\tau)(S(t_n - T) \phi_n \ | S(\tau)(S(t_n - T) \psi_n)|^2 \ dt \ dx
$$

$$
= \int_{T_0}^T e^{-2d(T-t)} \int_{|x| \geq m} S(\tau)(S(t_n - T) \phi_n \ | S(\tau)(S(t_n - T) \psi_n)|^2 \ dt \ dx
$$

$$
+ \int_{T_0}^T e^{-2d(T-t)} \int_{|x| \leq m} S(\tau)(S(t_n - T) \phi_n \ | S(\tau)(S(t_n - T) \psi_n)|^2 \ dt \ dx.
$$

(4.17)

We consider the first term on the right-hand side of (4.17). For given $\varepsilon > 0$, it follows from (2.35) that, for all $n \geq N_2(R, T)$ and $m \geq M(\varepsilon)$,

$$
\int_{T_0}^T e^{-2d(T-t)} \int_{|x| \geq m} S(\tau)(S(t_n - T) \phi_n \ | S(\tau)(S(t_n - T) \psi_n)|^2 \ dt \ dx
$$

$$
\leq \varepsilon \int_{T_0}^T e^{-2d(T-t)} \ dt \left( \int_{|x| \geq m} |S(\tau)(S(t_n - T) \phi_n)|^3 \ dx \right)^{1/3}
$$

$$
\times \left( \int_{|x| \geq m} |S(\tau)(S(t_n - T) \psi_n)|^6 \ dx \right)^{1/6}
$$

$$
\times \left( \int_{|x| \geq m} |S(\tau)(S(t_n - T) \psi_n)|^2 \ dx \right)^{1/2}
$$

$$
\leq \varepsilon \int_{T_0}^T e^{-2d(T-t)} \ dt \left( \int_{\mathbb{R}^n} |S(\tau)(S(t_n - T) \phi_n)|^3 \ dx \right)^{1/3}
$$

$$
\times \left( \int_{\mathbb{R}^n} |S(\tau)(S(t_n - T) \psi_n)|^6 \ dx \right)^{1/6}
$$

$$
\leq \varepsilon \int_{T_0}^T e^{-2d(T-t)} \ dt \ |S(\tau)(S(t_n - T) \phi_n)|_{\mu^1} \ |S(\tau)(S(t_n - T) \psi_n)|_{\mu^1} \ dt
$$

$$
\leq \varepsilon C \int_{T_0}^T e^{-2d(T-t)} \ dt
$$

(by (4.4) and (2.34) for $n \geq N_2(R, T)$ and $m \geq M(\varepsilon)$)

$$
= \frac{\varepsilon C}{2d} (1 - e^{-2d(T - T_0)}) \leq \frac{\varepsilon C}{2d}, \quad \forall n \geq N_2(R, T), \ m \geq M(\varepsilon). \quad (4.18)
$$
We now deal with the second term on the right-hand side of (4.17), for which we will prove as

\[
\begin{align*}
\left[ \int_{T_0}^T e^{-2\theta(T-t)} \int_{|x| \leq m} S(t)(S(t_n - T) \phi_n) |S(t)(S(t_n - T) \psi_n)|^2 \, dt \, dx \right] \\
&\rightarrow \left[ \int_{T_0}^T e^{-2\theta(T-t)} \int_{|x| \leq m} S(t) \phi_T |S(t) \psi_T|^2 \, dt \, dx. \right. \tag{4.19}
\end{align*}
\]

For every fixed \( \tau \in [T_0, T] \), by (4.5) and (3.2) we have

\[
S(t)(S(t_n - T) \psi_n, S(t_n - T) \phi_n, S(t_n - T) \theta_n) \\
\rightarrow S(t)(\psi_T, \phi_T, \theta_T) \quad \text{weakly in } H^1 \times H^1 \times H. \tag{4.20}
\]

Let \( \Omega_m = \{ x \in \mathbb{R}^n : |x| \leq m \} \). Then by the compactness of the Sobolev embedding \( H^1(\Omega_m) \subset L^2(\Omega_m) \), from (4.20) we infer that

\[
(S(t)(S(t_n - T) \psi_n), S(t)(S(t_n - T) \phi_n)) \\
\rightarrow (S(t) \psi_T, S(t) \phi_T) \quad \text{strongly in } L^2(\Omega_m) \times L^2(\Omega_m). \tag{4.21}
\]

By (4.4) and (2.34) we know

\[
S(t)(S(t_n - T) \psi_n, S(t_n - T) \phi_n, S(t_n - T) \theta_n) \in B, \\
\forall \tau \in [T_0, T], \quad n \geq N_2(R, T).
\]

Therefore, (4.19) follows from (4.21) and Lebesgue dominated convergence theorem. The details are omitted here.

