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Quasi-metrics and monotone normality ☆

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ABSTRACT

The purpose of this paper is to study which quasi-metrizable spaces are monotonically normal. In particular, we provide a sufficient condition for a quasi-metrizable space to be monotonically normal. This enables us to prove the monotone normality of a certain amount of interesting examples of quasi-metric spaces; for instance, we show that the continuous poset of formal balls of a metric space, endowed with the Scott topology, is a monotonically normal quasi-metrizable space.

1. Introduction

The notion of monotone normality was introduced in 1973 by Heath, Lutzer and Zenor [12] as a strengthening of normality. From that pioneering paper there has been an extensive literature on the topic; good sources of references are [1,2,9].

It deserves to be mentioned here that, with the exception of [15], and, more recently, [18] and [11] (see also [10]) monotonically normal spaces have always been considered to be T_1 spaces. This has several consequences, for example, in contrast with normality, monotone normality (when the T_1 axiom is a part of the definition) becomes a hereditary property. However, as it has already been proved in [11], this is not the case when the T_1 axiom is not considered to be a part of the definition.

On the other hand, because of the influence of theoretical computer science and its connections with domain theory, those spaces not satisfying T_1 axiom are playing a more important role, in particular, quasi-metric spaces. In fact, both in theoretical computer science and information theory, it is usual to work with sequences $(x_n)_n$ of objects of increasing information where the relation "$x_n \leqslant x_{n+1}$" is interpreted as that the object x_{n+1} contains all the information provided by the element x_n; actually $\leqslant$ is at least a partial order. These processes have a suitable topological model in the framework of non-T_1 quasi-metric spaces, where the partial order $\leqslant$ coincides with the specialization order of the associated quasi-metric space.

Metrizable spaces are not only normal but even monotonically normal and, of course, satisfy the T_1 axiom. However, it is not so easy to establish whether a quasi-metrizable space is normal or not. It is well known that not all quasi-metrizable spaces are normal, a typical example being the Sorgenfrey plane. It is natural to think then about the question

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of which quasi-metrizable spaces are normal, or perhaps monotonically normal. In this sense it could be mentioned, citing from [7], that whenever a space can be *explicitly* and *constructively* shown to be normal, then it is probably monotonically normal.

Kelly began in [14] a systematized study of bitopological spaces and, in particular, of the problem of quasi-metrization of bitopological spaces. Although many authors have investigated this problem, a quasi-metric analogue to the celebrated Nagata–Smirnov metrization theorem has still not been obtained, which is due in great part to the difficulty in obtaining a satisfactory notion of bitopological paracompactness (see [17, Chapter 10] for an excellent survey of the more important results on this topic). A nice solution to the bitopological quasi-metrization problem was obtained by Fox [6] (see [17, pp. 909–910]) as follows: A bitopological space (X, τ_1, τ_2) is quasi-metrizable if and only if it is pairwise stratifiable and τ_1 and τ_2 are quasi-metrizable topologies. Putting $\tau_1 = \tau_2$ in Fox's theorem it follows that a topological space is metrizable if and only if it is stratifiable and quasi-metrizable. This result cannot be generalized to monotonically normal spaces (recall that every stratifiable space is monotonically normal) because the Sorgenfrey line provides a paradigmatic example of a monotonically normal quasi-metrizable space which is not metrizable.

In the paper [11] the authors have extended a characterization of monotonically normal spaces in [12] to the non-T_1 setting (see [11, Proposition 3.2]). In this note we use that characterization to provide a sufficient condition for a quasi-metrizable space to be monotonically normal. Since every quasi-metric space which is a meet semilattice for its specialization order satisfies this condition, and many of the most important examples of quasi-metric spaces appearing in theoretical computer science are meet semilattices for the specialization order, then the property of monotone normality of these spaces will be a direct and natural consequence of the order-theoretic properties involved in the construction of such examples as we show in the last section of this note.

2. Preliminaries

Here we gather together some basic notions regarding quasi-metrics and monotonically normal spaces.

Definition 2.1. Let X be a non-empty set. A map $d: X \times X \to [0, +\infty)$ is a *quasi-metric* if the following two conditions hold for all $x, y, z \in X$:

(QM1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
(QM2) $d(x, y) \leqslant d(x, z) + d(z, y)$.

If d is a quasi-metric on X, then the map $d^{-1}: X \times X \to [0, +\infty)$ such that $d^{-1}(x, y) = d(y, x)$ is also a quasi-metric on X called *the conjugate of d*. A *quasi-metric space* is a pair (X, d) such that X is a non-empty set and d is a quasi-metric on X.

It is well known that every quasi-metric d on X generates a T_0 topology τ_d which has as a base the family of d-balls $\{B_d(x, \varepsilon)\colon\ x \in X,\ \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X\colon\ d(x, y) < \varepsilon\}$.

A topological space (X, τ) is said to be *quasi-metrizable* if there exists a quasi-metric d on X such that $\tau = \tau_d$. Let (X, d) be a quasi-metric space. Given $x \in X$ and $A \subseteq X$, we define $d(x, A) = \inf_{y \in A} d(x, y)$, as usual. Recall also that $d(x, \overline{\{y\}}) = d(x, y)$ for all $x, y \in X$.

Every quasi-metric d on a set X induces, in a natural way, a partial order $\leqslant_d$ on X (called the *specialization order*) defined by

$$y \leqslant_d x \quad \Longleftrightarrow \quad d(y, x) = 0 \quad \Longleftrightarrow \quad y \in \overline{\{x\}}.$$

For each $x \in X$ we shall also let $\downarrow_d x = \{y \in X\colon\ y \leqslant_d x\} = \overline{\{x\}}$.

We recall now the definition of monotonically normal spaces of [12] (see also [15] for the formulation presented below).

Definition 2.2. A topological space X is called *monotonically normal* if there exists a function G (a monotone normality operator) which assigns to each ordered pair (A, U) of subsets of X, with A closed, U open and $A \subseteq U$, an open set $G(A, U)$ such that

(1) $A \subseteq G(A, U) \subseteq \overline{G(A, U)} \subseteq U$;
(2) if B is closed, V open, $B \subseteq V$ and $A \subseteq B$ and $U \subseteq V$, then $G(A, U) \subseteq G(B, V)$.

Note that here, as in [15,18,11], we do not assume the T_1 axiom to be a part of the definition of monotone normality.

We recall the following result from [11] (which extends Lemma 2.2 in [12]):

Proposition 2.3. *([11, Proposition 3.2]) Let X be a topological space. The following are equivalent*:

(1) *X is monotonically normal*;
(2) *For each point x and open set U containing* $\overline{\{x\}}$ *we can assign an open set* $H(x, U)$ *such that*:

(H1) $\overline{\{x\}} \subseteq H(x,U) \subseteq U$;
(H2) *if* V *is open and* $\overline{\{x\}} \subseteq U \subseteq V$, *then* $H(x,U) \subseteq H(x,V)$;
(H3) *if* $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$, *then* $H(x, X \setminus \overline{\{y\}}) \cap H(y, X \setminus \overline{\{x\}}) = \varnothing$.

Note that the assignment in the previous proposition can be improved in the following way:

Corollary 2.4. *Let X be a topological space. The following are equivalent*:

(1) *X is monotonically normal*;
(2) *For each point x and open set U containing* $\overline{\{x\}}$ *we can assign an open set* $H^*(x,U)$ *such that*:
  (H1) $\overline{\{x\}} \subseteq H^*(x,U) \subseteq U$;
  (H2*) (i) *if* V *is open and* $\overline{\{x\}} \subseteq U \subseteq V$, *then* $H^*(x,U) \subseteq H^*(x,V)$ *and*
   (ii) *if* $x \in \overline{\{y\}} \subseteq U$, *then* $H^*(x,U) \subseteq H^*(y,U)$;
  (H3) *if* $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$, *then* $H^*(x, X \setminus \overline{\{y\}}) \cap H^*(y, X \setminus \overline{\{x\}}) = \varnothing$.

Proof. This follows immediately from Proposition 2.3 by considering $H^*(x,U) = \bigcup_{x' \in \overline{\{x\}}} H(x',U)$. □

When the space is T_1 such an assignment is sometimes called a *Borges monotone normality operator* or simply a *Borges operator*.