By (4.17)–(4.19) we find that for \( m \geq M(\varepsilon) \),

\[
\limsup_{n \to \infty} \int_{T_0}^T e^{-2\theta(T-t)} \int_{\mathbb{R}^n} S(t)(S(t_n - T) \phi_n) |S(t)(S(t_n - T) \psi_n)|^2 \, dt \, dx \\
\leq \varepsilon C + \int_{T_0}^T e^{-2\theta(T-t)} \int_{|x| \leq m} S(t) \phi_T |S(t) \psi_T|^2 \, dt \, dx. \tag{4.22}
\]

Letting \( m \to \infty \), we obtain from (4.22) that

\[
\limsup_{n \to \infty} \int_{T_0}^T e^{-2\theta(T-t)} \int_{\mathbb{R}^n} S(t)(S(t_n - T) \phi_n) |S(t)(S(t_n - T) \psi_n)|^2 \, dt \, dx \\
\leq \varepsilon C + \int_{T_0}^T e^{-2\theta(T-t)} \int_{\mathbb{R}^n} S(t) \phi_T |S(t) \psi_T|^2 \, dt \, dx + \varepsilon C. \tag{4.23}
\]
By (4.12), (4.15), (4.16), and (4.23), we finally obtain that
\[
\limsup_{n \to \infty} \int_{T_0}^T e^{-2\delta(T-t)} F(S(t_n-T) \psi_n), (S(t_n-T) \phi_n), (S(t_n-T) \theta_n)) \, dt \\
\leq -2 \int_{T_0}^T e^{-2\delta(T-t)} (2(\psi_T^2) + (\gamma - 2\delta) (S(t) \theta_T^2) \\
+ 2 \int_{T_0}^T e^{-2\delta(T-t)} \left( 2 \Im \int_{\mathbb{R}^n} f(S(t) \psi_T + 2(2\delta - \nu) \Re \int_{\mathbb{R}^n} f(S(t) \psi_T \\
+ \int_{\mathbb{R}^n} g(S(t) \theta_T) + 2(2\nu - \delta) \int_{T_0}^T e^{-2\delta(T-t)} \int_{\mathbb{R}^n} S(t) \psi_T |S(t) \psi_T|^2 + \varepsilon C \\
= \int_{T_0}^T e^{-2\delta(T-t)} F(S(t)(\psi_T, \phi_T, \theta_T)) \, dt + \varepsilon C \quad \text{(by (3.13)).} \tag{4.24}
\]

Then taking the limit of (4.10), by (4.11) and (4.24) we get
\[
\limsup_{n \to \infty} E(S(t_n)(\psi_n, \phi_n, \theta_n)) \\
\leq C e^{-2\delta(T-T_0)} + \int_{T_0}^T e^{-2\delta(T-t)} F(S(t)(\psi_T, \phi_T, \theta_T)) \, dt + \varepsilon C. \tag{4.25}
\]

On the other hand, by (4.6) and (4.9) we also have
\[
E(\psi, \phi, \theta) = E(S(T)(\psi_T, \phi_T, \theta_T)) \\
= e^{-2\delta(T-T_0)} E(S(T_0)(\psi_T, \phi_T, \theta_T)) \\
+ \int_{T_0}^T e^{-2\delta(T-t)} F(S(t)(\psi_T, \phi_T, \theta_T)) \, dt, \tag{4.26}
\]

Hence, it follows from (4.25)–(4.26) that
\[
\limsup_{n \to \infty} E(S(t_n)(\psi_n, \phi_n, \theta_n)) \\
\leq E(\psi, \phi, \theta) + C e^{-2\delta(T-T_0)} + \varepsilon C - e^{-2\delta(T-T_0)} E(S(T_0)(\psi_T, \phi_T, \theta_T)). \tag{4.27}
\]