3. Monotonically normal quasi-metrizable spaces

In what follows we will try to provide a sufficient condition for a quasi-metric space to be monotonically normal. We would like to emphasize at this point that Proposition 2.3 is the key result in order to obtain this condition. We start by proving the following:

Theorem 3.1. *Let* (X,d) *be a quasi-metric space satisfying*

$$\overline{\{x\}} \cap \overline{\{y\}} = \varnothing \quad \Longrightarrow \quad B_d\left(x', \frac{d(x',y)}{2}\right) \cap B_d\left(y', \frac{d(y',x)}{2}\right) = \varnothing \quad \forall x' \in \overline{\{x\}},\ y' \in \overline{\{y\}}. \tag{$*$}$$

Then (X, τ_d) *is monotonically normal.*

Proof. By Proposition 2.3, we only have to find, for each point x and open set U containing $\overline{\{x\}}$, an open set $H(x,U)$ such that conditions (H1), (H2) and (H3) are satisfied.

Let $x \in X$, U be open such that $\overline{\{x\}} \subseteq U$ and define

$$H(x,U) = \bigcup_{x' \in \overline{\{x\}}} B_d\left(x', \frac{d(x', X \setminus U)}{2}\right).$$

First of all, notice that $d(x', X \setminus U) > 0$ for all $x' \in \overline{\{x\}}$ and consequently $H(x,U)$ is well defined. Indeed, since U is open and $x' \in \overline{\{x\}} \subseteq U$, it follows that there exists $\varepsilon > 0$ such that $x' \in B_d(x', \varepsilon) \subseteq U$. Then, for each $y \in X \setminus U$, we have that $d(x',y) \geqslant \varepsilon$ and consequently $d(x', X \setminus U) \geqslant \varepsilon > 0$.

(H1): Clearly enough $\overline{\{x\}} \subseteq H(x,U)$. On the other hand, given $x' \in \overline{\{x\}}$ and $y \in X \setminus U$, we have that $d(x',y) \geqslant d(x', X \setminus U) > \frac{d(x', X\setminus U)}{2}$ and so $y \notin B_d(x', \frac{d(x', X\setminus U)}{2})$.

(H2): This is easy since, whenever U and V are two open sets such that $\overline{\{x\}} \subseteq U \subseteq V$, we have $d(x', X \setminus U) \leqslant d(x', X \setminus V)$ for each $x' \in \overline{\{x\}}$ and so $H(x,U) \subseteq H(x,V)$.

(H3): Assume $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$. Since $d(x', \overline{\{y\}}) = d(x',y)$, it follows that

$$H\big(x, X \setminus \overline{\{y\}}\big) = \bigcup_{x' \in \overline{\{x\}}} B_d\left(x', \frac{d(x',y)}{2}\right).$$

Consequently, $H(x, X \setminus \overline{\{y\}}) \cap H(y, X \setminus \overline{\{x\}}) = \varnothing$ follows immediately from condition $(*)$. □

Remark 3.2. If the quasi-metric considered in the previous proposition satisfies the symmetry condition, i.e., if it is indeed a metric, the condition $(*)$ above is obviously satisfied. In fact, this is precisely the Hausdorff condition. In this case the previous proposition is nothing but the well-known fact that metrizable spaces are monotonically normal.

It is also interesting to pay particular attention to the case when the space is T_1. Recall that the quasi-metric space (X,d) is T_1 if and only if the following condition is satisfied:

$$d(x,y) = 0 \quad \Longrightarrow \quad x = y \tag{T_1}$$

Recall that any quasi-metrizable space is T_0 and so (X,d) is T_1 if and only if $\overline{\{x\}} \subseteq U$ for each open U and each $x \in U$.

In this case we have the following characterization:

Proposition 3.3. *Let (X, d) be a T_1 quasi-metric space. The following are equivalent*:

(1) (X, τ_d) *is monotonically normal*;
(2) *There exists a map* $h : X \times (0, +\infty) \to (0, +\infty)$ *such that*:
  (h1) $0 < h(x, \varepsilon) \leqslant \varepsilon$;
  (h2) *if* $\varepsilon_1 < \varepsilon_2$, *then* $h(x, \varepsilon_1) \leqslant h(x, \varepsilon_2)$;
  (h3) *if* $x \neq y$, *then* $B_d(x, h(x, d(x, y))) \cap B_d(y, h(y, d(y, x))) = \varnothing$.

Proof. (1) $\Longrightarrow$ (2): Let H be a Borges monotone normality operator, i.e. a map assigning for each point x and open set U containing x an open set $H(x, U)$ such that:

(H1) $x \in H(x, U) \subseteq U$;
(H2) if V is open and $x \in U \subseteq V$, then $H(x, U) \subseteq H(x, V)$;
(H3) if $x \neq y$, then $H(x, X \setminus \{y\}) \cap H(y, X \setminus \{x\}) = \varnothing$.

For each $x \in X$ and $\varepsilon > 0$, we define

$$h(x, \varepsilon) = \bigvee \bigl\{\delta \in (0, \varepsilon):\ B_d(x, \delta) \subseteq H\bigl(x, B_d(x, \varepsilon)\bigr)\bigr\}.$$

(h1): Since $H(x, B_d(x, \varepsilon))$ is open, there exists some $\delta > 0$ such that $B_d(x, \delta) \subseteq H(x, B_d(x, \varepsilon))$. Hence $0 < h(x, \varepsilon) \leqslant \varepsilon$.

(h2): If $\varepsilon_1 < \varepsilon_2$ then $B_d(x, \varepsilon_1) \subseteq B_d(x, \varepsilon_2)$ and so $H(x, B_d(x, \varepsilon_1)) \subseteq H(x, B_d(x, \varepsilon_2))$. We conclude $h(x, \varepsilon_1) \leqslant h(x, \varepsilon_2)$.

(h3): First we notice that $B_d(x, h(x, \varepsilon)) \subseteq H(x, B_d(x, \varepsilon))$ for each $x \in X$ and $\varepsilon > 0$, indeed, if $d(x, y) < h(x, \varepsilon)$ then there exists $0 < \delta < \varepsilon$ such that $d(x, y) < \delta$ and $B_d(x, \delta) \subseteq H(x, B_d(x, \varepsilon))$. Hence $y \in H(x, B_d(x, \varepsilon))$.

Now let $x \neq y$. Since (X, d) is T_1, we have $d(x, y) > 0$ and so

$$B_d\bigl(x, h\bigl(x, d(x, y)\bigr)\bigr) \subseteq H\bigl(x, B_d\bigl(x, d(x, y)\bigr)\bigr) \subseteq H\bigl(x, X \setminus \{y\}\bigr).$$

Similarly $B_d(y, h(y, d(y, x))) \subseteq H(y, B_d(y, d(y, x))) \subseteq H(y, X \setminus \{x\})$. Hence

$$B_d\bigl(x, h\bigl(x, d(x, y)\bigr)\bigr) \cap B_d\bigl(y, h\bigl(y, d(y, x)\bigr)\bigr) \subseteq H\bigl(x, X \setminus \{y\}\bigr) \cap H\bigl(y, X \setminus \{x\}\bigr) = \varnothing.$$

(2) $\Longrightarrow$ (1): Let $x \in X$ and U be an open set containing x. Then $d(x, X \setminus U) > 0$ and hence we can define

$$H(x, U) = B_d\bigl(x, h\bigl(x, d(x, X \setminus U)\bigr)\bigr).$$

Then we have:

(H1): $x \in H(x, U) = B_d(x, h(x, d(x, X \setminus U))) \subseteq U$, since for all $y \in X \setminus U$ we have $d(x, y) \geqslant d(x, X \setminus U) \geqslant h(x, d(x, X \setminus U))$.

(H2): Let $x \in U \subseteq V$. Then $d(x, X \setminus U) \leqslant d(x, X \setminus V)$ and so $h(x, d(x, X \setminus U)) \leqslant h(x, d(x, X \setminus V))$. It follows that $H(x, U) \subseteq H(x, V)$.

(H3): If $x \neq y$, then $H(x, X \setminus \{y\}) = B_d(x, h(x, d(x, y)))$ and $H(y, X \setminus \{x\}) = B_d(y, h(y, d(y, x)))$. We conclude that

$$H\bigl(x, X \setminus \{y\}\bigr) \cap H\bigl(y, X \setminus \{x\}\bigr) = B_d\bigl(x, h\bigl(x, d(x, y)\bigr)\bigr) \cap B_d\bigl(y, h\bigl(y, d(y, x)\bigr)\bigr) = \varnothing. \qquad \Box$$

Corollary 3.4. *Let (X, d) be a T_1 quasi-metric space satisfying*

$$x \neq y \quad \Longrightarrow \quad B_d\bigl(x, k \cdot d(x, y)\bigr) \cap B_d\bigl(y, k \cdot d(y, x)\bigr) = \varnothing \qquad (*_{T_1})$$

for some $k \in (0, 1]$. Then (X, τ_d) is monotonically normal.