Since \((\psi_T, \phi_T, \theta_T) \in B \) and \( T_0 \geq T(B) \), by (2.34) we find that
\[
|e^{-2\delta(T-T_0)} E(S(T_0)(\psi_T, \phi_T, \theta_T))| \leq C e^{-2\delta(T-T_0)}.
\]
Then from (4.27) we have
\[ \limsup_{n \to \infty} E(S(t_n)(\psi_n, \phi_n, \theta_n)) \leq E(\psi, \phi, \theta) + Ce^{-2\delta(T-T_\epsilon)} + \varepsilon C. \] (4.28)

First taking the limit of (4.28) as \( T \to \infty \), and then letting \( \varepsilon \to 0 \), we obtain
\[ E(S(t_n)(\psi_n, \phi_n, \theta_n)) \leq E(\psi, \phi, \theta). \] (4.29)

In order to get an estimate on the norm of solutions in \( H^1 \times H^1 \times H \), we consider the nonlinear term in (4.29). We rewrite the nonlinear term as
\[ |R_n S(t_n)\psi_n| S(t_n)\psi_n|^2 \]
\[ = \int_{|x| \leq m} S(t_n) \phi_n |S(t_n)\psi_n|^2 + \int_{|x| \geq m} S(t_n) \phi_n |S(t_n)\psi_n|^2. \] (4.30)

By (4.3) we see that for every \( m > 0 \),
\[ (S(t_n)\psi_n, S(t_n)\phi_n) \to (\psi, \phi) \] strongly in \( L^2(\Omega_m) \times L^2(\Omega_m) \). (4.31)

We are now in a position to prove the convergence of the nonlinear term in (4.30), that is, we want to show
\[ \int_{\mathbb{R}^+} S(t_n) \phi_n |S(t_n)\psi_n|^2 dx \to \int_{\mathbb{R}^+} \phi |\psi|^2, \] as \( n \to \infty \). (4.32)

This means for every \( \varepsilon > 0 \), we want to prove there exists \( N(\varepsilon) \) such that for \( n \geq N(\varepsilon) \),
\[ \left| \int_{\mathbb{R}^+} (S(t_n) \phi_n |S(t_n)\psi_n|^2 - \phi |\psi|^2) dx \right| \leq \varepsilon. \] (4.33)

For \( \varepsilon > 0 \) given, let \( T(\varepsilon) \) and \( T(R) \) be the constant in (2.35) and (4.1), respectively. Then we find that
\[ \int_{|x| \geq M(T)} (|S(t)(S(T(R))\psi_n)|^2 + |S(t)(S(T(R))\phi_n)|^2) dx \leq \varepsilon^2, \]
\[ \forall t \geq T(\varepsilon). \] (4.34)
where $M(\varepsilon)$ is the constant in (2.35). Since $t_n \to \infty$, we know that there exists $N_3(\varepsilon)$ such that if $n \geq N_3(\varepsilon)$, then $t_n - T(R) \geq T(\varepsilon)$. Therefore, it follows from (4.34) that, for $n \geq N_3(\varepsilon)$,

$$
\int_{|x| \geq M(\varepsilon)} (|S(t_n) \psi_n|^2 + |S(t_n) \phi_n|^2) \, dx
$$

$$
= \int_{|x| \geq M(\varepsilon)} (|S(t_n - T(R))|S(T(R)) \psi_n|^2
$$

$$
+ |S(t_n - T(R))|S(T(R)) \phi_n|^2) \, dx \leq \varepsilon^2. \quad (4.35)
$$

Let $N_4(\varepsilon) = \max\{N_1(R), N_3(\varepsilon)\}$. Then by (4.2) and (4.35) we can easily prove that, for $n \geq N_4(\varepsilon)$,

$$
\int_{|x| \geq M(\varepsilon)} (S(t_n) \phi_n |S(t_n) \psi_n|^2 - \phi |\psi|^2) \, dx \leq \frac{\varepsilon}{2}. \quad (4.36)
$$

On the other hand, by the convergence (4.31), we find that there exists $N_5(\varepsilon)$ such that if $n \geq N_5(\varepsilon)$, then

$$
\int_{|x| \leq M(\varepsilon)} (|S(t_n) \psi_n - \psi|^2 + |S(t_n) \phi_n - \phi|^2) \, dx \leq \varepsilon. \quad (4.37)
$$