Remark 3.5. (1) In particular, if (X, d) is a metric space, the condition $(*_{T_1})$ above is obviously satisfied with $k = \frac{1}{2}$. Once again we obtain the well-known fact that metrizable spaces are monotonically normal.

(2) Notice that, under the T_1 assumption, condition $(*_{T_1})$ implies that the space (X, τ_d) is Hausdorff.[1] However, the converse is not true in general, i.e., there exist quasi-metrizable Hausdorff spaces which do not satisfy condition $(*_{T_1})$. For example the Sorgenfrey plane is quasi-metrizable and Hausdorff, but fails to be normal, and consequently doesn't satisfy condition $(*_{T_1})$.

It is also interesting to emphasize here that condition $(*_{T_1})$ is hereditary. In fact, given a quasi-metric space (X, d) satisfying $(*_{T_1})$ and a subset $A \subseteq X$, the quasi-metric space (A, d_A) clearly satisfy condition $(*_{T_1})$.

This fact shouldn't be a surprise taking into account that monotone normality (when the T_1 axiom is considered to be a part of its definition) is an hereditary property.

However, as pointed out in [11], monotone normality is not hereditary in general. Analogously, condition $(*)$ is not hereditary in general, as the following example shows:

[1] In fact this cannot be a surprise since normality plus T_1 axiom implies Hausdorff.

Example 3.6. Let (X,d) be a quasi-metric space with $d \leqslant 1$, which doesn't satisfy condition $(*)$. (There are many such spaces, for example the Sorgenfrey plane. In fact, any non-normal quasi-metrizable space can be used here.) Let $x^* \notin X$, $X^* = X \cup \{x^*\}$ and $d^*: X^* \times X^* \to [0,+\infty)$ defined by

$$d^*(x,y) = \begin{cases} d(x,y), & \text{if } x, y \in X; \\ 0, & \text{if } x = x^*; \\ 1, & \text{if } x \in X \text{ and } y = x^*. \end{cases}$$

Then the quasi-metric space (X^*, d^*) trivially satisfies the condition $(*)$ since $x^* \in \overline{\{x\}}$ for all $x \in X$. However, the restriction of the quasi-metric d^* to X coincides with d.

4. Examples

We have already pointed out that both Theorem 3.1 and Corollary 3.4 generalize the fact that metrizable spaces are monotonically normal. Now, we can use these results to prove the monotone normality of some well-known quasi-metrizable spaces.

(1) The Sorgenfrey line. Let $d: \mathbb{R} \times \mathbb{R} \to [0,+\infty)$ be defined as

$$d(x,y) = \begin{cases} \min\{y-x, 1\}, & \text{if } x \leqslant y, \\ 1, & \text{if } x > y. \end{cases}$$

Then, if $x > y$ (similarly if $y > x$) one has

$$B_d\big(x, d(x,y)\big) \cap B_d\big(y, d(y,x)\big) = [x, x+1) \cap \big[y, \min\{x, y+1\}\big) = \varnothing.$$

It follows from Corollary 3.4, with $k = 1$, the well-known fact that the Sorgenfrey line is monotonically normal.

(2) Let (X,d) be a quasi-metric space, with $d \leqslant 1$, and $M \subseteq X$. Let $d_M: X \times X \to [0,+\infty)$ be defined as $d_M(x,y) = d(x,y)$ if $x \in M$, $d_M(x,y) = 0$ if $x = y \notin M$ and $d_M(x,y) = 1$. It is easy to check that d_M is a quasi-metric on X and the d_M-balls (for $0 < \varepsilon \leqslant 1$) are the following:

$$B_{d_M}(x,\varepsilon) = B_d(x,\varepsilon) \quad \text{if } x \in M \quad \text{and} \quad B_{d_M}(x,\varepsilon) = \{x\} \quad \text{if } x \notin M.$$

If (X,d) satisfies condition $(*)$ then so does (X, d_M). Indeed, assume that (X,d) satisfies $(*)$ and $\overline{\{x\}}^M \cap \overline{\{y\}}^M \neq \varnothing$. Note that $\overline{\{x\}}^M \subseteq \overline{\{x\}}^X$ for all $x \in X$ and so $\overline{\{x\}}^X \cap \overline{\{y\}}^X \neq \varnothing$. Also, if $x \in M$, we have that $B_{d_M}(x, \frac{d_M(x,y)}{2}) = B_d(x, \frac{d(x,y)}{2})$ and, if $x \notin M$, then $B_{d_M}(x, \frac{d_M(x,y)}{2}) \subseteq \{x\}$. Hence

$$B_{d_M}\left(x, \frac{d_M(x,y)}{2}\right) \cap B_{d_M}\left(y, \frac{d_M(y,x)}{2}\right) \subseteq B_d\left(x, \frac{d(x,y)}{2}\right) \cap B_d\left(y, \frac{d(y,x)}{2}\right) = \varnothing.$$

As a particular case of this situation, consider $X = \mathbb{R}$, e the Euclidean metric on $\mathbb{R}$ and $M = \mathbb{Q}$. Then $(\mathbb{R}, \tau_{e_{\mathbb{Q}}})$ is the Michael line. Since obviously $(\mathbb{R}, e)$ satisfies $(*_{T_1})$, we conclude that the Michael line also satisfies $(*_{T_1})$ and hence is monotonically normal.

Another example could be obtained by considering (X,d) as the Sorgenfrey line and $M = \mathbb{Q}$. A basis of the resulting space is $\{[x,y)\colon x \in \mathbb{Q},\ x < y\} \cup \{\{x\}\colon x \notin \mathbb{Q}\}$. By the construction above it follows that this space satisfies $(*_{T_1})$ and hence is monotonically normal.

(3) Any quasi-metrizable space (X,τ) such that $\overline{\{x\}} \cap \overline{\{y\}} \neq \varnothing$ for all $x, y \in X$ (or, equivalently, $\downarrow_d x \cap \downarrow_d y \neq \varnothing$, for any compatible quasi-metric d) satisfies trivially condition $(*)$ and consequently is monotonically normal. In particular, the reals endowed with the right-order topology (Kolmogorov line) satisfy this condition.

(4) In [3], Edalat and Heckmann constructed a computational model for metric spaces based on the notion of a formal ball. (Terms and concepts from domain theory that are in the sequel may be found in [8].)

Given a metric space (X,d), the set of (closed) formal balls is given by $\mathbf{B}X := X \times [0,+\infty)$. Then $(\mathbf{B}X, \sqsubseteq)$ is a continuous partially ordered set, where $\sqsubseteq$ is the (partial) order given by

$$(x,r) \sqsubseteq (y,s) \quad \Longleftrightarrow \quad d(x,y) \leqslant r - s \quad \text{for all } (x,r), (y,s) \in \mathbf{B}X.$$

Maximal elements of $(\mathbf{B}X, \sqsubseteq)$ consist of all $(x,0)$, with $x \in X$. Furthermore, (X,d) is complete if and only if $(\mathbf{B}X, \sqsubseteq)$ is a continuous dcpo.

Later on, Heckmann [13] essentially proved, among other things, that the map $P: \mathbf{B}X \times \mathbf{B}X \to [0,+\infty)$ given by

$$P\big((x,r),(y,s)\big) = \max\big\{d(x,y), |r-s|\big\} + r + s,$$

is a partial metric on $\mathbf{B}X$ in the sense of [19], such that the topology generated by P coincides with the Scott topology on $(\mathbf{B}X, \sqsubseteq)$.

Since every partial metric p on a set X induces a quasi-metric q_p on X given by $q_p(x, y) = p(x, y) - p(x, x)$, and such that the topologies generated by p and q_p coincide [19], we deduce that the map $q_P : \mathbf{B}X \times \mathbf{B}X \to [0, +\infty)$ given by

$$q_P\big((x, r), (y, s)\big) = \max\big\{d(x, y), |r - s|\big\} + s - r,$$

is a quasi-metric on $\mathbf{B}X$ such that the topology τ_{q_P} generated by q_P coincides with the Scott topology on $\mathbf{B}X$. We shall show that the quasi-metric space $(\mathbf{B}X, q_P)$ is monotonically normal. Indeed, let $(x, r), (y, s) \in \mathbf{B}X$ and $t = d(x, y) + r + s$. Then $q_P((x, t), (x, r)) = q_P((x, t), (y, s)) = 0$ and so $(x, t) \in \overline{\{(x, r)\}} \cap \overline{\{(y, s)\}}$.