Set $N(\varepsilon) = \max\{N_4(\varepsilon), N_5(\varepsilon)\}$. Then from (4.2) and (4.37) we can deduce that for $n \geq N(\varepsilon)$,

$$
\int_{|x| \leq M(\varepsilon)} (S(t_n) \phi_n |S(t_n) \psi_n|^2 - \phi |\psi|^2) \, dx \leq \frac{\varepsilon}{2}. \quad (4.38)
$$

Therefore, from (4.36) and (4.38) we find (4.33) holds for $n \geq N(\varepsilon)$. Hence (4.32) is proved. Note that (4.3) implies

$$
\int_{\mathbb{R}^\ast} fS(t_n) \psi_n \to \int_{\mathbb{R}^\ast} f\psi. \quad (4.39)
$$

Taking the limit of (4.29), by (4.32) and (4.39) and the definition $E$ in (3.12), we finally get that

$$
\limsup_{n \to \infty} (2 \|S(t_n) \psi_n\|^2_{L^2} + (1 - \delta(\gamma - \delta)) \|S(t_n) \phi_n\|^2
$$

$$
+ \|\nabla S(t_n) \phi_n\|^2 + \|S(t_n) \theta_n\|^2) \leq 2 \|\psi\|^2_{L^2} + (1 - \delta(\gamma - \delta)) \|\phi\|^2 + \|\nabla \phi\|^2 + \|\theta\|^2.
Noting that the right-hand side of the above is equivalent to the norm of $H^1 \times H^1 \times H$, so, without loss of generality, we can assume that the norm of $H^1 \times H^1 \times H$ is defined by it. Then we have

$$\limsup_{n \to \infty} \| S(t_n)(\psi_n, \phi_n, \theta_n) \|_{H^1 \times H^1 \times H} \leq \| (\psi, \phi, \theta) \|_{H^1 \times H^1 \times H},$$

as desired in (4.8). Therefore, we get the strong convergence of $S(t_n)(\psi_n, \phi_n, \theta_n)$ to $(\psi, \phi, \theta)$ in $H^1 \times H^1 \times H$. The proof is complete.

We now state our main result in this section.

**Theorem 4.2.** Assume that $f$ and $g$ belong to $H$. Then, problem (2.1)-(2.4) possesses a global attractor in $H^1 \times H^1 \times H$ which is a compact invariant subset and attracts every bounded set of $H^1 \times H^1 \times H$ with respect to the norm topology.

**Proof.** The proof of this theorem is now obvious. Since we have established the existence of a bounded absorbing set in (2.32) and the asymptotic compactness for $S(t)$ in $H^1 \times H^1 \times H$ in Theorem 4.1, so Theorems 4.2 follows from Proposition 4.1.

The following result is the analogue of Theorem 4.1 in the space $H^k_{k+2} \times H^k_{k+2} \times H^k_{k+1}$.

**Theorem 4.3.** Assume that $f$ and $g$ belong to $H^k$ with $k \geq 0$. Then, the dynamical system $S(t)$ is asymptotically compact in $H^k_{k+2} \times H^k_{k+2} \times H^k_{k+1}$, that is, if $(\psi_n, \phi_n, \theta_n)$ is bounded in $H^k_{k+2} \times H^k_{k+2} \times H^k_{k+1}$ and $t_n \to \infty$, then $S(t_n)(\psi_n, \phi_n, \theta_n)$ is precompact in that space.

**Proof.** The proof of this theorem is similar to Theorem 4.1. In this case, we apply the bounded absorbing set $B_k$ in (2.33) and the energy equation in (3.56) instead of $B$ in (2.32) and the energy equation in (3.11), respectively. Since the idea is the same, we omit the details here.

The existence of the global attractor in $H^k_{k+2} \times H^k_{k+2} \times H^k_{k+1}$ is stated as follows.

**Theorem 4.4.** Assume that $f$ and $g$ belong to $H^k$ with $k \geq 0$. Then problem (2.1)-(2.4) possesses a global attractor in $H^k_{k+2} \times H^k_{k+2} \times H^k_{k+1}$ which is a compact invariant subset and attracts every bounded set of $H^k_{k+2} \times H^k_{k+2} \times H^k_{k+1}$ with respect to the norm topology.

**Proof.** This theorem follows from Theorem 4.3 and Proposition 4.1 and the existence of the bounded absorbing set $B_k$ in (2.33).
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