(5) Let d be a quasi-metric on X such that there is some $x_0 \in X$ with $x_0 \leqslant_d x$ for all $x \in X$. Then $x_0 \in \overline{\{x\}}$ for all $x \in X$. We conclude that $\overline{\{x\}} \cap \overline{\{y\}} \neq \varnothing$ for all $x, y \in X$.

Notice that the quasi-metric space (X^*, d^*) constructed in Example 3.6 is a particular case of this situation since $x^* \in \overline{\{x\}}$ for all $x \in X^*$.

Note also that several typical instances of quasi-metric spaces that appear in theoretical computer science are constructed from a partially ordered set $(X, \leqslant)$ having a bottom element $\bot$, in such a way that $\leqslant$ is exactly the specialization order of the corresponding quasi-metric. The following three paradigmatic examples illustrate this fact. It is also interesting to emphasize (cf. [23]) that the quasi-metric spaces of such examples are meet semilattices for the specialization order (recall that a meet semilattice is a partially ordered set $(X, \leqslant)$ such that for each $x, y \in X$, $x \wedge y$ exists). Hence they satisfy the condition of Example (3) above and, consequently, are monotonically normal quasi-metric spaces.

(6) The domain of words Σ^∞ ([16,19,21,23,24], etc.) consists of all finite and infinite sequences ("words") over an alphabet (a non-empty set) Σ, ordered by the prefix order $\sqsubseteq$ on Σ^∞, i.e. $x \sqsubseteq y \iff x$ is a prefix of y, where the empty sequence ϕ is an element of Σ^∞. Now for each $x, y \in \Sigma^\infty$ we define $x \sqcap y$ as the longest common prefix of x and y, and for each $x \in \Sigma^\infty$ we denote by $\ell(x)$ the length of x. Thus $\ell(x) \in [1, \omega]$ whenever $x \neq \phi$, and $\ell(\phi) = 0$. Then, the function $d : \Sigma^\infty \times \Sigma^\infty \to [0, +\infty)$ defined as

$$d(x, y) = 2^{-\ell(x \sqcap y)} - 2^{-\ell(x)}$$

is a quasi-metric on Σ^∞ whose specialization order coincides with $\sqsubseteq$. Since $\phi \leqslant_d x$ for all $x \in \Sigma^\infty$, it follows that (Σ^∞, d) is a monotonically normal quasi-metric space.

(7) The interval domain $I([0, 1])$ [4,5,19] consists of the non-empty compact intervals of $[0, 1]$ ordered by reverse inclusion. Then, the function $d : I([0, 1]) \times I([0, 1]) \to [0, +\infty)$ defined as

$$d\big([a, b], [c, d]\big) = (b \vee d) - (a \wedge c) - (b - a),$$

is a quasi-metric on $I([0, 1])$ whose specialization order coincides with the reverse inclusion order (compare [19,23,21], etc.). Since $[0, 1] \leqslant_d [a, b]$ for all $[a, b] \in I([0, 1])$, it follows that $(I([0, 1]), d)$ is a monotonically normal quasi-metric space.

(8) The complexity (quasi-metric) space $(\mathcal{C}, d_{\mathcal{C}})$ [22,20,21,23] consists of the set

$$\mathcal{C} = \left\{ f \in (0, +\infty]^\omega \colon \sum_{n=0}^{\infty} \frac{1}{2^n f(n)} < +\infty \right\},$$

endowed with the quasi-metric $d_{\mathcal{C}}$ given by

$$d_{\mathcal{C}}(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\left(\frac{1}{g(n)} - \frac{1}{f(n)} \right) \vee 0 \right).$$

If we denote by $\leqslant_p$ the usual pointwise order on $\mathcal{C}$, then $f \leqslant_p g \iff d_{\mathcal{C}}(f, g) = 0$, so the specialization order of $d_{\mathcal{C}}$ coincides with $\leqslant_p$. Since for each $f, g \in C$, we have that $\min\{f, g\} \in C$, it follows that $(C, \leqslant d_C)$ is a meet semilattice and thus (C, d_C) is monotonically normal. Finally, let f_∞ be the element of $\mathcal{C}$ defined by $f_\infty(n) = +\infty$ for all $n \in \omega$. Since $d_{\mathcal{C}}(f, f_\infty) = 0$ for all $f \in \mathcal{C}$, it follows that $(\mathcal{C}, (d_{\mathcal{C}})^{-1})$ is also a monotonically normal quasi-metric space.

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